EQUIVARIANT K-THEORY OF CELLULAR TORIC VARIETIES

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Abstract. In this article we describe the T_{comp}-equivariant topological K-ring of a T-*cellular* simplicial toric variety. We further show that $K^0_{T_{comp}}(X)$ is isomorphic as an $R(T_{comp})$-algebra to the ring of piecewise Laurent polynomial functions on the associated fan denoted $PLP(\Delta)$. Furthermore, we compute a basis for $K^0_{T_{comp}}(X)$ as a $R(T_{comp})$-module and multiplicative structure constants with respect to this basis.

1. Introduction

We shall consider varieties over the field of complex numbers unless otherwise specified.

A T-*cellular variety* is a T-variety X equipped with a T-stable algebraic cell decomposition. In other words there is a filtration

$$X = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m \supseteq Z_{m+1} = \emptyset \tag{1.1}$$

where each Z_i is a closed T-stable subvariety of X and $Z_i \setminus Z_{i+1} = Y_i$ is T-equivariantly isomorphic to the affine space $\mathbb{C}^{k_i}$ equipped with a linear action of T for $1 \leq i \leq m$. Furthermore, Y_i for $1 \leq i \leq m$, are the Bialynicki Birula cells associated to a generic one-parameter subgroup of T (see Definition 2.2).

Let $X = X(\Delta)$ be a toric variety associated to a simplicial (not necessarily complete) fan Δ in the lattice $\mathbb{Z}^n$ with the dense torus $T \simeq (\mathbb{C}^*)^n$. We further assume that X is T-*cellular*. There is a combinatorial characterization on the simplicial fan Δ so that the associated toric variety $X(\Delta)$ is T-cellular (see Theorem 2.5).

Our definition of cellular toric variety was motivated by the definition of a divisive weighted projective space due to Harada, Holm, Ray, and Williams (see [14]) and that of a retractable toric orbifold in [23]. We

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were also motivated by the definition of (possibly singular) varieties with *good decompositions* considered by Carrell in [7, Section 4.3] (see Remark 2.4).

There is also the notion of projective T-varieties which are $\mathbb{Q}$-filtrable (see [11, Definition 4.6, Theorem 4.7]). Here we assume that there exists a decreasing chain of T-stable closed subvarieties (1.1) of X such that the cells $Z_i \setminus Z_{i+1} = Y_i$ are rationally smooth (see [6], [11, Definition 3.4]) but not necessarily smooth. The equivariant cohomology of $\mathbb{Q}$-filtrable, T-skeletal varieties i.e. when X has finitely many T-fixed points and finitely many one dimensional T-orbits (see [11, Definition 2.4]) have been studied by Gonzales in [11].

Examples of T-cellular simplicial toric varieties include smooth projective toric varieties [9], [20], smooth semi-projective toric varieties defined in [15, Section 2] and the divisive weighted projective spaces defined in [14]. The retractable toric orbifolds considered in [23] includes projective simplicial toric varieties such that the T-equivariant rational Bialynicki Birula cells in the cellular decomposition of $X(\Delta)$ are in fact smooth integral cells (see Section 2.1, [9, pp. 101-105], [6], [11, Section 3]). There are smooth complete non-projective toric varieties which are T-cellular as seen in the example in [9, p. 71]. We give examples of simplicial non-smooth and non-complete toric varieties which are T-cellular (see Example 2.8).

Let $T_{comp} \simeq (S^1)^n$ denote the maximal compact subgroup of T. We consider the natural restricted action of $T_{comp} \subset T$ on $X(\Delta)$. Let K^0 denote the topological K-ring (see [2]) and for a compact connected Lie group G_{comp}, $K^0_{G_{comp}}$ denotes G_{comp}-equivariant K-ring. In particular, $K^0_{T_{comp}}$ denotes T_{comp}-equivariant topological K-ring (see [24]).

1.0.1. *Preliminaries on K-theory.* Let X be a compact G_{comp}-space for a compact Lie group G_{comp}. By $K^0_{G_{comp}}(X)$ we mean the Grothendieck ring of G_{comp}-equivariant topological vector bundles on X with the abelian group structure given by the direct sum and the multiplication given by the tensor product of equivariant vector bundles. In particular, $K^0_{G_{comp}}(pt)$, where G_{comp} acts trivially on pt, is the Grothendieck ring $R(G_{comp})$ of complex representations of G_{comp}. The ring $K^0_{G_{comp}}(X)$ has the structure of $R(G_{comp})$-algebra via the map $R(G_{comp}) \to K^0_{G_{comp}}(X)$ which takes $[V] \mapsto \mathbf{V}$, where $\mathbf{V} = X \times V$ is the trivial G_{comp}-equivariant vector bundle on X corresponding to the G_{comp}-representation V. Let pt be a G_{comp}-fixed point of X then the reduced equivariant K-ring $\widetilde{K}^0_{G_{comp}}(X)$ is the kernel of the map $K^0_{G_{comp}}(X) \to K^0_{G_{comp}}(pt)$, induced by the restriction of G_{comp}-equivariant vector bundles. If X is locally compact Hausdorff space but not compact we define $K^0_{G_{comp}}(X) = \widetilde{K}^0_{G_{comp}}(X^+)$ where X^+ denotes the one point compactification with the assumption that the point at infinity is G_{comp}-fixed. We let

$$K^{-n}_{G_{comp}}(X) = K^0_{G_{comp}}(X \times \mathbb{R}^n)$$

where G_{comp}-acts trivially on $\mathbb{R}^n$. Furthermore, we have the equivariant Bott periodicity $K^{-n}_{G_{comp}}(X) \simeq K^{-n-2}_{G_{comp}}(X)$ given via multiplication by the Bott element in $K^{-2}_{G_{comp}}(pt)$. (See [24] and [10].)

Let X be an algebraic variety with the action of an algebraic group G. Then $\mathcal{K}^0_G(X)$ denotes the Grothendieck ring of equivariant algebraic vector bundles on X and $\mathcal{K}^G_0(X)$ the Grothendieck group of equivariant coherent sheaves on X (see [26], [18]). The natural map $\mathcal{K}^0_G(X) \to \mathcal{K}^G_0(X)$ obtained by sending a class of a G-equivariant vector bundle $\mathcal{V}$ on X to the dual of its sheaf of sections is an isomorphism when X is smooth, but not in general. Moreover, when G is a complex reductive algebraic group and G_{comp} is a maximal compact subgroup of G, then any complex algebraic G-variety X is a G_{comp}-space. When X is smooth we have a natural map $\mathcal{K}^0_G(X) \to K^0_{G_{comp}}(X)$ obtained by viewing an algebraic G-vector bundle as a topological G_{comp}-vector bundle on X.

Recall that in [8, Section 5.5] we have an alternate notion of G-cellular variety X, where X admits a decreasing filtration (1.1) by G-stable closed subvarieties Z_i such that $Y_i = Z_i \setminus Z_{i+1}$ are complex affine spaces equipped with a linear G-action. However, Y_i are not assumed to be the cells of a Bialynicki-Birula decomposition of X. Thus for T-varieties, this notion of cellular is weaker than our definition (see Definition 2.2). It follows from [8, p. 272], that the map $\mathcal{K}^0_G(X) \to K^0_{G_{comp}}(X)$ is an isomorphism when X is smooth and G-cellular.

For singular G-varieties there are no natural isomorphisms $\mathcal{K}^0_G(X) \cong \mathcal{K}^G_0(X)$ and $\mathcal{K}^0_G(X) \cong K^0_{G_{comp}}(X)$. On the other hand, for a singular variety X, there is an analogous notion of algebraic operational equivariant K-theory denoted op $\mathcal{K}^0_G(X)$, which has been studied and described for projective simplicial toric varieties as piecewise Laurent polynomial functions on the associated fan (see [1, Theorem 1.6]). The results of Anderson and Payne [1] on algebraic operational T-equivariant K-ring of simplicial toric varieties generalize the results of Vezzosi and Vistoli [29] on the algebraic equivariant K-ring of smooth T-toric varieties. In [29], Vezzosi and Vistoli prove that the algebraic T-equivariant K-ring of a smooth toric variety is isomorphic to the Stanley-Reisner ring of the associated fan. (Also see [21] and [22] on the presentation of ordinary algebraic and topological K-theory of smooth projective toric varieties and toric bundles and smooth complete toric varieties using alternate techniques.) For singular varieties, the Grothendieck ring of algebraic vector bundles need not be finitely generated and can be quite large compared to the Grothendieck group of coherent sheaves (see [12]). However the algebraic operational equivariant K-ring seems to behave better in relation to the Grothendieck group of equivariant coherent sheaves (see [1, Theorem 1.3]) and also the topological equivariant K-ring (we are unable to find a precise reference for the latter assertion which is probably known).

