# Swap distance minimization beyond entropy minimization in word order variation

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**Abstract.** Here we consider the problem of all the possible orders of a linguistic structure formed by n elements, for instance, subject, direct object and verb (n=3) or subject, direct object, indirect object and verb (n=4). We investigate if the frequency of the n! possible orders is constrained by two principles. First, entropy minimization, a principle that has been suggested to shape natural communication systems at distinct levels of organization. Second, swap distance minimization, namely a preference for word orders that require fewer swaps of adjacent elements to be produced from a source order. Here we present average swap distance, a novel score for research on swap distance minimization, and investigate the theoretical distribution of that score for any n: its minimum and maximum values and its expected value in die-rolling experiments or when the word order frequencies are shuffled. We investigate whether entropy and average swap distance are significantly small in distinct linguistic structures with n=3or n=4 in agreement with the corresponding minimization principles. We find strong evidence of entropy minimization and swap distance minimization with respect to a die rolling experiment. The evidence of these two forces with respect to a Polya urn process is strong for n=4 but weaker for n=3. We still find evidence of swap distance minimization when word order frequencies are shuffled, indicating that swap distance minimization effects are beyond pressure to minimize word order entropy.

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#### 1. Introduction

Languages employ syntactic structures to communicate. Some example are the SOV structure, formed by a subject (S), an object (O) and a verb (V) [1, 2] and the nAND structure, a noun phrase consisting of a noun (n), an adjective (A), a numeral (N) and a demonstrative (D) [3, 4]. DNAn is the typical order of that noun phrase in English, as in "These three black horses". A structure formed by n elements has n! possible orders (n = 3 for SOV and n = 4 for nAND). Here we investigate the constraints that operate on the n! permutations of a syntactic structure as a statistical mechanics problem. First we review the principles that may constrain word order variation in that setting.

Entropy minimization has been argued to shape natural communication systems [5, 6]. The minimization of the entropy of words puts pressure to reduce the effective vocabulary size [5] and is one of the main ingredients of models that shed light on the origins of Zipf's law for word frequencies [7], the tendency of more frequent words to be older in a language [8], and vocabulary learning in children [9, 10]. The principle of word entropy minimization is justified by word frequency effects, namely the higher mental accessibility of more frequent words ([11, 12] and references therein). Such accessibility is maximized when only one word can be produced as then the frequency of the only word is maximum, that is, when entropy reaches its minimum value. When all words have the same probability (which is a small number if the number of word orders is sufficiently large), entropy yields its maximum value.

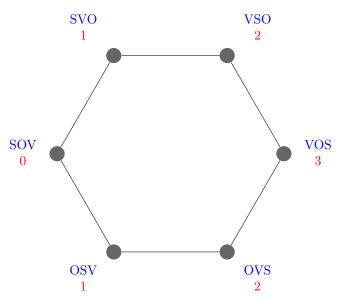
The principle of word entropy minimization can be extended to blocks of elements that form syntactic structures. We define  $p_i$  as the probability of the *i*-th order of a structure of n elements. Then, H, the entropy of the possible orders of a structure is defined as

$$H = -\sum_{i=1}^{n!} p_i \log p_i.$$

Block entropy minimization is justified in terms of frequency effects on blocks of words: word combinations that appear more often are easier to learn and to process [13]. This block entropy minimization predicts that using only one of the permutations is optimal. When applied to the SOV structure, it predicts that only one order, say SVO, has non-zero probability. This prediction matches to some degree the linguistic notion of canonical or basic order, namely the usual order in a language under certain conditions [14] or the suspected subjacent word order in case the usual order is not manifested [15]. Here we aim to test if the order of various kinds of syntactic structures is shaped by block entropy minimization.

The principle of word entropy minimization has been extended to the linear order of words in the sentence by the notion of conditional entropy [6]. In simple terms, a word whose processing cost needs to be reduced should be preceded by words that reduce its uncertainty. This principle of conditional entropy minimization has been used to predict the optimal placement of head words with respect to their dependents. An example is the nAND structure. The principle of conditional entropy minimization predicts that

**Figure 1.** The word order permutation ring for the structure SOV, that is an instantiation of the permutohedron of order 3. The number below each order indicates the swap distance to SOV.



the head should be placed first or last and this has been confirmed experimentally in conditions where that principle is less likely to suffer from interference from other word order principles [16, 17].

The principle of word entropy minimization predicts a tendency to reduce the effective size of the set of orders manifested but it is neutral concerning how orders other than the most frequent word order are going to be selected. Here we will confront entropy minimization over all permutations of a word order sequence against another word order principle that also predicts a preference for a canonical word order: swap distance minimization [18]. That principle states that given a certain word order (e.g. the canonical word order), variations that involve a smaller number of swaps of adjacent elements are cognitively easier [18]. Consider the case of the order of S, O and V. SOV requires just one swap of adjacent constituents to produce SVO (the swap of OV) or to produce OSV (the swap of SO), two swaps of adjacent constituents to become VSO or OVS and three swaps of adjacent constituents to produce VOS (Figure 1). It has been shown that the cognitive cost of processing an order of S, O and V is significantly correlated with its swap distance to the canonical order in languages whose canonical order is SOV [18]. The graph in Figure 1 is an instance of the permutohedron, a graph where vertices are the orders of a syntactic structure and an edge joining two vertices indicates that one vertex produces the other vertex by swapping a couple of adjacent word orders.

Here we introduce a new score for measuring the effect of swap distance minimization when the source order is unknown, unclear, or there may be multiple source orders from which word order variation is produced. This novel score, that we call average swap distance, is defined as

$$\langle d \rangle = \sum_{i=1}^{n!} \sum_{j=1}^{n!} d_{ij} p_i p_j \tag{1}$$

where  $d_{ij}$  is the swap distance between permutations i and j.

The remainder of this article is organized as follows. Section 2 presents the mathematical background and useful mathematical results for research on word order variation in syntactic structures. Section 2.1 presents the permutohedron of order n and its mathematical properties as a graph. Section 2.2 presents  $\langle d \rangle$  in further detail. Section 2.3 analyzes the mathematical properties of  $\langle d \rangle$ , focusing on its range of variation (lower and upper bounds). In particular,

$$0 \le \langle d \rangle \le \frac{n(n-1)}{4}$$

at least for  $n \leq 7$ .  $\langle d \rangle$  shares some mathematical properties with H: it is minimized when only one order is used and it is maximized when all words are equally likely. Section 2.4 presents the expectation of H and  $\langle d \rangle$  under a distinct null hypotheses. The expectation of  $\langle d \rangle$  under the null hypothesis of die rolling (dr), namely rolling a fair die to produce the orders is

$$\langle d \rangle_{dr} = \frac{F-1}{F} \frac{n(n-1)}{4},$$

where F is the total frequency of the word orders. The expected value of  $\langle d \rangle$  under the null hypothesis of a random permutation (rp), namely shuffling the actual word order frequencies, is

$$\langle d \rangle_{rp} = \bar{S} \frac{n!}{n! - 1} \frac{n(n-1)}{4},$$

where  $\bar{S} = 1 - S$  is the so-called dominant index and S is the Simpson index, that is defined as [19]

$$S = \sum_{i=1}^{n!} p_i^2.$$

In that setting, entropy remains constant. The random permutation null hypothesis aims to test for the presence of swap distance minimization beyond block entropy minimization taking into account that the frequency of a word order is likely to be determined by processing, learning and evolutionary constraints that are independent of swap distance minimization [20, 6, 21]. If we preserve the frequency of the word orders, namely we preserve the magnitude of those constraints as reflected in the distribution of word order frequencies, can we still observe a preference for swap distance minimization? Section 3 presents the syntactic structures and the datasets that will be employed to investigate the effect of entropy minimization and swap distance minimization in syntactic structures with n=3 or n=4 with H and  $\langle d \rangle$ . Section 5 tests for the manifestation of entropy minimization and swap distance minimization by means of the null hypotheses above and a Polya urn. This section shows statistically significant

effects of block entropy minimization and swap distance minimization and reveals swap distance minimization effects even when the entropy of orders remains constant, namely swap distance minimization captures constraints on word order that escape entropy minimization. Section 6 discusses the strength of entropy minimization and swap distance minimization effects and the depth of swap distance minimization.

#### 2. Theory

#### 2.1. The permutohedron

The permutohedron of order n is a graph where vertices are permutations of 1, 2, ..., n and an edge links two permutations if swapping to consecutive elements in one of the permutations leads to the other permutation. When n = 3, one obtains the permutation ring that has been used in word order research [22, 23, 24, 18] (Figure 1). When n = 4, one obtains the graph in Figure 2. The permutohedron has N = n! vertices and is a regular graph with vertex degree n - 1. Thus it has

$$\frac{n-1}{2}n!$$

edges. The permutohedron is a Hamiltonian graph, i.e. the graph has a path that visits every vertex once. The demonstration relies on algorithms that are able to generate all permutations (i.e. all vertices of the graph) by swapping a pair of adjacent elements to get to the next vertex in that path (i.e. following an edge in the permutohedron) [25].

We define d, the swap distance between two permutations, as their distance in the permutohedron. By definition,  $d \in [0, d_{max}]$ , where  $d_{max}$  is the diameter of the permutohedron. It is well-known that [26]

$$d_{max} = \binom{n}{2}. (2)$$

We define  $d_{ij}$  as the distance between permutations i and j in the permutohedron. We define two kinds of mean topological distances: the mean topological distance over all ordered pairs of vertices, that is defined as

$$\langle d \rangle_{op} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} \tag{3}$$

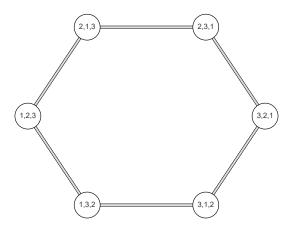
and the mean topological distance over all unordered pairs (excluding pairs formed by the same vertex), that is defined as

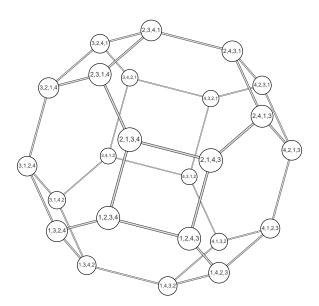
$$\langle d \rangle_{up} = \frac{1}{\binom{N}{2}} \sum_{i=1}^{N} \sum_{j=i+1}^{N} d_{ij}. \tag{4}$$

Since

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} d_{ij} = \frac{1}{2} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} - \sum_{i=1}^{N} d_{ii} \right)$$
$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij},$$

**Figure 2.** Permutohedra of order n with n=3 (top) and n=4 (bottom). Every vertex is a distinct permutation of the integer numbers between 1 and n. An edge joining two vertices indicates the one vertex yields the other vertex by swapping a pair of adjacent numbers.





