

Universal formal asymptotics for localized oscillation of a discrete mass-spring-damper system of time-varying properties, embedded into a one-dimensional medium described by the telegraph equation with variable coefficients

Serge N. Gavrilov¹, Ilya O. Poroshin², Ekaterina V. Shishkina¹, and Yulia A. Mochalova¹

¹Institute for Problems in Mechanical Engineering RAS, St. Petersburg, Russia

²Peter the Great St. Petersburg Polytechnic University (SPbPU), Russia

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Abstract

We consider a quite general problem concerning a linear free oscillation of a discrete mass-spring-damper system. This discrete sub-system is embedded into a one-dimensional continuum medium described by the linear telegraph equation. In a particular case, the discrete sub-system can move along the continuum one at a sub-critical speed. Provided that the dissipation in both discrete and continuum sub-systems is absent, if parameters of the sub-systems are constants, under certain conditions (the localization conditions), a non-vanishing oscillation localized near the discrete sub-system can be possible. In the paper we assume that the dissipation in the damper and the medium is small, and all discrete-continuum system parameters are slowly varying functions in time and in space (when applicable), such that the localization condition is fulfilled for the instantaneous values of the parameters in a certain neighbourhood of the discrete sub-system position. This general statement can describe a number of mechanical systems of various nature. We derive the expression for the leading-order term of a universal asymptotics, which describes a localized oscillation of the discrete sub-system. In the non-dissipative case, the leading-order term of the expansion for the amplitude is found in the form of an algebraic expression, which involves the instantaneous values of the system parameters. In the dissipative case, the leading-order term for the amplitude, generally, is found in quadratures in the form of a functional, which depends on the history of the system parameters, though in some exceptional cases the result can be obtained as a function of time and the instantaneous limiting values of the system parameters. In previous studies, several non-dissipative particular cases of the problem under consideration are investigated using a similar approach, provided that only one parameter of the discrete-continuum system is a slowly time-varying non-constant quantity, whereas all other parameters are constants. We show that asymptotics obtained in previous studies are particular cases of the universal asymptotics. The existence of a universal asymptotics in the form of a function is a non-trivial fact, which does not follow from the summarization of the previously obtained results. Finally, we have justified the universal asymptotics by numerical calculations for some particular cases.

Keywords — trapped mode, linear wave localization, asymptotics, method of multiple scales, WKB approximation, space-time ray method, anti-localization of non-stationary waves, moving load

1 Introduction

Consider a linear free oscillation of a discrete mass-spring system. The discrete system is embedded into a one-dimensional continuum medium, described by the linear Klein-Gordon partial differential equation (PDE). Such a coupled discrete-continuum system is very similar to the one considered in classical Lamb study [Lamb \[1900\]](#), where the wave equation was used instead of the Klein-Gordon one. Lamb showed that embedding a linear mass-spring system into the medium leads to transmission of energy outside from the discrete sub-system by running waves. As a result, free oscillation of the mass-spring system vanishes in the same way as it happens, when a damper is added into an isolated mass-spring system. In our case, when the medium is described by the Klein-Gordon equation, the dynamics of the discrete-continuum system essentially depends on the system parameters. Under certain conditions (the localization conditions) fulfilled in some domain in the problem parameter space (the localization domain), a free oscillation of the mass-spring system remains non-vanishing, as we observe in the case of an isolated discrete mass-spring system. This happens due to the existence of a trapped mode [Abramyan et al. \[1994\]](#), [Glushkov et al. \[2011\]](#),

Indeitsev and Mochalova [2006], Indeitsev et al. [2007], Kaplunov and Sorokin [1995], Kuznetsov et al. [2002], McIver et al. [2003], Mishuris et al. [2020], Pagneux [2013], Porter [2007], Porter and Evans [2014], Ursell [1951, 1987], Voo [2008], to which oscillation, spatially localized near the discrete sub-system, corresponds. The trapped mode frequency belongs to the stop-band of dispersion characteristics for the continuum system; thus we deal with so-called strong localization Luongo [1992, 2001]. For the discrete sub-system moving along the continuum one at a constant sub-critical speed, the trapped mode also can exist Gavrilov and Indeitsev [2002], Gavrilov et al. [2022]. If the localization conditions are not fulfilled, then we observe a power-law vanishing oscillation of the mass-spring system Shishkina et al. [2023].

In the present paper, we extend the discussed above discrete-continuum (or composite) system in several aspects:

1. A small viscous friction in the continuum sub-system is taken into account; thus, the Klein-Gordon PDE transforms into the telegraph PDE Myint-U and Debnath [2007];
2. A small viscous friction in the discrete sub-system is taken into account; thus, we deal with a mass-spring-damper system;
3. Parameters of the discrete mass-spring-damper system are slowly time-varying functions;
4. Parameters of the continuum system are slowly varying functions of time and the spatial co-ordinate;
5. In a particular case, the discrete sub-system can move along the continuum one at a sub-critical speed¹, which is also assumed to be a slowly time-varying function.

Thus, the problem for the composite system with constant parameters we have discussed above at the beginning of the Introduction is the zeroth order problem for such an extended problem formulation. All necessary results for this zeroth order problem are obtained, e.g., in Gavrilov et al. [2022]. We require that the localization conditions are always fulfilled for the instantaneous values of the system parameters in a certain neighbourhood of the discrete sub-system. Therefore, one can expect that the motion of the discrete sub-system is close to the oscillation with the same frequency as it is observed for zeroth order problem (i.e., with an instantaneous value of the trapped mode frequency). The amplitude of the oscillation is expected to be a slowly time-varying function.

The extended problem formulation suggested above can describe a number of physical systems of a various nature. Most often, when considering similar problems in the framework of the classical mechanics, see, e.g., Abramian et al. [2017], Glushkov et al. [2011], Indeitsev et al. [2007], Kaplunov and Sorokin [1995], Kaplunov and Muravskii [1986], Kruse et al. [1998], Roy et al. [2018], it is assumed that the motion of the continuum system corresponds to transverse oscillation of a string on an elastic foundation, though longitudinal Shatskyi et al. [2021] or rotational Kaplunov [1986] oscillation of a rod can be under investigation. The electric current oscillation in the transmission lines also is a possible application Voo [2008]. In the paper, we generally assume that we deal with a mechanical system, but the results can be applied to other physical systems. The possible applications can be related to ageing processes Abramian et al. [2017], design of dynamic materials Blekhman and Lurie [2000], Lurie [2007], Rousseau et al. [2011], ice-induced vibrations Abramian and Vakulenko [2019], rain-wind vibrations Zhou and Yin [2020]. The problems, which involve the motion of the discrete sub-system along the continuum one, are the classical moving loads problems Frýba [1972], which have engineering applications, e.g., related to railways Roy et al. [2021]. Note that the model of a string on the Winkler foundation is frequently used in mechanics of systems with moving loads to obtain the test analytic solutions DasGupta [2023], Frýba [1972], Gil et al. [2020], Kaplunov and Muravskii [1986], Kruse et al. [1998], Roy et al. [2018].

The simplest particular case of a system with time-varying parameters is an isolated mass-spring system with a time-varying stiffness. The behaviour of this system can be approximately described by WKB approach, which is initially based on studies by Carlini Carlini [1818], Liouville Liouville [1837] and Green Green [1838, 2014]. It was demonstrated that in the first approximation, the amplitude is proportional to the inverse of the square root of the natural frequency. This formula now is well-known as the Liouville–Green approximation, later rediscovered by Rayleigh Rayleigh [1912], and as (J)WKB approximation after studies Brillouin [1926], Jeffreys [1925], Kramers [1926], Wentzel [1926]. The historical aspects are discussed in Feschenko et al. [1967], Fröman and Fröman [2002], McHugh [1971]. In Nayfeh book Nayfeh [2008], a formal asymptotic procedure based on the general ideas of the method of multiple scales, which allows one to obtain such a result in a simple manner, is suggested. For the reader's convenience, we consider a discrete mass-spring-damper system with time-varying properties using the Nayfeh multiple scales approach in Appendix A. This simple example helps us to introduce and discuss some peculiarities of the problem, which play an important role when considering the composite discrete-continuum system with variable properties.

¹The absolute value of the load speed is less than the speed of sound

The similar asymptotic approach in spirit of the method of multiple scales was first time applied to describe non-stationary localized oscillation in a coupled discrete-continuum system with time-varying properties in [Gavrilov and Indeitsev \[2002\]](#), where an oscillation of an inertial load, non-uniformly moving along an infinite string on the Winkler foundation, was considered. The speed of the load was assumed to be a slowly time-varying function. Though the model of a string on the Winkler foundation is widely used in engineering, and such a problem has many important applications, before [Gavrilov and Indeitsev \[2002\]](#) there were no approximate solution of a simple structure. Indeed, this is an essentially non-stationary problem, which takes into account both an inertial character of the load [Kaplunov \[1986\]](#) and a non-uniform regime of motion [Gavrilov \[1999\]](#), [Kaplunov and Muravskii \[1986\]](#). The method proposed in [Gavrilov and Indeitsev \[2002\]](#) can be treated as an extension of the Nayfeh formal procedure of WKB approach, which deals with an ordinary differential equation (ODE), to the case of a dispersive hyperbolic PDE with two independent variables coupled with ODE. Now we have realized that the assumed representation for the solution in [Gavrilov and Indeitsev \[2002\]](#) (an asymptotic ansatz) has the structure very similar to the one used in the framework of the space-time ray method [Babich and Buldyrev \[2009\]](#), [Babich et al. \[2002\]](#). The free oscillation was assumed in the form, which we call a single-mode ansatz, corresponding to the frequency of a localized oscillation. To describe a non-stationary free oscillation in the discrete sub-system, a problem for a PDE, coupled with an ODE, in the first approximation was asymptotically reduced to a linear quite lengthy ODE with variable coefficients of the first order for the unknown amplitude. This is an essential simplification compared with the general space-time ray method procedure, which is related to the fact that we deal with an oscillation localized near the discrete sub-system only, and do not try to continue the solution outside. Finally, for an arbitrary law of the load speed variation, the oscillation amplitude was found in a form of a simple algebraic expression, which depends on the instantaneous value of the speed. The expression obtained involves the unknown initial amplitude and phase (or, equivalently, the complex amplitude). To find the unknown constant, the obtained asymptotic expansion was matched with the expression obtained by the method of stationary phase applied to the same system with constant parameters equal to the corresponding initial values. The obtained asymptotic solution was verified through the independent numerical calculations, and an excellent agreement was shown.

In [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#), the described above approach was applied to various particular cases of the general system considered in this paper, as well as to a more complicated problem where the continuum system was a Bernoulli-Euler beam [Shishkina et al. \[2019\]](#). The resonant case, i.e., passage through a resonance, was considered in [Shishkina et al. \[2020\]](#). In all these studies, the only one of the composite system parameters was assumed to be a function depending on time, whereas all other parameters were taken as constants. Again, the final expressions for the amplitude were found as the algebraic expressions, which are valid for an arbitrary law of the parameter variation. In particular, in [Gavrilov et al. \[2019b\]](#) a localized oscillation of a string with time-varying tension was considered. In the last case, the continuum system is described by PDE with time-varying coefficients, whereas in all other studies the corresponding PDE can be reduced to a PDE with constant coefficients. The asymptotics obtained in such a way is so-called formal asymptotics, i.e. it satisfies the corresponding equations and initial conditions up to the certain asymptotic accuracy. Moreover, when constructing the asymptotics, a number of not well grounded assumptions were used. The most voluntaristic of them is the possibility to represent a non-stationary oscillation of a system with time-varying parameters in the form of a single-mode ansatz, which corresponds to “a pure localized oscillation” at a given frequency. Dealing with PDE with time-varying coefficients, we do not even have the orthogonality conditions, which allow one to separate oscillations with different frequencies. Another question is the validity of the matching procedure for two asymptotics, which were obtained by entirely different approaches. Thus, in our opinion, the analytic results should be at least justified by independent numerical calculations. This was done in the framework of every problem for various regimes of parameters time-varying, and various loadings.

The current paper generalizes the analytic work in series of papers [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#). Compared to previous works, the novelty of this paper is assured by the following results:

1. In previous studies, *only one* parameter of the composite system was assumed to be a slowly time-varying quantity, whereas all other parameters were constants. Now *each parameter is an independent slowly time-varying function of time, and additionally each parameter of the continuum system is a slowly varying function of the spatial co-ordinate*. Nevertheless, in the non-dissipative case, we still obtain the leading-order term of the expansion for the amplitude of localized oscillation as *an algebraic expression, which involves the instantaneous limiting values of the system parameters independently varying accordingly to unknown arbitrary laws*. To justify the applicability of the formal asymptotics, a particular case of the composite system with two independently time-varying parameters is investigated numerically, and an excellent agreement was demonstrated, see Sect. 7.1.
2. In previous studies, we did not introduce the dissipation. Now, the small viscous friction in both sub-systems

is taken into account. In the dissipative case, we generally obtain the leading-order term of the expansion for the amplitude *in quadratures* as a functional, which depends on the history of the system parameters, though in some exceptional cases the result can be found as a function of time and the instantaneous limiting values of the system parameters. A particular case of the system with a time-varying parameter and a viscous friction in a discrete sub-system is investigated numerically, and an excellent agreement is demonstrated, see Sect. 7.2.

3. In the present paper, the corresponding asymptotics is obtained in the form of a universal formula. This formula involves all previous results of studies [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#) as very particular cases, see Sect. 6. This result is not the summarization of previously obtained results, but it is derived due to the mathematical trick firstly suggested by Poroshin in [Gavrilov et al. \[2022\]](#). The trick is related to the lucky choice of variables depending on the system parameters, which allows us to guess how to represent, in the non-dissipative case, the lengthy right-hand side of the first approximation equation in the form of the exact derivative for a function of the variables, see Sect. 6.6.
4. We demonstrate that the excellent practical applicability of the constructed formal asymptotics is related, in our opinion, with the phenomenon of the anti-localization of non-stationary linear waves [Gavrilov and Shishkina \[2024\]](#), [Gavrilov et al. \[2023\]](#), [Shishkina and Gavrilov \[2023\]](#), [Shishkina et al. \[2023\]](#), see Sect. 7.3.

To understand better the difference between the cases when the amplitude can be found as a function of instantaneous values of the system parameters, or as a functional depending on the history of the parameters, we recommend to the reader to look through the material of Appendix A. The fact that the leading-order term for the amplitude can be found in the form of a function in the case of multiple time-varying parameters is surprising for us. Apparently, from the mathematical point of view, it follows from some unobvious properties of the governing equations. On the other hand, from the engineer's perspective, the result in the form of a function, which is an algebraic expression, gives a comprehensive understanding of the system behaviour, whereas the result in the form of a functional requires additional calculations to visualize it even for a given history of the system parameters, see, e.g., Sect. 7.1, 7.2, respectively.

2 Mathematical formulation

Consider the following coupled system of equations:

$$\frac{d}{dt} \left(M \frac{d\mathcal{U}}{dt} \right) + 2\epsilon\Gamma \frac{d\mathcal{U}}{dt} + K\mathcal{U} = -P(t) + p(t), \quad (2.1)$$

$$\frac{\partial}{\partial x} \left(\mathcal{J} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial t} \left(\rho \frac{\partial u}{\partial t} \right) - 2\epsilon\gamma \frac{\partial u}{\partial t} - ku = -P(t)\delta(x - \ell(t)). \quad (2.2)$$

We assume that Eqs. (2.1), (2.2) describe the discrete and continuum sub-systems of the composite system under investigation, respectively. Equation (2.1) is an ordinary differential equation, which, clearly, describes a motion of a mass-spring-damper system. Here t is time, M is the mass, K is the stiffness, $\epsilon\Gamma$ is the damping. The quantity $\epsilon > 0$ is the dimensionless formal small parameter; thus, the damping is assumed to be small. The quantity $p(t)$ in the right-hand side of Eq. (2.1) is a given external force, whereas $P(t)$ is an unknown internal interaction force between the sub-systems. The assumed restrictions on function $p(t)$ are discussed in detail in what follows in this section. The parameters of the discrete sub-system are assumed to be given smooth slowly time-varying functions of ϵt :

$$K = K(\epsilon t), \quad M = M(\epsilon t), \quad \Gamma = \Gamma(\epsilon t), \quad (2.3)$$

such that

$$M \geq 0. \quad (2.4)$$

The first term in Eq. (2.1), accounting effects related to a time-varying mass, is written in the form, which assumes a mass supply into the discrete sub-system accompanied by zero momentum supply, see [Irschik and Holl \[2004\]](#), [Levi-Civita \[1928a,b\]](#), [Mescherskiy \[1952\]](#), [Zhilin \[2003\]](#). Analogous formulation is used, e.g., in studies [Abramian et al. \[2014\]](#), [Abramyan and Vakulenko \[2011\]](#).

Remark 2.1. Alternatively, one can assume that the term $2\epsilon\Gamma \frac{d\mathcal{U}}{dt}$ in the right-hand side of Eq. (2.1) is not related to a dissipation itself, but the sum of two terms

$$\frac{dM}{dt} \frac{d\mathcal{U}}{dt} + 2\epsilon\Gamma \frac{d\mathcal{U}}{dt} \quad (2.5)$$

in Eq. (2.1) represents a reactive force, which is proportional to the particle velocity of the discrete sub-system, see Eq. (3.22) in [Irschik and Holl \[2004\]](#), [Holl et al. \[1999\]](#). In such a way, we can take into account non-zero momentum supply into the discrete sub-system. In the framework of such a physical interpretation, quantity Γ is related to the rate of the momentum supply. Accordingly, in the paper, we generally do not restrict ourselves with the case $\Gamma \geq 0$, which corresponds to the presence of the viscous friction in the discrete sub-system.

In the same way, as in previous paper [Gavrilov et al. \[2022\]](#), we assume that the stiffness K can be positive (stabilizing), negative (destabilizing), or zero:

$$K \begin{matrix} \leq \\ \equiv \\ > \end{matrix} 0. \quad (2.6)$$

The destabilizing effect from the discrete spring with a negative stiffness can be compensated by stabilizing elastic foundation with $k > 0$. The stability condition for the coupled composite system is Eq. (B.19). Discussing destabilizing springs is quite natural in the context of localized oscillation, since the presence of a pure-spring inclusion with $K < 0$ can lead to linear waves localization, see again Eq. (B.19). A destabilizing spring can be considered as an oversimplified model of a crack, which can also cause the linear wave localization, see [Glushkov et al. \[2006\]](#), [Porter and Evans \[2014\]](#). Note, that destabilizing springs are used when constructing meta-materials [Chronopoulos et al. \[2015\]](#), [Danh and Ahn \[2014\]](#), [Huang et al. \[2014\]](#), [Li et al. \[2013\]](#), [Oyelade et al. \[2017\]](#), [Pasternak et al. \[2014\]](#), [Wu et al. \[2014\]](#).