In the topological direction, in [14], Harada, Holm, Ray and Williams study the topological equivariant K-ring of weighted projective spaces $\mathbb{P}(\chi_0, \chi_1, \ldots, \chi_n)$ such that χ_{j-1} divides χ_j for $1 \leq j \leq n$. They called such weighted projective spaces *divisive* and showed that they have a T_{comp}-invariant cellular structure (see [14, Section 2]). In [14, Theorem 5.5] they show that the T_{comp}-equivariant topological K-ring of a divisive weighted projective space is isomorphic to the ring of piecewice Laurent polynomial functions on the associated fan.

In [23], Sarkar and the author study the topological equivariant K-ring of a toric orbifold which has an invariant cellular structure. Such a toric orbifold was called *retractable* since the sufficient condition for the toric orbifold to have an invariant cellular structure was given by the notion of *a retraction sequence* of the associated simple poytope. Indeed the simplicial projective toric varieties which correspond to simplicial polytopal fans are examples of toric orbifolds. Therefore, by the results in [23, Section 4, Theorem 4.2], it follows that for a cellular simplicial projective toric variety the topological T_{comp}-equivariant K-ring is isomorphic to the $R(T_{comp})$-algebra of piecewise Laurent polynomial functions on the associated fan. This description agrees with the results of Anderson and Payne on the algebraic equivariant operational K-ring. The corresponding result for topological T_{comp}-equivariant K-ring of a smooth projective toric variety was proved in [14, Theorem 7.1, Corollary 7.2].

Our aim in this paper is to study the topological equivariant K-ring for certain classes of singular toric varieties which are T-cellular (see Definition 2.2).

Let $X = X(\Delta)$ be a complete cellular simplicial toric variety associated to the fan Δ. We first give a GKM type description of the

T_{comp}-equivariant topological K-ring of $X(\Delta)$ (see Theorem 4.2), using methods similar to those used by Mc Leod in [19, Section 1.5, Theorem 1.6] for the description of the topological T_{comp}-equivariant K-ring of a flag variety.

In Section 4.1, we further show that $K^0_{T_{comp}}(X)$ is isomorphic as an $R(T_{comp}) = R(T)$-algebra to the ring of piecewise Laurent polynomial functions on Δ which we denote by $PLP(\Delta)$ (see Theorem 4.6). We prove this result for any T-cellular complete toric variety with an additional combinatorial assumption on Δ (see Assumption 4.5). Since this assumption holds in particular for a T-cellular projective toric variety (see Remark 4.7), we recover the results in [23, Theorem 4.2].

By [29], for a smooth cellular T-toric variety $X(\Delta)$, $\mathcal{K}^0_T(X(\Delta)) \cong K^0_{T_{comp}}(X(\Delta))$ is isomorphic to the Stanley Reisner ring of Δ. Our results are therefore a generalization of the results of Vezzosi and Vistoli [29], in the direction of topological equivariant K-theory to the class of cellular simplicial toric varieties.

In Section 5 we determine a basis for $K^0_{T_{comp}}(X(\Delta))$ as an $R(T_{comp})$-module (see Theorem 5.1) (see [13, Proposition 4.3] and [11, Section 6] for similar results on the equivariant cohomology ring). Furthermore, in Section 5.1, we determine the multiplicative structure constants with respect to this basis (see Theorem 5.2, Corollary 5.3).

2. Bialynicki-Birula decomposition and cellular varieties

In this section we recall the notions of a filtrable Bialynicki-Birula decomposition and a T-cellular variety (see [5, Section 3] and [11, Section 2.2]).

Let X be a normal complex algebraic variety with the action of a complex algebraic torus T. We assume that the set of T-fixed points X^T is finite. A one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is said to be generic if $X^\lambda = X^T = \{x_1, \ldots, x_m\}$. Let

$$X_+(x_i, \lambda) = \{x \in X \mid \lim_{t\to 0} \lambda(t)x \text{ exists and is equal to } x_i\}. \tag{2.2}$$

Then $X_+(x_i, \lambda)$ is locally closed T-invariant subvariety of X and is called the *plus* stratum.

Definition 2.1. *The variety X with the action of T having finitely many fixed points $\{x_1, \ldots, x_m\}$ is called filtrable if it satisfies the following conditions:*

(i) X is the union of its plus strata $X_+(x_i, \lambda)$ for $1 \le i \le m$.

(ii) There exists a finite decreasing sequence of T-invariant closed subvarieties of X $X = Z_1 \supset Z_2 \cdots \supset Z_m \supset Z_{m+1} = \emptyset$ such that $Z_i \setminus Z_{i+1} = Y_i := X_+(x_i, \lambda)$ for $1 \leq i \leq m$. In particular, $\overline{Y_i} \subseteq Z_i = \bigcup_{j \geq i} Y_j$.

Definition 2.2. *Let X be a normal complex algebraic variety with an action of a complex algebraic torus T such that $X^T := \{x_1, \ldots, x_m\}$ is finite. We say that X is T-cellular if X is filtrable for a generic one-parameter subgroup λ and in addition each plus strata $Y_i := X_+(x_i, \lambda)$ is T-equivariantly isomorphic to a complex affine space $\mathbb{C}^{k_i}$ on which T-acts linearly for $1 \leq i \leq m$.*

Remark 2.3. Any smooth projective complex variety X with T-action having only finitely many T-fixed points is T-cellular (see [3], [5]).

Remark 2.4. In [7, Section 4.3] J. Carrell considers possibly singular varieties having *good decompositions*, that is filtrable Bialynicki-Birula cellular decomposition, and gives examples of such varieties, Springer fibres [7, Section 4.3.1] being one such example of a singular variety having good cellular decompositions given by Spaltenstein (see [25]). Carrell gives the intergal homology decomposition for such varieties in [7, Theorem 4.13]. The T-cellular varieties which we consider are varieties which admit good decompositions in the sense of Carrell.

2.1. **Cellular Simplicial Toric varieties.**

We begin by fixing some notations and conventions.

Let $X = X(\Delta)$ be the toric variety associated to a simplicial fan Δ in the lattice $N \simeq \mathbb{Z}^n$. Let $M := Hom(N, \mathbb{Z})$ be the dual lattice of characters of T. Let $\{v_1, \ldots, v_d\}$ denote the set of primitive vectors along the edges $\Delta(1) := \{\rho_1, \ldots, \rho_d\}$. Let $V(\gamma)$ denote the orbit closure in X of the T-orbit O_γ corresponding to the cone $\gamma \in \Delta$. Let $S_\sigma = \sigma^\vee \cap M$ and $U_\sigma := Hom_{sg}(S_\sigma, \mathbb{C})$ denote the T-stable open affine subvariety corresponding to a cone $\sigma \in \Delta$.

We further assume that all the maximal cones in Δ are n-dimensional, in other words Δ is *pure*. The T-fixed locus in X consists of the set of T-fixed points

$$\{x_1, x_2 \ldots, x_m\} \tag{2.3}$$

corresponding to the set of maximal dimensional cones

$$\Delta(n) := \{\sigma_1, \sigma_2, \ldots, \sigma_m\}. \tag{2.4}$$

Choose a *generic* one parameter subgroup $\lambda_v \in X_*(T)$ corresponding to a $v \in N$ which is outside the hyperplanes spanned by the $(n-1)$-dimensional cones, so that (2.3) is the set of fixed points of λ_v (see [5, §3.1]). For each x_i, we have the plus strata $Y_i := X_+(x_i, \lambda_v)$ (see (2.2)).

Consider a face γ of σ_i satisfying the property that the image of v in $N_{\mathbb{R}}/\mathbb{R}\gamma$ is in the relative interior of $\sigma_i/\mathbb{R}\gamma$. Since the set of such faces is closed under intersections, we can choose a minimal such face of σ_i which we denote by τ_i. We have

$$Y_i = \bigcup_{\tau_i \subseteq \gamma \subseteq \sigma_i} O_\gamma. \tag{2.5}$$

Moreover, if we choose a generic vector $v \in |\Delta|$, then

$$X = \bigcup_{i=1}^{m} Y_i$$

where Y_i, $1 \leq i \leq m$ are the cells of the *Bialynicki-Birula cellular decomposition* of the toric variety X corresponding to the one-parameter subgroup λ_v (see [15, Lemma 2.10]).