 $\langle d \rangle_{un}$  can be expressed equivalently as

$$\langle d \rangle_{up} = \frac{1}{2\binom{N}{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}$$

$$= \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}.$$
(5)

We define T(n, k) as the number of permutations of a string of length n at swap distance k. By definition,

$$\sum_{k>0} T(n,k) = n!. \tag{6}$$

T(n,k) is also referred to as Mahonian number [27]. The next property will help us to obtain the value of  $\langle d \rangle_{up}$  and  $\langle d \rangle_{op}$ .

Property 2.1. The Mahonian numbers satisfy

$$\sum_{k>0} T(n,k) \cdot k = \frac{n(n-1)}{4} n!. \tag{7}$$

*Proof.* We will prove this property by induction on n is as follows.

• Base case, n = 1. Trivially,

$$\sum_{k>0} T(0,k) \cdot k = \frac{1 \cdot 0}{4} 1! = 0.$$

- Induction hypothesis. Eq. 7.
- Induction step. We aim to show that

$$\sum_{k>0} T(n+1,k) \cdot k = \frac{n(n+1)}{4}(n+1)!$$

for n > 1.

As for the induction step, the well-known recursive definition of Mahonian numbers (Appendix A), in particular

$$T(n,k) = \sum_{j=0}^{n-1} T(n-1, k-j),$$

leads to

$$\sum_{k\geq 0} T(n+1,k) \cdot k = \sum_{k\geq 0} \left[ \sum_{j=0}^{n} T(n,k-j) \right] \cdot k$$
 def.  $T(n,k)$ 

Then, a series of mechanical algebraic manipulations yield

$$=\sum_{j=0}^{n}\sum_{k\geq 0}T(n,k-j)\cdot k \qquad \text{swap summations}$$

$$=\sum_{j=0}^{n}\sum_{k\geq j}T(n,k-j)\cdot k \qquad T(n,k)=0 \text{ for negative } k$$

$$=\sum_{j=0}^{n}\sum_{k\geq 0}T(n,k)\cdot (k+j) \qquad \text{change of variable}$$

$$=\sum_{j=0}^{n}\sum_{k\geq 0}T(n,k)\cdot k+\sum_{j=0}^{n}\sum_{k\geq 0}T(n,k)\cdot j \qquad \text{split}$$

$$=\sum_{j=0}^{n}\left[\frac{n(n-1)}{4}n!\right]+\sum_{j=0}^{n}n!\cdot j \qquad \text{induction hypothesis and Equation 6}$$

$$=\frac{n(n-1)}{4}(n+1)!+\frac{n}{2}(n+1)!$$

$$=\frac{n(n+1)}{4}(n+1)!.$$

**Property 2.2.** In a permutohedron of order n, the mean topological distances are

$$\langle d \rangle_{op} = \frac{1}{2} d_{max}$$

and

$$\langle d \rangle_{up} = \frac{N}{2(N-1)} d_{max}.$$

Proof. Thanks to Property 2.1,

$$\sum_{i=1}^{N} d_{ij} = \sum_{k>0} T(n,k) \cdot k = \frac{n(n-1)}{4} N.$$

Recalling the definition of  $d_{max}$  (Equation 2), we obtain

$$\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} = \frac{n(n-1)}{4} N^2 = \frac{d_{max}}{2} N^2.$$

Recalling the definition of  $\langle d \rangle_{op}$  (Equation 5), we obtain

$$\langle d \rangle_{op} = \frac{1}{2} d_{max},$$

while recalling the parallel definition of  $\langle d \rangle_{up}$  (Equation 5), we obtain

$$\langle d \rangle_{up} = \frac{N}{2(N-1)} d_{max}.$$

We define  $\bar{i}$  as the reverse of order i. Then  $d_{\bar{i}i} = d_{max}$ . As  $d_{ij}$  is a metric distance between two vertices i and j in the permutohedron graph, the triangular inequality yields

$$d_{\bar{i},j} \leq d_{\bar{i}i} + d_{ij} = d_{max} + d_{ij}$$
.

The following property gives an exact relationship between  $d_{i,j}$  and  $d_{\bar{i},j}$ .

#### Property 2.3.

$$d_{\bar{i}j} = d_{max} - d_{ij}.$$

*Proof.* Consider a permutation  $\pi = \pi_1, ..., \pi_a, ..., \pi_n$  of the integers in [1, n].  $inv(\pi)$ , the number of inversions of  $\pi$ , ias defined as the number of pairs a and b, such that  $1 \le a < b \le n$  and  $\pi_a > \pi_b$ . It is easy to see that

$$inv(i) + inv(\bar{i}) = \binom{n}{2} = d_{max}.$$

It is well-known that the number of inversions is equivalent to  $d(\pi)$ , the swap distance of  $\pi$  to the identity permutation, i.e. I = 1, 2, 3, ..., n. ‡ Thus

$$d_{Ii} + d_{I\bar{i}} = \binom{n}{2}.$$

If one takes vertex j of the permutohedron as the reference permutation instead of the identity permutation, one obtains the desired result, i.e.

$$d_{ji} + d_{j\bar{i}} = \binom{n}{2}.$$

The last step follows by the vertex transitivity of the permutohedron.

#### 2.2. A swap distance score

In this article, we use the terms permutation and order (sequential order) interchangeably. An n-tuple has n! possible orderings. We define the mean swap distance conditioning on a source order j as

$$\langle d|i\rangle = \sum_{j=1}^{N} d_{ij} p_i. \tag{8}$$

One could investigate swap distance minimization with  $\langle d|i\rangle$  and choosing i as the source order. The choice of the source could be determined by that ordering being canonical, dominant or underlying word order. However, using a single word order

‡ The number of inversions of a permutation equals the minimal number of simple transpositions, namely transpositions of adjacent elements, needed to write the permutation https://www.findstat.org/StatisticsDatabase/St000018/. That number is, in turn, our definition of swap distance. Indeed, the swap distance is the Coxeter length in the symmetric group, which is known to be equivalent to the number of inversions of a permutation [28, Proposition 1.5.2].

as source assumes that all word order variation is produced from it. However, other word orders, specially high frequency ones, could be sources from which word order variation is produced. Then we define the mean swap distance as the mean swap distance conditioning on all possible source orders, i.e.

$$\langle d \rangle = \sum_{i=1}^{N} p_i \langle d | i \rangle \tag{9}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} p_i p_j. \tag{10}$$

Thus  $\langle d \rangle$  can be seen as the expectation of  $d_{ij}$  with respect to the joint probability of orders i and j, i.e.

$$\langle d \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} p_{ij}. \tag{11}$$

under the assumption that orders i and j are statistically independent and then  $p_{ij} = p_i p_j$ .  $\langle d \rangle$  is a weighted average distance from every order to every other order, as if each order could potentially radiate to all other word orders following trajectories in the permutohedron. With respect to using just  $\langle d|i\rangle$ ,  $\langle d\rangle$  solves the problem of the choice of the most frequent word order as the canonical in case the canonical order is unclear or weak as it happens in the case of couples of primary alternating word orders [1].

The following property indicates that the value of  $\langle d|i\rangle$  is the complementary of that of  $\langle d|\bar{i}\rangle$  with respect to the diameter.

#### Property 2.4.

$$\langle d|i\rangle + \langle d|\bar{i}\rangle = d_{max},$$
 (12)

*Proof.* Property 2.3 transforms Equation 8 into

$$\langle d|\bar{i}\rangle = \sum_{j=1}^{N} p_j (d_{max} - d_{ij})$$
  
=  $d_{max} - \langle d|i\rangle$ .

 $\langle d \rangle$  can be rewritten equivalently as

$$\langle d \rangle = \sum_{d=0}^{d_{max}} P(d)d = \sum_{d=1}^{d_{max}} P(d)d, \tag{13}$$

where P(d) is the probability mass of a certain distance d, i.e.

$$P(c) = \sum_{i=1}^{N} \sum_{\substack{j=1\\d_{ij}=c}}^{N} p_i p_j.$$
(14)

Obviously,

$$\sum_{d=0}^{d_{max}} P(d) = 1. (15)$$

Notice that

$$P(0) = \sum_{i=1}^{N} p_i \sum_{\substack{j=1\\d_{ij}=0}}^{N} p_j$$
$$= \sum_{i=1}^{N} p_i^2 = S.$$

Notice also that P(0) > 0 because all the  $p_i$ 's cannot be zero.

2.3. Lower and upper bounds of the swap distance score

2.3.1. The minimum value of  $\langle d \rangle$  First, tight lower bound of  $\langle d \rangle$  is straightforward.

**Property 2.5.**  $0 < \langle d \rangle$  with equality if and only if just one of the  $p_i$ 's is non-zero.

Proof.

Step 1. Show that  $\langle d \rangle \geq 0$ . Trivial since  $0 \leq d$  by definition.

Step 2. Show that  $\langle d \rangle = 0$  if and only if all probability mass is on just one order. If  $p_i > 0$  for some  $1 \le i \le N$  and  $p_j = 0$  for  $j \ne i$ , there are no word orders at distance d > 0 and thus Eq. 10 gives  $\langle d \rangle = 0$ . If there are orders at d > 0 (namely there is some i and j such that  $d_{ij} > 0$  and  $p_i, p_j > 0$ ), then  $\langle d \rangle > 0$ .

2.3.2. An upper bound of  $\langle d \rangle$  Since the distance to a source order is a number d such that  $0 \le d \le d_{max}$ , where  $d_{max}$  is the diameter of the permutohedron, it follows trivially that  $0 \le \langle d \rangle \le d_{max}$ . However, we will show that  $d_{max}$  is a too loose upper bound.

Property 2.6.

$$\langle d \rangle \le d_{max} \bar{S}.$$

*Proof.* By definition,

$$\langle d \rangle = \sum_{d=1}^{d_{max}} P(d)d$$

$$= \sum_{d=1}^{d_{max}-1} P(d)d + d_{max} \left(1 - \sum_{d=0}^{d_{max}} P(d)\right)$$

$$= d_{max}(1 - P(0)) - \sum_{d=1}^{d_{max}-1} (d_{max} - d)P(d)$$

$$\leq d_{max}(1 - P(0))$$

$$= d_{max}\bar{S}.$$

2.3.3. The maximum value of  $\langle d \rangle$  H achieves its maximum value when all orders are equally likely. We will show first that the value of  $\langle d \rangle$  achieved in that situation is also achieved by another configuration according to the following property. Later on, we will show that this value is also maximum for  $\langle d \rangle$ .

