Estimation for SLS models: finite sample guarantees

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Abstract

This note continues and extends the study from Spokoiny (2023a) about estimation for parametric models with possibly large or even infinite parameter dimension. We consider a special class of stochastically linear smooth (SLS) models satisfying three major conditions: the stochastic component of the log-likelihood is linear in the model parameter, while the expected log-likelihood is a smooth and concave function. For the penalized maximum likelihood estimators (pMLE), we establish several finite sample bounds about its concentration and large deviations as well as the Fisher and Wilks expansions and risk bounds. In all results, the remainder is given explicitly and can be evaluated in terms of the effective sample size n and effective parameter dimension $\mathbb{p}$ which allows us to identify the so-called *critical parameter dimension*. The results are also dimension and coordinate-free. Despite generality, all the presented bounds are nearly sharp and the classical asymptotic results can be obtained as simple corollaries. Our results indicate that the use of advanced fourth-order expansions allows to relax the critical dimension condition $\mathbb{p}^3 \ll n$ from Spokoiny (2023a) to $\mathbb{p}^{3/2} \ll n$. Examples for classical models like logistic regression, log-density and precision matrix estimation illustrate the applicability of general results.

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Contents

1 Introduction

This paper presents some general results describing the properties of the *penalized Maximum Likelihood Estimator* (pMLE). Our starting point is a parametric assumption about the distribution $\boldsymbol{I\!P}$ of the data $\boldsymbol{Y}$: $\boldsymbol{I\!P}$ belongs a given parametric family $(\boldsymbol{I\!P}_{\boldsymbol{\upsilon}}, \boldsymbol{\upsilon} \in \Upsilon)$ dominated by a sigma-finite measure $\boldsymbol{\mu}_0$. This assumption is usually an idealization of reality and the true distribution $\boldsymbol{I\!P}$ is not an element of $(\boldsymbol{I\!P}_{\boldsymbol{\upsilon}})$. However, a parametric assumption, even being wrong, may appear to be very useful, because it yields the method of estimation. Namely, the MLE $\widetilde{\boldsymbol{\upsilon}}$ is defined by maximizing the log-likelihood function $L(\boldsymbol{Y}, \boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) = \log \frac{d\boldsymbol{I\!P}_{\boldsymbol{\upsilon}}}{d\boldsymbol{\mu}_0}(\boldsymbol{Y})$ over the parameter set Υ:

$$\widetilde{\boldsymbol{\upsilon}} = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \Upsilon} L(\boldsymbol{\upsilon}).$$

For a penalty function $\operatorname{pen}_G(\boldsymbol{\upsilon})$ on Υ, the *penalized MLE* $\widetilde{\boldsymbol{\upsilon}}_G$ is defined by maximizing the penalized MLE $L_G(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \operatorname{pen}_G(\boldsymbol{\upsilon})$:

$$\widetilde{\boldsymbol{\upsilon}}_G = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \Upsilon} L_G(\boldsymbol{\upsilon}) = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \Upsilon} \bigl\{ L(\boldsymbol{\upsilon}) - \operatorname{pen}_G(\boldsymbol{\upsilon}) \bigr\}.$$

The sub-index G in the penalty refers to its quadratic structure:

$$\operatorname{pen}_G(\boldsymbol{\upsilon}) = \frac{1}{2} \| G \boldsymbol{\upsilon} \|^2$$

for a symmetric $p \times p$ positive definite matrix $G \in \mathfrak{M}_p$.

1.1 Challenges of the classical parametric theory

The classical Fisher parametric theory assumes that Υ is a subset of a finite-dimensional Euclidean space $I\!R^p$, the underlying data distribution $\boldsymbol{I\!P}$ indeed belongs to the considered parametric family $(\boldsymbol{I\!P}_{\boldsymbol{\upsilon}})$, that is, $\boldsymbol{Y} \sim \boldsymbol{I\!P} = \boldsymbol{I\!P}_{\boldsymbol{\upsilon}^*}$ for some $\boldsymbol{\upsilon}^* \in \Upsilon$. In addition, some regularity of the family $(\boldsymbol{I\!P}_{\boldsymbol{\upsilon}})$, or, equivalently, of the log-likelihood function $L(\boldsymbol{\upsilon})$ is assumed. This, in particular, enables us to apply the third order Taylor expansion of $L(\boldsymbol{\upsilon})$ around the point of maximum $\widetilde{\boldsymbol{\upsilon}}$ and to obtain a Fisher expansion

$$\widetilde{\boldsymbol{\upsilon}} - \boldsymbol{\upsilon}^* \approx I\!\!F^{-1} \nabla L(\boldsymbol{\upsilon}^*).$$