Furthermore, the Bialynicki-Birula decomposition for X corresponding to λ_v is *filtrable* if

$$\overline{Y_i} \subseteq \bigcup_{j \geq i} Y_j \tag{2.6}$$

for every $1 \leq i \leq m$. It can be seen that (2.6) is equivalent to the following combinatorial condition in Δ:

$$\tau_i \subseteq \sigma_j \ \text{ implies } \ i \leq j. \tag{2.7}$$

Now, from (2.5) it follows that $Y_i = V(\tau_i) \cap U_{\sigma_i}$. Let $k_i := \dim(\tau_i)$ for $1 \leq i \leq m$. Thus Y_i is a T-stable affine open set in the toric variety $V(\tau_i)$, corresponding to the maximal dimensional cone $\overline{\sigma_i}$ in the fan $(\mathrm{star}(\tau_i), N(\tau_i) = N/N_{\tau_i})$ (see [9, Chapter 3]). Thus Y_i is isomorphic to the complex affine space $\mathbb{C}^{n-k_i}$ if and only if the $(n-k_i)$-dimensional cone $\overline{\sigma_i} := \sigma_i/N_{\tau_i}$ is a smooth cone in $(\mathrm{star}(\tau_i), N(\tau_i))$ for $1 \leq i \leq m$.

It follows by (2.6) that under the above conditions on Δ, namely (2.7) and that $\overline{\sigma_i}$ is a smooth cone in $\mathrm{star}(\tau_i)$, $Z_i := \bigcup_{j \geq i} Y_j$ are T-stable closed subvarieties, which form a chain

$$X = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m = V(\tau_m) \tag{2.8}$$

such that $Z_i \setminus Z_{i+1} = Y_i \simeq \mathbb{C}^{n-k_i}$ for $1 \leq i \leq m$.

We summarize the above discussion in the following theorem which gives a combinatorial characterization on Δ for $X(\Delta)$ to admit a filtrable Bialynicki-Birula decomposition or equivalently for $X(\Delta)$ to be a T-cellular toric variety (see Definition 2.2).

Theorem 2.5. *The toric variety $X(\Delta)$ is T-cellular if and only if the following combinatorial conditions hold in Δ:*

(i) Δ admits an ordering of the maximal dimensional cones

$$\sigma_1 < \sigma_2 < \cdots < \sigma_m \tag{2.9}$$

such that the distinguished faces $\tau_i \subseteq \sigma_i$ defined above satisfy the following property:

$$\tau_i \subseteq \sigma_j \quad \textit{implies} \quad i \leq j \tag{2.10}$$

(ii) $\overline{\sigma_i} := \sigma_i/N_{\tau_i}$ is a smooth cone in the fan $(star(\tau_i), N(\tau_i))$ for $1 \leq i \leq m$.

Remark 2.6. We shall assume without loss of generality that v belongs to the relative interior of σ_1 so that $\tau_1 = \{0\}$. In particular, this implies that σ_1 is a smooth cone in Δ so that $Y_1 = U_{\sigma_1} \cap V(\tau_1) \simeq \mathbb{C}^n$ is smooth in $X = V(\tau_1)$.

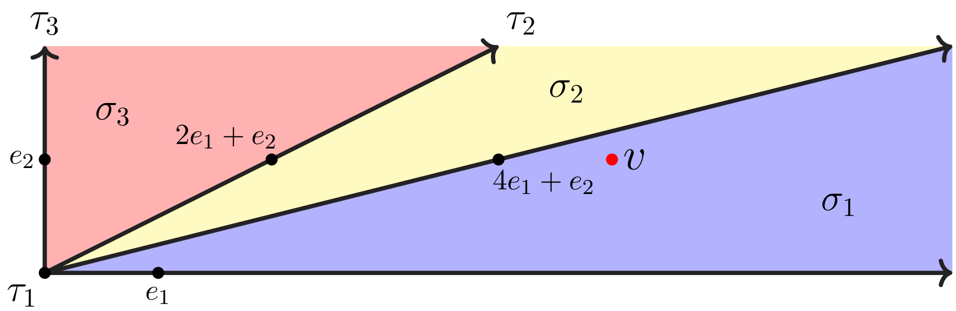
Remark 2.7. When $X = X(\Delta)$ is a smooth projective T-toric variety, then the corresponding fan Δ is smooth and polytopal. Recall that we have a filtrable Bialynicki-Birula cellular decomposition of X corresponding to a generic one-parameter subgroup λ_v on T (see [4] and [5]). This in turn gives a generic height function on the polytope P dual to Δ, which comes from the linear form $\langle\ ,v\rangle$ on M defined by $v \in N$. Let $u(\sigma_i)$ denote the vertex of the polytope corresponding to the maximal dimensional cone σ_i. Then the ordering of the maximal dimensional cones of Δ such that $\langle u_{\sigma_1}, v\rangle < \langle u_{\sigma_2}, v\rangle < \cdots < \langle u_{\sigma_m}, v\rangle$ satisfies the condition (2.7) (see [9, page 101]). Moreover, since x_{σ_i} is a smooth point in X it is smooth in $V(\tau_i)$ for $1 \leq i \leq m$, verifying the cellular condition.

The toric variety corresponding to the following fan is a 2-dimensional non-smooth, non-complete cellular toric variety.

Example 2.8. Let Δ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ consist of the 2-dimensional cones $\sigma_1 = \langle e_1, 4e_1+e_2\rangle$, $\sigma_2 = \langle 2e_1+e_2, 4e_1+e_2\rangle$ and $\sigma_3 = \langle e_2, 2e_1+e_2\rangle$ and their faces. We denote the edge generated by e_1 by ρ_1, the edge generated by $4e_1 + e_2$ by ρ_2, the edge generated by $2e_1 + e_2$ by ρ_3, the edge generated by e_2 by ρ_4. Let $v = 5e_1 + e_2$. Then v belongs to the relative interior of σ_1 so that $\tau_1 = \{0\}$. We now check that

$\tau_2 = \langle 2e_1 + e_2 \rangle = \rho_3$ and $\tau_3 = \langle e_2 \rangle = \rho_4$. Note that $\overline{v} = \overline{3e_1}$ in $N_{\mathbb{R}}/\mathbb{R} \cdot (2e_1 + e_2)$. On the other hand $\overline{\sigma_2} = \sigma_2/\mathbb{R} \cdot (2e_1 + e_2)$ is a 1-dimensional cone generated by the class of $\overline{4e_1 + e_2} = \overline{2e_1}$. Since $\overline{e_1}$ is the primitive vector along $\overline{\sigma_2}$ it follows that $\overline{v}$ is in the relative interior of $\overline{\sigma_2}$ in star$(\langle 2e_1 + e_2 \rangle)$. Moreover, since v is not in the relative interior of σ_2 it follows that $\tau_2 = \langle 2e_1 + e_2 \rangle$. Similarly the image of $\overline{v} = \overline{5e_1}$ in $N_{\mathbb{R}}/\mathbb{R} \cdot e_2$. Now, $\overline{\sigma_3} = \sigma_3/\mathbb{R}e_2$ is the 1-dimensional cone spanned by $\overline{2e_1 + e_2}$ and hence generated by the primitive vector $\overline{e_1}$. Thus $\overline{v}$ is in the relative interior of $\overline{\sigma_3}$ in star$(\langle e_2 \rangle)$. Since v is not in the relative interior of σ_3 it follows that $\tau_3 = \langle e_2 \rangle$. We further observe that the ordering $\sigma_1 < \sigma_2 < \sigma_3$ satisfies (2.7). It therefore follows that $X(\Delta)$ is a T-cellular toric variety.

The below picture illustrates the above example.



Henceforth we assume that $X := X(\Delta)$ is a T-cellular toric variety (see Definition 2.2). In particular, we assume that Δ satisfies the conditions (i) and (ii) of Theorem 2.5.

3. Topological equivariant K-theory of T-cellular varieties

Let $T \simeq (\mathbb{C}^*)^n$ and $T_{comp} \simeq (S^1)^n \subseteq T$.