Equation (2.2) can describe a number of mechanical systems of a various nature. Most often, see, e.g., [Abramian et al. \[2017\]](#), [DasGupta \[2023\]](#), [Glushkov et al. \[2011\]](#), [Indeitsev et al. \[2007\]](#), [Kaplunov and Sorokin \[1995\]](#), [Kaplunov and Muravskii \[1986\]](#), [Kruse et al. \[1998\]](#), [Roy et al. \[2018\]](#) it is assumed that the motion of the continuum system corresponds to transverse oscillation of a string on an elastic foundation, though longitudinal [Shatskyi et al. \[2021\]](#) or rotational [Kaplunov \[1986\]](#) oscillation of a rod can be under investigation. The corresponding displacement is $u(x, t)$, where x is a spatial co-ordinate. The parameters of the continuum sub-system are assumed to be given smooth, slowly varying functions of ϵx and ϵt :

$$\mathcal{T} = \mathcal{T}(\epsilon x, \epsilon t), \quad \rho = \rho(\epsilon x, \epsilon t), \quad k = k(\epsilon x, \epsilon t). \quad \gamma = \gamma(\epsilon x, \epsilon t), \quad (2.7)$$

such that

$$\mathcal{T} > 0, \quad \rho \geq 0, \quad k > 0. \quad (2.8)$$

Here k has the meaning of the elastic foundation stiffness, ρ is the mass density (or the moment of inertia) depending on the nature of physical processes under consideration. Quantity \mathcal{T} is the string tension or the corresponding elastic modulus (depending on the physical interpretation). Additionally, the damping in foundation per unit length $\epsilon\gamma$ is assumed to be a small quantity. Again, the second term in Eq. (2.2) accounting effects related to a time-varying mass per unit length $\rho(\epsilon x, \epsilon t)$, is formulated in the form, which assumes supply of mass accompanied by zero supply of momentum, or the reactive force proportional to the particle velocity $\frac{du}{dt}$ as discussed in Remark 2.1 is under investigation. In the paper, we generally do not restrict ourselves with the case $\gamma \geq 0$, which corresponds to the presence of the viscous friction in the continuum sub-system.

In the paper, the case of the system under consideration, where

$$\Gamma = 0, \quad \gamma = 0 \quad (2.9)$$

is referred to as “the non-dissipative case”. The case, when Eq. (2.9) is not fulfilled, i.e.,

$$\Gamma \neq 0 \quad \text{or} \quad \gamma \neq 0 \quad (2.10)$$

is referred to as “the dissipative case”, even if $\Gamma < 0$ and/or $\gamma < 0$.

Though, in our previous papers [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#), it was assumed that we dealt with a transverse oscillation of a taut string, now we consciously do not specify the physical nature of the continuum sub-system. One of the reasons is that the case when \mathcal{T} depends on ϵx does not have a simple interpretation in the framework of such a model (and does have if we deal with a longitudinal or rotational oscillation). Accordingly, we do not specify dimensions, which can be different depending on the interpretation, when working with dimensional physical quantities. We do not use non-dimensional formulation since it is not useful in the context of problems with several varying parameters. Another reason is that in previous papers [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#) we used various kinds of the non-dimensionalization.

At the instant $t = 0$ an external given force $p(t)$ emerges, which is applied to the discrete sub-system. Simultaneously, the discrete sub-system starts to move along the string according to the following law:

$$\ell(t) = \ell(0) + \int_0^t v(\hat{\epsilon}t) d\hat{t}. \quad (2.11)$$

Here, $\ell(0)$ is a given initial position for the discrete sub-system and

$$v = v(\epsilon t) \quad (2.12)$$

is the speed of the sub-system (a given smooth slowly time-varying function). We assume a sub-critical regime for the motion of the discrete sub-system, i.e., for all t the following inequality must be satisfied in a certain neighbourhood of $x = \ell(t)$:

$$|v(\epsilon t)| < c(\epsilon x, \epsilon t), \quad (2.13)$$

where

$$c = \sqrt{\frac{\mathcal{J}}{\rho}} \quad (2.14)$$

is the local instantaneous value for the speed of the wave propagation (the speed of sound). The discrete and continuum sub-systems are kinematically coupled, namely, it is assumed that

$$\mathcal{U}(t) = u(\ell(t), t). \quad (2.15)$$

Accordingly, the unknown internal force on the continuum sub-system is expressed by the term $-P(t)\delta(x - \ell(t))$ in the right-hand side of Eq. (2.2), where $\delta(\cdot)$ is the Dirac delta-function. Thus, for $v \neq 0$ our problem transforms into a kind of moving load problem Frýba [1972].

Consider Eqs. (2.1), (2.2) in the co-moving with the discrete sub-system co-ordinates:

$$\xi = x - \ell(t), \quad \tau = t. \quad (2.16)$$

One has

$$\begin{aligned} \frac{\partial}{\partial x} &= (\cdot)', & \frac{\partial}{\partial t} &= (\dot{\cdot}) - v(\cdot)'; \\ \frac{\partial^2}{\partial x^2} &= (\cdot)'' , & \frac{\partial^2}{\partial t^2} &= (\ddot{\cdot}) + v^2(\cdot)'' - 2v(\cdot)' - \dot{v}(\cdot)', & \frac{\partial^2 u}{\partial x \partial t} &= (\dot{\cdot})' - v(\cdot)'' , \end{aligned} \quad (2.17)$$

where prime and overdot denote the partial derivatives with respect to ξ and τ , respectively. Now, equations (2.1), (2.2) can be rewritten as

$$(M\dot{\mathcal{U}})' + 2\epsilon\Gamma\dot{\mathcal{U}} + K\mathcal{U} = -P(\tau) + p(\tau), \quad (2.18)$$

$$(\mathcal{J} - \rho v^2)u'' + ((\mathcal{J} - \rho v^2)' + (\rho v)')u' + 2\rho v\dot{u}' - (\dot{\rho} - \rho'v)u - \rho\ddot{u} - 2\epsilon\gamma(\dot{u} - vu') - ku = -P(\tau)\delta(\xi), \quad (2.19)$$

where

$$\mathcal{U}(\tau) = u(0, \tau). \quad (2.20)$$

According to formulae, which are inverse of Eq. (2.17), for an arbitrary quantity $\mu(\epsilon x, \epsilon t)$ one has

$$\mu' = O(\epsilon), \quad \dot{\mu} = O(\epsilon), \quad \mu'' = O(\epsilon^2), \quad \ddot{\mu} = O(\epsilon^2), \quad \dot{\mu}' = O(\epsilon^2), \quad (2.21)$$

provided that Eq. (2.12) is true. Thus, provided that Eqs. (2.7), (2.12) are true, we can assume that quantities in Eq. (2.19) are such that

$$\mathcal{J} = \mathcal{J}(\epsilon\xi, \epsilon\tau), \quad \rho = \rho(\epsilon\xi, \epsilon\tau), \quad k = k(\epsilon\xi, \epsilon\tau), \quad \gamma = \gamma(\epsilon\xi, \epsilon\tau). \quad (2.22)$$

In what follows in the paper, we deal with basic equations (2.18), (2.19) formulated in the co-moving co-ordinates.

The following Hugoniot conditions must be satisfied at $\xi = 0$:

$$[u] = 0, \quad (2.23)$$

$$[u'] = -\frac{P(\tau)}{\mathcal{J} - \rho v^2}. \quad (2.24)$$

Here and in what follows $[\mu] \equiv \mu(\xi + 0) - \mu(\xi - 0)$ for any arbitrary quantity $\mu(\xi, \tau)$. The initial conditions for Eq. (2.19) can be formulated in the following form, which is conventional for distributions (or generalized functions) Vladimirov [1971]:

$$u|_{\tau < 0} \equiv 0. \quad (2.25)$$

In the paper, we assume that the external loading p is a pulse, which acts during some time, i.e., a given real integrable function (or a generalized function Vladimirov [1971]) of compact support Nikol'skii:

$$p(\tau) \equiv 0, \quad \tau < 0 \quad \text{or} \quad \tau > \tau_0 \quad (2.26)$$

for certain $\tau_0 > 0$.

Remark 2.2. In [Gavrilov et al. \[2019b\]](#) (see Sect. 6.3) we apply our method in the case, where $p(\tau)$ is an exponentially vanishing function and get an excellent agreement. Thus, in principle, one can try to use our results for a wider class of external pulse loadings.

In zeroth order approximation $\epsilon = 0$, Eq. (2.2) transforms into a linear Klein-Gordon equation with constant coefficients. The latter equation in the co-moving co-ordinate system (2.16), namely Eq. (2.19), has alternated coefficients and an additional term $2\rho v\dot{u}'$ in the left-hand side compared with Eq. (2.2). Equation (2.18) coincides with Eq. (2.1), wherein $t = \tau$. Considering Eqs. (2.18), (2.19) at $\epsilon = 0$, we see that the properties of the corresponding system can be completely characterized by an n -tuple \mathcal{P} of six ($n = 6$) real parameters, namely

$$\begin{aligned} \mathcal{P} &\stackrel{\text{def}}{=} (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6) \in \mathbb{R}^6; \\ \mathcal{P}_1 &= K, \quad \mathcal{P}_2 = M, \quad \mathcal{P}_3 = \mathcal{T}, \quad \mathcal{P}_4 = \rho, \quad \mathcal{P}_5 = k, \quad \mathcal{P}_6 = v. \end{aligned} \quad (2.27)$$

Under certain conditions (the localization conditions, see [Remark B.3](#)), which are fulfilled in a certain domain (the localization domain) $\mathbb{L} \subset \mathbb{R}^6$ in the six-dimensional problem parameter space, the free oscillation of the mass-spring system remains non-vanishing, as we observe in the case of an isolated discrete mass-spring system. This happens to due to the existence of a unique trapped mode, to which a spatially localized near the discrete sub-system non-stationary oscillation corresponds, see [Appendix B.2](#). For $\xi = 0$, provided that the external loading is a pulse, this localized oscillation asymptotically dominates over all other motions [Shishkina et al. \[2023\]](#). The frequency of this localized oscillation is $\Omega_0 > 0$ defined by Eq. (B.21). The necessary auxiliary results related to the stationary and non-stationary problems concerning a localized oscillation in the system described by Eqs. (2.18)–(2.19) are derived, e.g., in [Gavrilov et al. \[2022\]](#). For convenience of the reader, in [Appendix B](#), we reproduce the basic formulae in the dimensional form used in the present paper.

To characterize completely all variable properties of the system described by Eqs. (2.18), (2.19) in the case $\epsilon > 0$, additionally to \mathcal{P} , we need to introduce a $\tilde{\mathcal{P}}$ -tuple ($\tilde{n} = 2$)

$$\tilde{\mathcal{P}} \stackrel{\text{def}}{=} (\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2) \in \mathbb{R}^2; \quad \tilde{\mathcal{P}}_1 = \Gamma, \quad \tilde{\mathcal{P}}_2 = \gamma. \quad (2.28)$$

In the latter case, the tuples \mathcal{P} and $\tilde{\mathcal{P}}$ are functions of ϵx and $\epsilon \tau$.

Remark 2.3. We consider the case of a small dissipation only, since we want the existence of the trapped mode to be possible in the zeroth order system with $\epsilon = 0$.

We plan to evaluate the displacements $\mathcal{U}(\tau) = u(0, \tau)$ of the discrete subsystem only. Namely, we look for the asymptotic solution for \mathcal{U} under the following conditions:

- $\epsilon = o(1)$;
- $\tau = O(\epsilon^{-1})$, thus,

$$T \stackrel{\text{def}}{=} \epsilon \tau = O(1); \quad (2.29)$$

- The instantaneous values of the zeroth order system parameters are such that

$$\mathcal{P}(\epsilon \xi, \epsilon \tau) \in \mathbb{L} \subset \mathbb{R}^6 \quad (2.30)$$

for ξ from certain neighbourhood of zero and all $\tau \geq 0$.

Provided that these assumptions are true, we assume that the localized oscillation with frequency $\Omega_0(T)$ is an asymptotically dominant component of $u(0, \tau)$ as well as it is for $\epsilon = 0$. The frequency $\Omega_0(T)$ is defined by the same equation (B.21), as well as in the case $\epsilon = 0$, wherein the instantaneous values of system parameters, taken at $\xi = 0$ for the continuum sub-system, are under consideration.

If the instantaneous value of the vector of the zeroth order system parameters (2.30) at $\xi = 0$ leaves the localization domain, then we expect very fast vanishing of the displacements $\mathcal{U}(t)$. In a particular case, this was demonstrated numerically in [Gavrilov et al. \[2022\]](#). Later, we have shown that at least for $\epsilon = 0$ the displacement $\mathcal{U}(t)$ in the composite system vanishes asymptotically faster than in the corresponding pure continuum system. This happens due to the existence of the wave phenomenon, which we have called the anti-localization of non-stationary linear waves [Gavrilov and Shishkina \[2024\]](#), [Gavrilov et al. \[2023\]](#), [Shishkina and Gavrilov \[2023\]](#), [Shishkina et al. \[2023\]](#).

Remark 2.4. In [Gavrilov et al. \[2022\]](#) (see Sect. 6.5), besides a free localized oscillation, it was considered the forced oscillation caused by a class of loadings, which can be represented as a superposition of a pulse $\hat{p}(\tau)$, which satisfies Eq. (2.26), and a number of harmonics with slowly time-varying frequencies and amplitudes:

$$p(\tau, T) = H(\tau) \left(\frac{\hat{p}(\tau)}{2} + \sum_{i=1}^N p^{(\Omega_i)}(T) \exp \left(-i \int_0^\tau \Omega_i(\hat{T}) d\hat{T} \right) \right) + \text{c.c.}, \quad (2.31)$$

where the notation c.c. denotes the complex conjugate terms for the whole right-hand side, $H(\cdot)$ is the Heaviside step-function,

$$\Omega_i \geq 0, \quad (2.32)$$

$$\Omega_i \neq \Omega_j \quad \forall t \quad \text{if} \quad i \neq j \quad \text{for} \quad i, j = \overline{1, N}. \quad (2.33)$$

Here, the amplitudes $p^{(\Omega_i)}(T)$ ($i = \overline{1, N}$) are given smooth complex-valued functions. The results concerning the forced oscillation can be straightforwardly transferred to the problem under consideration in this paper. On the other hand, taking into account the forced oscillation complicates the calculations. Thus, in this paper, we deal with pulse loadings and a free localized oscillation only.

Remark 2.5. The special limiting case

$$\rho = 0 \quad (2.34)$$

corresponds to a discrete sub-system attached to an inertialess waveguide. In this case, the continuum sub-system behaves as an additional distributed spring attached to the discrete sub-system. The similar models were used in studies [Dyńiewicz and Bajer \[2009\]](#), [Gavrilov et al. \[2016a\]](#), [Smith \[1964\]](#).

Remark 2.6. The formal limiting case when

$$\mathcal{T} = 0, \quad \rho = 0, \quad k = 0, \quad \gamma = 0 \quad (2.35)$$

corresponds to an isolated uncoupled mass-spring-damper system with time-varying properties described by Eq. (2.1) with $P = 0$. A free oscillation in such a system is considered in Appendix A. We recommend the reader to look through the material in Appendix A before reading the rest of the paper.

3 Derivation of the first approximation equation

For $\xi > 0$ and $\xi < 0$ we look for the solution in the form, which we called in [Gavrilov et al. \[2022\]](#) “the single-frequency ansatz”, or better to say, “the single mode ansatz” corresponding to the trapped mode frequency $\Omega_0(T)$:

$$u(\xi, \tau) = W(X, T) \exp \phi(\xi, \tau) + \text{c.c.}, \quad (3.1)$$

where

$$X = \epsilon \xi, \quad T = \epsilon \tau \quad (3.2)$$

are the slow spatial co-ordinate and the slow time; $\phi(\xi, \tau)$ such that

$$\phi' = i\omega(X, T), \quad \dot{\phi} = -i\Omega(X, T), \quad (3.3)$$

$$\lim_{X \rightarrow 0} \Omega = \Omega_0(T) \quad (3.4)$$

is the fast phase;

$$W(X, T) = \sum_{j=0}^{\infty} \epsilon^j W_j(X, T) \quad (3.5)$$

such that

$$\mathcal{W}(T) \stackrel{\text{def}}{=} \lim_{X \rightarrow 0} W(X, T), \quad (3.6)$$

$$\mathcal{W}_j(T) \stackrel{\text{def}}{=} \lim_{X \rightarrow 0} W_j(X, T) \quad (3.7)$$

is the amplitude. The wave-number $\omega(X, T)$ and the frequency $\Omega(X, T)$ should satisfy dispersion relation (B.4) and equation

$$\Omega'_X + \omega'_T = 0 \quad (3.8)$$

that follows from (3.3) for all X and T in a neighbourhood of $X = 0$. In this case, the phase $\phi(\xi, \tau)$ can be defined by the formula

$$\phi = i \int (\omega d\xi - \Omega d\tau). \quad (3.9)$$

Additionally, we require that

$$[W_j] = 0, \quad [\phi] = 0. \quad (3.10)$$

Remark 3.1. Equation (3.8) can be recognized as the space-time eikonal equation, see Eq. (A.6.3) in [Babich and Buldyrev \[2009\]](#).

Note that the analytic expressions for quantities u , W , W_j , ϕ , Ω , ω are generally different for $\xi \leq 0$. Therefore, we additionally require that u satisfy vanishing boundary conditions at infinity ($\xi \rightarrow \pm\infty$). Accordingly, we need to choose for $\xi \geq 0$ different roots (B.5) of dispersion relation (B.4):

$$u(\xi, \tau) = \sum_{(\pm)} H(\pm\xi) W^{(\pm)}(X, T) \exp \phi^{(\pm)}(\xi, \tau) + \text{c.c.} \quad (3.11)$$

In Eqs. (3.1)–(3.11) and in what follows, we drop superscript (\pm) for the aim of simplicity. For $\epsilon = 0$ the single-mode ansatz (3.1)–(3.11) should transform into the solution (B.24) of the corresponding zeroth order problem with an accuracy to an unknown constant multiplier.