Here $I\!\!F = I\!\!F(\boldsymbol{\upsilon}^*)$ is the total Fisher information at $\boldsymbol{\upsilon}^*$ defined as a negative Hessian of the expected log-likelihood function $I\!\!E L(\boldsymbol{\upsilon})$:

$$I\!\!F(\boldsymbol{\upsilon}) = -\nabla^2 I\!\!E L(\boldsymbol{\upsilon}).$$

Under standard parametric assumptions, $I\!\!F(\boldsymbol{\upsilon})$ is symmetric positive definite, $I\!\!F(\boldsymbol{\upsilon}) \in \mathfrak{M}_p$. Moreover, if the data $\boldsymbol{Y}$ is generated as a sample of independent random variables $Y_1, \ldots, Y_n$, then the log-likelihood has an additive structure: $L(\boldsymbol{\upsilon}) = \sum_{i=1}^{n} \ell(Y_i, \boldsymbol{\upsilon})$. This allows to establish asymptotic standard normality of the standardized score $\boldsymbol{\xi} \stackrel{\text{def}}{=} I\!\!F^{-1/2} \nabla L(\boldsymbol{\upsilon}^*)$ and hence, to state Fisher and Wilks Theorems: as $n \to \infty$

$$
\begin{aligned}
I\!\!F^{1/2}\bigl(\widetilde{\boldsymbol{\upsilon}} - \boldsymbol{\upsilon}^*\bigr) &\approx \boldsymbol{\xi} \qquad \xrightarrow{d} \boldsymbol{\gamma}, \\
2L(\widetilde{\boldsymbol{\upsilon}}) - 2L(\boldsymbol{\upsilon}^*) &\approx \|\boldsymbol{\xi}\|^2 \xrightarrow{d} \|\boldsymbol{\gamma}\|^2 \sim \chi_p^2 ,
\end{aligned}
\tag{1.1}
$$

where $\boldsymbol{\gamma}$ is a standard Gaussian vector in $I\!\!R^p$ and χ_p^2 is a chi-squared distribution with p degrees of freedom. These results are fundamental and build the basis for most statistical applications like analysis of variance, canonical and correlation analysis, uncertainty quantification and hypothesis testing etc. We refer to van der Vaart (1998) for a comprehensive discussion and a historical overview of the related results including the general LAN theory by L. Le Cam. Modern statistical problems require to extend the classical results in several directions.

Model misspecification and bias Very often, the underlying data generating measure $I\!\!P$ is not an element of the family $(I\!\!P_{\boldsymbol{\upsilon}}, \boldsymbol{\upsilon} \in \varUpsilon)$. This means that the used log-likelihood function is not necessarily a true log-likelihood. In particular, the condition $I\!\!E \exp L(\boldsymbol{\upsilon}) = 1$ does not hold. Also the target of estimation $\boldsymbol{\upsilon}^*$ has to be redefined as the maximizer of the expected log-likelihood:

$$
\boldsymbol{\upsilon}^* \stackrel{\text{def}}{=} \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \varUpsilon} I\!\!E L(\boldsymbol{\upsilon})
$$

leading to some modelling bias as the distance between $I\!\!P$ and $I\!\!P_{\boldsymbol{\upsilon}^*}$. This also concerns the use of a penalty leading to some *penalization bias*. When operating with the penalized log-likelihood $L_G(\boldsymbol{\upsilon})$, the target of estimation becomes

$$
\boldsymbol{\upsilon}_G^* \stackrel{\text{def}}{=} \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \varUpsilon} I\!\!E L_G(\boldsymbol{\upsilon}),
\tag{1.2}
$$

which might be significantly different from $\boldsymbol{\upsilon}^*$. This requires to carefully evaluate the penalization bias $\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*$.