Theorem 3.1. *Let X be a T-cellular variety (see Definition 2.1). Then $K^0_{T_{comp}}(X)$ is a free $R(T_{comp})$-module of rank m-which is the number of cells. Furthermore, we have $K^{-1}_{T_{comp}}(X) = 0$.*

Proof. By [24, Proposition 2.6, Definition 2.7, Definition 2.8, Proposition 3.5] it follows that we have a long exact sequence of T_{comp}-equivariant K-groups which is infinite in both directions:

$$\begin{aligned} \cdots \to K^{-q}_{T_{comp}}(Z_i, Z_{i+1}) \to K^{-q}_{T_{comp}}(Z_i) \to K^{-q}_{T_{comp}}(Z_{i+1}) \to \\ \to K^{-q+1}_{T_{comp}}(Z_i, Z_{i+1}) \to \cdots \end{aligned} \tag{3.11}$$

for $1 \leq i \leq m$ and $q \in \mathbb{Z}$.

Moreover, by [24, Proposition 2.9] and [24, Proposition 3.5] we have

$$\begin{aligned} K^q_{T_{comp}}(Z_i, Z_{i+1}) &= K^q_{T_{comp}}(Z_i \setminus Z_{i+1}) \\ &= K^q_{T_{comp}}(\mathbb{C}^{k_i}) \simeq K^q_{T_{comp}}(x_i) \\ &= \widetilde{K}^{-q}_{T_{comp}}(x_i^+) \\ &= \widetilde{K}^0_{T_{comp}}(S^q) = R(T_{comp}) \otimes \widetilde{K}^0(S^q(x_i^+)) \end{aligned}$$

for $1 \leq i \leq m$ (see [24, Proposition 2.2]). Thus when q is even $K^{-q}_{T_{comp}}(Z_i, Z_{i+1}) = R(T_{comp})$ and when q is odd $K^{-q}_{T_{comp}}(Z_i, Z_{i+1}) = 0$. Here x_i^+ is the sum of the T_{comp}-fixed point x_i and a base point $\mathfrak{o}$ which is also T_{comp}-fixed (see [24, p. 135]).

Alternately, we can also identify $K^q_{T_{comp}}(Z_i, Z_{i+1}) = \widetilde{K}^q_{T_{comp}}(Z_i/Z_{i+1})$ where $Z_i/Z_{i+1} = Th(Y_i) \simeq Y_i^+ \simeq S^{2k_i}$ the Thom space of the trivial equivariant vector bundle Y_i over the T-fixed point x_i so that the above isomorphism is the Thom isomorphism (see [17]). Indeed for any integer q we have $\widetilde{K}^{-q}_{T_{comp}}(S^{2k_i}) \simeq \widetilde{K}^0_{T_{comp}}(S^{q+2k_i})$ [24, p. 136].

Thus when q is even $K^{-q}_{T_{comp}}(Z_i, Z_{i+1}) \simeq \widetilde{K}^0_{T_{comp}}(S^{q+2k_i}) = R(T_{comp})$ since $q + 2k_i$ is even. Further, when q is odd $K^{-q}_{T_{comp}}(Z_i, Z_{i+1}) = \widetilde{K}^0_{T_{comp}}(S^{q+2k_i}) = R(T_{comp}) \otimes \widetilde{K}^0(S^{q+2k_i}) = 0$ since $q + 2k_i$ is odd (see [24, Section 2, Proposition 3.5, Proposition 2.2] and [2, Lemma 2.7.4]).

Moreover, since $Z_m = Y_m$ and $X_{m+1} = \emptyset$ we have $K^0_{T_{comp}}(Z_m) = R(T_{comp})$ and $K^{-1}_{T_{comp}}(Z_m) = K^{-1}_{T_{comp}}(x_m) = \widetilde{K}^{-1}_{T_{comp}}(x_m^+) = \widetilde{K}^0_{T_{comp}}(S^1) = 0$ where $x_m^+ = x_m \sqcup \mathfrak{o}$ where both x_m and the base point $\mathfrak{o}$ are T_{comp}-fixed (see [24, p. 135]).

Now, by decreasing induction on i suppose that $K^0_{T_{comp}}(Z_{i+1})$ is a free $R(T_{comp})$-module for $1 \leq i \leq m$ of rank $m-i$ and $K^{-1}_{T_{comp}}(Z_{i+1}) = 0$. We can start the induction since $K^0_{T_{comp}}(Z_m) = R(T_{comp})$ and $K^{-1}_{T_{comp}}(Z_m) = 0$.

It then follows that from the (3.11) we get the following split short exact sequence of $R(T_{comp})$-modules

$$0 \to K^0_{T_{comp}}(Z_i, Z_{i+1}) \to K^0_{T_{comp}}(Z_i) \xrightarrow{\alpha^*} K^0_{T_{comp}}(Z_{i+1}) \to 0 \tag{3.12}$$

for $1 \leq i \leq m$.

Hence $K^0_{T_{comp}}(Z_i)$ is a free $R(T_{comp})$-module of rank $m-i+1$. By induction it follows that $K^0_{T_{comp}}(X=Z_1)$ is a free $R(T_{comp})$-module of rank m.

Since $K^{-1}_{T_{comp}}(Z_i, Z_{i+1}) = 0$ as shown above and by induction assumption $K^{-1}_{T_{comp}}(Z_{i+1}) = 0$, it follows from (3.11) that $K^{-1}_{T_{comp}}(Z_i) = 0$. Thus $K^{-1}_{T_{comp}}(X_1) = K^{-1}_{T_{comp}}(X) = 0$ by induction on i.

□

We have the following corollary for any T-cellular variety X.

Let $\iota : X^{T_{comp}} = X^T \hookrightarrow X$ denote the inclusion of the set of T-fixed points in X.

Corollary 3.2. *The canonical restriction map*

$$K^0_{T_{comp}}(X) \stackrel{\iota^*}{\to} K^0_{T_{comp}}(X^{T_{comp}}) \cong R(T_{comp})^m$$

is injective where $m = |X^{T_{comp}}|$.

Proof. Since the prime ideal (0) of $R(T_{comp})$ has support T_{comp}, by localizing at (0) (see [24, Proposition 4.1]), we have that

$$K^0_{T_{comp}}(X) \otimes_{R(T_{comp})} Q(T_{comp}) \to K^0_{T_{comp}}(X^{T_{comp}}) \otimes_{R(T_{comp})} Q(T_{comp})$$

is an isomorphism where $Q(T_{comp}) := R(T_{comp})_{\{0\}}$ is the quotient field of the integral domain $R(T_{comp})$. This further implies that the restriction map $K^0_{T_{comp}}(X) \to K^0_{T_{comp}}(X^{T_{comp}})$ is injective, since by Theorem 3.1, $K^0_{T_{comp}}(X)$ is a free $R(T_{comp})$-module of rank m. □

4. GKM theory of $X(\Delta)$

In this section we shall assume that Δ is a complete fan so that $X = X(\Delta)$ is a complete toric variety.

In this section we give a GKM type description for $K^0_{T_{comp}}(X)$ as a $K^0_{T_{comp}}(pt) = R(T_{comp})$-algebra.

Recall that (see [8]) we have the isomorphism

$$R(T) \simeq R(T_{comp}). \tag{4.13}$$

Since X is a cellular toric variety, recall that we have a decreasing sequence of T-invariant closed subvarieties of X given by (2.8) such that $Z_i \setminus Z_{i+1} = Y_i \simeq \mathbb{C}^{n-k_i}$. Moreover, $x_i \in Y_i \subseteq Z_i$, $1 \le i \le m$. Since $Y_i \simeq \mathbb{C}^{n-k_i}$ are T-stable we have a T-representation $\rho_i = (V_i, \pi_i, x_i)$, which can alternately be viewed as a T-equivariant complex vector bundle over the T-fixed point x_i for $1 \leq i \leq m$.

Let $\mathcal{A}$ be the set of all $(a_i)_{1\leq i\leq m} \in R(T_{comp})^m$ such that $a_i \equiv a_j \pmod{(1-e^{\chi})}$ whenever the maximal dimensional cones σ_i and σ_j share a wall $\sigma_i \cap \sigma_j$ in Δ and $\chi \in (\sigma_i \cap \sigma_j)^{\perp} \cap M$. In other words, the T-fixed points x_i and x_j lie in the closed T-stable irreducible curve $C_{ij} := V(\sigma_i \cap \sigma_j)$ in $X(\Delta)$ and T (and hence T_{comp}) acts on C_{ij} through the character χ. Moreover, $\mathcal{A}$ is a $R(T_{comp})$-algebra where $R(T_{comp})$ is identified with the subalgebra of $\mathcal{A}$ consisting of the diagonal elements $(a, a, \ldots, a)$.