Remark 3.2. The structure of the single-mode ansatz (3.1)–(3.11) assumes that

- Dispersion relation (B.4) holds for all T and X , in particular for $X = \pm 0$;
- Frequency equation (B.17) for the trapped mode holds for all T .

According to the procedure of the method of multiple scales [Nayfeh \[2008\]](#), the slow variables X , T , and the fast phase ϕ are assumed to be independent variables. In this way, we represent the differential operators with respect to time and the spatial co-ordinate in the following form:

$$\begin{aligned} (\dot{\cdot}) &= -i\Omega\partial_\phi + \epsilon\partial_T, & (\cdot)' &= i\omega\partial_\phi + \epsilon\partial_X, \\ (\ddot{\cdot}) &= -\Omega^2\partial_{\phi\phi}^2 - 2\epsilon i\Omega\partial_{\phi T}^2 - \epsilon i\Omega_T'\partial_\phi + O(\epsilon^2), \\ (\cdot)'' &= -\omega^2\partial_{\phi\phi}^2 + 2\epsilon i\omega\partial_{\phi X}^2 + \epsilon i\omega_X'\partial_\phi + O(\epsilon^2), \\ (\cdot)'\prime &= \omega\Omega\partial_{\phi\phi}^2 - \epsilon i\Omega\partial_{\phi X}^2 - \epsilon i\Omega_X'\partial_\phi + \epsilon i\omega\partial_{\phi T}^2 + O(\epsilon^2). \end{aligned} \quad (3.12)$$

To derive the first approximation equation, we substitute the single-mode ansatz (3.1)–(3.11) as well as the representations for differential operators (3.12) into three equations. These are

1) The second Hugoniot condition (2.24), where quantity $P(\tau)$ in the right-hand side is expressed according to Eq. (2.18); and

2,3) PDE under investigation in the form of Eq. (2.19) considered at $\xi \rightarrow -0$ and $\xi \rightarrow +0$ (two additional equations).

For all resulting equations, the corresponding zeroth order approximation is satisfied automatically. The joint consideration of these three equations allows one to derive the first approximation equation in the form of an ODE.

At first, consider the second Hugoniot condition (2.24) for $\tau > \tau_0$, where τ_0 is defined by Eq. (2.26). The internal force $P(\tau)$ in the right-hand side can be expressed according to Eq. (2.18) wherein $p = 0$. Thus, we get

$$[i\omega W + \epsilon W_X'] + O(\epsilon^2) = \frac{M(-\Omega_0^2 W - 2\epsilon i\Omega_0 W_T' - \epsilon i\Omega_0 T' W) - \epsilon iM_T' \Omega_0 W - 2\epsilon i\Gamma \Omega_0 W + KW}{\mathcal{T} - \rho v^2} \Big|_{X=0}. \quad (3.13)$$

After equating of the coefficients of like powers ϵ one can find that the zeroth order approximation is identically satisfied due to the frequency equation (B.17). The first approximation is

$$[W_0' X] = - \frac{2iM\Omega_0 W_0 T' + i(M\Omega_0 T' + M_T' \Omega_0 + 2\Gamma \Omega_0) W_0}{\mathcal{T} - \rho v^2} \Big|_{X=0}. \quad (3.14)$$

Note that the last equation does not involve the terms, which depend on W_1 , since the procedure of the method of multiple scales guaranties that the common multiplier before all such terms equals zero due to the frequency equation (B.17). The left-hand side of Eq. (3.14) equals the magnitude of the jump discontinuity $[W_0' X]$, whereas the right-hand side does not involve any spatial derivatives.

Now we plan to calculate $[W_0' X]$ in an independent way using PDE (2.19) considered at $\xi = \pm 0$ (or, equivalently, at $X = \pm 0$). To do this, we substitute (3.1) and (3.12) into Eq. (2.19), taking into account that the right-hand side of Eq. (2.19) is zero for all $\xi \neq 0$. After equating of the coefficients of like powers ϵ , one can find that zeroth order approximation is identically satisfied due to the dispersion relation (B.4). For the first order approximation, we obtain:

$$\begin{aligned} &((\mathcal{T} - \rho v^2)2\omega - 2\rho v\Omega_0)W_0' X + \rho(2v\omega + 2\Omega_0)W_0 T' \\ &+ ((\mathcal{T} - \rho v^2)\omega_X' + 2\rho v\omega_T' + \rho\Omega_0 T' + ((\mathcal{T} - \rho v^2)' X + (\rho v)' T)\omega + (\rho_T' - \rho_X' v)\Omega_0 + 2\gamma(\Omega_0 + v\omega))W_0 = 0 \end{aligned} \quad (3.15)$$

or

$$\begin{aligned} & ((\mathcal{J} - \rho v^2)2\omega - 2\rho v\Omega_0)W_0'{}_X + (2\rho v\omega + 2\rho\Omega_0)W_0'{}_T \\ & + \left(((\mathcal{J} - \rho v^2)\omega - \rho v\Omega_0)'{}_X + (\rho v\omega + \rho\Omega_0)'{}_T + 2\gamma(\Omega_0 + v\omega) \right) W_0 = 0. \end{aligned} \quad (3.16)$$

Again, the last equation does not involve the terms, which depend on W_1 , since the procedure of the method of multiple scales guaranties that the common multiplier before all such terms equals zero due to the dispersion equation (B.4). Using Eqs. (B.5), (B.6) we obtain:

$$\begin{aligned} (\mathcal{J} - \rho v^2)\omega - \rho v\Omega &= (\mathcal{J} - \rho v^2)B \pm i(\mathcal{J} - \rho v^2)S - \rho v\Omega = (\mathcal{J} - \rho v^2)\frac{\rho v\Omega}{\mathcal{J} - \rho v^2} \pm i(\mathcal{J} - \rho v^2)S - \rho v\Omega \\ &= \rho v\Omega \pm i(\mathcal{J} - \rho v^2)S - \rho v\Omega = \pm i(\mathcal{J} - \rho v^2)S. \end{aligned} \quad (3.17)$$

Thus, we can rewrite Eq. (3.16) as follows:

$$\begin{aligned} & \pm 2i(\mathcal{J} - \rho v^2)S W_0'{}_X + 2(\rho v\omega + \rho\Omega_0)W_0'{}_T \\ & + \left((\pm i(\mathcal{J} - \rho v^2)S)'{}_X + (\rho v\omega + \rho\Omega_0)'{}_T + 2\gamma(\Omega_0 + v\omega) \right) W_0 = 0 \end{aligned} \quad (3.18)$$

or

$$W_0'{}_X = -\frac{(2\rho v\omega + 2\rho\Omega_0)W_0'{}_T + \left((\pm i(\mathcal{J} - \rho v^2)S)'{}_X + (\rho v\omega + \rho\Omega_0)'{}_T + 2\gamma(\Omega_0 + v\omega) \right) W_0}{\pm 2i(\mathcal{J} - \rho v^2)S}. \quad (3.19)$$

Taking into account the eikonal equation (3.8) and Eq. (B.5), we get

$$\begin{aligned} \pm i((\mathcal{J} - \rho v^2)S)'{}_X &= -(\mathcal{J} - \rho v^2)(\pm iS'_\Omega B'_T - S'_\Omega S'_T) \pm i((\mathcal{J} - \rho v^2)S)'_\rho \rho'_X \\ & \quad \pm i((\mathcal{J} - \rho v^2)S)'_{\mathcal{J}} \mathcal{J}'_X \pm i((\mathcal{J} - \rho v^2)S)'_k k'_X. \end{aligned} \quad (3.20)$$

Remark 3.3. We remind that formulae (3.15)–(3.20) are written for $X = \pm 0$.

Calculating the magnitude $[W_0'{}_X]$ of jump discontinuity yields:

$$[W_0'{}_X] = -\frac{(2\rho vB + 2\rho\Omega_0)W_0'{}_T + \left((\mathcal{J} - \rho v^2)S'_\Omega S'_T + (\rho vB + \rho\Omega_0)'{}_T + 2\gamma(\Omega_0 + vB) \right) W_0}{i(\mathcal{J} - \rho v^2)S} \Big|_{X=0}. \quad (3.21)$$

Finally, equating the right-hand sides of Eqs. (3.14) and (3.21) results in the first approximation equation for $W_0(T)$ defined by Eq. (3.7) in the form, which does not involve any spatial derivatives:

$$\begin{aligned} \frac{2M\Omega_0 W_0'{}_T + (M\Omega_0'{}_T + M'_T \Omega_0 + 2\Gamma\Omega_0)W_0}{(\mathcal{J} - \rho v^2)} &= \\ -\frac{(2\rho vB + 2\rho\Omega_0)W_0'{}_T + \left((\mathcal{J} - \rho v^2)S'_\Omega S'_T + (\rho vB + \rho\Omega_0)'{}_T + 2\gamma(vB + \Omega_0) \right) W_0}{(\mathcal{J} - \rho v^2)S}. \end{aligned} \quad (3.22)$$

One has $\mathcal{J} \neq \rho v^2$, due to Eqs. (2.13), (2.14), and $S \neq 0$ due to Eqs. (B.8), (B.22). Thus, Eq. (3.22) can be rewritten in the form

$$\begin{aligned} 2M\Omega_0 S W_0'{}_T + (M\Omega_0'{}_T + M'_T \Omega_0 + 2\Gamma\Omega_0)S W_0 &= \\ - (2\rho vB + 2\rho\Omega_0)W_0'{}_T - \left((\mathcal{J} - \rho v^2)S'_\Omega S'_T + (\rho vB + \rho\Omega_0)'{}_T + 2\gamma(vB + \Omega_0) \right) W_0, \end{aligned} \quad (3.23)$$

which is equivalent to

$$\begin{aligned} (2\rho vB + 2\rho\Omega_0 + 2M\Omega_0 S)W_0'{}_T &= \\ - \left((\mathcal{J} - \rho v^2)S'_\Omega S'_T + (\rho vB + \rho\Omega_0)'{}_T + (M\Omega_0'{}_T + M'_T \Omega_0)S + 2\Gamma\Omega_0 S + 2\gamma(vB + \Omega_0) \right) W_0 \end{aligned} \quad (3.24)$$

or

$$\begin{aligned} (2\rho vB + 2\rho\Omega_0 + 2M\Omega_0 S)W_0'{}_T &= \\ - \left((\mathcal{J} - \rho v^2)S'_\Omega S'_T - M\Omega_0 S'_T + (\rho vB + \rho\Omega_0 + M\Omega_0 S)'{}_T + 2\Gamma\Omega_0 S + 2\gamma(vB + \Omega_0) \right) W_0. \end{aligned} \quad (3.25)$$

The last equation can be transformed to the following one:

$$\frac{W_0'{}_T}{W_0} = -\frac{1}{2} \frac{(\mathcal{J} - \rho v^2)S'_\Omega - M\Omega_0}{\rho vB + \rho\Omega_0 + M\Omega_0 S} S'_T - \frac{1}{2} \frac{(\rho vB + \rho\Omega_0 + M\Omega_0 S)'{}_T}{\rho vB + \rho\Omega_0 + M\Omega_0 S} - \frac{\Gamma\Omega_0 S + \gamma(vB + \Omega_0)}{\rho vB + \rho\Omega_0 + M\Omega_0 S}. \quad (3.26)$$

4 Analysis of possible approaches to solve the first approximation equation

By construction, the first approximation equation (3.26), as well as Eq. (A.11) considered in Appendix A, can be equivalently transformed to the equation of the following structure:

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \sum_{i=1}^n F_i(\mathcal{P}) \mathcal{P}_{iT}' + \left\{ \sum_{i=1}^{\tilde{n}} \tilde{F}_i(\mathcal{P}) \tilde{\mathcal{P}}_i \right\}, \quad (4.1)$$

Here $F_i(\mathcal{P})$, $\tilde{F}_i(\mathcal{P})$ are certain functions of \mathcal{P} ; \mathcal{P} and $\tilde{\mathcal{P}}$ are system parameters introduced by Eqs. (2.27), (2.28). The first sum in the right hand-side of Eq. (4.1) emerges since the coefficients of Eqs. (2.18), (2.19) are slowly time-varying functions. The second sum in the curly brackets is due to coefficients of Eqs. (2.18), (2.19), which are proportional to ϵ . Provided that the following conditions

$$\left\{ \sum_{i=0}^{\tilde{n}} \tilde{F}_i(\mathcal{P}) \tilde{\mathcal{P}}_i \right\}' = 0 \Leftrightarrow \sum_{i=0}^{\tilde{n}} \tilde{F}_i(\mathcal{P}) \tilde{\mathcal{P}}_i = \Lambda = \text{const}, \quad (4.2)$$

$$\frac{\partial F_i}{\partial \mathcal{P}_j} = \frac{\partial F_j}{\partial \mathcal{P}_i}, \quad i \neq j, \quad (4.3)$$

are satisfied, Eq. (4.1) can be rewritten in the form of the equation

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = J_T'(\mathcal{P}) + \Lambda, \quad (4.4)$$

where the expression in the right-hand side is the exact derivative of a function $J(\mathcal{P}) + \Lambda T$. Now, Eq. (4.4) can be integrated:

$$\mathcal{W}_0 = \mathcal{C} \exp(J(\mathcal{P}) + \Lambda T). \quad (4.5)$$

Here \mathcal{C} is an arbitrary complex constant. Thus, the solution is obtained in the form of a function, depending on the instantaneous values of the system parameters $\mathcal{P}(0, T)$ and the current slow time T in the explicit way. Such a solution is valid for an arbitrary history of $\mathcal{P}(0, \hat{T})$, $\hat{T} \leq T$. In the particular case

$$\Lambda = 0, \quad (4.6)$$

Eq. (4.1) can be rewritten as

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = J_T'(\mathcal{P}), \quad (4.7)$$

and the corresponding solution can be obtained in the form of a function, depending on the current values of the system parameters $\mathcal{P}(0, T)$ only:

$$\mathcal{W}_0 = \mathcal{C} \exp J(\mathcal{P}). \quad (4.8)$$

If Λ is not a constant, i.e., condition (4.2) is not satisfied, the solution of Eq. (4.4) is the functional

$$\mathcal{W}_0 = \mathcal{C} \exp \left(J(\mathcal{P}) + \int_0^T \sum_{i=1}^{\tilde{n}} \tilde{F}_i(\mathcal{P}) \tilde{\mathcal{P}}_i d\hat{T} \right), \quad (4.9)$$

which depends on the history of $\mathcal{P}(0, \hat{T})$ and $\tilde{\mathcal{P}}(0, \hat{T})$. Finally, if both conditions (4.2) and (4.3) are not satisfied, the solution of Eq. (4.1) is the functional

$$\mathcal{W}_0 = \mathcal{C} \exp \left(\int_0^T \left(\sum_{i=1}^n F_i(\mathcal{P}) \mathcal{P}_{iT}' + \sum_{i=1}^{\tilde{n}} \tilde{F}_i(\mathcal{P}) \tilde{\mathcal{P}}_i \right) d\hat{T} \right), \quad (4.10)$$

which again depends on the history of $\mathcal{P}(0, \hat{T})$ and $\tilde{\mathcal{P}}(0, \hat{T})$. In last two cases (4.9) and (4.10), if the history is not specified, the solution of (4.1) can be found in quadratures only. If the history of $\mathcal{P}(0, \hat{T})$, $\tilde{\mathcal{P}}(0, \hat{T})$ is known, the solution is an explicit function of slow time T .

Now, let us note that for reasons of convenience, it may be useful to use in calculations, instead of \mathcal{P} , some new variables:

$$\Psi \stackrel{\text{def}}{=} (\Psi_1(\mathcal{P}), \dots, \Psi_\nu(\mathcal{P})), \quad (4.11)$$

where ν is not necessary equal to n . The first approximation for the amplitude (4.1) in terms of variables Ψ is

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \sum_{i=1}^{\nu} \Phi_i(\Psi) \Psi_{i'T}' + \sum_{i=1}^{\tilde{n}} \tilde{\Phi}_i(\Psi) \tilde{\mathcal{P}}_i, \quad (4.12)$$

where $\Phi_i(\Psi)$, $\tilde{\Phi}_i(\Psi)$ are certain functions of the variables Ψ . If Eq. (4.12) can be rewritten in the form

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \mathcal{J}'_T(\Psi) + \Lambda, \quad (4.13)$$

where

$$\Lambda = \sum_{i=0}^{\tilde{n}} \tilde{\Phi}_i(\Psi) \tilde{\mathcal{P}}_i = \text{const}, \quad (4.14)$$

then conditions (4.3) are definitely satisfied, and the solution can be again obtained in the form of a function, depending on the current values of the system parameters $\mathcal{P}(0, T)$ and current slow time T in the explicit way:

$$\mathcal{W}_0 = \mathcal{C} \exp(\mathcal{J}(\Psi(\mathcal{P})) + \Lambda T). \quad (4.15)$$

In the particular case (4.6), Eq. (4.1) can be rewritten as

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \mathcal{J}'_T(\Psi), \quad (4.16)$$

and the corresponding solution can be obtained in the form of a function, depending on the current values of the system parameters $\mathcal{P}(0, T)$ only:

$$\mathcal{W}_0 = \mathcal{C} \exp \mathcal{J}(\Psi). \quad (4.17)$$

Let us note that the possibility to represent a given equation with the structure of Eq. (4.1) in the form of Eq. (4.13) or Eq. (4.16) is absolutely unobvious. Of course, one can try to check conditions (4.3) directly. However, this can be a difficult problem due to a lengthy structure of the right-hand side of the equation under consideration. On the other hand, a lucky choice of the variables $\Psi(\mathcal{P})$ can essentially simplify the calculations and help to obtain the representation of Eq. (4.1) in the form of Eq. (4.13) or Eq. (4.16).