Finite samples, general likelihood, effective sample size Another important issue is a possibility of relaxing the assumption of i.i.d. or independent observations which ensures an additive structure of the function $L(\boldsymbol{\upsilon})$. Below we operate with the general likelihood, its structure does not need to be specified. We can even proceed with just one observation. However, for stating our results about accuracy of estimation, we need a notion of *effective sample size* n. This is given via the so-called Fisher information matrix. Everywhere we use the notation

$$I\!\!F(\boldsymbol{\upsilon}) = -\nabla^2 I\!\!E L(\boldsymbol{\upsilon}), \qquad I\!\!F_G(\boldsymbol{\upsilon}) = -\nabla^2 I\!\!E L_G(\boldsymbol{\upsilon}) = I\!\!F(\boldsymbol{\upsilon}) + G^2.$$

We also write $I\!\!F = I\!\!F(\boldsymbol{\upsilon}^*_G)$, $I\!\!F_G = I\!\!F_G(\boldsymbol{\upsilon}^*_G) = I\!\!F + G^2$. If the Y_i's are i.i.d. then $I\!\!F(\boldsymbol{\upsilon})$ is proportional to n. Therefore, we use the value $n = \|I\!\!F^{-1}\|^{-1}$ as a proxy for the "sample size".

Effective parameter dimension and critical dimension One more important issue is the parameter dimension p. The classical theory assumes p fixed and n large. We aim at relaxing both conditions by allowing a large/huge/infinite parameter dimension and a small or moderate n. It appears that all the results below rely on the so-called *effective dimension* $\mathbb{p}_G$ defined as

$$\mathbb{p}_G \stackrel{\text{def}}{=} \operatorname{tr}\bigl\{ I\!\!F_G^{-1} \operatorname{Var}(\nabla L(\boldsymbol{\upsilon}^*_G)) \bigr\}.$$

This quantity replaces the original dimension p and it can be small or moderate even for p infinite. One of the main intentions of our study is to understand the range of applicability of the mentioned results in terms of the effective parameter dimension $\mathbb{p}_G$ and the effective sample size n. It appears that most of the results ahead about concentration of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ apply under the condition $\mathbb{p}_G \ll n$ which replaces the classical signal-to-noise relation: the effective number of parameters to be estimated is smaller in order than the effective sample size. More advanced results like Fisher and Wilks expansions and sharp risk bounds for a low dimensional sub-vector of $\boldsymbol{\upsilon}$ may require stronger conditions $\mathbb{p}_G^2 \ll n$ or $\mathbb{p}_G^{3/2} \ll n$.

1.2 Main steps of study

Now we briefly describe our setup and the main focus of our analysis. Below we limit ourselves to a special class of *stochastically linear smooth* (SLS) statistical models. The major feature of such models is that the stochastic component $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - I\!\!E L(\boldsymbol{\upsilon})$ of the log-likelihood $L(\boldsymbol{\upsilon})$ is linear in parameter $\boldsymbol{\upsilon}$. We also assume that the expected log-likelihood is a concave and smooth function of the parameter $\boldsymbol{\upsilon}$. This class includes popular Generalized Linear Models but it is much larger. In particular, by extending the parameter space, one can consider many nonlinear models including nonlinear regression or nonlinear inverse problems as a special case of SLS; see Spokoiny (2019). We also focus on the case of a quadratic penalization $\mathrm{pen}_G(\boldsymbol{\upsilon}) = \|G\boldsymbol{\upsilon}\|^2/2$. This would not affect the SLS conditions. The assumption of stochastic linearity helps to avoid heavy tools of empirical process theory which is typically used in the analysis of pMLE $\widetilde{\boldsymbol{\upsilon}}_G$; see e.g. Birgé and Massart (1998), van der Vaart (1998), Geer (2000), Kosorok (2005), Ginè and Nickl (2015) among many others. We only need some accurate deviation bounds for quadratic forms of the errors; see Section B in the appendix. Our aim is to establish possibly sharp and accurate results under realistic assumptions on a SLS model and the amount of data. The study includes several steps.

Concentration of the pMLE The first step of our analysis is to establish a concentration result for the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ defined by maximization of $L_G(\boldsymbol{\upsilon})$. If the expected log-likelihood $\mathbb{E}L_G(\boldsymbol{\upsilon})$ is strictly concave and smooth in $\boldsymbol{\upsilon}$ then $\widetilde{\boldsymbol{\upsilon}}_G$ well concentrates in a small elliptic vicinity $\mathcal{A}_G$ of the "target" $\boldsymbol{\upsilon}_G^*$ from (1.2):

$$\mathbb{P}\bigl(\|\mathbb{F}_G^{1/2}(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*)\| > \mathtt{r}_G + \sqrt{2\mathtt{x}}\bigr) \le 3\mathrm{e}^{-\mathtt{x}},$$