Let $\iota : X^T \hookrightarrow X$ be the inclusion of the set of T-fixed points in X.

The following lemma is a refinement of Corollary 3.2 for a T-cellular toric variety X. For a similar result when X is a full flag variety see [19, Lemma 2.3].

Lemma 4.1. *Let X be a T-cellular toric variety. Then we have the following inclusion of $R(T_{comp})$-algebras $\iota^*(K^0_{T_{comp}}(X)) \subseteq \mathcal{A}$.*

Proof. Let the T-fixed points x_i and x_j lie in the closed T-stable irreducible curve C_{ij} and T (and hence T_{comp}) act on C_{ij} through the character χ. The composition

$$K^0_{T_{comp}}(X) \stackrel{\iota^*}{\to} K^0_{T_{comp}}(X^T) \to K^0_{T_{comp}}(\{x_i, x_j\})$$

equals

$$K^0_{T_{comp}}(X) \to K^0_{T_{comp}}(C_{ij}) \stackrel{\iota^*_{C_{ij}}}{\to} K^0_{T_{comp}}(\{x_i, x_j\}).$$

Since X is a T-toric variety, $C_{ij} = V(\sigma_i \cap \sigma_j)$ is a 1-dimensional toric subvariety of X containing two distinct T-fixed points x_i and x_j. In particular, C_{ij} is smooth and isomorphic to $\mathbb{P}^1$. We now claim that the image of $\iota^*_{C_{ij}}$ consists of pairs of elements $(f, g) \in R(T_{comp}) \oplus R(T_{comp})$ such that $f \equiv g \pmod{(1-e^{-\chi})}$ where T acts on C_{ij} through the weight χ.

The proof of this fact follows by the same arguments as can be found in [19, proof of Lemma 2.3, p. 322] and [27, Theorem 1.3, p. 274] so we do not repeat them (also see [5, Section 3.4]).

Thus if $x \in K^0_{T_{comp}}(X)$ then $\iota^*(x)_i - \iota^*(x)_j$ is divisible by $(1 - e^{-\chi})$. hence the lemma.

$\square$

Theorem 4.2. *Let $X = X(\Delta)$ be a complete cellular toric variety. The ring $K^0_{T_{comp}}(X)$ is isomorphic to $\mathcal{A}$ as an $R(T_{comp})$-algebra.*

Proof. By Corollary 3.2 and Lemma 4.1 we know that the image of $\iota^*(K^0_{T_{comp}}(X)) \subseteq \mathcal{A}$. It remains to show that $\iota^*(K^0_{T_{comp}}(X)) \supseteq \mathcal{A}$. We shall follow the methods in [19, Section 2.5].

Let $1 < i_1, \ldots, i_n \leq m$ be such that $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_n} \in \Delta(n)$ share a wall with σ_i. Without loss of generality we assume that $i < i_j$ for $1 \leq j \leq n - k_i$ and $i > i_j$ for $n - k_i < j \leq n$.

Since $\overline{\sigma_i}$ is a smooth cone of dimension $n-k_i$ in $\mathrm{star}(\tau_i)$, the primitive vectors $\overline{v}_{i_1}, \ldots, \overline{v}_{i_{n-k_i}}$ in $N(\tau_i)$ along the edges of $\overline{\sigma_i}$ form a basis for the lattice $N(\tau_i)$.

By our assumption σ_{i_j} for $1 \leq j \leq n-k_i$ are the maximal dimensional cones which share a wall with σ_i satisfying $i_j > i$. Moreover, since $V(\tau_i)$ is a complete simplicial toric subvariety of $X(\Delta)$, $\mathrm{star}(\tau_i)$ is a complete simplicial fan in $N(\tau_i)$. Thus $\overline{\sigma_i}$ shares a wall with $n - k_i$ cones in $\mathrm{star}(\tau_i)(n - k_i)$, each of which correspond to a maximal dimensional cone in Δ containing τ_i. Thus by (2.7), it follows that the maximal dimensional cones in $\mathrm{star}(\tau_i)$ which share a wall with $\overline{\sigma_i}$ are precisely $\overline{\sigma_{i_j}}$ for $1 \leq j \leq n - k_i$.

Furthermore, since the walls of $\overline{\sigma_i}$ are $\langle \overline{v}_{i_1}, \ldots, \widehat{\overline{v}_{i_l}}, \ldots, \overline{v}_{i_{n-k_i}} \rangle$, we shall assume without loss of generality that this wall is $\overline{\sigma_i} \cap \overline{\sigma_{i_l}}$.

Now, $\overline{\sigma_i}^{\vee}$ is generated by the dual basis $u_{i_1}, \ldots, u_{i_{n-k_i}}$ for the lattice $M(\tau_i) := M \cap \tau_i^{\perp}$. Thus any $x \in U_{\overline{\sigma_i}}$ can be uniquely expressed as $\displaystyle\sum_{j=1}^{n-k_i} x(u_{i_j}) x^{i_j}$ where $x^{i_t}(u_{i_s}) = \delta_{t,s}$ where δ denotes the Kronecker delta. Indeed x^{i_j} can be identified with the distinguished point $x_{\sigma_{i_j} \cap \sigma_i}$. Thus x^{i_j} spans a 1-dimensional representation ρ_{i_j} of ρ_i corresponding to the character χ_{i_j} where $\chi_{i_j}(t) = t(u_{i_j})$ and $\rho_i = \displaystyle\bigoplus_{j=1}^{n-k_i} \rho_{i_j}$. We can therefore write $\rho_i = \displaystyle\bigoplus_{l>i} \rho_{il}$ where $\rho_{ii_j} = \rho_{i_j}$ is the one dimensional complex representation $\mathbb{C}_{\chi_{i_j}}$ determined by the character χ_{i_j} if $1 \leq i_j \leq n - k_i$ and $\rho_{il} = 0$ otherwise.

Since $\overline{v}_{i_1}, \cdots, \overline{v}_{i_{n-k_i}}$ form a $\mathbb{Z}$-basis for $N(\tau_i)$, the dual basis vectors $u_{i_1}, \ldots, u_{i_{n-k_i}}$ of $M(\tau_i)$ are primitive and pairwise linearly independent. Since u_{i_j} is primitive and non-zero for every $1 \leq j \leq n - k_i$, the

K-theoretic equivariant Euler class $e^T(\rho_{i_j}) = 1 - e^{-u_{i_j}}$ is a non-zero divisor in the unique factorization domain $R(T_{comp}) = \mathbb{Z}[e^u : u \in M]$. Moreover, since u_{i_j} and $u_{i_{j'}}$ are linearly independent, it follows that $1 - e^{-u_{i_j}}$ and $1 - e^{-u_{i_{j'}}}$ are relatively prime in $R(T_{comp})$. Also if $l > i$ and $l \neq i_j$ for $1 \leq j \leq n - k_i$, then $\rho_{il} = 0$ so that $e^{T_{comp}}(\rho_{il}) = 1$.

By the Thom isomorphism $K^0_{T_{comp}}(V_i)$ is a free $R(T_{comp})$-module on one generator g_i which restricts under the restriction map $K^0_{T_{comp}}(V_i) \rightarrow K^0_{T_{comp}}(x_i)$ to $\displaystyle\prod_{j=1}^{n-k_i}(1 - e^{-u_{i_j}})$.