If Λ is not a constant, i.e., condition (4.14) is not satisfied, the solution of Eq. (4.4) is the functional:

$$\mathcal{W}_0 = \mathcal{C} \exp \left(\mathcal{J}(\Psi) + \int_0^T \sum_{i=1}^{\tilde{n}} \tilde{\Phi}_i(\Psi) \tilde{\mathcal{P}}_i d\hat{T} \right). \quad (4.18)$$

Remark 4.1. If the tuple $\tilde{\mathcal{P}}$ is empty, and only one \mathcal{P}_j is a function of time T , whereas all other \mathcal{P}_i , $i \neq j$, are constants, then Eq. (4.1) can be rewritten as

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = F_j(\mathcal{P}) \mathcal{P}_{j'T}' \quad (4.19)$$

Accordingly, conditions (4.2), (4.3) are satisfied automatically, and the solution can be obtained in the form of the following function:

$$\mathcal{W}_0 = \mathcal{C} \exp \left(\int_0^{\mathcal{P}_j} F_j(\mathcal{P}_1, \dots, \hat{\mathcal{P}}_j, \dots, \mathcal{P}_n) d\hat{\mathcal{P}}_j \right). \quad (4.20)$$

Though Eq. (4.20) always provides the solution, the corresponding calculations can be difficult. In practical applications, the calculations of the function $F_j(\mathcal{P})$ and the expression in the right-hand side of (4.19) may be quite lengthy. Instead, it may be useful to introduce n -tuple of new variables Ψ such that Eq. (4.12) can be rewritten in the form of the following equation

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \sum_{i=1}^{\nu} \Phi_i(\Psi_i) \Psi_{i'T}', \quad (4.21)$$

where variables Ψ in the right-hand side are separated. The last equation can be integrated as follows:

$$\mathcal{W}_0 = \mathcal{C} \exp \left(\sum_{i=1}^{\nu} \int_0^{\Psi_i} \Phi_i(\hat{\Psi}_i) d\hat{\Psi}_i \right). \quad (4.22)$$

5 Solving of the first approximation equation

The first approximation equation (3.26) has the structure of Eq. (4.12) wherein Ψ are the variables defined by Eq. (4.11) with $\nu = 2$:

$$\Psi_1 = S, \quad \Psi_2 = \rho v B + \rho \Omega_0 + M \Omega_0 S; \quad (5.1)$$

$$\Lambda = -\frac{\Gamma \Omega_0 S + \gamma(vB + \Omega_0)}{\rho v B + \rho \Omega_0 + M \Omega_0 S}. \quad (5.2)$$

Noticing that the following identity

$$\frac{(\mathcal{T} - \rho v^2) S'_\Omega - M \Omega_0}{\rho v B + \rho \Omega_0 + M \Omega_0 S} = -\frac{1}{S}, \quad (5.3)$$

is true Gavrilov et al. [2022], see Appendix B.4, we, finally, rewrite the first approximation equation in the form of Eq. (4.13):

$$\frac{\mathcal{W}_{0T}'}{\mathcal{W}_0} = \frac{1}{2} \frac{S'_T}{S} - \frac{1}{2} \frac{(\rho v B + \rho \Omega_0 + M \Omega_0 S)'_T}{\rho v B + \rho \Omega_0 + M \Omega_0 S} - \left\{ \frac{\Gamma \Omega_0 S + \gamma(vB + \Omega_0)}{\rho v B + \rho \Omega_0 + M \Omega_0 S} \right\} \quad (5.4)$$

wherein

$$\mathcal{J} = \ln \sqrt{\frac{S}{\rho v B + \rho \Omega_0 + M \Omega_0 S}}. \quad (5.5)$$

In the general (dissipative) case, the solution of Eq. (5.4) has the structure of the functional defined by Eq. (4.18):

$$\mathcal{W}_0 = \mathcal{C} \mathcal{A}(T) \exp(-\mathcal{D}(T)), \quad (5.6)$$

where \mathcal{C} is an arbitrary complex constant, and the functions $\mathcal{A}(T)$ and $\mathcal{D}(T)$ are

$$\mathcal{A} = \sqrt{\frac{S}{\rho v B + \rho \Omega_0 + M \Omega_0 S}}, \quad \mathcal{D} = \int_0^T \frac{\Gamma \Omega_0 S + \gamma(vB + \Omega_0)}{\rho v B + \rho \Omega_0 + M \Omega_0 S} d\hat{T}. \quad (5.7)$$

Substituting (B.6) and (B.8) into the expression (5.7), one gets the equivalent forms

$$\mathcal{A} = \sqrt{\frac{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2}}{\Omega_0 M \sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} + \mathcal{T}\rho\Omega_0}}, \quad \mathcal{D} = \int_0^T \frac{\Gamma \sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} + \gamma\mathcal{T}}{M \sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} + \mathcal{T}\rho} d\hat{T} \quad (5.8)$$

or

$$\mathcal{A} = \sqrt{\frac{\sqrt{\Omega_*^2 - \Omega_0^2}}{\Omega_0 M \sqrt{\Omega_*^2 - \Omega_0^2} + \sqrt{\mathcal{T}\rho}\Omega_0}}, \quad \mathcal{D} = \int_0^T \frac{\Gamma \sqrt{\Omega_*^2 - \Omega_0^2} + \gamma \sqrt{\mathcal{T}/\rho}}{M \sqrt{\Omega_*^2 - \Omega_0^2} + \sqrt{\mathcal{T}\rho}} d\hat{T}, \quad (5.9)$$

where Ω_* is the cut-off frequency given by Eq. (B.9). Taking into account the frequency equation (B.17), we obtain one more equivalent form

$$\mathcal{A} = \sqrt{\frac{M\Omega_0^2 - K}{\Omega_0(M^2\Omega_0^2 - KM + 2\mathcal{T}\rho)}}, \quad \mathcal{D} = \int_0^T \frac{\Gamma(M\Omega_0^2 - K) + 2\gamma\mathcal{T}}{M^2\Omega_0^2 - KM + 2\mathcal{T}\rho} d\hat{T}. \quad (5.10)$$

For a non-dissipative system where Eq. (2.9) is fulfilled, we obtain the solution for the amplitude in the form of an algebraic expression (4.17), depending on current values of the system parameters $\Psi(\mathcal{P})$ only, and not involving the slow time in an explicit way. Such a solution is valid for an arbitrary history of $\mathcal{P}(0, \hat{T})$. For a dissipative system where Eq. (2.10) is true, and additionally

$$\dot{\Lambda} \neq 0, \quad (5.11)$$

the solution for the amplitude generally is a functional, which depends on the history of $\mathcal{P}(0, T)$ and $\hat{\mathcal{P}}(0, T)$. If the history is not specified, the solution of Eq. (5.4) can be found in quadratures only. If the history of $\mathcal{P}(0, T)$ and $\hat{\mathcal{P}}(0, T)$ is known, the solution is an explicit function of slow time T . The analogous results are valid for a mass-spring-damper system with time-varying parameters (see Appendix A).

Remark 5.1. The possibility to obtain the solution of Eq. (3.26) in the non-dissipative case (2.9) in the form of an algebraic expression follows not only from the fact that the right-hand side of Eq. (5.4) is the exact derivative of some function, but, additionally, from the fact that the function is the logarithm of an algebraic expression.

One can see from Eq. (5.10) that in the formal limiting case (2.35) of an isolated mass-spring-damper system, we have

$$\mathcal{W}_0 \rightarrow \frac{\mathcal{C}}{\sqrt{M\Omega_0}} \exp\left(-\int_0^T \frac{\Gamma}{M} d\hat{T}\right). \quad (5.12)$$

Since, due to Eq. (B.21), $\Omega_0 \rightarrow \sqrt{\frac{K}{M}}$ in this case, we get:

$$\mathcal{W}_0 \rightarrow \frac{\mathcal{C}}{\sqrt[4]{MK}} \exp\left(-\int_0^T \frac{\Gamma}{M} d\hat{T}\right). \quad (5.13)$$

The last expression coincides with the corresponding result (A.14) for the problem concerning a free non-stationary oscillation of a mass-spring-damper system with time-varying parameters (see Appendix A).

In the exceptional cases, where

$$\Lambda = \text{const} \neq 0 \quad (5.14)$$

the solution can be found in the form of Eq. (4.15), depending on current values of the system parameters $\Psi(\mathcal{P})$ and the slow time T . We can underline at least the following such cases:

1. Considered in detail in Appendix A limiting case (2.35) of an isolated mass-spring-damper system with $M = \text{const}$, $\Gamma = \text{const}$, where K is variable quantity;
2. Limiting case (2.34) with $M = \text{const}$, $\Gamma = \text{const} \neq 0$, $\gamma = 0$, where K , k , \mathcal{T} are variable quantities, see Eq. (5.10);
3. The case $M = 0$, $K = \text{const}$, $\Gamma = \text{const} \neq 0$, $\mathcal{T} = \text{const}$, $\rho = \text{const}$, $\gamma = 0$, where k and v are variable quantities, see Eq. (5.10);
4. The case $M = 0$, $\Gamma = 0$, $\rho = \text{const}$, $\gamma = \text{const} \neq 0$, where K, \mathcal{T}, k, v are variable quantities, see Eq. (5.10).

Combining the obtained solution with the complexly conjugated one following to Eqs. (3.1)–(3.11), we get the asymptotic solution for \mathcal{U} in the real form:

$$\mathcal{U} = 2|\mathcal{C}|A(T) \exp(-\mathcal{D}(T)) \cos\left(\int_0^T \Omega_0(\hat{T}) d\hat{T} - \arg \mathcal{C}\right). \quad (5.15)$$

Remark 5.2. The unknown complex constant \mathcal{C} should be found by the matching at $\xi = 0$ and $\tau = 0$ the expression (5.15) for \mathcal{U} with results (B.26) obtained by the method of stationary phase applied to the same system with $\epsilon = 0$ and constant parameters, which equal the corresponding initial values. The procedure is completely analogous to the one presented in Sect. 4.3 of Gavrilov et al. [2022]. The result is

$$|\mathcal{C}| = \frac{|\mathcal{F}\{p\}(\Omega_0)|}{2} \sqrt{\frac{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2}}{\Omega_0 M \sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} + \mathcal{T}\rho\Omega_0}} \Bigg|_{T=0}, \quad (5.16)$$

$$\arg \mathcal{C} = \arg \mathcal{F}\{p\}(\Omega_0) - \frac{\pi}{2},$$

where symbol $\mathcal{F}\{p\}(\Omega_0)$ denotes the value of the Fourier transform for the loading $p(\tau)$ calculated at the frequency Ω_0 .

6 Analysis of previous studies

In this section, we discuss in the chronological order the basic results of all our previous studies, which deal with the particular cases of Eqs. (2.18), (2.19) with a single time-varying parameter. We show that the earlier analytic results follow from universal formulae (5.6), (5.7) and discuss the method of solving of the first approximation equation, which was applied in every study.

For all problems considered in this section we assume the absence of dissipation, i.e., Eq. (2.9) is fulfilled. All coefficients for the particular forms of Eq. (2.19) were assumed to be functions of time τ only (but not ξ), i.e.,

$$\mathcal{T}' \equiv 0, \quad \rho' \equiv 0, \quad k' \equiv 0. \quad (6.1)$$

Recall, see Remark 4.1, that in such a case, the right-hand side of the first approximation equation can always be represented in the form of the exact derivative. Therefore, the solution is definitely can be obtained in the form of a function depending on the current values of the system parameters. At the same time, while considering different cases with a single time-varying parameter, we supposed that it would be rather impossible to obtain the result in the form of a function in the case of multiple time-varying parameters, since, in the latter case, conditions (4.3) are not satisfied automatically any more.

6.1 A non-uniformly moving inertial load

In Gavrilov and Indeitsev [2002], the suggested asymptotic approach was first time applied to describe a localized oscillation in a coupled discrete-continuum system with time-varying properties. Namely, an oscillation of an inertial load, non-uniformly moving along an infinite string on the Winkler foundation, was considered. To obtain the equations, which coincide with the ones considered in Gavrilov and Indeitsev [2002] one should formally put in Eqs. (2.18), (2.19)

$$K = 0, \quad \mathcal{T} = 1, \quad \rho = c^{-2}. \quad (6.2)$$

The unique time-varying parameter was the speed v , i.e.,

$$\dot{M} \equiv 0, \quad \dot{\rho} \equiv 0, \quad \dot{k} \equiv 0. \quad (6.3)$$

In the problem under consideration, we can also put, without loss of generality, that

$$k = 1, \quad c = 1. \quad (6.4)$$

This assumption is used in what follows in Sect. 6.1 for the aim of simplicity.

Though the model of a string on the Winkler foundation is widely used in engineering, and such a problem has many important applications, before Gavrilov and Indeitsev [2002] there was no approximate solution of a simple structure valid for a non-uniform regime of an inertial load motion. Indeed, this is an essentially non-stationary problem, which takes into account both an inertial character of the load Kaplunov [1986] and a non-uniform regime of motion Gavrilov [1999], Kaplunov and Muravskii [1986]. The free oscillation was assumed in the form of the single-mode ansatz corresponding to the trapped mode frequency. In this way, the problem describing the localized oscillation in a distributed discrete-continuum system with time-varying parameters was first time reduced to the first approximation for the amplitude in the form of an ODE. The procedure of solving of the equation is quite complicated, since expressions (B.6) for B and (B.8) for S , respectively, were substituted into the right-hand side of the equation at a very early stage of solving. Finally, the equation had transformed into the form of Eq. (4.21) wherein

$$\Psi_1 = v, \quad \Psi_2 = \Omega_0, \quad \Psi_3 = v^2, \quad \Psi_4 = M^2\Omega_0^2 + 2, \quad \Psi_5 = 1 + M^2k(1 - v^2) \quad (6.5)$$

and integrated according to Eq. (4.22). Finally, the solution for the amplitude was expressed in the form of a product of eleven integrals.² After simplification, the result was obtained in the form of the simple algebraic expression:

$$\mathcal{W}_0 = \mathcal{C} \sqrt{\frac{1 - v^2}{\Omega_0(M^2\Omega_0^2 + 2)}}, \quad (6.6)$$

see Eq. (5.15) in Gavrilov and Indeitsev [2002]. The expression obtained in such a way describes a free localized oscillation and involves an unknown constant C (the initial complex amplitude). The external force $p(\tau)$ was taken in the form of the weight of the load, i.e., a suddenly applied constant force. Finally, the solution was taken in the form called the multi-frequency ansatz Gavrilov et al. [2022],³ which takes into account both free and forced components of the motion. To find the constant C , the solution was matched with the results obtained by the method of stationary phase applied to the same system with constant parameters, which are equal to the corresponding initial values. The constructed analytic solution was verified by numerics based on solving a Volterra integral equation of the second kind for the internal force $P(\tau)$. An excellent agreement with results of the analytic approach was demonstrated for the case of a uniformly accelerated motion of the load, excepting the case when the trapped mode frequency approaches the cut-off frequency Ω_* . Nevertheless, later, in Gavrilov et al. [2022], we showed that formula (6.6) is erroneous, and clearly identified the error in calculations. This formula cannot be obtained as a particular case of general formulae (5.6), (5.7). The results given by the correct formula derived later in Gavrilov et al. [2022], see also Sect. 6.5, are very close to ones given by (6.6) and differ considerably only for large enough mass M (see Sect. 5.2.2 of Gavrilov et al. [2022]).

²The coefficients Φ_i were sums of several expressions.

³Or, better to say, the multi-modes ansatz.

6.2 An inclusion of a time-varying mass

In [Indeitsev et al. \[2016\]](#), the particular case where

$$K = 0, \quad \mathcal{T} = 1, \quad \rho = c^{-2}, \quad v = 0 \quad (6.7)$$

was considered. The unique time-varying parameter was the mass M , i.e.,

$$\dot{\rho} \equiv 0, \quad \dot{k} \equiv 0. \quad (6.8)$$

In the problem under consideration we again, for the aim of simplicity, can accept assumptions (6.4).

Again, expressions (B.6) for B and (B.8) for S were substituted into the first approximation at a very early stage of solving. This enough lengthy equation was reduced to one having the structure of Eq. (4.21), wherein

$$\Psi_1 = m, \quad \Psi_2 = \Omega_0^2 \quad (6.9)$$

and then integrated.⁴ The obtained expression for the amplitude has the form:

$$\mathcal{W}_0 = e^{\frac{\sqrt{2} \sqrt[4]{M} \sqrt[8]{z-2}}{\sqrt{z} \sqrt[8]{z+2}}}, \quad (6.10)$$

see Eq. (46) in [Indeitsev et al. \[2016\]](#). Here, C is an arbitrary constant,

$$z = M^2 \Omega_0^2 + 2, \quad (6.11)$$

$$\Omega_0 = \sqrt{\frac{2(\sqrt{1+M^2}-1)}{M^2}} \quad (6.12)$$

is the root of frequency equation (B.17) wherein identities (6.7) and (6.4) are taken into account:

$$M\Omega^2 = 2\sqrt{1-\Omega^2}. \quad (6.13)$$

Due to Eq. (6.13), one gets

$$z = M^2 \Omega_0^2 + \frac{M\Omega_0^2}{\sqrt{1-\Omega_0^2}} = \frac{M\Omega_0^2(M\sqrt{1-\Omega_0^2}+1)}{\sqrt{1-\Omega_0^2}}. \quad (6.14)$$

Taking into account Eqs. (6.11),(6.14), one can show that Eq. (6.10) can be transformed as follows:

$$\mathcal{W}_0 = e^{\sqrt{2} \sqrt{\frac{\sqrt{1-\Omega_0^2}}{\Omega_0(1+M\sqrt{1-\Omega_0^2})}} \frac{1}{\sqrt[4]{\Omega_0} \sqrt[8]{M^2\Omega_0^2+4}}} \quad (6.15)$$

According to Eq. (6.12) we get

$$\frac{1}{\sqrt[4]{\Omega_0} \sqrt[8]{M^2\Omega_0^2+4}} = \frac{1}{\sqrt[4]{2}}. \quad (6.16)$$

Hence,

$$\mathcal{W}_0 = \tilde{C} \sqrt{\frac{\sqrt{1-\Omega_0^2}}{\Omega_0(1+M\sqrt{1-\Omega_0^2})}}, \quad (6.17)$$

where $\tilde{C} = \sqrt[4]{2}C$. One can see that Eq. (6.17) coincides with Eq. (5.8) wherein Eqs. (2.9), (6.4), (6.7) are assumed (with an accuracy to an arbitrary constant multiplier).

The external force $p(\tau)$ was taken in the form

$$p(\tau) = p_*(H(\tau) - H(\tau - \tau_0)), \quad (6.18)$$

where $p_*, \tau_0 > 0$ are constants. Thus, introducing the multi-modes ansatz was not necessary. The analytical solution in the case of exponentially vanishing mass M was verified by numerical calculations, based on a Volterra integral equation of the second kind for $P(\tau)$. An excellent agreement with results of the analytic approach was demonstrated for the case of an exponentially vanishing mass M , excepting the case when the trapped mode frequency approaches the cut-off frequency Ω_* , i.e., $M \simeq 0$.