where $\mathtt{r}_G^2 \asymp \mathbb{p}_G$. The result becomes sensible provided that $\mathbb{p}_G \ll n$ with $n^{-1} \asymp \|\mathbb{F}_G^{-1}\|$. In the classical parametric theory, such results about concentration of pMLE involve some advanced tools from the empirical process theory. The use condition $(\boldsymbol{\zeta})$ about linearity of the stochastic component $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \mathbb{E}L(\boldsymbol{\upsilon})$ allows to reduce the analysis to deviation bounds of the quadratic form $\|\mathbb{F}_G^{-1/2}\nabla\zeta\|^2$; cf. condition $(\boldsymbol{\nabla\zeta})$. Section B presents several results in this direction under different assumptions on the stochastic gradient $\nabla\zeta$.

3G Fisher and Wilks expansions Having established the concentration of $\widetilde{\boldsymbol{\upsilon}}_G \in \mathcal{A}_G$, we can restrict the analysis to this vicinity and use the Taylor expansion of the penalized log-likelihood function $L_G(\boldsymbol{\upsilon})$. This helps to derive rather precise approximations for $\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*$ and $L_G(\widetilde{\boldsymbol{\upsilon}}_G) - L_G(\boldsymbol{\upsilon}_G^*)$:

$$\begin{aligned} \bigl\|\mathbb{F}_G^{1/2}(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*) - \boldsymbol{\xi}_G\bigr\| &\le \frac{3\tau_3}{4}\,\bigl\|\boldsymbol{\xi}_G\bigr\|^2, \\ \Bigl|L_G(\widetilde{\boldsymbol{\upsilon}}_G) - L_G(\boldsymbol{\upsilon}_G^*) - \frac{1}{2}\bigl\|\boldsymbol{\xi}_G\bigr\|^2\Bigr| &\le \tau_3\,\bigl\|\boldsymbol{\xi}_G\bigr\|^3, \end{aligned} \tag{1.3}$$

where

$$\boldsymbol{\xi}_G \stackrel{\mathrm{def}}{=} \mathbb{F}_G^{-1/2}\nabla L_G(\boldsymbol{\upsilon}_G^*) = \mathbb{F}_G^{-1/2}\nabla\zeta,$$

and $\nabla\zeta = \nabla\zeta(\boldsymbol{\upsilon})$ does not depend on $\boldsymbol{\upsilon}$ due to linearity of $\zeta(\boldsymbol{\upsilon}) = L_G(\boldsymbol{\upsilon}) - \mathbb{E}L_G(\boldsymbol{\upsilon})$. The accuracy of approximation is controlled by the value τ_3 which describes the accuracy of the third-order Taylor expansion of the function $f_G(\boldsymbol{\upsilon}) = \mathbb{E} L_G(\boldsymbol{\upsilon})$ in terms of the third directional derivative of f_G. In typical examples $\tau_3 \asymp \sqrt{1/n}$. The presented results require $\tau_3^2 \, \mathbb{p}_G \ll 1$ which again leads to the condition $\mathbb{p}_G \ll n$. The first result in (1.3) about the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ will be referred to as *the Fisher expansion*, while the second one about $L_G(\widetilde{\boldsymbol{\upsilon}}_G)$ is called *the Wilks expansion*. The main technical tool for deriving these expansions is the so-called *basic lemma*; see Proposition A.9.

These two expansions provide a finite sample analog of the asymptotic statements (1.1) and are informative even in the classical parametric situation. In fact, under standard assumptions, the normalized score vector $\boldsymbol{\xi}_G$ is asymptotically normal $\mathcal{N}(0, \Sigma_G)$ with $\Sigma_G = \mathbb{F}_G^{-1/2} V^2 \mathbb{F}_G^{-1/2} \in \mathfrak{M}_p$ and $V^2 = \operatorname{Var}\bigl(\nabla L(\boldsymbol{\upsilon})\bigr) \in \mathfrak{M}_p$. Stochastic linearity implies that the matrix V^2 does not depend on the point $\boldsymbol{\upsilon}$. If the model is correctly specified, then Σ_G approaches the identity as $n \to \infty$, and we obtain the classical results (1.1). Note that the use of stochastic linearity allows us to obtain much more accurate bounds than in Spokoiny (2012) or Spokoiny (2017).