Recall from (3.12) that for every $1 \leq i \leq m$ we have a split exact sequence of $R(T_{comp})$-modules

$$0 \rightarrow K^0_{T_{comp}}(Y_i) \xrightarrow{j_i} K^0_{T_{comp}}(Z_i) \xrightarrow{k_i} K^0_{T_{comp}}(Z_{i+1}) \rightarrow 0 \tag{4.14}$$

where $K^0_{T_{comp}}(Y_i) = K^0_{T_{comp}}(Z_i, Z_{i+1})$. These induce the following maps induced by inclusions

$$\begin{array}{ccccccccc}
K^0_{T_{comp}}(x_1) & & K^0_{T_{comp}}(x_2) & & & & K^0_{T_{comp}}(x_{m-1}) & & K^0_{T_{comp}}(x_m) \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
K^0_{T_{comp}}(Z_1 = X) & \xrightarrow{k_1} & K^0_{T_{comp}}(Z_2) & \xrightarrow{k_2} & \cdots & \rightarrow & K^0_{T_{comp}}(Z_{m-1}) & \xrightarrow{k_{m-1}} & K^0_{T_{comp}}(Z_m) \\
\uparrow j_1 & & \uparrow j_2 & & & & \uparrow j_{m-1} & & \uparrow j_m \\
K^0_{T_{comp}}(Y_1) & & K^0_{T_{comp}}(Y_2) & & & & K^0_{T_{comp}}(Y_{m-1}) & & K^0_{T_{comp}}(Y_m)
\end{array}$$

In the above diagram k_i are all surjective and j_i are all injective maps.

Thus we can choose $f_i \in K^0_{T_{comp}}(X)$ such that $k_{i-1} \circ \cdots k_1(f_i) = j_i(g_i)$. Thus $\iota^*(f_i)_i = \displaystyle\prod_{j=1}^{n-k_i}(1 - e^{-u_{i_j}})$ and $\iota^*(f_i)_l = 0$ if $l > i$ since $k_i \circ j_i = 0$.

Let $a \in \mathcal{A}$. We prove by induction on i that if $a_l = 0$ for $l \geq i$ then $a \in \iota^*(K^0_{T_{comp}}(X))$. This will complete the proof of the theorem.

For $i = 1$, since $a = 0$ the claim is obvious.

Assume by induction that $a_l = 0$ for $l \geq i + 1$. We know by the above arguments that $\sigma_i \in \Delta(n)$ shares a wall with $\sigma_{i_j} \in \Delta(n)$ and $i_j > i$ for $1 \leq j \leq n - k_i$. Moreover, the wall $\sigma_i \cap \sigma_{i_j}$ is orthogonal to the character u_{i_j}. Alternately the T-fixed points x_i and x_{i_j} lie on a T-invariant curve C_{i,i_j} in $X(\Delta)$ on which T acts through the character u_{i_j}. Since $i_j > i$ by induction hypothesis $a_{i_j} = 0$ for all $1 \leq j \leq n - k_i$. Now, since $a \in \mathcal{A}$ we have $(1 - e^{-u_{i_j}})$ divides $a_i - a_{i_j} = a_i$ for

$1 \leq j \leq n-k_i$. This implies that $\prod_{1\leq j\leq n-k_i}(1-e^{-u_{i_j}})$ divides a_i since $R(T_{comp})$ is a unique factorization domain and the factors are relatively prime. Let $a_i = c_i \cdot \prod_{1\leq j\leq n-k_i}(1-e^{-u_{i_j}})$ where $c_i \in R(T_{comp})$. Thus $a-\iota^*(c_i \cdot f_i) \in \mathcal{A}$ by Lemma 4.1. Also $(a-\iota^*(c_i\cdot f_i))_i = 0$. Thus we get $(a-\iota^*(c_i\cdot f_i))_l = 0$ for $l \geq i$. By induction there exists $q \in K^0_{T_{comp}}(X)$ such that $\iota^*(q) = a-\iota^*(c_i\cdot f_i)$. Thus we get $a = \iota^*(q + c_i \cdot f_i)$. Hence the theorem. $\square$

Remark 4.3. Note that we can define the constant map $\mathrm{f}_{ij} : x_i \to x_j$ between any two T_{comp}-fixed points of X such that $i<j$, which satisfy $\mathrm{f}_{ik} = \mathrm{f}_{ij}\circ \mathrm{f}_{jk}$ for $m \geq k > j > i \geq 1$. Thus we have the pull-back map of equivariant K-theory $\mathrm{f}^*_{ij} : K^0_{T_{comp}}(x_j) \to K^0_{T_{comp}}(x_i)$ which satisfies $\mathrm{f}^*_{ik} = \mathrm{f}^*_{ij}\circ \mathrm{f}^*_{jk}$ for $m \geq k > j > i \geq 1$. Thus we have the pull-back isomorphisms

$$\mathrm{f}^*_{im} : K^0_{T_{comp}}(x_m) \to K^0_{T_{comp}}(x_i)$$

for $1 \leq i \leq m$. This gives $\prod_{i=1}^{m} K^0_{T_{comp}}(x_i)$ a canonical $K^0_{T_{comp}}(x_m)$-algebra structure via the inclusion defined by $(\mathrm{f}^*_{im}(a))$ for $a \in K^0_{T_{comp}}(x_m)$. Also by identifying $K^0_{T_{comp}}(x_i)$ with $R(T_{comp})$ for each $i = 1,\ldots,m$ we see that f^*_{ij} corresponds to the identity map of $R(T_{comp})$ for every $1 \leq i < j \leq m$ and the inclusion (f^*_{im}) corresponds to the diagonal embedding of $R(T_{comp})$ in $R(T_{comp})^m$.

Remark 4.4. The results in this section can alternately be obtained by applying the theorem of Harada Henriques and Holm [13, Theorem 3.1] as done in [23] for the case of cellular toric orbifolds, but we give a self-contained proof here using methods similar to those used in [19].

4.1. **Piecewise Laurent polynomial functions on Δ.** Let $X(\Delta)$ be a complete simplicial T-cellular toric variety.

Assumption 4.5. *We make the following additional combinatorial assumption on Δ. Let $\tau \in \Delta(k)$ and σ and $\sigma' \in \Delta(n)$ such that $\tau \preceq \sigma, \sigma'$. There is a sequence $\sigma = \sigma_1, \ldots, \sigma_r = \sigma'$ of cones in $\Delta(n)$ each containing τ, such that $\sigma_j \cap \sigma_{j+1} \in \Delta(n-1)$ for $1 \leq j \leq r-1$.*

In this section we show that $K^0_{T_{comp}}(X)$ is isomorphic as an $R(T_{comp})$-algebra to the ring of piecewise Laurent polynomial functions on Δ. We follow methods similar to those used in [14, Theorem 5.5] and [23, Theorem 4.2].

Assumption 4.5 is used in the proof of Theorem 4.6. The geometric interpretation of this assumption is that for every $\tau \in \Delta$, any two T-fixed points in $V(\tau)$, can be connected by a finite chain of T-invariant curves, each of which is a projective line joining a pair of T-fixed points lying in $V(\tau)$.

Let $\sigma \in \Delta(k)$. Since σ is simplicial, $\sigma^{\perp} \cap M$ is a rank $n-k$ sublattice of M. (Indeed the primitive vectors $v_1, \ldots, v_k$ along $\sigma(1)$ define a $\mathbb{Z}$-linear map $\mathbb{Z}^n \to \mathbb{Z}^k$ of $\mathbb{Q}$-rank k whose kernel is a sublattice of $\mathbb{Z}^n$ of rank $n-k$.) Let $u_1, \ldots, u_{n-k}$ be a basis of $\sigma^{\perp} \cap M$. Let

$$\mathcal{K}_\sigma := R(T_{comp})/J_\sigma$$

where J_σ is the ideal generated by $e^{T_{comp}}(u_1), \ldots, e^{T_{comp}}(u_{n-k})$ where $e^{T_{comp}}(u) := 1 - e^{-u}$ denotes the T_{comp}-equivariant K-theoretic Euler class of the 1-dimensional T_{comp}-representation corresponding to u. Moreover, since u_i's are pairwise linearly independent, $e^{T_{comp}}(u_i)$ for $1 \leq i \leq n-k$ are pairwise relatively prime in $R(T_{comp})$. We further note that $e^{T_{comp}}(u) \in J_\sigma$ for every $u \in \sigma^{\perp} \cap M$.

For $\sigma, \sigma' \in \Delta$, whenever σ is a face of σ' we have the inclusion $\sigma'^{\perp} \subseteq \sigma^{\perp}$. This gives an inclusion $J_{\sigma'} \subseteq J_\sigma$ which induces a ring homomorphism $\psi_{\sigma,\sigma'} : \mathcal{K}_{\sigma'} \to \mathcal{K}_\sigma$.