⁴Note that [Indeitsev et al. \[2016\]](#) is a short communication paper. The details how we reduced the first approximation equation to the form of Eq. (4.21) are not presented there.

6.3 A string with time-varying tension

In Gavrilov et al. [2019b], the particular case where

$$K = -2, \quad M = 0, \quad \mathcal{T} = c^2, \quad \rho = 1, \quad k = 1, \quad v = 0 \quad (6.19)$$

was considered. The unique time-varying parameter was the speed of sound c . The motivations for such a model were related to geophysical applications and a new model of a seismic source suggested by Prof. D.A. Indeitsev in private communications (see conference paper Gavrilov et al. [2016b]). This paper is the first of our studies in the series, where the partial differential equation describing the continuum sub-system cannot be reduced to an equation with constant coefficients. Accordingly, we had some doubts about the applicability of a single-mode ansatz to approximate non-stationary free oscillation in a continuum system with time-varying coefficients. Indeed, for a PDE with time-varying coefficients we do not even have the orthogonality conditions, which allow one to separate oscillations with different frequencies.

In this study, we first time considered the case where the existence of the trapped mode in zeroth order approximation was ensured by a discrete spring of a negative stiffness, but not by the discrete inertial inclusion. Considering the discrete sub-system in the form of a pure spring significantly simplified the first approximation for the amplitude, which was obtained in the very short form:

$$\frac{d\mathcal{W}_0}{\mathcal{W}_0} = -\frac{d\Omega_0}{2\Omega_0} - \frac{d\Omega_0^2}{4(1-\Omega_0^2)} - \frac{dc}{2c}, \quad (6.20)$$

see Eqs. (48), (53) in Gavrilov et al. [2019b]. Equation (6.20) has the structure of Eq. (4.21) wherein

$$\Psi_1 = \Omega_0, \quad \Psi_2 = \Omega_0^2, \quad \Psi_3 = c. \quad (6.21)$$

This results in

$$\mathcal{W}_0 = c \frac{\sqrt[4]{1-\Omega_0^2}}{\sqrt{c\Omega_0}}, \quad (6.22)$$

see Eq. (54) in Gavrilov et al. [2019b]. Here Ω_0 can be found by the second formula of (B.21):

$$\Omega_0^2 = 1 - c^{-2}, \quad (6.23)$$

see Eq. (23) in Gavrilov et al. [2019b]. Provided that Eqs. (2.9), (6.19) are true, it is easy to see that Eq. (6.22) is the particular case of Eq. (5.8).

Our previous numeric approach based on the Volterra integral equations is not applicable for such a system. Therefore, to verify the solution, we applied the finite difference method based on a numerical scheme conserving the energy Donninger and Schlag [2011], Strauss and Vazquez [1978]. Various laws for $c(et)$ were considered, in particular a uniform one, as well as a periodic non-monotonous one. The external loading was taken in the form of a finite step function (6.18) or an exponentially vanishing loading. An excellent agreement with analytic results was shown. It was demonstrated that Liouville-Green approximation (A.13) for a single degree of freedom system can be clearly distinguished as a wrong solution compared to Eq. (6.22) (Fig. 5 in Gavrilov et al. [2019b]).

We considered such a problem as the first step in the investigation of the problem for a Bernoulli-Euler beam with a defect compressed by a time-varying force, where the trapped modes also can be observed Gavrilov et al. [2016b], Indeitsev et al. [2015]. Though the simplest model problem for an uncompressed beam with time-varying parameters was solved in Shishkina et al. [2019], the problem for a compressed beam remains unsolved despite our several attempts. For now, we realize that the last problem is much more difficult, than the reader can think looking through Shishkina et al. [2019].

6.4 A discrete mass-spring sub-system of time-varying stiffness

In Gavrilov et al. [2019a], the particular case where

$$\mathcal{T} = 1, \quad \rho = 1, \quad k = 1, \quad v = 0 \quad (6.24)$$

was considered. The unique time-varying parameter was the spring stiffness K , i.e.,

$$\dot{M} \equiv 0. \quad (6.25)$$

The equation for the first approximation was obtained in the simple form

$$\frac{d\mathcal{W}_0}{\mathcal{W}_0} = -\frac{d\Omega_0}{2\Omega_0} - \frac{\Omega_0 d\Omega_0}{2(1-\Omega_0^2)(1+M\sqrt{1-\Omega_0^2})}, \quad (6.26)$$

where Ω_0 is the trapped mode frequency, see Eq. (62) in Gavrilov et al. [2019a]. The final result coincides with Eq. (6.17). To verify the solution numerically, we again applied the finite difference method. We have shown that the analytic and numerical results are in excellent agreement. It was demonstrated that the results are clearly visually distinguishable with the solution (A.13), which describes the single degree of freedom system with time-varying spring. The latter problem is a limiting case of the problem under consideration as $M \rightarrow \infty$, $K \rightarrow \infty$, $K/M = \Omega_0^2$.

6.5 A non-uniformly moving discrete mass-spring sub-system

In Gavrilov et al. [2022], the particular case where

$$\mathcal{T} = 1, \quad \rho = 1, \quad k = 1 \quad (6.27)$$

was considered. The unique time-varying parameter was the speed v , i.e.,

$$\dot{K} \equiv 0, \quad \dot{M} \equiv 0. \quad (6.28)$$

This problem, being an extension of the problem considered in the first paper Gavrilov and Indeitsev [2002], possesses more rich dynamics than the one considered before. Indeed, in the latter case the critical speed $v = 1$ generally does not coincide with the speed v satisfying condition (B.20), at which the trapped mode disappears. The case $K < 0$ was also included into consideration. Besides a free localized oscillation, the forced oscillation caused by a class of loadings, which can be represented as a superposition of a pulse loading, which acts during some time, and a number of harmonics with slowly time-varying frequencies and amplitudes is considered, see Eqs. (2.31)–(2.33). The solution was looked for in the form of a superposition of a free oscillation with frequency Ω_0 and forced oscillations with frequencies Ω_i , $i = \overline{1, N}$, i.e., in the form of the multi-modes ansatz. The forced oscillations can be found by zero order approximation in the framework of the method of multiple scales. Considering the equation for the first approximation in the form of Eq. (3.22) where Eqs. (2.9), (6.27), (6.28) are taken into account, and expressions (B.6) for B and (B.8) for S are not substituted, we did not use the approach discussed in Remark 4.1, which was applied in all previous papers. Instead, new variables Ψ (5.1) were introduced. Utilizing of variables (5.1) allowed us to reduce the equation for the first approximation to the form of Eq. (4.16) due to unobvious relationship (5.3) between the system parameters. Thus, the expression for the amplitude was obtained in the form of Eqs. (5.6), (5.7) wherein Eqs. (2.9), (6.27) are taken into account. The obtained result corrects the error introduced in Gavrilov and Indeitsev [2002], see Sect. 6.1.

The constructed analytic solution was verified by numerics based on solving a Volterra integral equation of the second kind for the internal force $P(\tau)$. An excellent agreement with results of analytic approach was demonstrated for the cases of a pure free and forced oscillation.

6.6 Discussion

After the publication of paper Gavrilov et al. [2022], we have realized that the procedure suggested there allows one to obtain the solution for the amplitude in the form of a function for an extended non-dissipative problem, where all the parameters are assumed to be time-varying quantities, which is considered in this paper. The results of this paper is an essential generalization of Gavrilov et al. [2022], which becomes possible due to a new approach based on discovered in Gavrilov et al. [2022] lucky choice (5.1) of variables Ψ . The fact that for the multi-parametric case of Eq. (5.4) conditions (4.3) are fulfilled, and the logarithmic structure of the right-hand side is still observed (see Remark 5.1), is surprising for us. Apparently, from the mathematical point of view, it follows from some unobvious properties of governing equations (2.18), (2.19).

The previous studies Gavrilov and Indeitsev [2002], Gavrilov et al. [2019a,b, 2022], Indeitsev et al. [2016] still keep the importance after this paper, in particular, due to numerical work done there. Indeed, in previous studies the practical applicability of the formal asymptotics to a wide class of systems with various single time-varying parameters, various regimes of time-varying, and various loadings was demonstrated in the case when the hodograph of the tuple $\mathcal{P}(0, \tau)$ introduced by Eqs. (2.27)–(2.30) does not approach the boundaries of the localization domain \mathbb{L} . The suggested approach allows us to investigate analytically the practically important problems, where non-uniformly moving mass-spring system is under consideration Gavrilov and Indeitsev [2002], Gavrilov et al. [2022]. The same approach was also applied Gavrilov [2006] to find the law of motion for a discrete moving load subjected to a drag force and interacting with the continuum sub-system by so-called “wave pressure force”, see Andrianov [1993], Brillouin [1925], Denisov et al. [2012], Gavrilov [2002], Gavrilov et al. [2016a], Havelock [1924], Nicolai [1912, 1925], Rayleigh [1902], Slepyan [2017a,b], Vesnitski et al. [1983], or “external configurational force” Cherepanov [1985], Gurtin [2000].

The formal asymptotics generally breaks for a system with parameters such that the trapped mode frequency becomes close to the cut-off frequency, i.e., when $\mathcal{P}(0, \tau)$ approaches for certain values of τ the boundary of \mathbb{L}

defined by Eq. (B.20). In another limiting case, where the trapped mode frequency becomes zero, i.e., when $\mathcal{P}(0, \tau)$ approaches the boundary of \mathbb{L} defined by Eq. (B.19), the formal asymptotics usually describes the solution quite well until the loss of stability, which leads to a localized buckling. The cases when $\mathcal{P}(0, \tau)$ approaches the boundaries (B.20) or (B.19) are discussed in more detail in what follows in the present paper, see Sect. 7.3.

7 Numerics

As well, as we have already discussed in the Introduction, the obtained asymptotics is so-called formal asymptotics, i.e. it satisfies the corresponding equations and initial conditions up to the certain asymptotic accuracy. Moreover, when constructing the asymptotics, a number of not well grounded assumptions were used. The most voluntaristic of them is the possibility to represent a non-stationary oscillation of a system with time-varying parameters in the form of a single-mode ansatz, which corresponds to “a pure localized oscillation” at a given frequency. Dealing with PDE with time-varying coefficients, we do not even have the orthogonality conditions, which allow one to separate oscillations with different frequencies. Another question is the validity of the matching procedure for two asymptotics, which were obtained by entirely different approaches. Thus, in our opinion, the analytic results should be at least justified by independent numerical calculations.

As it is mentioned in Sect. 6, there are two basic numeric approaches, which can be applied. The first approach is related with a Volterra integral equation of the second kind, to which the problem under consideration can be reduced in some particular cases [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2022\]](#), [Indeitsev et al. \[2016\]](#). A more general alternative method is based on the finite difference schemes used in [Gavrilov et al. \[2019a,b\]](#). The various particular cases, where the only one parameter varies in time, were considered in the previous studies. To verify the constructed single-mode asymptotics, in this paper, we consider only two particular cases. The first one is the case of a system with two simultaneously time-varying parameters (Sect. 7.1). The second one is related with the dissipative case (Sect. 7.2). Both cases require the second approach.

Remark 7.1. We restrict ourselves with the symmetric with respect to ξ problems, since only for such problems we have the debugged code for now. The non-symmetric cases, e.g., the cases where $v \neq 0$ or some parameters of the continuum sub-system are spatially varying functions, require considerable additional programming work, which is beyond the scope of this paper.

Additionally, in Sect. 7.3 we consider several examples of systems, where the hodograph of the tuple $\mathcal{P}(0, \tau)$ introduced by Eq. (2.30) approaches the boundary of the localization domain \mathbb{L} . In particular, in such a way, we show that the good practical applicability of the constructed single-mode asymptotics is related with the phenomenon of the anti-localization of non-stationary linear waves [Gavrilov and Shishkina \[2024\]](#), [Gavrilov et al. \[2023\]](#), [Shishkina and Gavrilov \[2023\]](#), [Shishkina et al. \[2023\]](#).

The external force $p(\tau)$ for numerics is taken in the form of Eq. (6.18) wherein $p_* = 1/\tau_0$. Thus, $p(\tau) \rightarrow \delta(\tau)$ in the weak sense. The corresponding asymptotics is taken for the case $p(\tau) = \delta(\tau)$. In all following examples, the value of the small parameter is

$$\epsilon = 0.01, \tag{7.1}$$

and

$$\tau_0 = 0.01. \tag{7.2}$$

7.1 Two time-varying parameters in a non-dissipative system

Consider the particular case where equations

$$M = 1, \quad \rho = 1, \quad k = 1, \quad v = 0 \tag{7.3}$$

and Eq. (2.9) are fulfilled. The time-varying parameters are \mathcal{J} and K :

$$\mathcal{J}(T) = 1.0 - 0.9 \sin(0.4T), \tag{7.4}$$

$$K(T) = -0.6 - 0.8 \sin(1.5T). \tag{7.5}$$

The plots of \mathcal{J} , K and the corresponding value of the trapped mode frequency Ω_0 are presented in Fig. 1. In Fig. 2 we compare the corresponding asymptotic and numerical solutions. One can see that the solutions are in an excellent agreement.

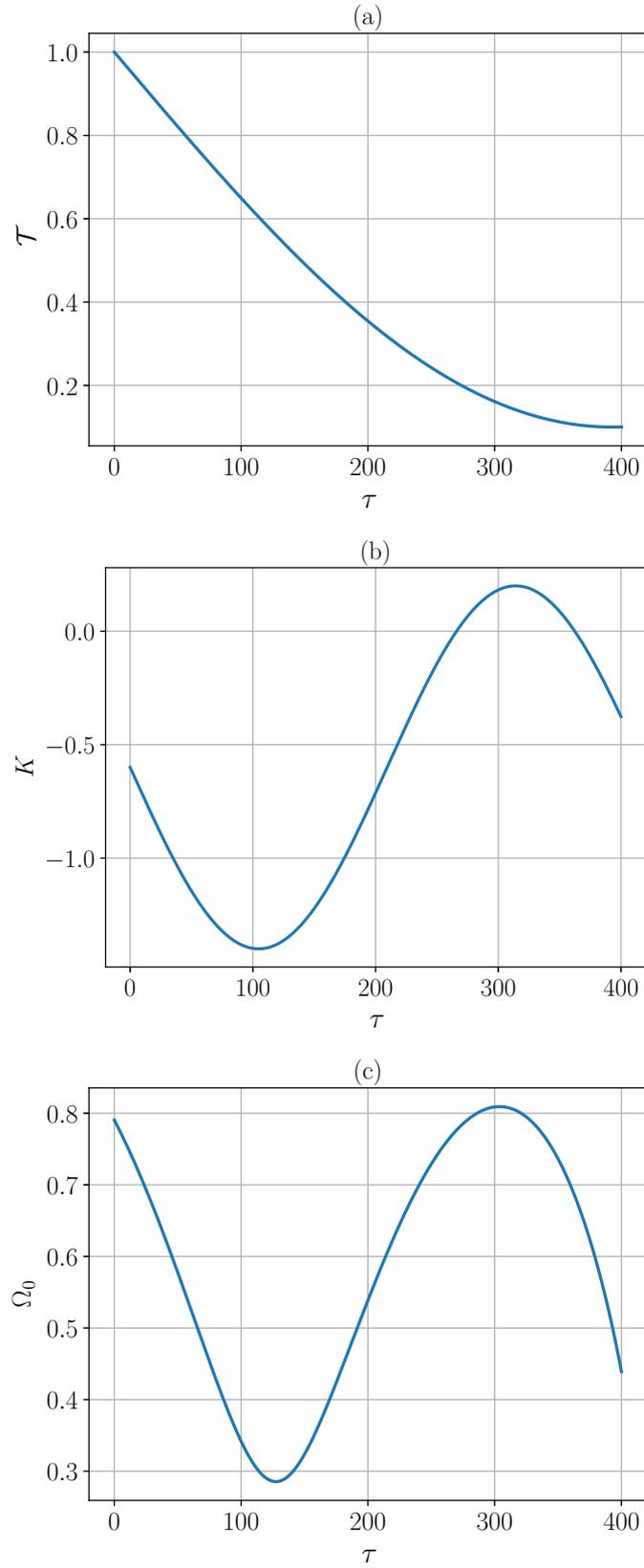


Figure 1: Given time-varying parameters (a) \mathcal{T} , (b) K , and (c) the trapped mode frequency Ω_0 for the case when parameters are taken according Eqs. (7.3), (7.4), (7.5)

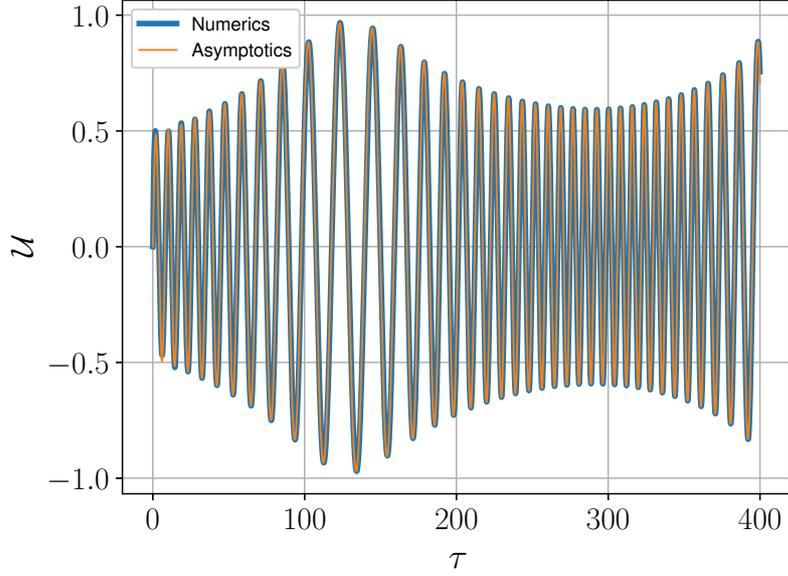


Figure 2: Oscillation of the discrete sub-system for the case when parameters are taken according Eqs. (2.9), (7.3), (7.4), (7.5)

7.2 A dissipative system

Consider the particular case where equations

$$M = 1, \quad \mathcal{T} = 1, \quad \rho = 1, \quad k = 1, \quad v = 0, \quad \Gamma = 1, \quad \gamma = 0 \quad (7.6)$$

are satisfied. The only time-varying parameter is K :

$$K(T) = K_0 + K_1 T. \quad (7.7)$$

Note that K should satisfy restriction

$$-2 < K < 1, \quad (7.8)$$

which follows from inequalities (B.19), (B.20), wherein Eq. (7.6) is used. For squared frequency Ω_0 according to Eq. (B.21) we have

$$\Omega_0^2 = K - 2 + 2\sqrt{2 - K}. \quad (7.9)$$

Taking into account the above expression, from Eqs. (5.6), (5.10) one gets:

$$\mathcal{W}_0 = \mathcal{C} \sqrt{\frac{\sqrt{2 - K} - 1}{\sqrt{2 - K} \sqrt{2\sqrt{2 - K} - (2 - K)}}} \exp\left(-\int_0^T \left(1 - \frac{1}{\sqrt{2 - K(\hat{T})}}\right) d\hat{T}\right). \quad (7.10)$$

Substituting Eq. (7.7) into the expression for \mathcal{W}_0 , we derive:

$$\mathcal{W}_0 = \mathcal{C} \sqrt{\frac{\sqrt{2 - K} - 1}{\sqrt{2 - K} \sqrt{2\sqrt{2 - K} - (2 - K)}}} \exp\left(-T - \frac{2}{K_1} \left(\sqrt{2 - K_0 - K_1 T} - \sqrt{2 - K_0}\right)\right). \quad (7.11)$$

Remark 7.2. Note that for $K_1 \rightarrow 0$ one has:

$$\lim_{K_1 \rightarrow 0} \left(-T - \frac{2}{K_1} \left(\sqrt{2 - K_0 - K_1 T} - \sqrt{2 - K_0}\right)\right) = -T \left(1 - \frac{1}{\sqrt{2 - K_0}}\right). \quad (7.12)$$

Thus, the right-hand side of (7.11) is properly defined for $K_1 = 0$.