#### Property 2.7.

$$\langle d \rangle = \frac{1}{2} d_{max}$$

in at least two conditions

- (i)  $p_i = 1/N$  for any i such that  $1 \le i \le N$ .
- (ii)  $p_i = 0$  except for two orders, k and l, such that  $d_{kl} = d_{max}$  and  $p_k = p_l = 1/2$ .

*Proof.* (i) Equation 10 gives

$$\langle d \rangle = d_{up} = \frac{1}{2} d_{max}.$$

(ii) Equation 10 gives

$$\langle d \rangle = 2d_{max}p_kp_l = \frac{1}{2}d_{max}.$$

 $\langle d \rangle$  can be defined as product involving the vector of probabilities  $p = (p_1, ..., p_i, ..., p_n)$  and  $D = \{d_{ij}\}$ , the matrix of vertex-vertex distances in the permutohedron, i.e.

$$\langle d \rangle = p^T D p. \tag{16}$$

Recall that  $\sum_{i} p_i = 1$ .

In that setting, the following lemma will help us to obtain tight upper bounds of  $\langle d \rangle$ .

**Lemma 2.1.** Let Q be a real symmetric matrix of size  $N \times N$  and let k be a real constant. Consider the quadratic optimization problem of the form  $\max_{x \in \mathbb{R}^N} x^T Q x$  under the constraint  $\sum x_i = 1$ . Let O be the all-ones matrix of the same size as Q. If the matrix Q - Ok is negative semidefinite then

$$\max_{x \in \mathbb{R}^N} x^T Q x \le k.$$

*Proof.* We have that

$$\max x^T (Q - Ok)x = \max x^T Qx - k,$$

because  $x^T(Ok)x = kx^TOx = k$  thanks to  $\sum x_i = 1$ .

If (Q-Ok) is a negative semidefinite matrix, by definition of negative semidefinite, we have that for every x,  $x^T(Q-Ok)x \leq 0$ , which implies that  $\max x^T(Q-Ok)x = \max x^TQx - k \leq 0$  and finally  $\max_{x \in \mathbb{R}^N} x^TQx \leq k$ .

We conjecture that

$$\langle d \rangle \leq \frac{d_{max}}{2}$$

for any  $n \ge 1$ . If the conjecture is true it will always yield a tight upper bound thanks to Property 2.7 The following property states that this is true at least up to n = 6.

Property 2.8. For  $1 \le n \le 7$ ,

$$\langle d \rangle \leq \frac{d_{max}}{2}$$

with equality if the  $p_i$ 's satisfy one of the conditions indicated in Property 2.7.

*Proof.* The conjecture is trivially true for  $n \leq 2$  (recall Property 2.7). For n > 2, the proof consists of two steps

- (i) Apply Property 2.7 to obtain configurations such that  $\langle d \rangle = d_{max}/2$ .
- (ii) Show that  $\langle d \rangle \leq d_{max}/2$ . The argument is based on the equivalence between a negative semi-definite matrix and a matrix whose all eigenvalues are non-positive. The procedure consists of the following steps:
  - (a) Computing D by means of a breadth-first traversal of the permutohedron of order n from an initial arbitrary order.
  - (b) Computing the eigenvalues of the matrix  $D' = D \frac{d_{max}}{2}O$ , where O is the all-one matrix of the same size as D.
  - (c) Checking that D' is negative semi-definite by checking that all its eigenvalues are non-positive. We compute the eigenvalues of D' by means of the Implicitly Restarted Arnoldi Method (IRAM). §
  - (d) Invoking Lemma 2.1, to conclude that  $d_{max}/2$  is an upper bound of  $\langle d \rangle$ .

Next we apply Step ii to specific values of n. When n = 3,  $d_{max}/2 = 3/2$ , D can be defined as

$$D = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 0 & 2 & 1 & 3 & 2 \\ 1 & 2 & 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 & 2 & 1 \\ 2 & 3 & 1 & 2 & 0 & 1 \\ 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}$$

and then the eigenvalues of the matrix

$$D - \frac{3}{2}O = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

§ In particular, we called the function scipy.linalg.eigvals available in the scipy Python library. The function is a wrapper for the ARPACK implementation of the method.

eigenvalues  $d_{max}/2$ group 1 group 2 group 3  $\lambda_{1,2} = -4$   $\lambda_3 = -1$   $\lambda_{1,2,3} = -20$   $\lambda_{4,5,6} = -4$ 3 1.5  $\lambda_{4,5,6} = 0$  $\lambda_{7....24} = 0$ 4 3  $\lambda_{1,2,3,4} = -120$   $\lambda_{4,\dots,10} = -20$   $\lambda_{11,\dots,120} = 0$ 5 5  $\lambda_{1,...5} = -840$   $\lambda_{6,...15} = -120$   $\lambda_{16,...720} = 0$ 6 7.57 10.5  $\lambda_{1,\dots,6} = -6720$   $\lambda_{7,\dots,21} = -840$   $\lambda_{22,\dots,5040} = 0$ 

**Table 1.** The eigenvalues of the matrix  $D - Od_{max}/2$  as a function of n, the order of the permutohedron.

are

$$\lambda_1 = \lambda_2 = -4$$

$$\lambda_3 = -1$$

$$\lambda_4 = \lambda_5 = \lambda_6 = 0.$$

Table 1 shows that the eigenvalues of the matrix  $D - Od_{max}/2$  are all negative for  $3 \le n \le 7$ .

Appendix B presents complementary results for n = 3: a computationally efficient compact formula and alternative ways of deriving upper bounds and the theoretical maximum for n = 3.

2.4. Expectation of the swap distance score under distinct null hypotheses

2.4.1. Die rolling experiment We define  $\mathcal{M}(F, N)$  as a multinomial distribution where F is the number of trials, N is the number of bins and the probability of each bin is 1/N. We define f as the vector  $f = f_1, ..., f_i, ..., f_N$  where  $f_i$  is the number of times bin f has been chosen.

**Lemma 2.2.** When  $f \sim \mathcal{M}(F, N)$ , the expected value of the product  $f_i f_j$  when  $i \neq j$  is

$$\mathbb{E}_{i\neq j}\left[f_i f_j | f \sim \mathcal{M}(F, N)\right] = \frac{F(F-1)}{N^2}.$$

*Proof.* By definition

$$f_i = \sum_{k=1}^F a_{ik},$$

where  $a_{ik}$  is a Bernoulli random variable that indicates if the *i*-th bin has received a ball in the *k*-th trial. Then

$$f_i f_j = \sum_{k=1}^F a_{ik} \sum_{l=1}^F a_{jl}.$$

By the linearity of expectation,

$$\mathbb{E}_{i\neq j}[f_i f_j] = \sum_{k=1}^F \sum_{l=1}^F \mathbb{E}_{i\neq j}[a_{ik} a_{jl}].$$

When l = k,  $\mathbb{E}_{i \neq j}[a_{ik}a_{jl}] = 0$ . When  $l \neq k$ ,  $a_{ik}a_{jl} = 1$  with probability  $1/N^2$ . Hence

$$\mathbb{E}_{i \neq j}[f_i f_j] = \sum_{k=1}^{F} \sum_{\substack{l=1 \ l \neq k}}^{F} \frac{1}{N^2}$$
$$= \frac{F(F-1)}{N^2}.$$

**Property 2.9.** The expected value of  $\langle d \rangle$  in a permutohedron of order n when f is generated by rolling a die F times (F > 0) is

$$\langle d \rangle_{dr} = \mathbb{E}[\langle d \rangle] = \frac{F - 1}{F} \frac{d_{max}}{2}.$$
 (17)

*Proof.* By the linearity of expectation over the definition of  $\langle d \rangle$  (Equation 1) with N = n! and  $p_i = f_i/T$  and thanks to  $d_{ii} = 0$ , we obtain

$$\mathbb{E}[\langle d \rangle] = \frac{1}{F^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \ j \neq i}}^{N} d_{ij} \mathbb{E}_{i \neq j} \left[ f_i f_j | f \sim \mathcal{M}(F, N) \right].$$

By invoking Lemma 2.2, we obtain

$$\mathbb{E}[\langle d \rangle] = \frac{1}{F^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \ j \neq i}}^{N} d_{ij} \frac{F(F-1)}{N^2}$$
$$= \frac{F-1}{F} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \ j \neq i}}^{N} d_{ij}$$
$$= \frac{F-1}{F} \langle d \rangle_{op}.$$

Thanks to Property 2.2, we finally obtain

$$\mathbb{E}[\langle d \rangle] = \frac{F - 1}{F} \frac{d_{max}}{2}.$$

#### 2.4.2. Random permutation

**Property 2.10.** The expected value of  $\langle d \rangle$  when the original p is replaced by a uniformly random permutation of it is given by the expression

$$\langle d \rangle_{rp} = \mathbb{E}[\langle d \rangle] = \langle d \rangle_{up} \bar{S},$$
 (18)

where  $\bar{S}$  is the dominance index and  $\langle d \rangle_{up}$  is the average distance in the permutohedron over unordered pairs.