3G risk bounds The loss of $\widetilde{\boldsymbol{\upsilon}}_G$ can be naturally expanded as

$$\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^* = \widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^* + \boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^* . \tag{1.4}$$

Due to the Fisher expansion (1.3),

$$\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^* \approx \mathbb{F}_G^{-1} \nabla \zeta .$$

This expansion is based on the basic lemma; see Proposition A.9. Another application of this result yields an expansion of the bias:

$$\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^* \approx -\mathbb{F}_G^{-1} G^2 \boldsymbol{\upsilon}^* .$$

Putting together these two expansions leads to the so-called bias-variance decomposition of the squared risk: for any linear mapping $Q \colon \mathbb{R}^p \to \mathbb{R}^q$

$$\mathbb{E} \bigl\| Q(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*) \bigr\|^2 \approx \mathscr{R}_Q ,$$

where $\mathscr{R}_Q$ is the squared risk in the approximating linear model:

$$\mathscr{R}_Q \stackrel{\text{def}}{=} \bigl\| Q \, \mathbb{F}_G^{-1} (\nabla \zeta - G^2 \boldsymbol{\upsilon}^*) \bigr\|^2 = \operatorname{tr} \operatorname{Var}(Q \mathbb{F}_G^{-1} \nabla \zeta) + \bigl\| Q \, \mathbb{F}_G^{-1} G^2 \boldsymbol{\upsilon}^* \bigr\|^2 .$$

Theorem 2.7 provides sufficient conditions allowing to state a sharp risk bound:

$$(1 - \alpha_Q)^2 \mathscr{R}_Q \le \mathbb{E} \bigl\| Q(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*) \bigr\|^2 \le (1 + \alpha_Q)^2 \mathscr{R}_Q .$$

Of course, this result is only meaningful if $\alpha_Q \ll 1$. It appears that this value strongly depends on the dimension q of the mapping Q. If $Q = I\!\!I_p$ or $Q = I\!\!F_G^{1/2}$ then $\mathbb{p}_G \ll n$ is sufficient to ensure $\alpha_Q \ll 1$. In the case of a low dimensional target with $q \asymp 1$, the condition $\alpha_Q \ll 1$ translates into $\mathbb{p}_G^2 \ll n$.

4G expansions and risk bounds The critical dimension condition $\mathbb{p}_G^2 \ll n$ can be very limiting. Fourth-order smoothness conditions on $f_G(\boldsymbol{v})$ allow us to improve the accuracy of expansion (1.4) by accounting for the third-order term and thus, relax the critical dimension bound. Consider the third-order tensor $\mathcal{T}(\boldsymbol{u}) = \frac{1}{6}\langle \nabla^3 f(\boldsymbol{v}_G^*), \boldsymbol{u}^{\otimes 3}\rangle$. Let $\nabla\mathcal{T}(\boldsymbol{u}) = \frac{1}{2}\langle \nabla^3 f(\boldsymbol{v}_G^*), \boldsymbol{u}^{\otimes 2}\rangle$ be its gradient. Define the vectors $\boldsymbol{n}_G$ and $\boldsymbol{m}_G$ by

$$
\begin{aligned}
\boldsymbol{n}_G &= I\!\!F_G^{-1}\bigl\{\nabla\zeta + \nabla\mathcal{T}(I\!\!F_G^{-1}\nabla\zeta)\bigr\}\,, \\
\boldsymbol{m}_G &= I\!\!F_G^{-1}G^2\boldsymbol{v}^* + I\!\!F_G^{-1}\nabla\mathcal{T}(I\!\!F_G^{-1}G^2\boldsymbol{v}^*)\,.
\end{aligned}
\tag{1.5}
$$

Theorem 2.11 states the following bound:

$$
\|Q\,(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^* - \boldsymbol{n}_G + \boldsymbol{m}_G)\| \le \|Q I\!\!F_G^{-1/2}\| \left(\frac{\tau_4}{3} + \tau_3^2\right) \bigl(\|I\!\!F_G^{-1/2}\nabla\zeta\|^3 + \mathsf{b}_G^3\bigr), \tag{1.6}
$$

where $\mathsf{b}_G = \|I\!\!F_G^{-1/2}G^2\boldsymbol{v}^*\|$ and τ_4 controls the fourth derivative of f_G. Typically $\tau_4 \asymp n^{-1}$ and (1.6) is an improvement of (1.3) because the full dimensional error term in the right-hand side of (1.6) is of order $\mathbb{p}_G^{3/2}/n$ compared to $\mathbb{p}_G^2/n$ in (1.3). Therefore, the corrections from (1.5) improves the critical dimension condition from $\mathbb{p}_G^2 \ll n$ to $\mathbb{p}_G^{3/2} \ll n$. An interesting question of using a higher order expansion of f_G for a further relaxation of the critical dimension condition is still open because even for 4G case, a closed-form solution of the corresponding 4G approximation problem is not available.