In Figs. 3, 4 we show the plots of the stiffness K , the trapped mode frequency Ω_0 and compare asymptotic and numerical solutions for the case when parameters of the system are taken according to Eqs. (7.6), (7.7) wherein $K_0 = 0.5$, $K_1 = -1.0$ and $K_0 = -1.5$, $K_1 = 1.0$, respectively. One can see that the asymptotic and numerical solutions are in an excellent agreement, everywhere excepting the case when $\Omega_0 \simeq \Omega_* = 1$. The analytic solutions, which correspond to the same system with $\Gamma = 0$, are also shown.

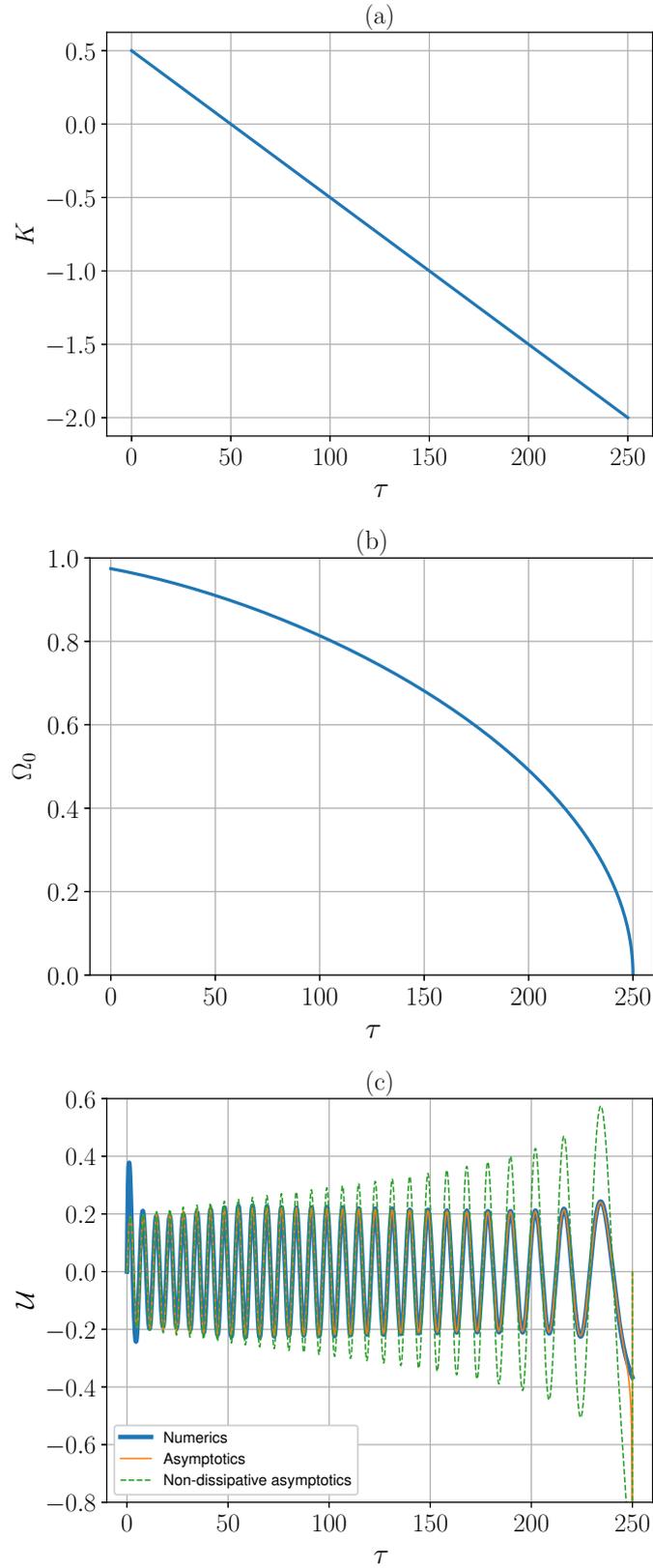


Figure 3: (a) Given time-varying parameter K , (b) the trapped mode frequency Ω_0 , and (c) the displacement \mathcal{U} of the discrete sub-system for the case when parameters are taken according Eq. (7.6), (7.7) wherein $K_0 = 0.5$, $K_1 = -1.0$

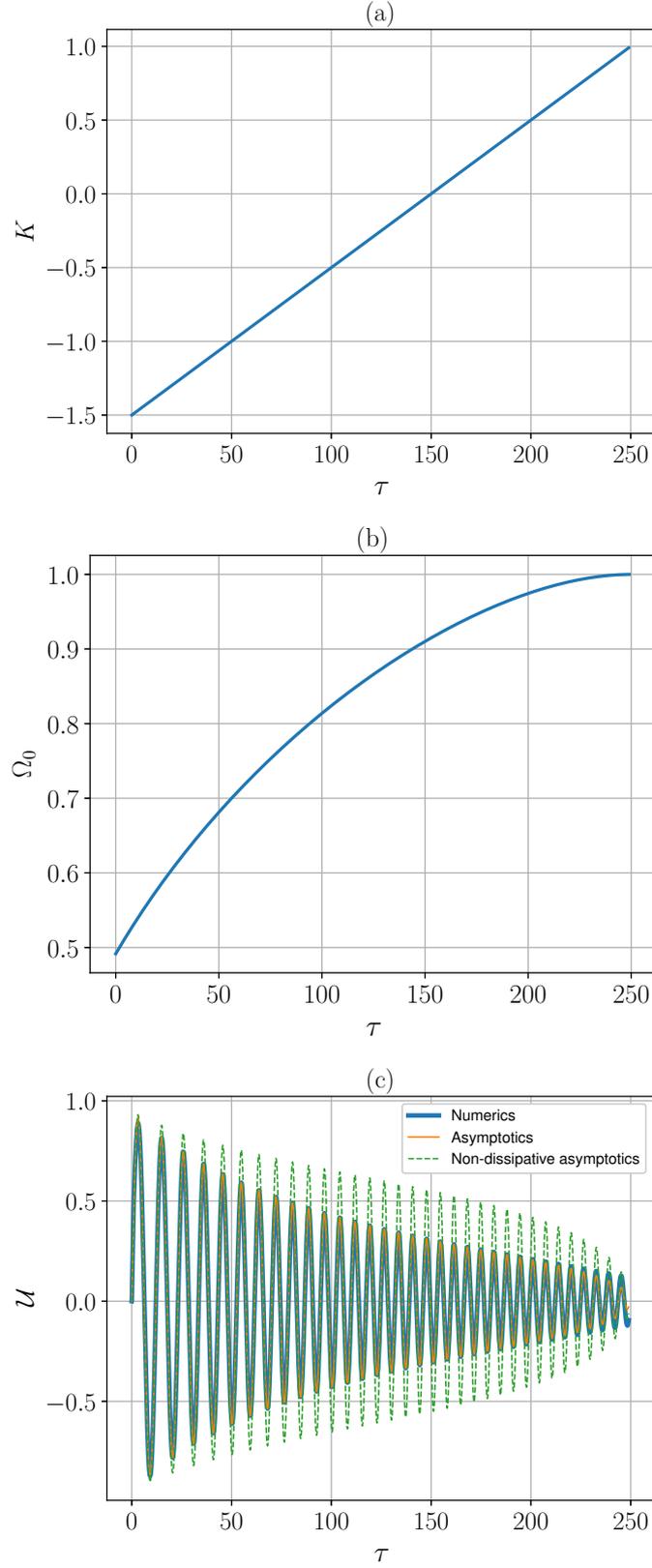


Figure 4: (a) Given time-varying parameter K , (b) the trapped mode frequency Ω_0 , and (c) the displacement \mathcal{U} of the discrete sub-system for the case when parameters are taken according Eq. (7.6), (7.7) wherein $K_0 = -1.5$, $K_1 = 1.0$

7.3 Systems with parameters approaching and crossing the boundaries of the localization domain

Here we consider two examples of the systems where the constructed single-mode asymptotics is not practically applicable. These examples are related to situations, when the hodograph of the tuple $\mathcal{P}(0, \tau)$ introduced by Eq. (2.30) approaches the boundaries of the localization domain \mathbb{L} defined by inequalities (B.19), (B.20). Finally, we plan to discuss why the single-mode approximation is so good for systems with instantaneous values of parameters lying always far from the boundaries of the localization domain \mathbb{L} .

The first example corresponds to the case when the trapped mode frequency reaches an almost zero positive value, and then again moves away from zero. This corresponds to the case when the hodograph of the tuple $\mathcal{P}(0, \tau)$ approaches a neighbourhood of the boundary of the localization domain \mathbb{L} defined by inequality (B.19) and then returns inside \mathbb{L} without any intersection. Consider a particular case of such a system, where Eqs. (7.3)–(7.4) are fulfilled and the relationship

$$K(T) = -0.6 - 0.95 \sin(1.5T) \quad (7.13)$$

is used instead of Eq. (7.5). The plots of K and the corresponding value of the trapped mode frequency Ω_0 are presented in Fig. 5(a–b) (the plot of \mathcal{T} is shown in Fig. 1(a)). One can see that at $\tau \approx 125$ the trapped mode frequency approaches zero (the minimal value is approximately 0.03) and again increases. In Fig. 5(c) we compare the corresponding asymptotic and numerical solutions. Looking at the plot of the numerical solution, one can observe essential increasing of the oscillation amplitude after $\tau \approx 125$, which is not described by the asymptotic solution. Note that repeating the numerical calculations with decreased both space and time step sizes by factor of 2 yields the same results, thus, the observed effect, apparently, is not related to a numerical instability. For larger values of time ($\tau \gtrsim 125$), when the parameters are again far enough from the boundary of the localization domain, the asymptotic solution again correctly predicts the phase value and the profile of the amplitude evolution, but the value of amplitude remains wrong.

Remark 7.3. It is well known that in the limiting case of the mass-spring system (2.35) the special considerations are necessary to obtain the correct asymptotics in the neighbourhood of points, where $\Omega^2 = 0$, i.e., the turning points, see, e.g., Jeffreys [1925], McHugh [1971]. We do not consider in this paper such subtle effects, and we are not sure that it could be meaningful in the case of the composite system under consideration.

The second example is more interesting. As we already noted in Sect. 6.6, in previous studies Gavrilov and Indeitsev [2002], Gavrilov et al. [2019a,b, 2022], Indeitsev et al. [2016] it was observed that the formal single-mode asymptotics generally breaks for a system with parameters such that the trapped mode frequency becomes close to the cut-off frequency. Let us consider an example of such a system in more details. Consider the system with a single time-varying parameter, where Eq. (2.9) and equations

$$M = 1, \quad \mathcal{T} = 1, \quad \rho = 1, \quad k = 1, \quad v = 0, \quad (7.14)$$

$$K = 1 - \exp(-3T) \quad (7.15)$$

are fulfilled. The plots of K and the corresponding value of the trapped mode frequency Ω_0 are presented in Fig. 6(a–b). One can see that for large values of time, the trapped mode frequency approaches the cut-off frequency $\Omega_* = 1$, see Eqs. (B.9), (B.21). Comparing in Fig. 6(c) the corresponding asymptotic and numerical solutions, one can observe two stages. At the first stage ($\tau \lesssim 50$), asymptotics describes the numerical solution quite well. At the next stage, the amplitude of the constructed single mode asymptotics, clearly, approaches zero faster than the corresponding numerical solution. Namely, according to Eqs. (5.6), (5.8), (5.15), (5.16) one has

$$\mathcal{W} = C_1 \exp\left(-\frac{3}{2}\epsilon\tau\right) + o\left(\exp\left(-\frac{3}{2}\epsilon\tau\right)\right), \quad \tau \rightarrow \infty; \quad C_1 \approx 0.4. \quad (7.16)$$

At the same time, one can qualitatively estimate the asymptotic decay rate for the amplitude of the numerical solution at the second stage as

$$\mathcal{W} = \frac{C_2}{\sqrt{\tau}} + o\left(\frac{1}{\sqrt{\tau}}\right), \quad \tau \rightarrow \infty; \quad C_2 = \text{const} \neq 0. \quad (7.17)$$

(see Fig. 6(c)). It is natural to assume that at the second stage the system under consideration can be treated as a system with almost constant parameters lying at the boundary of the localization domain where $\Omega_0 \simeq \Omega_*$. Actually, in such a system with constant parameters, the order of decay is $\tau^{-1/2}$ Shishkina et al. [2023]. Thus, apparently, the approximate description for \mathcal{U} can be obtained as a matched asymptotics, which stages correspond to the single-mode approximation for the system with time-varying parameters and asymptotics for the same system with constant parameters, respectively.

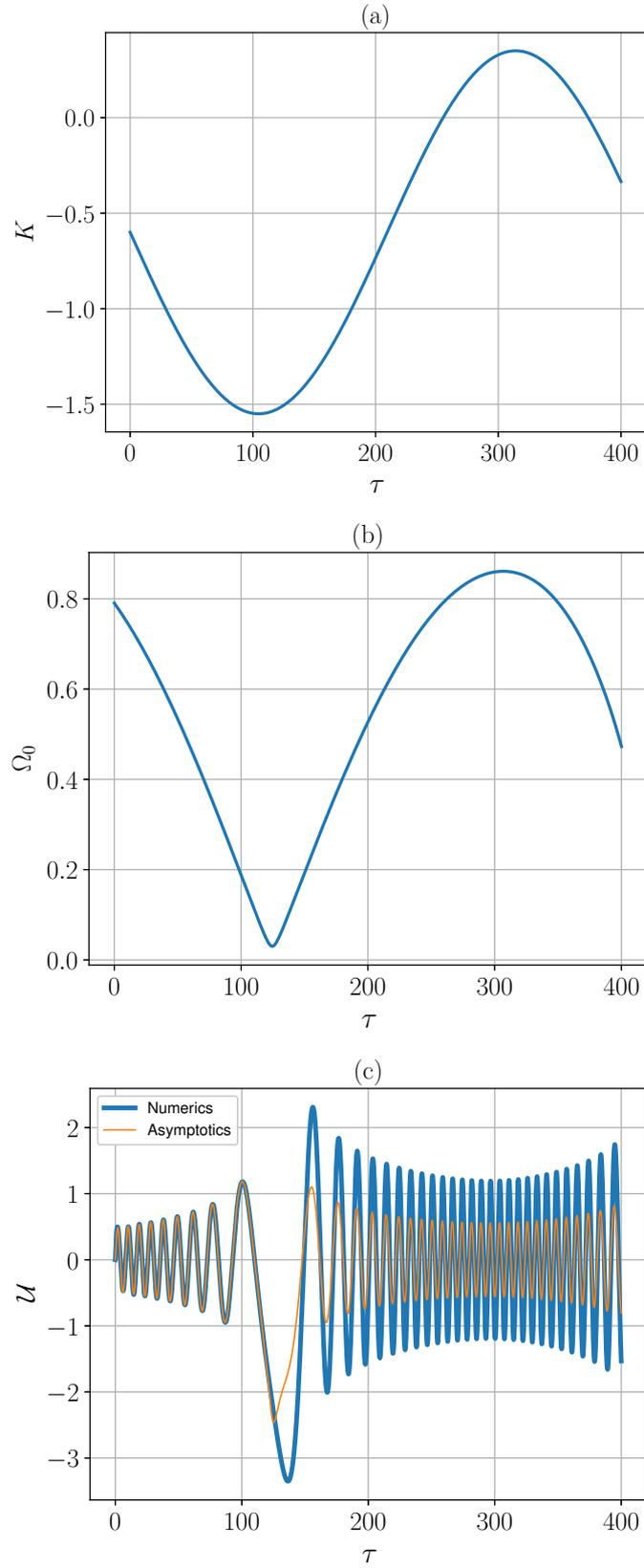


Figure 5: (a) Given time-varying parameter K , (b) the trapped mode frequency Ω_0 , (c) the displacement \mathcal{U} of the discrete sub-system for the case when parameters are taken according Eqs. (7.3), (7.4), (7.13)