*Proof.* Recall that N is the number of vertices of the permutohedron of order n (N = n!). We define  $\pi$  as a permutation function, namely a one-to-one mapping between natural numbers in [1, N] and numbers in [1, N]. By definition of  $\langle d \rangle$  and by the linearity of expectation

$$\mathbb{E}[\langle d \rangle] = \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} \mathbb{E}[p_i p_j],$$

where

$$\mathbb{E}[p_i p_j] = \frac{1}{N!} \sum_{\pi} p_{\pi(i)} p_{\pi(j)}$$

If i = j,  $d_{ij} = 0$ , which implies that  $d_{ij}\mathbb{E}[p_i p_j] = 0$ . If  $i \neq j$ , then

$$\sum_{\pi} p_{\pi(i)} p_{\pi(j)} = 2 \cdot \frac{N!}{N(N-1)} \sum_{i=1}^{N} p_i \sum_{j=i+1}^{N} p_j$$

$$= 2 \cdot \frac{N!}{N(N-1)} \frac{1}{2} \left( \sum_{i=1}^{N} \sum_{j=i}^{N} p_i p_j - \sum_{i=1}^{N} p_i^2 \right)$$

$$= \frac{N!}{N(N-1)} \left( 1 - \sum_{i=1}^{N} p_i^2 \right)$$

$$= \frac{N!}{N(N-1)} \bar{S},$$

which in turn gives

$$\mathbb{E}[p_i p_j] = \frac{1}{N!} \sum_{\pi} p_{\pi(i)} p_{\pi(j)}$$
$$= \frac{1}{N(N-1)} \bar{S}.$$

Finally,

$$\mathbb{E}[\langle d \rangle] = \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} \mathbb{E}[p_i p_j]$$
$$= \frac{1}{N(N-1)} \bar{S} \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}$$
$$= \langle d \rangle_{un} \bar{S}.$$

#### 2.5. Dominance index

We have seen that  $\bar{S}$  appears in various expressions about  $\langle d \rangle$  and thus it is worth asking about its range of variation. As  $0 \leq p_i \leq 1$ , it is easy to see that

$$0 < \bar{S} < 1.$$

However, 1 is not a tight upper bound for  $\bar{S}$  as we will show. In addition, as the value of n increases, languages will have difficulties to fill space of possible permutations. Therefore, we aim for a simple upper bound of  $\bar{S}$  that involves just m, namely the number of  $p_i$ 's that are not zero (m is the size of the support set;  $m \leq N$ ). The next property indicates that S is minimized when all orders in the support set are equally likely.

#### Property 2.11.

$$S \ge \frac{1}{m}$$

*Proof.* Let q be a vector of the same length as p. By the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{N} p_i q_i\right)^2 \le \sum_{i=1}^{N} p_i^2 \sum_{i=1}^{N} q_i^2.$$

Suppose that  $q_i = 1$  if  $p_i > 0$  and  $q_i = 0$  otherwise. Then, recalling that

$$\sum_{i=1}^{N} p_i = 1,$$

the inequality becomes

$$\sum_{i=1}^{N} p_i^2 \ge \frac{1}{m}.$$

Then  $\bar{S}$  is bounded above by the proportion of non-zero probability orders, i.e.

$$\bar{S} \le 1 - \frac{1}{m}.\tag{19}$$

### 2.6. Entropy

It is well-known that [29]

$$0 \le H \le H_{max} = \log m.$$

Here we compute H using the plug-in estimator, namely

$$H = -\sum_{i=1}^{N} \frac{f_i}{F} \log \frac{f_i}{F}.$$
 (20)

We use that estimator for three reasons: (a) simplicity, (b) our main goal is not to estimate the true value of entropy but to determine if entropy is significantly small and (c) the plug-in estimator is very strongly correlated with advanced entropy estimators [30].

The following property gives an expression for the expectation of H under the die rolling null hypothesis.

**Property 2.12.** When  $f \sim \mathcal{M}(F, N)$  and H is computed with the plug-in estimator, the expected value of H is

$$H_{dr} = \mathbb{E}[H] = \log F - \frac{N}{F} \sum_{a=0}^{F} p(F, 1/N, a) a \log a,$$

where

$$p(F, q, a) = {F \choose a} q^a (1 - q)^{F-a}.$$

*Proof.* Expanding Equation 20 one obtains

$$H = \log F - \frac{1}{F} \sum_{i=1}^{N} f_i \log f_i$$
 (21)

and then

$$\mathbb{E}[H] = \log F - \frac{1}{F} \sum_{i=1}^{N} \mathbb{E}[f_i \log f_i]$$
$$= \log F - \frac{N}{F} \mathbb{E}[f_i \log f_i].$$

 $\mathbb{E}[f_i \log f_i]$  is the expected value of  $f_i \log f_i$  knowing that  $f_i$  follows a binomial distribution with parameters F and 1/N. Then

$$\mathbb{E}[f_i \log f_i] = \sum_{a=2}^{F} p(F, N, a) a \log a,$$

where p(F, q, a) is the probability of a successes over N trials with a probability of success of q.

#### 3. Materials

We obtain the frequency of a word order from two kind of sources: the frequency of a dominant or preferred order in a collection of languages or the frequency of each order in a corpus. The syntactic structures we consider are

- SOV, formed by subject (S), direct object (O) and verb (V).
- VOX, formed by verb (V), direct object (O) and oblique (X).
- OVI, formed by direct object (O), verb (V) and indirect object (I).
- SOVI, formed by subject (S), direct object (O), verb (V) and an indirect object (I).

• nAND, namely noun phrases formed by a noun (n), an adjective (A), a numeral (N) and a demonstrative (D).

Figure 3 shows permutohedra for SOVI and nAND structures.

#### 3.1. Frequency of dominant word orders

The frequency of the dominant order, measured in languages, of SOV and VOX according to WALS are borrowed from [1] and [32], respectively. Word order frequencies, measured in languages and in families, are borrowed from Hammarström [2]. The frequency of the dominant orders of nAND, measured in languages, genera and adjusted number of languages, are borrowed from Dryer [3].

#### 3.2. Frequency of word order in corpora

We obtain the corpus frequency of the orders of SOV, OVI and SOVI from research on word order by Leela and collaborators [31, 33, 34]. The frequency of the orders of SOV, OVI and SOVI in Hindi-Urdu spontaneous speech was obtained from Table 1 of [31]. In that table, one finds VSO (with frequency 50) and VSOc (with frequency 17). The subindex c indicates that the object O is a subordinate phrase || that is known as Complementizer Phrase (CP) in the terminology of generative grammar CP [33]. We offer two analyses: one where they are merged as VSO (with frequency 67) and another where VSOc is discarded. The one where they are merged is marked with the label CP. In table 1 of [31], one also finds ISVOc (with frequency 9)¶. We offer two analyses: one where it was replaced by ISVO (with frequency 9) and another where it was removed. The corpus frequency of the orders of SOV in Hindi-Urdu was obtained also from child directed speech (CDS), namely adult speech directed to children (Table 2 of [31]). This dataset is marked as CDS. The corpus frequency of the order of SOV in Malayalam was obtained from Table 3 of [31]. In Table 1 of [31], the order IOV appears twice, one with frequency 42 and the other with frequency 12. Those numbers were summed to produce 52 as the frequency of OIV. + Finally, we also include the absolute frequencies of the orders of the SOV structure from Lev-Ari's word order experiments [35].

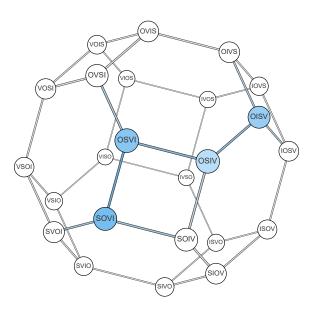
Tables 3 and 4 summarize the features of the whole dataset.

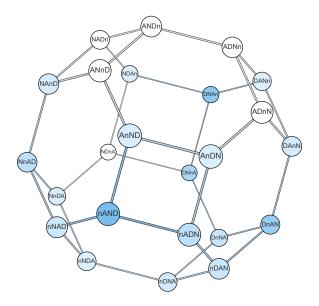
<sup>||</sup> Anna Gavarró, personal communication (19 February 2024); Maya Leela, personal communication (14 April 2024)

 $<sup>\</sup>P$  Indeed, it appears with the notation DI S V DO where DI is I in our notation and DO is O in our notation

<sup>&</sup>lt;sup>+</sup> According to Maya Leela (personal communication, 16 April 2024), there was a typo in the original table and she told us to sum both numbers to fix the problem.

Figure 3. The permutohedra of order 4 that result from the SOVI structure (top) and the nAND structure (bottom). Vertex or edge colors reflect their weight. For a vertex i, the weight is  $p_i$ , the relative frequency of the word order. If  $p_i = 0$  then the color is white. If  $p_i > 0$  then the intensity of blue reflects  $p_i$ . For an edge between vertex i and vertex j we use  $p_i + p_j$  as the weight. The choice of that weight ensures that the edge has some color even when one  $p_i = 0$  or  $p_j = 0$ . For the SOVI structure, the frequency of each order in a corpus of Hindi-Urdu is borrowed from Table 1 of [31] excluding CP constituents. For the nAND structure, the frequency of each dominant order is its adjusted number of languages according to [3].





#### 4. Methods

Given a structure of order n, we define  $f_i$  as the frequency of the i-th word order, and F as the total frequency, i.e.

$$F = \sum_{i=1}^{N} f_i.$$

For a certain structure in a dataset, we compute F, H, the entropy of the word orders, and  $\langle d \rangle$ , the average swap distance between the possible orders of that structure. H is computed using the plug-in estimator (Equation 20). To reduce numerical error, we compute H via the expanded equivalent expression (Equation 21) and  $\langle d \rangle$  via

$$\langle d \rangle = \frac{1}{F^2} \sum_{i=1}^{N} f_i \sum_{j=1}^{N} f_j d_{ij},$$

where  $d_{ij}$  is the swap distance between orders i and j, namely the distance in edges between i and j in the permutohedron.

#### 4.1. Null hypotheses

We determine if H is significantly small as expected by the entropy minimization principle or if  $\langle d \rangle$  is significantly small as expected by the swap distance minimization principle using the following null hypotheses, which are sorted in order of increasing strength:

- (i) A die rolling experiment (dr). The experiment consists of rolling a fair die F times. The die has n! sides, each corresponding to one of the orders of the structure. The frequency of each word order is the frequency of the corresponding side over the F rolls.
- (ii) A Polya urn experiment (Pu) with balls of n! distinct colors. Each color corresponds to a word order. The experiment consists of initializing the urn with n! balls of distinct colors and then choosing a ball with replacement and adding a copy of it to the urn until the urn contains F balls.
- (iii) A random permutation (rp), that consists of shuffling the n! word order frequencies. That null hypothesis preserves the empirical distribution of word order frequencies and thus preserves H.
- (iv) Random walk. Orders are chosen by performing an unbiased random walk on the permutohedron graph. Each random walk starts on some vertex of that graph and lasts for an arbitrary number of steps.

Notice that the three first null hypotheses neglect the structure of the permutohedron in the sense that word order frequencies are produced ignoring the permutohedron structure. However, as the permutohedron is a regular graph (all vertices have the same degree), it is easy to see that the probability of landing on a certain vertex is the same

for all vertices regardless of the initial vertex or the length of the walk [36]. Therefore, the random walk null hypothesis is equivalent to the die rolling null hypothesis.