Tools The presented results are based on two kinds of statements. The results about concentration of the PMLE heavily rely on deviation bounds for quadratic forms of a centered and standardized score vector. Such results are collected in Section B. We separately study the cases of Gaussian errors, sub-Gaussian errors, and sub-exponential errors. The other important technical element of the proofs is the so-called "basic lemma". It describes the solution of a convex optimization problem after a linear perturbation. This result only relies on the smoothness and convexity of the objective function and it can be proved by elementary tools like a third of fourth-order Taylor expansions. Section A presents this and similar results.

1.3 Examples

Section 2 presents some general theoretical results. Section 3 illustrates how the general conditions of Section 2 can be checked for the classical setups like logistic regression, log-density, and precision matrix estimation. This enables us to apply the results of Section 2 to such models which improve and extend the similar results from the earlier paper Spokoiny (2023a). All the mentioned examples are particular cases of Generalized Linear Models (GLM). However, the SLS approach goes far beyond the GLM setup. In particular, the paper Spokoiny (2023b) explains, how the so-called calming device can be used to bring a nonlinear regression problem to the SLS setup. The developed results can be applied to models like deep neuronal networks, nonlinear inverse problems, etc. One more class of examples is given by error-in-operator models. This class includes random design regression, instrumental regression, functional data analysis, diffusion, and McKean-Vlasov models, etc. The calming trick applies here as well; see Puchkin et al. (2023) for the case of a high-dimensional random design. The other examples include effective dimension reduction, Gaussian mixture estimation, low-rank matrix recovery, covariance and precision matrix estimation, smooth functional estimation, among others. However, a rigorous treatment of each problem requires a separate study with a careful check of the conditions and specific results and will be done elsewhere.

2 Properties of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$

This section collects general results about concentration and expansion of the pMLE which substantially improve the bounds from Spokoiny and Panov (2021) and Spokoiny (2023a). We assume to be given a pseudo log-likelihood random function $L(\boldsymbol{\upsilon})$, $\boldsymbol{\upsilon} \in \varUpsilon \subseteq \mathbb{R}^p$, $p < \infty$. Given a quadratic penalty $\|G\boldsymbol{\upsilon}\|^2/2$, define

$$L_G(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \|G\boldsymbol{\upsilon}\|^2/2.$$

Consider in parallel three optimization problems defining the penalized MLE $\widetilde{\boldsymbol{\upsilon}}_G$, its population counterpart $\boldsymbol{\upsilon}_G^*$, and the background truth $\boldsymbol{\upsilon}^*$:

$$\widetilde{\boldsymbol{\upsilon}}_G = \operatorname*{argmax}_{\boldsymbol{\upsilon}} L_G(\boldsymbol{\upsilon}), \qquad \boldsymbol{\upsilon}_G^* = \operatorname*{argmax}_{\boldsymbol{\upsilon}} \mathbb{E} L_G(\boldsymbol{\upsilon}), \qquad \boldsymbol{\upsilon}^* = \operatorname*{argmax}_{\boldsymbol{\upsilon}} \mathbb{E} L(\boldsymbol{\upsilon}).$$

The corresponding Fisher information matrix $\mathbb{F}_G(\boldsymbol{\upsilon})$ is given by

$$\mathbb{F}(\boldsymbol{\upsilon}) = -\nabla^2 \mathbb{E} L(\boldsymbol{\upsilon}), \qquad \mathbb{F}_G(\boldsymbol{\upsilon}) = -\nabla^2 \mathbb{E} L_G(\boldsymbol{\upsilon}) = \mathbb{F}(\boldsymbol{\upsilon}) + G^2.$$

We assume $I\!\!F_G(\boldsymbol{\upsilon})$ to be positive definite for all considered $\boldsymbol{\upsilon}$. By $D_G(\boldsymbol{\upsilon})$ we denote a positive symmetric matrix with $D_G^2(\boldsymbol{\upsilon}) = I\!\!F_G(\boldsymbol{\upsilon})$, and $I\!\!F_G = I\!\!F_G(\boldsymbol{\upsilon}_G^*)$, $D_G = I\!\!F_G^{1/2}$.