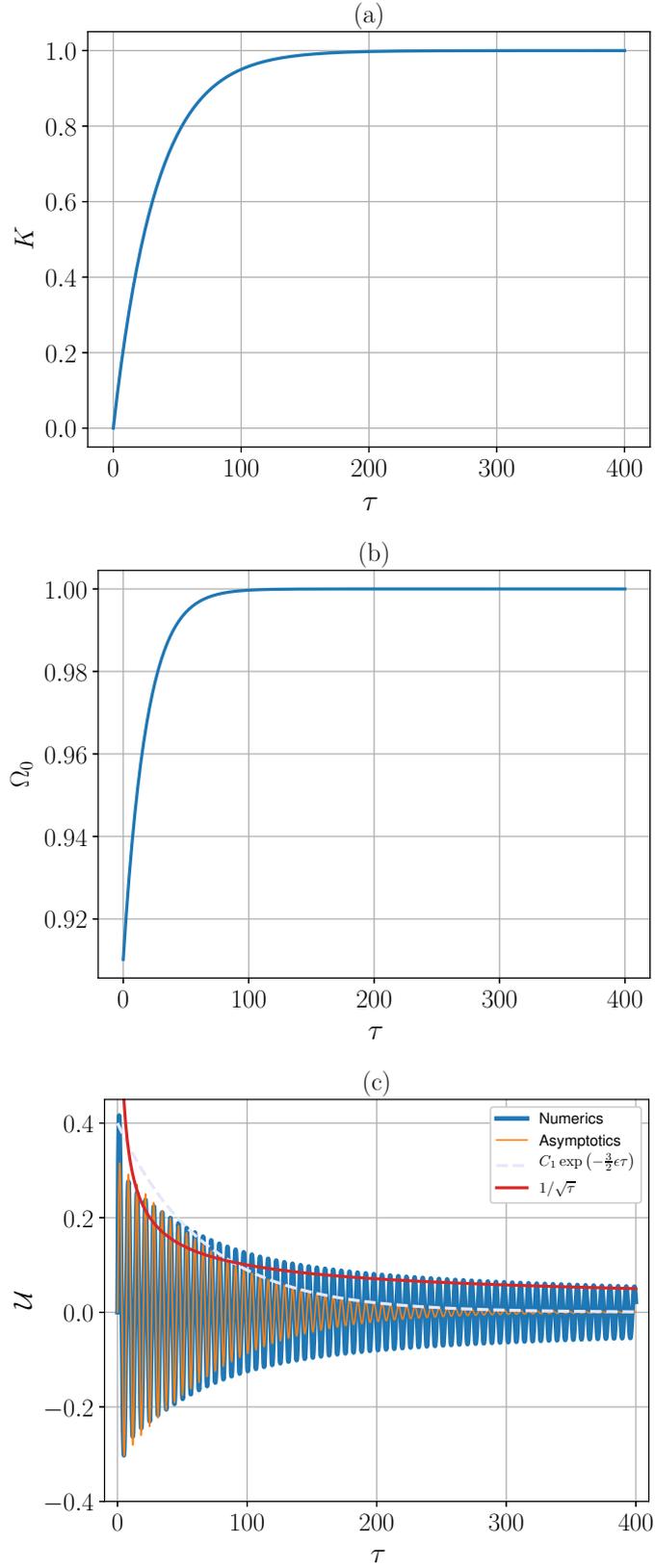


Figure 6: (a) Given time-varying parameter K , (b) the trapped mode frequency Ω_0 , (c) the displacement \mathcal{U} of the discrete sub-system for the case when parameters are taken according Eqs. (7.14), (7.15)

One can see that in Fig. 6(c) oscillation with the asymptotic rate of decay $\tau^{-1/2}$ is clearly recognizable. So the question arises why such an oscillation is not recognizable in Figs. 2, 3(b), 4(b) and in all corresponding plots obtained in previous papers [Gavrilov and Indeitsev \[2002\]](#), [Gavrilov et al. \[2019a,b, 2022\]](#), [Indeitsev et al. \[2016\]](#). Why is the single-mode approximation so good inside the localization domain? Indeed, we expect that in an infinite spatially uniform one-dimensional continuum system without trapped modes, the perturbations decay with the rate $\tau^{-1/2}$ [Slepyan \[1972\]](#), [Whitham \[1999\]](#).⁵ Of course, this is true for a spatially uniform pure continuum system. Nevertheless, for continuum systems with inclusions, the corresponding decaying as $\tau^{-1/2}$ oscillation is also observed everywhere between the leading wave-fronts excepting a neighbourhood of a discrete inclusion. However, the mode decaying with the rate $\tau^{-1/2}$ always has zero amplitude near the inclusion, excepting the case, when the system parameters correspond to the boundary of the localization domain where $\Omega_0 \rightarrow \Omega_*$. The particular case of such a system is a uniform pure continuum system. We have called this wave phenomenon the anti-localization of non-stationary linear waves [Gavrilov and Shishkina \[2024\]](#), [Gavrilov et al. \[2023\]](#), [Shishkina and Gavrilov \[2023\]](#), [Shishkina et al. \[2023\]](#). Thus, just the existence of the anti-localization makes the single-mode asymptotics to be so good approximation for a system with parameters inside the localization domain far from its boundaries.

On the other hand, for a system where the hodograph of $\mathcal{P}(0, T)$ transversely crosses the boundary of \mathbb{L} where $\Omega_0 \rightarrow \Omega_*$, we expect fast decaying of the oscillation amplitude \mathcal{W} . Indeed, due to the anti-localization, the rate of decay near the discrete sub-system for the corresponding system with constant parameters lying outside \mathbb{L} is $\tau^{-3/2}$ [Shishkina et al. \[2023\]](#). In Fig. 7 we illustrate this fact for the system where Eqs. (2.9), (7.14) and

$$K = 0.5 + 0.5T \tag{7.18}$$

are fulfilled.

Remark 7.4. For the particular case of the discrete sub-system in the form of pure inertial inclusion, the result, which shows that the decay rate for the vanishing component is $\tau^{-3/2}$ was obtained by Kaplunov in [Kaplunov \[1986\]](#), see also earlier studies [Hemmer \[1959\]](#), [Kashiwamura \[1962\]](#), [Müller \[1962\]](#), [Müller and Weiss \[2012\]](#), [Rubin \[1963\]](#) concerning infinite discrete systems. In paper [Shishkina et al. \[2023\]](#), we first time have demonstrated that the emergence and the intensity of the anti-localization are related not with its non-uniformity itself, but with the position of \mathcal{P} far enough from the boundary of the localization domain where $\Omega_0 \rightarrow \Omega_*$.

8 Conclusion

In the paper, we have obtained the leading-order term of a universal asymptotics, which can approximate the free non-stationary response \mathcal{U} of a discrete mass-spring-damper system embedded into a continuum system governed by the telegraph PDE. All parameters of such a composite system are assumed to be slowly time-varying functions. Additionally, the parameters of the continuum sub-system can be spatially varying functions. In a particular case, the discrete sub-system can move along the continuum one at a sub-critical speed, which is also assumed to be a slowly time-varying function. We propose the solution in the form of the single-mode ansatz (3.1)–(3.11), which corresponds to the dominant frequency in the non-stationary response of the corresponding system with constant coefficients, i.e., the trapped mode frequency (B.21). We require that the localization conditions, see Remark (B.3), are always fulfilled for the instantaneous values of the system parameters in a certain neighbourhood of the discrete sub-system. The most important result of the paper is formula (5.6) for the leading-order term $\mathcal{W}_0(T)$ of the non-stationary oscillation amplitude. The quantities involved into Eq. (5.6) should be found by Eqs. (5.7), (5.16). The asymptotics for the response \mathcal{U} in the real form is given by Eq. (5.15).

Though mathematical technique used in the current paper and suggested first time in [Gavrilov and Indeitsev \[2002\]](#) is inspired by the formal procedure of obtaining the Liouville-Green (or WKB) approximation by the method of multiple scales [Nayfeh \[2008\]](#), the asymptotic ansatz that we use is quite similar to one used in the space-time ray method [Babich and Buldyrev \[2009\]](#), [Babich et al. \[2002\]](#). The essential difference from the first approach is that we deal with a PDE coupled with an ODE, whereas the Liouville-Green approximation was suggested for an ODE. Comparing our approach with the space-time ray method, one can see that we deal with a pure non-stationary solution. Even in zeroth order approximation, the non-stationary solution is a Fourier integral over all possible frequencies. Therefore, we need an additional step to satisfy the initial conditions, where we match (see Remark 5.2) the single-mode asymptotics with the results obtained by the method of stationary phase applied to the same system with constant parameters, which are equal to the corresponding initial values. In the framework of the space-time ray method the initial conditions are usually formulated in terms of quantities involved in the ansatz (the amplitude and the phase), therefore the matching is not required. Another important difference of our problem, compared with problems where

⁵The rate of decay $\tau^{-1/2}$ corresponds to the contribution from a stationary point in the representation of the time-domain fundamental solution in the form of the inverse Fourier transform.

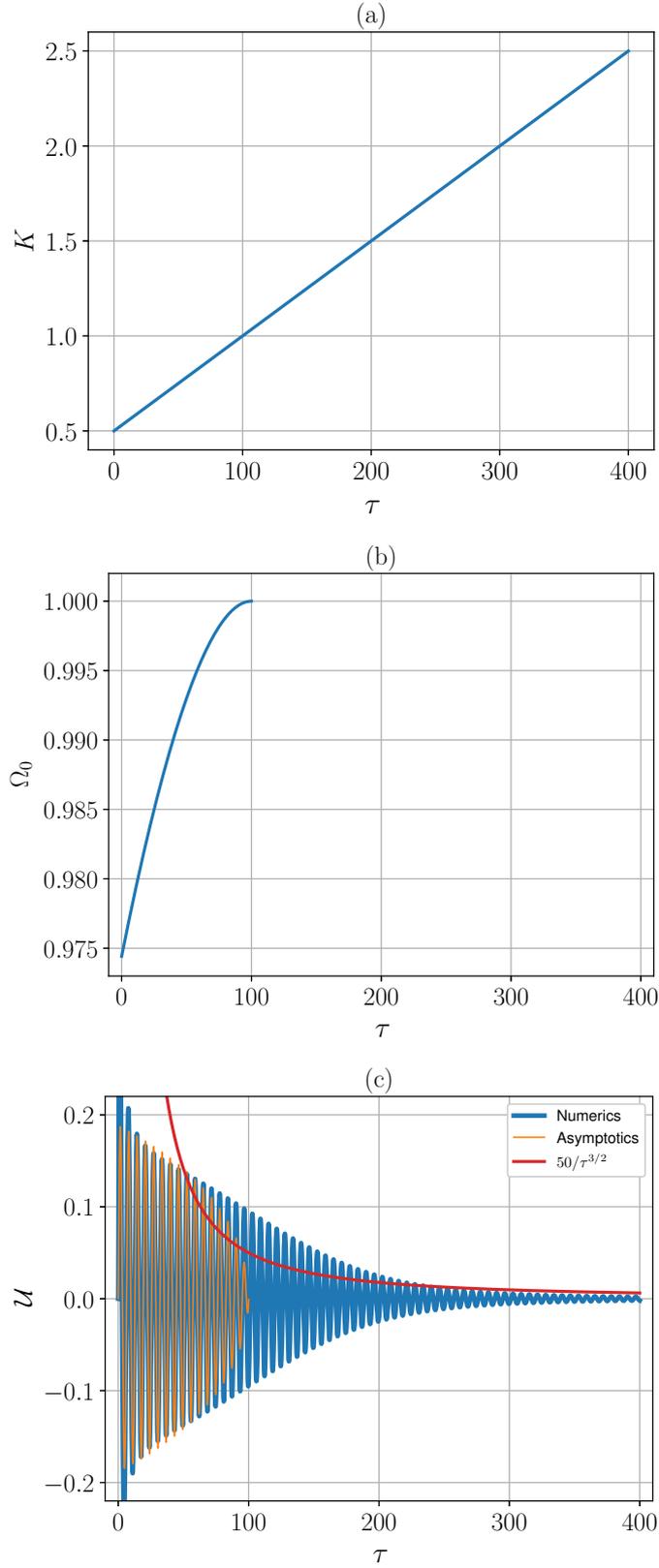


Figure 7: (a) Given time-varying parameter K , (b) the trapped mode frequency Ω_0 , (c) the displacement \mathcal{U} of the discrete sub-system for the case when parameters are taken according Eqs. (7.14), (7.15)

the space-time ray method is usually used, is that the equations for the amplitude (the transport equations in the framework of the space-time ray method) in our case can be equivalently reduced, at least in the first approximation, to not a PDE, but to a single ODE.

According to Eqs. (5.7)–(5.10), in the non-dissipative case (2.9), the leading-order term of the expansion for the amplitude of oscillation is obtained in the form of an algebraic expression, which involves the instantaneous limiting values of the system parameters independently varying accordingly to unknown arbitrary laws. In the dissipative case (2.10), we, generally, obtain the leading-order term of the expansion for the amplitude in quadratures as a functional, which depends on the history of the system parameters, though in some exceptional cases the result can be obtained as a function of time and the instantaneous limiting values of the system parameters. From the physical point of view, the obtained results are an essential extension, related to the localized oscillation, of the classical result (A.15) concerning the oscillation of a mass-spring-damper system with time-varying parameters, see Appendix A.

By construction, the first approximation equation (3.26) has the structure of Eq. (4.1), where \mathcal{P} and $\tilde{\mathcal{P}}$ are system parameters introduced by Eqs. (2.27), (2.28). The most general form of the solution of Eq. (4.1) is functional (4.10), which depends on the history of the system parameters. Nevertheless, if the right-hand side of Eq. (4.1) (or Eq. (3.26)) can be represented in the form of the exact derivative of a certain function of the system parameters, see Eq. (4.7), then the corresponding solution is a function, which depends on the instantaneous limiting values of the system parameters, see Eq. (4.8). Such a representation is possible if and only if conditions (4.2) and (4.3) are fulfilled. Note that Eq. (4.2) is always true for any non-dissipative system under consideration. For non-dissipative problems with a single time-varying parameter, considered in our previous studies Gavrilov and Indeitsev [2002], Gavrilov et al. [2019a,b, 2022], Indeitsev et al. [2016], condition (4.3) is also always true, and the solution of the first approximation equation can be found in the form of certain functions, see Remark 4.1. In the non-dissipative case of several time-varying parameters considered in this paper, the possibility to represent a given equation with the structure of Eq. (4.1) in the form of Eq. (4.7) is absolutely unobvious. We can try to check conditions (4.3) directly. However, this can be a difficult problem due to a lengthy structure of the right-hand side of the equation under consideration. On the other hand, it is possible to introduce new variables $\Psi(\mathcal{P})$ instead of \mathcal{P} and equivalently rewrite the first approximation equation in the form of Eq. (4.12). A lucky choice of the variables $\Psi(\mathcal{P})$ can essentially simplify the calculations and help to obtain the representation of Eq. (4.1) in the form of Eq. (4.16). This trick was suggested by Poroshin in Gavrilov et al. [2022] and the same approach is applied in this paper. The existence of such a representation immediately guarantees us that the amplitude can be obtained as a function, not as a functional, and gives us the possibility to calculate this function in an explicit form. Generally, it is not clear for us, due to which properties of governing equations (2.18), (2.19) with variable coefficients, we have obtained the first approximation equation for the amplitude \mathcal{W}_0 with the exact derivative in the right-hand side. Note that magically the exact derivatives can be met often when dealing with WKB approximations. For example, the right-hand sides of the odd successive WKB approximations for ODE describing a mass-spring system with time-varying stiffness, i.e., for Eq. (A.20) wherein $M = \text{const}$ and $\Gamma = 0$, are the exact derivatives Sukumar [2023], see Remark A.4. Thus, the possibility to obtain the result for the amplitude in a closed form for system with several simultaneously varying parameters is not related with the summarization of the obtained in previous papers Gavrilov and Indeitsev [2002], Gavrilov et al. [2019a,b], Indeitsev et al. [2016] results, but with the trick suggested in the recent paper Gavrilov et al. [2022], see Sect. 6.6.

The practical applicability of the obtained in such a way formal single-mode asymptotics should be verified numerically. This has been done in our previous papers Gavrilov and Indeitsev [2002], Gavrilov et al. [2019a,b, 2022], Indeitsev et al. [2016] for a number of single-parameter perturbed non-dissipative systems. In the current paper, a two-parameter perturbed system was considered, see Sect. 7.1, as well as a dissipative one, see Sect. 7.2. In all cases, an excellent agreement was demonstrated. The fact that the single-mode approximation is in an excellent agreement with numerics, is related, in our opinion, with the phenomenon of the anti-localization of non-stationary linear waves, discovered recently Gavrilov and Shishkina [2024], Gavrilov et al. [2023], Shishkina and Gavrilov [2023], Shishkina et al. [2023], see Sect. 7.3.

One of the practically important problems, for which the asymptotics obtained in this paper appears to be applicable, is the problem concerning a mass-spring-damper system non-uniformly moving along the wave-guide with the spatially varying stiffness k . Since this is a non-symmetrical with respect to ξ problem, for now, we are not ready to present the corresponding numerical calculations, see Remark 7.1. On the other hand, this problem has many specific important for applications peculiarities, which can make it the subject for a separate study.

Acknowledgement

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memory of Prof. D.A. Indeitsev. The research is supported by the Ministry of Science and Higher Education of the Russian Federation (project 124040800009-8).

A Oscillation of a mass-spring-damper system with time-varying parameters

Consider, mostly following the formal procedure used by Nayfeh (Sect. 7.1.6 in Nayfeh [2008]), a free non-stationary oscillation of a mass-spring-damper system. The viscosity of the dumper is a small quantity $\epsilon\Gamma$.

Remark A.1. Nayfeh did not use the last assumption. We use it, to make the problem similar to one formulated in Sect. 2, see also Remark 2.3.

The stiffness $K(\epsilon\tau)$, the mass $M(\epsilon\tau)$, and the viscosity $\epsilon\Gamma(\epsilon\tau)$ are slowly time-varying quantities. The governing equation is

$$(M(\epsilon\tau)\dot{\mathcal{U}})' + \left\{ 2\epsilon\Gamma(\epsilon\tau)\dot{\mathcal{U}} \right\} + K(\epsilon\tau)\mathcal{U} = 0, \quad (\text{A.1})$$

where \mathcal{U} is the displacement.

Remark A.2. Generally, Nayfeh (Nayfeh [2008]) assumed that coefficients M , Γ , and K are given in the form of asymptotic power series in ϵ :

$$M = \sum_n M_n(\epsilon\tau)\epsilon^n, \quad \Gamma = \sum_n \Gamma_n(\epsilon\tau)\epsilon^n, \quad K = \sum_n K_n(\epsilon\tau)\epsilon^n. \quad (\text{A.2})$$

Here, we assume that the only non-zero terms in expansions (A.2) correspond to $M_0 \equiv M$, $K_0 \equiv K$, and $\Gamma_0 \equiv \Gamma$. In (A.1) the only term in the left-hand side, which is proportional to ϵ is shown in the curly brackets.