The frequencies of the orders of nAND structures are not integer numbers when measured in adjusted number of languages [3]. In that case, F is rounded to the nearest integer for the die rolling and the Polya urn experiments.

For each structure in Table 4, the expected value of  $\langle d \rangle$  for the 1st null hypothesis,  $\langle d \rangle_{dr}$ , is calculated with Equation 17; the expected value of  $\langle d \rangle$  for the 3rd null hypothesis,  $\langle d \rangle_{rp}$ , is calculated with Equation 18. The expected value of  $\langle d \rangle$  for the 2nd null hypothesis,  $\langle d \rangle_{Pu}$ , is estimated by means of a Monte Carlo procedure that consists of running T Polya urn experiments. Given a null hypothesis, we run T experiments and estimate  $\mathbb{P}$ , the left p-value, as the proportion of experiments such that  $\langle d \rangle' \leq \langle d \rangle$ , where  $\langle d \rangle'$  is the value of  $\langle d \rangle$  in a experiment. If the p-value estimated by the Monte Carlo procedure is 0, the estimate is replaced by 1/T, a likely upper bound of the actual p-value. For the permutation null hypothesis and n=3,  $\mathbb{P}$  is computed exactly by generating all permutations of the vector p.

To control for multiple comparisons, we apply a Holm correction to p-values obtained with the same null hypothesis. We use a < sign to indicate p-values that are a likely upper bound before or after applying the Holm correction.

As the manifestation of the principles (entropy minimization or swap distance minimization) may be weak, we also consider a test that, given a score x (H or  $\langle d \rangle$ ) and some null hypothesis, compares x, the value of the score against x', the value of the score under the null hypothesis across all cases by means of a Wilcoxon signed-rank test [37]. This test is often used to check if a certain medical treatment has a significant effect on patients. The test is paired in the sense that one has two paired (or matched) samples; a measurement on a patient before the treatment in one sample is paired with a measurement on the same patient after the treatment in the other sample. Here we are comparing the original value of the score against its paired value after randomization by some null hypothesis that preserves just some information about original case (e.g., the three first null hypotheses just preserves F, the random permutation hypothesis preserves the multiset of word frequencies). If the original values came from the null hypothesis, no significant difference between x and its randomized counterpart x' should be found. Here we used a left-sided Wilcoxon signed-rank test as we assume that, if the principle is acting, a tendency towards x < x' is expected. Put differently, the test checks if the randomization of the null hypothesis has some effect towards larger values. If it does, that implies that the corresponding optimization principle (entropy minimization or swap distance minimization) has some effect.

In this article, we assume a significance level of 0.05 just as an orientation for discussion.

#### 4.2. The p-value of the random permutation test

We aim to find lower bounds of  $\mathbb{P}$ , the left p-value of left-sided permutation test to understand the statistical power of the permutation test, namely its prior capacity to reject the null hypothesis. By definition,

$$\mathbb{P} \ge \frac{1}{N!}.$$

However, we will show that  $\mathbb{P}$  is predetermined to be "large" specially when n=3.

Given a vector of probabilities p, every permutation can be represented by a 0-1 matrix  $\Pi$  with exactly one 1 per row and column, where the permuted vector is  $\Pi p$ . We call a permutation  $\Pi$  "unchanging" if  $\Pi^T D\Pi = D$ , where D is the distance matrix of the permutohedron.

**Property 4.1.** If a permutation  $\Pi$  is unchanging, then  $\langle d \rangle$  computed on p is equal to  $\langle d \rangle$  computed on  $\Pi p$ .

*Proof.* We have already seen that  $\langle d \rangle$  can be expressed in matrix form as in Equation 16. It is easy to see that  $\langle d \rangle$  does not change if p is replaced by  $\Pi p$ , that is

$$(\Pi p)^T D(\Pi p) = p^T \Pi^T D \Pi p$$
$$= p^T D p.$$

We define  $\mathbb{P}_{=}$  as the proportion of permutations with same  $\langle d \rangle$  as in the original vector p, namely  $\mathbb{P}_{=}$  is the proportion of unchanging permutations. We define  $\mathbb{P}_{m}$  as the proportion of unchanging permutations over all the vectors p such that they have m non-zero probabilities and the multiset of pairwise swap distances is the same as in the original configuration. We define  $\mathbb{P}_{A}$ , as the proportion of permutations of the vertices that are automorphisms of the permutohedron graph. Recall that an automorphism of a graph is a permutation of its vertices which brings the graph into itself.

The following property explains why the permutation test lacks statistical power when n=3 by showing that  $\mathbb{P} \geq \frac{1}{60}$ .

**Property 4.2.** The left p-value of the permutation test obeys the following chain of inequalities

$$\mathbb{P} \ge \mathbb{P}_{=} \ge \mathbb{P}_{m} \ge \mathbb{P}_{A} \ge \frac{1}{N!}$$

When n = 3,

$$\mathbb{P}_A = \frac{1}{60}.$$

When n=4,

$$\mathbb{P}_A = \frac{2}{23!} \approx 7.74 \cdot 10^{-23}.$$

Let m be the number of non-zero probability orders in p. The value of  $\mathbb{P}_m$  as a function of m is shown in Table 2 for n=3 and n=4.

Proof.

$$\mathbb{P}_A = \frac{A}{N!},$$

where A is the number of automorphisms. Here we compute A using the BLISS algorithm [38]. \* When n = 3, A = 16 and then

$$\mathbb{P}_A = \frac{1}{60}.$$

Alternatively, A = 16 can be derived as the number of symmetries of a hexagon. Without loss of generality, suppose that the permutohedron of order 3 is plotted as a regular hexagon in a two dimensional space. That hexagon has six rotational symmetries and six reflection symmetries, which constitute the dihedral group  $D_6$  [39, Chapter 2, p. 25]. When n = 4, A = 48 and then

$$\mathbb{P}_A = \frac{2}{23!}.$$

Given a value of m,  $\mathbb{P}_m$  was calculated by means of an exhaustive method over the space of distinct vectors p' that are permutations of a vector of the form

$$p = (p_1, \dots, p_m, 0, \dots, 0).$$

For each p', we compare the coefficients of the polynomial  $\langle d \rangle' = p'^T D p'$  against the coefficients of the polynomial for the original  $\langle d \rangle = p^T D p$ . Then  $\mathbb{P}_m$  is the proportion of these vectors where the comparison yields equality. Table 2 shows the value of  $\mathbb{P}_m$  as a function of m for n=3 and n=4. For certain values of m,  $\mathbb{P}_m$  depends on the location of the non-zero probability vertices in the permutohedron. Therefore, these tables also indicate the multiset of pairwise swap distances among the non-zero probability vertices.

By definition,  $\mathbb{P} \geq \mathbb{P}_{=}$ . We have  $\mathbb{P}_{=} \geq \mathbb{P}_{m}$  because the former comprises a wider set of permutations. We have  $\mathbb{P}_{m} \geq \mathbb{P}_{A}$  instead of  $\mathbb{P}_{m} = \mathbb{P}_{A}$  because  $\mathbb{P}_{A}$  does not take into account that vertices that have zero probability are another source of symmetries. Indeed, Table 2 shows that  $\mathbb{P}_{A} = \mathbb{P}_{m}$  requires m = N or m = N - 1 when n = 3.

#### 5. Results

m, the number of orders that have non-zero frequency is maximum (N = m) in most cases when n = 3 (Table 3). When n = 4, there is a big gap between m and N specially for the SOIV structure (Table 3), suggesting some difficulty of languages to cover the whole space of possible permutations. The next subsections shed light on the possible nature of such difficulty.

\* We used the implementation available in the library igraph for R, https://igraph.org/r/doc/automorphisms.html

**Table 2.**  $\mathbb{P}_m$  as a function of m for  $1 \leq m \leq N$ .  $\mathbb{P}_m$  depends on the distribution of the non-zero probability orders in the permutohedron and sometimes also on the multisets of pairwise swap distances that achieve each  $\mathbb{P}_m$ , that are also shown when relevant. — indicates that the multiset of swap distances is not relevant.

n	m	$\mathbb{P}_m$	Multisets of swap distances
3	1	1	-
	2	1/5	{3}.
		2/5	$\{1\}, \{2\}.$
	3	1/10	-
	4	1/30	-
	5	1/60	-
	6	1/60	-
4	1	1	-
	2	1/23	$\{6\}.$
		3/23	$\{1\}, \{5\}$
		5/23	$\{2\}, \{4\}$
		6/23	{3}
	3	3/506	$\{1, 5, 6\}$
		5/506	$\{2,4,6\}$
		6/506	$\{1,1,2\}, \{1,4,5\}, \{2,5,5\}, \{3,3,6\}$
		9/506	$\{1,2,3\}, \{1,3,4\}, \{2,3,5\}, \{3,4,5\}$
		10/506	${2,2,2}, {2,2,4}, {2,4,4}, {4,4,4}$
		12/506	${2,3,3}, {3,3,4}$

#### 5.1. Entropy

Table 3 shows the actual entropy (H), its expected value in a die rolling experiment  $(H_{dr})$  and in a Polya urn experiment  $(H_{Pu})$  for all syntactic structures. In all cases,  $H < H_{Pu} < H_{dr}$ . The fact  $H_{dr}$  is always the largest is not very surprising as one expects that,  $H_{dr} \approx H_{max}$ , where  $H_{max} = \log N$  is the theoretical maximum H. Then, it is not surprising that H is always significantly small with respect to die rolling. Under the Polya urn null hypothesis, H is always significantly small when n = 4 and never when n = 4 (Table 3). It is not very surprising to find that entropy minimization is captured in all cases when n = 4 as the space of possible permutations is wider. When n = 3,  $H < H_{Pu}$  in all 10 cases. How likely is it that this has happened by chance? A one-sided Wilcoxon signed rank test supports a tendency for  $H < \langle d \rangle_{Pu}$  as expected by entropy minimization (V = 0, p-value  $= 9.8 \cdot 10^{-4})$ .

Summary of the statistical information by database, kind, the sequence length n, the structure (struct.) and unit of measurement ucy: F, the total frequency,  $\bar{S}$ , the dominance index, H, the entropy,  $H_{Pu}$ , the expected value of H in a Polya urn experiment, expected value of  $\langle d \rangle$  in a die rolling experiment,  $H_{max} = \log(n!)$ , the maximum value of H, and the p-values ( $\mathbb{P}$ ) of left sided H: Polya urn (Pu) and die rolling (H). Each column of H-values has been adjusted with a Holm correction; the original H-value Table 3.