We have the following isomorphism

$$R(T_{comp})/J_\sigma \simeq K^0_{T_{comp}}(T/T_\sigma) = R(T_\sigma) \tag{4.15}$$

(see [14, Example 4.12]) which further gives the identifications

$$\mathcal{K}_\sigma \simeq K^0_{T_{comp}}(O_\sigma) = K^0_{T_{comp}}(T/T_\sigma) = R(T_\sigma) \tag{4.16}$$

for $\sigma \in \Delta$ where O_σ is the T-orbit in X corresponding to σ and $T_\sigma \subseteq T$ is the stabilizer of O_σ.

Further, since $X^*(T_\sigma) = M/M \cap \sigma^{\perp}$ we can identify $R(T_\sigma) \simeq \mathcal{K}_\sigma$ with $\mathbb{Z}[M/M \cap \sigma^{\perp}]$ which is the ring of Laurent polynomial functions on σ, for $\sigma \in \Delta$. Furthermore, whenever σ is a face of $\sigma' \in \Delta$, the homomorphism $\psi_{\sigma,\sigma'}$ can be identified with the natural homomorphism $\mathbb{Z}[M/\sigma'^{\perp} \cap M] = R(T_{\sigma'}) \to \mathbb{Z}[M/\sigma^{\perp} \cap M] = R(T_\sigma)$, given by restriction of Laurent polynomial functions on σ' to σ.

Let

$$PLP(\Delta) := \{(f_\sigma) \in \prod_{\sigma \in \Delta} \mathbb{Z}[M/M \cap \sigma^{\perp}] \mid \psi_{\sigma,\sigma'}(f_{\sigma'}) = f_\sigma \text{ whenever } \sigma \preceq \sigma' \in \Delta.\} \tag{4.17}$$

Then $PLP(\Delta)$ is a ring under pointwise addition and multiplication and is called the ring of piecewise Laurent polynomial functions on

Δ. Moreover, we have a canonical map $R(T_{comp}) = \mathbb{Z}[M] \to PLP(\Delta)$ which sends f to the constant tuple $(f)_{\sigma\in\Delta}$. This gives $PLP(\Delta)$ the structure of $R(T_{comp})$-algebra.

The fan Δ corresponds to a small category $\mathcal{C}(\Delta)$ with objects $\sigma \in \Delta$ and morphisms given by inclusions $i_{\tau,\sigma} : \tau \subseteq \sigma$ whose initial object is the zero cone $\{0\}$. Then $\sigma \mapsto \mathcal{K}_\sigma$ defines a contravariant diagram from this small category to the category of graded commutative $R(T_{comp})$-algebras $\mathcal{C}(alg)$. The colimit of this diagram exists in $\mathcal{C}(alg)$ and can be identified with $PLP(\Delta)$ (see [14, Section 4] and [23, Section 4]).

Theorem 4.6. *The ring $K^0_{T_{comp}}(X)$ is isomorphic to $PLP(\Delta)$ as an $R(T_{comp})$-algebra.*

Proof: Using the universal property of colimits we prove this by finding compatible homomorphisms $h_\sigma : \mathcal{A} \to \mathcal{K}_\sigma$ for every $\sigma \in \Delta$. Let $y = (y_i) \in \mathcal{A}$. On the maximal dimensional cones of Δ we define $h_{\sigma_j}(y) := y_j$ for each $1 \le j \le m$. On the $(n-1)$-dimensional cones we let $h_\gamma(y) := y_i \bmod J_\gamma$ in $\mathcal{K}_\gamma$ for any maximal dimensional cone σ_i containing γ. If $\gamma = \sigma_i \cap \sigma_j$ for $i \neq j$ then $y_i \equiv y_j \pmod{e^{T_{comp}}(u_{i_j})}$ where $u_{i_j} \in \gamma^\perp \cap M$ and $e^{T_{comp}}(u_{i_j}) \in J_\gamma$. Thus h_γ is well defined on the $(n-1)$-dimensional cones. Now, let $\gamma \in \Delta(n-k)$ for $k \ge 2$. We can similarly define $h_\gamma(y) := y_i \bmod J_\gamma$ where $\gamma \prec \sigma_i \in \Delta(n)$. To show that h_γ is well defined we need to show that if $\gamma \in \sigma_j$ for $j \neq i$ then $y_i \equiv y_j \pmod{J_\gamma}$. Now, $\gamma \prec \sigma_i \cap \sigma_j$. By Assumption 4.5 we have a sequence of cones $\sigma_i = \sigma_{i_1}, \ldots, \sigma_{i_r} = \sigma_j$ in $\Delta(n)$ all containing γ such that $\dim(\sigma_{i_t} \cap \sigma_{i_{t+1}}) = n-1$ for $1 \le t \le r-1$. Since $J_{\sigma_{i_t}\cap\sigma_{i_{t+1}}} \subseteq J_\gamma$ we have $y_{i_t} - y_{i_{t+1}} \in J_\gamma$ for $1 \le t \le r-1$. Thus $y_i - y_j = \displaystyle\sum_{t=1}^{r-1} y_{i_t} - y_{i_{t+1}} \in J_\gamma$.

Now to check the compatibility of h_γ for $\gamma \in \Delta$. For $\gamma \prec \gamma'$ we have $\psi_{\gamma,\gamma'} : \mathcal{K}_{\gamma'} \to \mathcal{K}_\gamma$. We need to verify that $\psi_{\gamma,\gamma'} \circ h_{\gamma'} = h_\gamma$. For $y \in \mathcal{A}$, $h_{\gamma'}(y) = y_i \bmod J_{\gamma'}$ for $\gamma' \prec \sigma_i \in \Delta(n)$. Now, by definition of $\psi_{\gamma,\gamma'}$ we get $\psi_{\gamma,\gamma'} \circ h_{\gamma'}(y) = y_i \bmod J_\gamma = h_\gamma(y)$. This proves the compatibility of $\{h_\gamma\}_{\gamma\in\Delta}$.Thus $\{h_\gamma\}$ induces a well defined ring homomorphism h from $\mathcal{A}$ to $PLP(\Delta)$. We shall now show that h is an isomorphism. Let $y \neq y' \in \mathcal{A}$. Then $y_i \neq y'_i$ for at least one $1 \le i \le m$. Thus $h_{\sigma_i}(y) \neq h_{\sigma_i}(y')$ in $R(T_{comp})$. Hence h is injective. Now, let $(a_\gamma)_{\gamma\in\Delta} \in PLP(\Delta)$. This in particular determines an element $(a_{\sigma_i}) \in (R(T_{comp}))^m$. We claim that $(a_{\sigma_i}) \in \mathcal{A}$. Let σ_i share a wall $\gamma = \sigma_i \cap \sigma_j$ with σ_j. Then we see that $\psi_{\gamma,\sigma_i}(a_{\sigma_i}) = a_\gamma = \psi_{\gamma,\sigma_j}(a_{\sigma_j})$. It implies that $a_{\sigma_i} - a_{\sigma_j} \equiv \pmod{J_\gamma}$ whenever σ_i and σ_j share a wall γ in Δ. Since J_γ is generated by $e^{T_{comp}}(u_{i_j})$ where u_{i_j} generates the lattice $\gamma^\perp \cap M$, it follows that

$e^{T_{comp}}(u_{i_j})$ divides $a_{\sigma_i} - a_{\sigma_j}$. This implies by (4.2) that $(a_{\sigma_i}) \in \mathcal{A}$ proving the surjectivity of h. $\square$

Remark 4.7. We note that Assumption 4.5 holds for a projective simplicial toric variety. We can see this as follows. When $X(\Delta)$ is projective then Δ is the normal fan of a simple convex polytope P in the dual lattice M. For $\tau \in \Delta$, $V(\tau) = X(\mathrm{star}(\tau))$ is a projective simplicial toric subvariety of $X(\Delta)$. Moreover, $\mathrm{star}(\tau)$ is the normal fan of a face P_τ of P, which is again a simple convex polytope. Since the 1-skeleton of P_τ is connected, any two vertices can be joined by a sequence of edges lying in P_τ. Consider the moment map $\mu : V(\tau) \to P_\tau$. The vertices of P_τ are the images of the T-fixed points in $V(\tau)$ and the edges joining the vertices are the images of the T-stable curves (which are projective lines) in $V(\tau)$ joining the T-fixed points. Thus our assertion follows by considering the inverse image of the sequence of edges connecting two vertices in P_τ under μ. Taking $\tau = \{0\}$ this follows in particular for $X(\Delta)$ (see [9]).