In the same way as we have done it, see Eqs. (2.27), (2.28), for the full system, we can characterize the properties of the system described by (A.1) by n -tuple ($n = 2$)

$$\mathcal{P}(\epsilon\tau) \stackrel{\text{def}}{=} (\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{R}^2, \quad \mathcal{P}_1 = K, \quad \mathcal{P}_2 = M, \quad (\text{A.3})$$

of parameters for the zero-order system, and an additional \tilde{n} -tuple ($\tilde{n} = 1$)

$$\tilde{\mathcal{P}}(\epsilon\tau) \stackrel{\text{def}}{=} (\tilde{\mathcal{P}}_1) \in \mathbb{R}^1, \quad \tilde{\mathcal{P}}_1 = \Gamma, \quad (\text{A.4})$$

of parameters, which are necessary to consider additionally to \mathcal{P} for the system with $\epsilon > 0$.

We introduce the slow time $T = \epsilon\tau$ and represent the solution in the form of the following ansatz

$$\mathcal{U} = \mathcal{W} \exp \phi + \text{c.c.}, \quad (\text{A.5})$$

where

$$\mathcal{W}(T) = \sum_i \epsilon^i \mathcal{W}_i(T), \quad (\text{A.6})$$

$$\dot{\phi}(T) = -i\Omega_0(T) \quad (\text{A.7})$$

are the amplitude and the phase, respectively;

$$\Omega_0(T) = \sqrt{\frac{K(T)}{M(T)}} \quad (\text{A.8})$$

is the natural frequency. Then, we consider T and ϕ as independent time-like variables, and use the corresponding representations (3.12) for differential operators with respect to time. Substituting Eqs. (A.5)–(A.8) into Eq. (A.1) results in the following first approximation equation for the leading-order term $\mathcal{W}_0(T)$:

$$2\sqrt{MK}\mathcal{W}_0'T + M \left(\sqrt{\frac{K(T)}{M(T)}} \right)'_T \mathcal{W}_0 + M'_T \sqrt{\frac{K(T)}{M(T)}} \mathcal{W}_0 + \left\{ 2\Gamma \sqrt{\frac{K(T)}{M(T)}} \mathcal{W}_0 \right\} = 0, \quad (\text{A.9})$$

which is written in terms of variables \mathcal{P} , $\tilde{\mathcal{P}}$. Here and in what follows in Appendix A, the contribution from the term in the left-hand side of (A.1), which is proportional to ϵ , is again shown in the curly brackets. Taking into account that

$$\left(\sqrt{\frac{K(T)}{M(T)}} \right)'_T = \frac{K'_T}{2\sqrt{KM}} - \frac{M'_T \sqrt{K}}{2M^{3/2}}, \quad (\text{A.10})$$

after simplification, one can rewrite Eq. (A.9) in the following form:

$$\frac{\mathcal{W}_0'}{\mathcal{W}_0} = -\frac{K'_T}{4K} - \frac{M'_T}{4M} - \left\{ \frac{\Gamma}{M} \right\}. \quad (\text{A.11})$$

The last equation has the structure of Eq. (4.1). Provided that $\Gamma = 0$, we get the solution of Eq. (4.1) in the form of a function, which depends on the instantaneous values of the system parameters \mathcal{P} only:

$$\mathcal{W}_0(T) = \frac{\mathcal{C}}{\sqrt[4]{M(T)K(T)}} = \frac{\mathcal{C}}{\sqrt{M(T)\Omega_0(T)}}. \quad (\text{A.12})$$

In the particular case $\dot{M} = 0$, the amplitude \mathcal{W}_0 is proportional to the inverse of the square root of the natural frequency.

$$\mathcal{W}_0(T) = \frac{\bar{\mathcal{C}}}{\sqrt{\Omega_0(T)}}, \quad (\text{A.13})$$

where $\bar{\mathcal{C}} = \mathcal{C}/\sqrt{M}$ is an arbitrary constant. This formula now is well-known as the Liouville – Green or WKB approximation [Nayfeh \[2008\]](#), see also [Feschenko et al. \[1967\]](#), [McHugh \[1971\]](#) for historical aspects. If $\Gamma = \text{const} \neq 0$ and $M = \text{const}$, we again obtain the solution in the form of a function, which depend on current values of the system parameters \mathcal{P} , and, in the latter case, the current value of time T :

$$\mathcal{W}_0(T) = \frac{\mathcal{C} \exp\left(-\frac{\Gamma T}{M}\right)}{\sqrt{M\Omega_0(T)}} = \frac{\bar{\mathcal{C}} \exp\left(-\frac{\Gamma T}{M}\right)}{\sqrt{\Omega_0(T)}}. \quad (\text{A.14})$$

Finally, if any of Γ or M is not a constant, one obtains the solution in quadratures in the form of a functional:

$$\mathcal{W}_0(T) = \frac{\mathcal{C} \exp\left(-\int_0^T \frac{\Gamma(\hat{T}) d\hat{T}}{M(\hat{T})}\right)}{\sqrt[4]{M(T)K(T)}} = \frac{\mathcal{C} \exp\left(-\int_0^T \frac{\Gamma(\hat{T}) d\hat{T}}{M(\hat{T})}\right)}{\sqrt{M(T)\Omega_0(T)}}. \quad (\text{A.15})$$

Remark A.3. The right-hand side of (A.11) satisfies condition (4.3) since

$$\frac{\partial F_0}{\partial \mathcal{P}_1} = \frac{\partial\left(-\frac{1}{4\mathcal{P}_0}\right)}{\partial \mathcal{P}_1} = 0, \quad \frac{\partial F_1}{\partial \mathcal{P}_0} = \frac{\partial\left(-\frac{1}{4\mathcal{P}_1}\right)}{\partial \mathcal{P}_0} = 0. \quad (\text{A.16})$$

Note that for the case of Eq. (A.1), instead of \mathcal{P} , it is useful to take new variables:

$$\Psi_1(T) = \Omega_0(T), \quad \Psi_2(T) = M(T). \quad (\text{A.17})$$

Instead of Eq. (A.9) we get

$$2M\Omega_0\mathcal{W}_0' + M\Omega_0'\mathcal{W}_0 + M'_T\Omega_0\mathcal{W}_0 + \{2\Gamma\Omega_0\mathcal{W}_0\} = 0 \quad (\text{A.18})$$

or

$$\frac{\mathcal{W}_0'}{\mathcal{W}_0} = -\frac{\Omega_0'}{2\Omega_0} - \frac{M'_T}{2M} - \left\{ \frac{\Gamma}{M} \right\}. \quad (\text{A.19})$$

Provided that $\Gamma = 0$, Eq. (A.19) has the structure of Eq. (4.16), and the solution is given by Eq. (A.12). The solution in case $\Gamma \neq 0$ can be obtained analogously, in the same way, as it was done when deriving formulae (A.14)–(A.15). One can see that in terms of variables Ψ defined by Eq. (A.17) the calculations are essentially simpler than for the original parameters \mathcal{P} .

Remark A.4. Equation (A.1) with time-varying coefficients K and Γ is considered in Nayfeh book [Nayfeh \[2008\]](#). In [Feschenko et al. \[1967\]](#) equation

$$M(\epsilon\tau)\ddot{u} + 2\Gamma(\epsilon\tau)\dot{u} + K(\epsilon\tau)u = 0 \quad (\text{A.20})$$

is discussed instead of Eq. (A.1). This equation can also be reduced to the form of Eq. (A.1) with only time-varying coefficients K and Γ . Equation (A.1) with $\Gamma = 0$ is discussed in book [Kevorkian and Cole \[1996\]](#) (Sect. 4.3.3), where solution (A.12) is found. In all these studies, the question if the expression for the leading-order term \mathcal{W}_0 of the amplitude of the solution of a certain ODE with variable coefficients can be represented as a function, depending on the current values of the system parameters only, or as a functional, is not discussed. Moreover, we did not manage to find any studies where this problem was noticed, although there may be some. Note that, the right-hand sides of successive odd WKB approximations for Eq. (A.1) wherein $M = \text{const}$ and $\Gamma = 0$ are the exact derivatives, see [Sukumar \[2023\]](#), where such an equation treated as the one-dimensional time-independent Schrödinger equation with a spatial variable τ .

B The properties of the zeroth order system

In Appendix B, we provide the necessary formulae for the system under consideration with $\epsilon = 0$, i.e., for the composite system with constant parameters \mathcal{P} introduced by Eq. (2.27) and zero dissipation, see Eq. (2.9). Accordingly, Eqs. (2.18) and (2.19) can be rewritten as

$$M\ddot{u} + K\mathcal{U} = -P(\tau) + p(\tau), \quad (\text{B.1})$$

$$(\mathcal{T} - \rho v^2)u'' + 2v\rho u' - \rho\ddot{u} - ku = -P(\tau)\delta(\xi). \quad (\text{B.2})$$

All results are obtained in the dimensionless form in Gavrilov et al. [2022]. Some particular results were obtained in Gavrilov et al. [2019a], Glushkov et al. [2011], Kaplunov and Sorokin [1995], Kaplunov [1986], Kruse et al. [1998].

B.1 The dispersion relation

Assuming that $p(\tau) = 0$ and

$$u = \mathcal{W}e^{-i(\Omega\tau - \omega\xi)} \quad (\text{B.3})$$

in intervals $\xi \geq 0$, we get the dispersion relation for the operator in the left-hand side of Eq. (B.2):

$$\omega^2 - 2B(\Omega)\omega + A^2(\Omega) = 0. \quad (\text{B.4})$$

Here ω is the wave-number:

$$\omega = B(\Omega) \pm iS(\Omega), \quad (\text{B.5})$$

$$B(\Omega) = \frac{v\Omega\rho}{\mathcal{T} - \rho v^2} \quad (\text{B.6})$$

$$A^2(\Omega) = \frac{k - \rho\Omega^2}{\mathcal{T} - \rho v^2}, \quad (\text{B.7})$$

$$S(\Omega) = \sqrt{A^2(\Omega) - B^2(\Omega)} = \frac{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}}{\mathcal{T} - \rho v^2} = \frac{k}{\sqrt{\rho\mathcal{T}}} \frac{\sqrt{\Omega_*^2 - \Omega^2}}{\Omega_*^2}. \quad (\text{B.8})$$

The quantity Ω_* is the cut-off (boundary) frequency

$$\Omega_* = \frac{k}{\rho} - \frac{kv^2}{\mathcal{T}} = \frac{k}{\rho\mathcal{T}}(\mathcal{T} - \rho v^2), \quad (\text{B.9})$$

which separates the pass-band $\Omega^2 > \Omega_*^2$, where the solution is a sinusoidal wave and the stop-band $0 < \Omega^2 < \Omega_*^2$, where the solution is an inhomogeneous wave.

Remark B.1. Note that according Eqs. (2.8), (2.13), (2.14) $\Omega_* > 0$.

B.2 Spectral problem for a trapped mode

Put $p = 0$ and consider the spectral problem concerning free oscillation of the system described by Eqs. (B.1), (B.2). We consider only sub-critical speeds, i.e., Eq. (2.13) is fulfilled. Let

$$u = W(\xi)e^{-i\Omega\tau}, \quad (\text{B.10})$$

$$\mathcal{U} = \mathcal{W}e^{-i\Omega\tau}. \quad (\text{B.11})$$

Trapped modes are modes with finite energy, i.e., we require

$$\int_{-\infty}^{+\infty} W^2 d\xi < \infty, \quad \int_{-\infty}^{+\infty} W'^2 d\xi < \infty. \quad (\text{B.12})$$

By substituting Eqs. (B.10), (B.11) into Eqs. (B.1), (B.2), one gets

$$(\mathcal{T} - \rho v^2)(W'' - 2iB(\Omega)W' - A^2(\Omega)W) = -(M\Omega^2 - K)W\delta(\xi). \quad (\text{B.13})$$

The solution of Eq. (B.13) can be written as follows:

$$W(\xi) = (M\Omega^2 - K)WG(\xi, \Omega), \quad (\text{B.14})$$

where $G(\xi, \Omega)$ is the Green function in the frequency domain. For $\Omega^2 < \Omega_*^2$ one has

$$G(\xi, \Omega) = \frac{\exp(iB\xi - S|\xi|)}{2(\mathcal{T} - \rho v^2)S}. \quad (\text{B.15})$$

Taking into account Eqs. (B.6), (B.8), one gets:

$$W(\xi) = \frac{(M\Omega^2 - K)\mathcal{W}}{2\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}} \exp\left(\frac{iv\Omega\rho\xi - \sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}|\xi|}{\mathcal{T} - \rho v^2}\right). \quad (\text{B.16})$$

Calculating Eq. (B.16) at $\xi = 0$ yields the frequency equation for the trapped mode frequency Ω_0 :

$$2\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} = M\Omega_0^2 - K. \quad (\text{B.17})$$

The last equation can be equivalently rewritten in the following form:

$$2\sqrt{\Omega_*^2 - \Omega_0^2} = \frac{M\Omega_0^2 - K}{\sqrt{\mathcal{T}\rho}}. \quad (\text{B.18})$$

The positive roots $\Omega_0^2 > 0$ of the frequency equation (B.18) are trapped modes frequencies. It can be shown that provided that Eqs. (2.4), (2.8), (2.13) as well as the following inequalities

$$-2\sqrt{\mathcal{T}\rho}|\Omega_*| < K, \quad (\text{B.19})$$

$$K < M\Omega_*^2 \quad (\text{B.20})$$

are true, there exists a unique trapped mode with the frequency

$$\begin{aligned} \Omega_0^2 &= \frac{KM - 2\mathcal{T}\rho + 2\sqrt{\mathcal{T}\rho(\mathcal{T}\rho - KM) + M^2k(\mathcal{T} - \rho v^2)}}{M^2}, & M > 0; \\ \Omega_0^2 &= \Omega_*^2 - \frac{K^2}{4\mathcal{T}\rho}, & M = 0. \end{aligned} \quad (\text{B.21})$$

Here Ω_* is defined by Eq. (B.9).

Remark B.2. The root (B.21) of frequency equation (B.17) satisfies inequalities

$$\begin{aligned} \frac{K}{M} < \Omega_0^2 < \Omega_*^2, & \quad M > 0 \text{ and } K \geq 0, \\ 0 < \Omega_0^2 < \Omega_*^2, & \quad K < 0. \end{aligned} \quad (\text{B.22})$$

Remark B.3. The domain defined by Eqs. (2.4), (2.8), (2.13), (B.19), (B.20) in the six-dimensional parameter space with co-ordinates (2.27) is the localization domain \mathbb{L} . Condition (B.19) has the meaning of the stability condition, which is important for $K \leq 0$. Violating Eq. (B.19) leads to zeroing of the trapped mode frequency $\Omega_0 = 0$ and a localized buckling of the continuum sub-system. Approaching of the system parameters to the boundary (B.20) corresponds to approaching the trapped mode frequency Ω_0 to the cut-off frequency Ω_* :

$$\lim_{K \rightarrow M\Omega_* - 0} \Omega_0 = \Omega_* \quad (\text{B.23})$$

i.e., to the upper boundary of the stop-band, and, finally, to the disappearing of the trapped mode.

Remark B.4. Conditions (2.4), (2.8) are some pre-assumed physical restrictions, which are not directly related to the existence of the trapped mode. Moreover, as shown in Glushkov et al. [2011] for $M < 0$ even two trapped modes can exist. Nevertheless, we do not know any physical realization for a system with $M < 0$ (though there may be some) and do not take into account such a case.

Remark B.5. It can be shown that the expression for Ω_0^2 defined by (B.21) is continuous at $M = +0$.

Remark B.6. The real solution of Eq. (B.1) wherein $p = 0$ and Eq. (B.2), which correspond to the trapped mode, is

$$u = C \exp(-S(\Omega_0)|\xi|) \cos(\Omega_0\tau - B(\Omega_0)\xi - D), \quad (\text{B.24})$$

where C, D are arbitrary real constants.

B.3 Non-stationary free oscillation

Consider now the non-stationary problem with the external loading $p(\tau)$ satisfying Eq. (2.26). The solution can be obtained in the form of the (generalized) inverse Fourier transform

$$\mathcal{U}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathcal{F}\{p\}(\Omega) e^{-i\Omega\tau} d\Omega}{2\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2} + K - M\Omega^2}, \quad (\text{B.25})$$

where symbol $\mathcal{F}\{p\}(\Omega)$ denotes the value of the Fourier transform for the loading $p(\tau)$ calculated at a frequency Ω . The large time asymptotics for the right-hand side of Eq. (B.25) can be found by the method of stationary phase Fedoryuk [1977] and has the following form Gavrilov et al. [2022]:

$$\mathcal{U}(\tau) = \frac{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2} |\mathcal{F}\{p\}(\Omega_0)|}{\Omega_0 (\mathcal{T}\rho + M\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega_0^2})} \sin(\Omega_0\tau - \arg(\mathcal{F}\{p\}(\Omega_0))) + o(1), \quad \tau \rightarrow \infty. \quad (\text{B.26})$$

Hence, for the large times, the oscillation with the trapped mode frequency Ω_0 is dominant and only non-vanishing component of the non-stationary response for the system under consideration.

B.4 Derivation of identity (5.3)

According to Eq. (B.8), we have

$$S'_\Omega = \frac{-\mathcal{T}\rho\Omega}{(\mathcal{T} - \rho v^2)\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}} \quad (\text{B.27})$$

and, therefore,

$$(\mathcal{T} - \rho v^2)S'_\Omega - M\Omega = -\frac{\mathcal{T}\rho\Omega + M\Omega\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}}{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}}. \quad (\text{B.28})$$

Due to Eqs. (B.6), (B.8) we get:

$$\rho v B + \rho\Omega + M\Omega S = \frac{\mathcal{T}\rho\Omega + M\Omega\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}}{\mathcal{T} - \rho v^2}. \quad (\text{B.29})$$

Substituting Eqs. (B.28), (B.29) into the left-hand side of Eq. (5.3), and using Eq. (B.8), we, finally, obtain:

$$\frac{(\mathcal{T} - \rho v^2) S'_\Omega - M\Omega}{\rho v B + \rho\Omega + M\Omega S} = -\frac{\mathcal{T} - \rho v^2}{\sqrt{k\mathcal{T} - k\rho v^2 - \mathcal{T}\rho\Omega^2}} = -\frac{1}{S}. \quad (\text{B.30})$$

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