Table of free $H_{dr}$ , t	<b>Table 3.</b> Summary of the statistical information by database, kind, the sequence length $n$ , the structure (struct.) and unit of measurer of frequency: $F$ , the total frequency, $\overline{S}$ , the dominance index, $H$ , the entropy, $H_{Pu}$ , the expected value of $H$ in a Polya urn experim $H_{dr}$ , the expected value of $A$ in a die rolling experiment, $H_{max} = \log(n!)$ , the maximum value of $H$ , and the $p$ -values ( $\mathbb{P}$ ) of left structure of $H$ in $H$ i	e star $1$ freq of $\langle d \rangle$	tistical infinency, $\vec{S}$ , uency, $\vec{S}$ , in a die	ormation by c the dominan rolling experi	$\frac{1}{1}$	, kind $\vdots$ , $H$ , $t$ $t_{max} = t$	the sector $\frac{1}{2}$ the $\frac{1}{2}$ the $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	quence ropy, $I$ ropy, $I$ , the	$I_{Pu}$ , the maxim	n, the e expection value $n$	$\frac{1}{1}$	e (struct.) and the of $H$ in a Pc, and the $p$ -variance.	the sequence length $n$ , the structure (struct.) and unit of measurer the entropy, $H_{Pu}$ , the expected value of $H$ in a Polya urn experim $= \log(n!)$ , the maximum value of $H$ , and the $p$ -values ( $\mathbb{P}$ ) of left structure $H$ .
tesus c is shov Database	tests on $H$ : Forya urn $(Fu)$ and the forming $(ar)$ . Each commit of $p$ -values has been adjusted with a from correction; the original $p$ -vision below the adjusted $p$ -value. The unit "adj. langs." stands for adjusted number of languages.  Kind $n$ Struct. Unit $F$ $m$ $\bar{S}$ $H$ $H_{Pu}$ $H_{max}$ $\mathbb{P}_{Pu}$	$\frac{a}{a}$ and $\frac{a}{b}$	ra die ron value. Tł Struct.	ronnig ( <i>ar.</i> ). Eac The unit "adj." t. Unit	$^{\prime\prime}$ adj. langs." $^{\prime\prime}$	stands $\frac{p}{m}$	For a $S$	$\begin{array}{c} \text{mas b} \\ \text{ijusted} \\ H \end{array}$	stands for adjusted number of languages. $m=\bar{S}$ $H$ $H_{Pu}$ $H_{dr}$ $H_{max}$	$\begin{array}{c} \text{usted v} \\ \text{r of lar} \\ H_{dr} \end{array}$	guages. $H_{max}$	$\mathbb{P}_{P_u}$	the original $p$ -v $\mathbb{P}_{dr}$
WALS	dominant order	3	SOV	langs.	1187	9	9.0	1.06	1.45	1.79	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.03	$< 1 \cdot 10^{-7}$
WALS	dominant order	က	VOX	langs.	333	5	0.56	1.09	1.46	1.78	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.03	$< 1 \cdot 10^{-7}$
Hammarström	dominant order	33	AOS	langs.	5128	9	0.62	1.13	1.45	1.79	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.05	$< 1 \cdot 10^{-7}$
Hammarström	dominant order	က	SOV	families	340	9	0.47	0.94	1.46	1.78	1.79	0.1	$< 1.5 \cdot 10^{-6}$
												$9.6 \cdot 10^{-3}$	$< 1 \cdot 10^{-7}$
Lev-Ari	experiments	က	AOS	frequency	13985	9	0.58	1.13	1.45	1.79	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.05	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CP	က	SOV	frequency	3206	9	0.51	1.06	1.45	1.79	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.03	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	က	SOV	frequency	3139	5	0.48	0.98	1.45	1.79	1.79	0.14	$< 1.5 \cdot 10^{-6}$
												0.02	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CDS	က	SOV	frequency	398	9	0.64	1.25	1.46	1.79	1.79	0.24	$< 1.5 \cdot 10^{-6}$
												0.12	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	3	OVI	frequency	138	4	0.61	1.08	1.47	1.77	1.79	0.21	$< 1.5 \cdot 10^{-6}$
												0.03	$< 1 \cdot 10^{-7}$
Leela	Malayalam	က	SOV	frequency	3763	4	0.74	1.37	1.45	1.79	1.79	0.28	$< 1.5 \cdot 10^{-6}$
												0.28	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	langs.	929	18	0.83	2.13	2.8	3.16	3.18	$1.3 \cdot 10^{-4}$	$< 1.5 \cdot 10^{-6}$
												$1 \cdot 10^{-5}$	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	genera	322	18	98.0	2.28	2.81	3.14	3.18	$8.4 \cdot 10^{-4}$	$< 1.5 \cdot 10^{-6}$
												$7 \cdot 10^{-5}$	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	adj. langs.	212	18	0.88	2.42	2.83	3.12	3.18	$6.2\cdot 10^{-3}$	$< 1.5 \cdot 10^{-6}$
												$5.6\cdot 10^{-4}$	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CP	4	SOVI	frequency	320	5	0.74	1.42	2.81	3.14	3.18	$< 1.5 \cdot 10^{-6}$	$< 1.5 \cdot 10^{-6}$
												$< 1 \cdot 10^{-7}$	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	4	SOVI	frequency	311	4	0.72	1.33	2.81	3.14	3.18	$< 1.5 \cdot 10^{-6}$	$< 1.5 \cdot 10^{-6}$
												$< 1 \cdot 10^{-7}$	$< 1 \cdot 10^{-7}$

#### 5.2. Average swap distance

Table 4 shows the actual value of  $\langle d \rangle$  and its expected value in a die rolling experiment  $(\langle d \rangle_{dr})$ , a Polya urn experiment  $(\langle d \rangle_{Pu})$  and a random permutation  $(\langle d \rangle_{rp})$  for all syntactic structures. In all cases,  $\langle d \rangle < \langle d \rangle_{Pu} < \langle d \rangle_{dr}$ . The finding that  $\langle d \rangle < \langle d \rangle_{dr}$  is not very surprising because the expected values for a die rolling experiment are close to  $\langle d \rangle_{max}$ , the theoretical maximum  $\langle d \rangle$  ( $\langle d \rangle_{max} = 1.5$  when n = 3 and  $\langle d \rangle_{max} = 3$  when n = 4).

When it comes to significance, we find that  $\langle d \rangle$  is always significantly small with respect to die rolling, never significant with respect to a random permutation and only significantly small with respect to a Polya urn when the structure is SOVI with and without CP in Hindi Urdu and borderline significant in the SOV structure without CP in Hindi-Urdu (Table 4). However, notice that, for the SOV structure in Malayalam, the raw p-value of the permutation test (the p-value before Holm's correction), is  $1/30 \approx 0.03$ , which matches the minimum p-value for n=3 and m=4 according to Table 2 and the theoretical analyses in Section 4.2.

As for the Polya urn null hypothesis, it is not very surprising to find the swap distance minimization is captured more easily when n=4 (SOVI structure) as the space of possible permutations is bigger (Figure 3). When n=3, the statistical tests never capture swap distance minimization but  $\langle d \rangle < \langle d \rangle_{Pu}$  in 10 out of 10 cases. How likely is it that this has happened by chance? A one-sided Wilcoxon signed rank test supports a tendency for  $\langle d \rangle < \langle d \rangle_{Pu}$  as expected by swap distance minimization (V=0, p-value =  $9.8 \cdot 10^{-4}$ ).

Regarding the random permutation null hypothesis,  $\langle d \rangle$  was never significantly low after Holm's correction but before it was in a few cases: the SOV structure including CP in Hindi-Urdu, the SOV structure in Malayalam and SOVI structures with and without CP in Hindi-Urdu (Fig. 3). The finding of swap distance minimization for the SOV structure in Malayalam confirms previous findings on that corpus using a correlation statistic that assumes that SOV is the source order [18] while  $\langle d \rangle$  does not make any assumption about the existence of a primary source order or the number of sources. Notice also that  $\langle d \rangle < \langle d \rangle_{rp}$  in 11 out 15 cases (the failures were the VOX structure in WALS and the nAND structure independently of the unit of measurement of frequency). A one-sided Wilcoxon signed rank test supports a tendency for  $\langle d \rangle < \langle d \rangle_{rp}$  as expected by swap distance minimization (V = 17, p-value =  $6.2 \cdot 10^{-3}$ ).

Table 5 shows the outcome of the Wilcoxon signed rank test over all conditions, including conditions that we have not selected above for the sake of completeness and to shed some light on the robustness of the tests beyond the critical cases discussed in the main text.