5. Basis for $K_{T_{comp}}(X)$ as an $R(T_{comp})$ − *module*

Let $X = X(\Delta)$ be a T-cellular toric variety. In this section we shall give a basis for $K^0_{T_{comp}}(X)$ as a $R(T_{comp})$-module.

We shall follow the notations developed in Theorem 4.2 and its proof.

Let $f_i \in K^0_{T_{comp}}(X)$ such that $k_{i-1} \circ \cdots \circ k_1(f_i) = j_i(g_i)$. Thus $\iota^*(f_i)_i = \prod_{i=1}^{n-k_i}(1 - e^{-u_{i_j}})$ and $\iota^*(f_i)_l = 0$ if $l > i$ since $k_i \circ j_i = 0$.

We shall let

$$f\,|_{x_l} := \iota^*(f)_l$$

denote the restriction of an element $f \in K_{T_{comp}}(X)$ to the T-fixed point x_l for $1 \leq l \leq m$.

Theorem 5.1. *The elements f_i for $1 \leq i \leq m$ form a basis for $K^0_{T_{comp}}(X)$ as a $R(T_{comp})$-module.*

Proof. We shall prove by downward induction on i that the set $\{\overline{f_i}\}_{r=i}^m$ span $K^0_{T_{comp}}(Z_i)$ as $R(T_{comp})$-module. Here $\overline{f_r} := k_{i-1} \circ \cdots \circ k_1(f_r) \in K^0_{T_{comp}}(Z_i)$. The proposition will follow since $Z_1 = X$.

When $r = m$, $Z_m = Y_m$ since $Z_{m+1} = \emptyset$. Thus the inclusion j_m is an isomorphism. Now, g_m spans for $K^0_{T_{comp}}(Y_m)$. Hence $j_m(g_m) = k_{m-1} \circ \cdots \circ k_1(f_m) = \overline{f_m}$ spans $K^0_{T_{comp}}(Z_m)$.

Suppose by induction that the claim holds for $K^0_{T_{comp}}(Z_{i+1})$. We shall now prove it for $K^0_{T_{comp}}(Z_i)$.

By induction assumption $\overline{f_r} := k_i \circ \cdots \circ k_1(f_r)$ for $i+1 \leq r \leq m$ span $K^0_{T_{comp}}(Z_{i+1})$.

Let $f \in K^0_{T_{comp}}(Z_i)$. Then $k_i(f) = \displaystyle\sum_{r=i+1}^{m} a_r \cdot k_i \circ \cdots \circ k_1(f_r)$ where $a_r \in R(T_{comp})$ for $i+1 \leq r \leq m$.

Thus $f - \displaystyle\sum_{r=i+1}^{m} a_r \cdot k_{i-1} \circ \cdots \circ k_1(f_r) \in \ker(k_i)$. From (4.14) we get that $f - \displaystyle\sum_{r=i+1}^{m} a_r \cdot k_{i-1} \circ \cdots \circ k_1(f_r) = j_i(a_i g_i)$ for some $a_i \in R(T_{comp})$. Here we recall that g_i denotes the Thom class of the T_{comp}-equivariant vector bundle V_i in $K^0_{T_{comp}}(V_i)$. Thus by definition of f_i it follows that $f - \displaystyle\sum_{r=i+1}^{m} a_r \cdot k_{i-1} \circ \cdots \circ k_1(f_r) = a_i \cdot k_{i-1} \circ \cdots \circ k_1(f_i)$. Thus it follows that $k_{i-1} \circ \cdots \circ k_1(f_r)$ for $i \leq r \leq m$ span $K^0_{T_{comp}}(Z_i)$. By induction it follows that $\{f_i\}_{i=1}^m$ span $K^0_{T_{comp}}(X)$ as a free $R(T_{comp})$-module.

It only remains to show that $\{f_r\}_{r=1}^m$ are linearly independent. Suppose $\displaystyle\sum_{r=1}^{m} a_r \cdot f_r = 0$. Then restricting to the fixed point x_m it follows that $a_m \cdot \displaystyle\prod_{j=1}^{n-k_m} (1 - e^{-u_{m_j}}) = 0$ since $f_r \mid_{x_m} = 0$ for $1 \leq r < m$ and $f_m \mid_{x_m} = \displaystyle\prod_{j=1}^{n-k_m} (1 - e^{-u_{m_j}})$. Since $\displaystyle\prod_{j=1}^{n-k_m} (1 - e^{-u_{m_j}})$ is a non-zero divisor in $R(T_{comp})$ this implies that $a_m = 0$. Now, by induction we assume that $a_r = 0$ for $i+1 \leq r \leq m$. We shall now show that $a_i = 0$. The proof will then follow by induction.

Now, restricting $\displaystyle\sum_{r=1}^{m} a_i \cdot f_i = 0$ to x_i we get $a_i \cdot \displaystyle\prod_{j=1}^{n-k_i} (1 - e^{-u_{i_j}}) = 0$. This is because by induction assumption and by the choice of f_r, $a_r = 0$ for $i+1 \leq r \leq m$, $f_r \mid_{x_i} = 0$ for $1 \leq r \leq i-1$ and $f_i \mid_{x_i} = \displaystyle\prod_{j=1}^{n-k_i} (1 - e^{-u_{i_j}})$, which is a non-zero divisor in $R(T_{comp})$. This implies that $a_i = 0$. $\square$

5.1. **Structure constants.** We have the following theorem which gives a closed formula for the coefficients of an element $f \in K_T^0(X)$ with respect to the basis $\{f_i\}_{i=1}^m$.

Theorem 5.2. *Let $f \in K^0_{T_{comp}}(X)$. Let*

$$f = \sum_{j=1}^{m} a_j \cdot f_j \tag{5.18}$$

where $a_j \in R(T_{comp})$ for $1 \leq j \leq m$. We have the following closed formula for the coefficients which is determined iteratively

$$a_i = \frac{\left[f - \sum_{j=i+1}^{m} a_j \cdot f_j\right] |_{x_i}}{\prod_{r=1}^{n-k_i}(1 - e^{-u_{ir}})} \tag{5.19}$$

for $1 \leq i \leq m$.

Proof. Recall that $f_j |_{x_i} = 0$ for $i > j$ and $f_j |_{x_j} = \displaystyle\prod_{r=1}^{n-k_j}(1 - e^{-u_{jr}})$.

Thus from (5.18) we get that:

$$f |_{x_m} = \sum_{j=1}^{m} a_j \cdot f_j |_{x_m} = a_m \cdot \prod_{r=1}^{n-k_m}(1 - e^{-u_{mr}}).$$

Since $\displaystyle\prod_{r=1}^{n-k_m}(1 - e^{-u_{mr}})$ is a non-zero divisor in $R(T_{comp})$ the theorem follows for $i = m$. Proceeding by descending induction, having determined the coefficients $a_m, \ldots, a_{i+1}$, we have the following from (5.18):

$$(f - \sum_{j=i+1}^{m} a_j \cdot f_j) |_{x_i} = a_i \cdot \prod_{r=1}^{n-k_i}(1 - e^{-u_{ir}}).$$

Since $\displaystyle\prod_{r=1}^{n-k_i}(1 - e^{-u_{ir}})$ is a non-zero divisor in $R(T_{comp})$, the theorem follows for i.

□

We have the following corollary which gives a formula for the multiplicative structure constants with respect to the basis $\{f_i\}_{i=1}^m$.

Corollary 5.3. *Let* $f_i \cdot f_j = \sum_{p=1}^{m} a_{i,j}^{p} \cdot f_p$ *where* $a_{i,j}^{p} \in R(T_{comp})$. *Then the structure constants* $a_{i,j}^{l}$ *can be determined by the following closed formula iteratively, by using descending induction on* l*:*

$$a_{i,j}^{l} = \frac{\left[f_i \cdot f_j - \sum_{p=l+1}^{m} a_{i,j}^{p} \cdot f_p\right] |_{x_l}}{\prod_{r=1}^{n-k_l}(1 - e^{-u_{l_r}})} \tag{5.20}$$

for $1 \leq l \leq m$.

Proof. The corollary follows readily by letting $f = f_i \cdot f_j$ in Theorem 5.2. □

Remark 5.4. In Theorem 5.2 and Corollary 5.3, although a priori the expressions on the right hand side of (5.19) and (5.20) respectively belong to $Q(T_{comp})$, as a consequence of Theorem 5.1, it follows that in fact they belong to $R(T_{comp})$.

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