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	<b>Table 4.</b> Summary of the statistical information by database, kind, the sequence length $n$ , structure (struct.) and unit of measurement of frequency: $F$ , the total frequency, $m$ , the number of non-zero probability orders, $\langle d \rangle$ , the average swap distance, $\langle d \rangle_{rp}$ , the expected value of $\langle d \rangle$ in a Polya urn experiment, $\langle d \rangle_{dr}$ , the expected value of $\langle d \rangle$ in a permutation of order $n$ , and the left $p$ -values ( $\mathbb{P}$ ) of tests on $\langle d \rangle$ : randon permutation ( $rp$ ), Polya urn ( $Pu$ ) and die rolling ( $dr$ ). Each column of $p$ -values has been adjusted with a Holm correction; the original $p$ -value is shown below the adjusted $p$ -value. The unit "adj. Jangs." stands for adjusted number of Janguages.	y of 1 e tot dom $\langle d \rangle_{m_1}$ olya	the statist. all frequen permutati $_{xx}$ , the me urn $(Pu)$ he adjuste	ical informat cy, $m$ , the n ion, $\langle d \rangle_{Pu}$ , th aximum valu and die rolli d $p$ -value. T	umber of umber of ne expect e of $\langle d \rangle$ and $\langle dr \rangle$ .	ataba f non- ed va in a f Eack	se, kinc-zero pr lue of $\langle$ bermutc colum	1, the secobability of $d$ in a subsection of $p^{-1}$ stands	equence ty order Polya un of orde values h	length $r$ s, $\langle d \rangle$ , t rn exper r $n$ , and as been steed number of $r$ is the different contraction of the steed number of the steed number of $r$ is the	, the sequence length $n$ , structure (struct obability orders, $\langle d \rangle$ , the average swap d $d \rangle$ in a Polya urn experiment, $\langle d \rangle_{dr}$ , the enhedron of order $n$ , and the left $p$ -values $n$ of $p$ -values has been adjusted with a $H$ stands for adjusted number of languages	re (stru- e swap $\lambda_{dr}$ , the $p$ -values with a	Summary of the statistical information by database, kind, the sequence length $n$ , structure (struct.) and unit of measuremer y: $F$ , the total frequency, $m$ , the number of non-zero probability orders, $\langle d \rangle$ , the average swap distance, $\langle d \rangle_{rp}$ , the expected in a random permutation, $\langle d \rangle_{Pu}$ , the expected value of $\langle d \rangle$ in a Polya urn experiment, $\langle d \rangle_{dr}$ , the expected value of $\langle d \rangle$ in a permutohedron of order $n$ , and the left $p$ -values ( $\mathbb{P}$ ) of tests on $\langle d \rangle$ : randon $(rp)$ , Polya urn ( $Pu$ ) and die rolling ( $dr$ ). Each column of $p$ -values has been adjusted with a Holm correction; the original hown below the adjusted $p$ -value. The unit "adi. larges" stands for adjusted number of languages.	f measuremen, the expecte; of $\langle d \rangle$ in a d on $\langle d \rangle$ : randon; the originant
Database	Kind		Struct.	Unit	F	m m	$\langle p \rangle$		$\langle d \rangle_{Pu}$	$\langle d \rangle_{dr}$	$\langle d \rangle_{max}$	$\mathbb{P}_{rp}$	$\mathbb{P}_{Pu}$	$\mathbb{P}_{dr}$
WALS	dominant order	က	SOV	langs.	1187	9	0.76	1.08	1.29	1.5	1.5	⊣	0.12	$< 1.5 \cdot 10^{-6}$
												0.18	0.01	$<1\cdot 10^{-7}$
WALS	dominant order	3	VOX	langs.	333	ಬ	1.08	1	1.29	1.5	1.5	1	0.59	$< 1.5 \cdot 10^{-6}$
												0.7	0.12	$< 1 \cdot 10^{-7}$
Hammarström	dominant order	3	SOV	langs.	5128	9	0.82	1.12	1.29	1.5	1.5	П	0.16	$< 1.5 \cdot 10^{-6}$
												0.18	0.02	$< 1 \cdot 10^{-7}$
Hammarström	dominant order	3	SOV	families	340	9	0.74	0.85	1.29	1.5	1.5	Н	0.1	$< 1.5 \cdot 10^{-6}$
												0.27	$9.2 \cdot 10^{-3}$	$< 1 \cdot 10^{-7}$
Lev-Ari	experiments	3	SOV	frequency	13985	9	0.83	1.04	1.29	1.5	1.5	П	0.16	$< 1.5 \cdot 10^{-6}$
												0.2	0.02	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CP	3	SOV	frequency	3206	9	0.72	0.91	1.29	1.5	1.5	0.43	0.1	$< 1.5 \cdot 10^{-6}$
												0.03	$8.6 \cdot 10^{-3}$	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	က	SOV	frequency	3139	5	0.67	0.87	1.29	1.5	1.5	0.73	0.07	$< 1.5 \cdot 10^{-6}$
												0.07	$5.3\cdot 10^{-3}$	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CDS	3	SOV	frequency	398	9	0.99	1.15	1.29	1.5	1.5	П	0.39	$< 1.5 \cdot 10^{-6}$
												0.17	0.06	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	3	OVI	frequency	138	4	0.83	1.1	1.29	1.49	1.5	П	0.15	$< 1.5 \cdot 10^{-6}$
												0.2	0.02	$< 1 \cdot 10^{-7}$
Leela	Malayalam	က	SOV	frequency	3763	4	1.16	1.34	1.29	1.5	1.5	0.43	0.59	$< 1.5 \cdot 10^{-6}$
												0.03	0.2	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	langs.	249	18	2.84	2.59	2.89	2.99	3	1	0.59	$< 1.5 \cdot 10^{-6}$
												6.0	0.24	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	genera	322	18	2.83	2.69	2.89	2.99	3	1	0.59	$< 1.5 \cdot 10^{-6}$
												0.78	0.19	$< 1 \cdot 10^{-7}$
Dryer	dominant order	4	$_{ m nAND}$	adj. langs.	212	18	2.81	2.77	2.89	2.99	3	⊣	0.59	$< 1.5 \cdot 10^{-6}$
												0.55	0.12	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu CP	4	SOVI	frequency	320	$\mathbf{c}$	1.43	2.31	2.89	2.99	3	0.28	$< 1.5 \cdot 10^{-6}$	$< 1.5 \cdot 10^{-6}$
												0.02	$< 1 \cdot 10^{-7}$	$< 1 \cdot 10^{-7}$
Leela	Hindi-Urdu	4	SOVI	frequency	311	4	1.28	2.27	2.89	2.99	3	0.22	$< 1.5 \cdot 10^{-6}$	$< 1.5 \cdot 10^{-6}$
												0.01	$< 1 \cdot 10^{-7}$	$< 1 \cdot 10^{-7}$

**Table 5.** Summary of the Wilcoxon signed rank test for each inequality over the whole dataset ("all") and for distinct subgroups determined by structure size (n) or the nature of the word order frequencies ("dominant" for word order preferences across languages and "corpus" for word order frequencies for the corpus of a specific language). V is the value of the statistic of the Wilcoxon signed rank test and p is the corresponding p-value. Given an inequality, we do not control for multiple corrections (with a Holm correction) as the aim of this table is not to fish for significance across conditions; this table is essentially provided for the sake of completeness.

		all		n = 3	n	=4	d	ominant		corpus
	$\overline{V}$	p	$\overline{V}$	p	$\overline{V}$	p	$\overline{V}$	p	$\overline{V}$	p
$\overline{H < H_{Pu}}$										
$\langle d \rangle < \langle d \rangle_{Pu}$	0	$3.1\cdot 10^{-5}$	0	$9.8\cdot 10^{-4}$	0	0.03	0	$7.8\cdot 10^{-3}$	0	$3.9\cdot 10^{-3}$
$\langle d \rangle < \langle d \rangle_{rp}$	17	$6.2\cdot10^{-3}$	1	$2\cdot 10^{-3}$	6	0.41	12	0.41	0	$3.9\cdot10^{-3}$

#### 6. Discussion

#### 6.1. Theory

We have presented entropy minimization and swap distance minimization as separate principles but they are related via the scores that we have used for each. H,  $\bar{S}$ , and  $\langle d \rangle$  are diversity scores: they hit zero, their minimum value when diversity is minimum (only one word is used) and hit their maximum value when orders are equally likely (this is true for  $\langle d \rangle$  at least for  $n \leq 7$ ). However,  $\langle d \rangle$  hits is maximum value in conditions where H is not maximum (Property 2.7). H and  $\bar{S}$  care only about the probabilities of each order while  $\langle d \rangle$  takes also into account the structure of the permutothedron. The relationship between entropy minimization and swap distance minimization can be understood via  $\bar{S}$ , that is another diversity score that only cares about word order probabilities as H does.  $\bar{S}$  has appeared in various properties of  $\langle d \rangle$ : involved in an upper bound for  $\langle d \rangle$  (Property 2.6) or in the expected value of  $\langle d \rangle$  (Property 2.10). Indeed, both H and  $\langle d \rangle$  are related via the Rényi entropy, that is defined as [40]

$$R_{\alpha} = \frac{1}{1-\alpha} \log \left( \sum_{i=1}^{n} p_i^{\alpha} \right).$$

H and S (or  $\bar{S}$ ) are particular cases of Rényi entropy because

$$\lim_{\alpha \to 1} R_{\alpha} = H$$

$$R_2 = -\log(S).$$

Very similar arguments can be made with Tsallis entropy [41]. Therefore, entropy minimization in word order could be an implication of the more fundamental principle of swap distance minimization. Alternatively, the principle of entropy minimization and the principle of swap distance minimization may be implications of the minimization of some generalized entropy that is aware of the permutohedron structure. These possibilities should be the subject of future research.

#### 6.2. Experiments

Here we have investigated the manifestation of word entropy minimization and swap distance minimization in word order applying a novel score for swap distance minimization.

Entropy minimization We have found super strong support for entropy minimization as expected by generalizing the principle of word entropy minimization [5, 6] with respect to the die rolling null hypothesis and also, when n = 4 with respect to the Polya urn null hypothesis. When n = 3, there is still a significant signal of entropy minimization under the Polya urn null hypothesis that is unveiled by a global analysis over all structures with n = 3.

Swap distance minimization With respect to the null hypothesis of a die rolling experiment, we have found super strong support for swap distance minimization but this is a rather trivial result because the word orders of a certain syntactic structure are not equally likely and we have already provided support for entropy minimization in word order variation. Interestingly, the Polya urn is able to produce a more skewed distribution of word orders but its dynamics is not driven explicitly by any cost minimization principle. According to the Polya urn null hypothesis, we have found a weak signal of the principle in triplets but a strong signal for the SOIV structure. In triplets, the actual  $\langle d \rangle$  was significantly small with respect to a Polya urn experiment only in the SOV structure in Hindu-Urdu. Such a signal is weak but significant because of a tendency for  $\langle d \rangle < \langle d \rangle_{Pu}$  when n=3. In quadruplets, the actual  $\langle d \rangle$  was significantly small compared to a Polya experiment for the SOVI structure with and without CP [3].

The fact that entropy or  $\langle d \rangle$  is significantly small with respect to die rolling or Polya urn does not imply that word orders are constrained to have low H or  $\langle d \rangle$ . It could simply mean that actual word order frequencies are more skewed than expected by these stochastic processes. As researchers have tried to reduce the explanation of the frequency distribution of linguistic units in languages to simplistic stochastic processes since G. K. Zipf's times [42, 43, 44], we invite skeptical researchers about the reality of swap distance minimization to use any mechanism they wish for the origins of the frequency distribution. Here we have taken the best model for the N word order probabilities, i.e. the N relative frequencies obtained from the dataset, to build the random permutation null hypothesis and to demonstrate that swap distance minimization effects cannot be reduced to the distribution of word order frequencies.

From a cognitive or linguistic standpoint, the random permutation null hypothesis is able to control for other word order principles or mechanisms. We do not need to know exactly which. In particular, the random permutation null hypothesis addresses the following question: if we accept that word order frequencies are determined by other word order principles (principles other than swap distance minimization) and

accordingly preserve the frequency distribution, can we still see the action of swap distance minimization? The finding is that there is a weak but significant sign of the action of swap distance minimization across conditions. Therefore, we conclude that, even when the entropy of word orders is kept constant as in the random permutation null hypothesis, we still detect the effect of swap distance minimization. Therefore, neither the general principle of entropy minimization [5, 6] nor alternatives to swap distance minimization can fully explain word order preferences.

It could still be argued that swap distance minimization is a rather weak principle compared to entropy minimization. Indeed, the p-values produced by the permutation test (Table 4) were very high for n=3. However, this may be caused by a lack of statistical power of the test for n=3. In section 4.2, we have shown that the minimum p-value is 1/60 because of the large number of symmetries and that this lower bound increases as m reduces from m=5 down to 1. Thus, symmetries and the difficulty of languages to cover the whole permutation space may be shadowing the action of swap distance minimization in languages. The paradox is that, in case that swap distance was minimized, namely  $\langle d \rangle = 0$ , the left p-value of the permutation test would be  $\mathbb{P}=1$  independently of the value of n. Thus the permutation test cannot detect swap distance minimization if m is small (Property 4.2), although that small number can be interpreted as a consequence of swap distance minimization. Even when n=4, we failed to find evidence of swap distance minimization for the nAND structure (Table 4). The fact that  $\langle d \rangle > \langle d \rangle_{rp}$  in nAND structures suggests that swap distance minimization is not acting in these structures. However, notice that the two most frequent orders, nAND and its mirror, DNAn are located at maximum distance in the permutohedron (Fig. 3). We hypothesize that the principle is still acting but that  $\langle d \rangle$  is not well-suited to capture its effects when variants are produced from distant sources. Remember that  $\langle d \rangle$  is maximized when only two orders have non-zero probability and they are located at maximum distance (Property 2.7). For the nAND structure,  $\langle d \rangle > 2.8$  while, theoretically,  $\langle d \rangle \leq 3$  (Property 2.8). That hypothesis should be the subject of future research.

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#### Appendix A. Mahonian numbers

Here we revisit a well-known property of Mahonian numbers that we have not found proven in standard publications.

Property Appendix A.1. T(n,k) can be defined as

$$T(n,k) = \begin{cases} 1 & \text{if } n = 1 \text{ and } k = 0 \\ 0 & \text{if } n = 1 \text{ and } k \neq 0 \\ \sum_{j=0}^{n-1} T(n-1, k-j) & \text{otherwise} \end{cases}$$

*Proof.* This property can be inferred from a more complex definition [45]#,

$$T(n,k) = \begin{cases} 1 & \text{if } n = 1 \text{ and } k = 0 \\ 0 & \text{if } n < 0, k < 0 \text{ or } k > \binom{n}{2} \\ \sum_{j=0}^{n-1} T(n-1, k-j) \\ T(n, k-1) + T(n-1, k) - T(n-1, k-n). \end{cases}$$

Here we provide a proof for the statement of the property. First, the Mahonian numbers T(n, k) are defined as the coefficients of the expansion of the product [45]

$$\prod_{k=0}^{n-1} \sum_{j=0}^{k} x^{j},$$

namely

$$\prod_{k=0}^{n-1} \sum_{j=0}^{k} x^{j} = \sum_{i} T(n,k) x^{k}.$$

For the case n = 1, the LHS equals 1, which implies for the RHS that T(1,0) = 1 and T(1,k) = 0 for every other  $k \neq 0$ . For n > 1,

 $\sharp$  See the section Formula and the note "From Andrew Woods, Sep 26 2012, corrected by Peter Kagey, Mar 18 2021"

$$\sum_{k} T(n,k)x^{k} = \prod_{k=0}^{n-1} \sum_{j=0}^{k} x^{j}$$

$$= \left(\prod_{k=0}^{n-2} \sum_{j=0}^{k} x^{j}\right) \cdot \left(\sum_{j=0}^{n-1} x^{j}\right)$$

$$= \left(\sum_{k} T(n-1,k)x^{k}\right) \cdot \left(\sum_{j=0}^{n-1} x^{j}\right)$$
substitution
$$= \sum_{k} \sum_{j=0}^{n-1} T(n-1,k)x^{k+j}$$

$$= \sum_{k} \sum_{j=0}^{n-1} T(n-1,k-j)x^{k}$$
change of variable

and then, as the coefficients of the polynomials on both sides of the equality must be the same, we finally obtain

$$T(n,k) = \sum_{j=0}^{n-1} T(n-1, k-j).$$

## Appendix B. A compact expression for $\langle d \rangle$ when n=3.

Next we will introduce a new expression for  $\langle d \rangle$  for n=3 that is computationally efficient and that leads to straightforward tight upper bounds for  $\langle d \rangle$ .

**Property Appendix B.1.** In triplets,  $\langle d \rangle$  can be expressed equivalently as

$$\langle d \rangle = \frac{3 - \Delta}{2},$$

where  $\Delta$  can be defined in two equivalent ways. First,

$$\Delta = 3[P(0) - P(3)] + P(1) - P(2).$$

Second, let  $q_i = p_i - p_{\bar{i}}$ . Then

$$\Delta = 3 \sum_{i=1}^{3} q_i^2 - 2(q_1 q_3 - q_1 q_2 - q_2 q_3)$$

$$= \sum_{i=1}^{3} q_i^2 + (q_1 + q_2)^2 + (q_2 + q_3)^2 + (q_1 - q_3)^2.$$
(B.1)

*Proof.* In triplets,  $\langle d \rangle$  is

$$\langle d \rangle_a = P(1) + 2P(2) + 3P(3).$$

Since

$$\langle d \rangle + 2P(1) + P(2) = 3[P(1) + P(2) + P(3)]$$
  
= 3[1 - P(0)].

 $\langle d \rangle$  is also

$$\langle d \rangle_b = 3[1 - P(0)] - 2P(1) + P(2).$$

Therefore

$$\langle d \rangle = \frac{1}{2} (\langle d \rangle_a + \langle d \rangle_b)$$

$$= \frac{1}{2} (3 - \Delta),$$

where

$$\Delta = 3[P(0) - P(3)] + P(1) - P(2).$$

In triplets, the definition of  $\langle d \rangle$  can be split as

$$\langle d \rangle_1 = \sum_{i=1}^3 p_i \langle d|i \rangle + \sum_{i=4}^6 p_i \langle d|i \rangle.$$

Applying Property 2.4 to the second summation,  $\langle d \rangle$  can be expressed as

$$\langle d \rangle_1 = \sum_{i=1}^3 p_i \langle d|i \rangle + \sum_{i=4}^6 p_i (3 - \langle d|\bar{i} \rangle)$$

$$= \sum_{i=1}^3 p_i \langle d|i \rangle + \sum_{i=1}^3 p_{\bar{i}} (3 - \langle d|i \rangle)$$

$$= \sum_{i=1}^3 \left[ (p_i - p_{\bar{i}}) \langle d|i \rangle + 3p_{\bar{i}} \right].$$

Applying Property 2.4 to the first summation,  $\langle d \rangle$  becomes

$$\langle d \rangle_2 = \sum_{i=1}^3 p_i (3 - \langle d|i \rangle) + \sum_{i=4}^6 p_i \langle d|i \rangle$$
$$= \sum_{i=1}^3 p_i (3 - \langle d|i \rangle) + \sum_{i=1}^3 p_{\bar{i}} \langle d|\bar{i} \rangle$$
$$= \sum_{i=1}^3 \left[ (p_{\bar{i}} - p_i) \langle d|\bar{i} \rangle + 3p_i \right].$$

Let

$$\Delta' = \sum_{i=1}^{3} (p_i - p_{\bar{i}})(\langle d|i\rangle - \langle d|\bar{i}\rangle).$$

Then

$$\langle d \rangle_1 + \langle d \rangle_2 = \Delta' + 3 \sum_{i=1}^3 (p_i + p_{\bar{i}})$$
$$= \Delta' + 3 \sum_{i=1}^6 p_i$$
$$= \Delta' + 3.$$

Finally, the fact that  $\langle d \rangle_1 = \langle d \rangle_2$ , leads to a new expression for  $\langle d \rangle$  as

$$\langle d \rangle = \frac{1}{2} (\langle d \rangle_1 + \langle d \rangle_2)$$

$$= \frac{3 + \Delta'}{2},$$

where

$$\Delta' = (p_1 - p_4)[p_2 + 2p_3 + 3p_4 + 2p_5 + p_6 - (3p_1 + 2p_2 + p_3 + p_5 + 2p_6)] + (p_2 - p_5)[p_1 + p_3 + 2p_4 + 3p_5 + 2p_6 - (2p_1 + 3p_2 + 2p_3 + p_4 + p_6)] + (p_3 - p_6)[2p_1 + p_2 + p_4 + 2p_5 + 3p_6 - (p_1 + 2p_2 + 3p_3 + 2p_4 + p_5)]$$

$$= -3(p_1 - p_4)^2 - 3(p_2 - p_5)^2 - 3(p_3 - p_6)^2 + (p_1 - p_4)[(p_3 - p_6) - (p_2 - p_5)] + (p_2 - p_5)[-(p_1 - p_4) - (p_3 - p_6)] + (p_3 - p_6)[(p_1 - p_4) - (p_2 - p_5)]$$

The substitution  $q_i=p_i-p_{\bar{i}}$  the last expression for  $\Delta'$  into

$$\Delta' = q_1(q_3 - q_2) + q_2(-q_1 - q_3) + q_3(q_1 - q_2) - 3\sum_{i=1}^{3} q_i^2$$
$$= 2(q_1q_3 - q_1q_2 - q_2q_3) - 3\sum_{i=1}^{3} q_i^2.$$

The substitutions

$$(q_1 + q_2)^2 = q_1^2 + 2q_1q_2 + q_2^2$$
$$(q_2 + q_3)^2 = q_2^2 + 2q_2q_3 + q_3^2$$
$$(q_1 - q_3)^2 = q_1^2 - 2q_1q_3 + q_3^2$$

lead to

$$\Delta' = -\left[ (q_1 + q_2)^2 + (q_2 + q_3)^2 + (q_1 - q_3)^2 + \sum_{i=1}^3 q_i^2 \right]$$
  
=  $-\Delta$ 

after some algebra.

The compact formula for  $\langle d \rangle$  above yields an alternative proof of Property 2.8 when n=3. Notice that

$$\langle d \rangle = \frac{3 - \Delta}{2} \le \frac{3}{2}$$

because  $\Delta \geq 0$  (recall Equation B.1).