2.1 Basic conditions

Now we present our major conditions. The most important one is about linearity of the stochastic component $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - I\!\!E L(\boldsymbol{\upsilon}) = L_G(\boldsymbol{\upsilon}) - I\!\!E L_G(\boldsymbol{\upsilon})$.

$(\boldsymbol{\zeta})$ *The stochastic component* $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - I\!\!E L(\boldsymbol{\upsilon})$ *of the process* $L(\boldsymbol{\upsilon})$ *is linear in* $\boldsymbol{\upsilon}$. *We denote by* $\nabla\zeta \equiv \nabla\zeta(\boldsymbol{\upsilon}) \in I\!\!R^p$ *its gradient* .

Below we assume some concentration properties of the stochastic vector $\nabla\zeta$. More precisely, we require that $\nabla\zeta$ obeys the following condition; see (B.51) of Theorem B.15.

$(\boldsymbol{\nabla\zeta})$ *Let* $V^2 \geq \operatorname{Var}(\nabla\zeta)$, $D_G^2 = D_G^2(\boldsymbol{\upsilon}_G^*)$, $\mathbb{p}_G = \operatorname{tr}(D_G^{-2} V^2)$, *and* $\lambda_G = \| D_G^{-1} V^2 D_G^{-1} \|$. *Then for any considered* $\mathtt{x} > 0$

$$
I\!\!P\bigl(\| D_G^{-1} \nabla\zeta \| \geq \mathtt{r}_G(\mathtt{x}) \bigr) \leq 3 \mathrm{e}^{-\mathtt{x}}, \tag{2.1}
$$

$$
\mathtt{r}_G(\mathtt{x}) \stackrel{\text{def}}{=} \sqrt{\mathbb{p}_G} + \sqrt{2\mathtt{x}\lambda_G} . \tag{2.2}
$$

This condition can be effectively checked if the errors in the data exhibit sub-gaussian or sub-exponential behavior; see Section B.3. The important value $\mathbb{p}_G = \operatorname{tr}(D_G^{-2} V^2)$ can be called the *effective dimension*; see Spokoiny (2017).

We also assume that the penalized log-likelihood $L_G(\boldsymbol{\upsilon})$ or, equivalently, its deterministic part $I\!\!E L_G(\boldsymbol{\upsilon})$ is a concave function. It can be relaxed using localization; see Spokoiny (2023b).

$(\boldsymbol{\mathcal{C}_G})$ *The function* $I\!\!E L_G(\boldsymbol{\upsilon})$ *is concave on* Υ *which is open and convex set in* $I\!\!R^p$.

Later we will also need some smoothness conditions on the function $f(\boldsymbol{\upsilon}) = I\!\!E L(\boldsymbol{\upsilon})$. The class of models satisfying the conditions $(\boldsymbol{\zeta})$, $(\boldsymbol{\nabla\zeta})$ with a smooth function $f(\boldsymbol{\upsilon}) = I\!\!E L(\boldsymbol{\upsilon})$ will be referred to as *stochastically linear smooth* (SLS). This class includes linear regression, generalized linear models (GLM), and log-density models; see Spokoiny and Panov (2021) or Spokoiny (2023a). However, this class is much larger. For instance, nonlinear regression and nonlinear inverse problems can be adapted to the SLS framework by an extension of the parameter space; see Spokoiny (2023b).

2.2 Concentration of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$. 2G expansions

This section discusses some concentration properties of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G = \operatorname{argmax}_{\boldsymbol{\upsilon}} L_G(\boldsymbol{\upsilon})$ under second-order smoothness conditions.

2.2.1 Local smoothness conditions

Given $\mathtt{x}$ and $\mathtt{r}_G = \mathtt{r}_G(\mathtt{x})$ from (2.2), define for some $\nu < 1$ the set $\mathcal{U}_G$ by

$$\mathcal{U}_G \stackrel{\text{def}}{=} \bigl\{ \boldsymbol{u} \colon \| D_G \boldsymbol{u} \| \le \nu^{-1} \mathtt{r}_G \bigr\}. \tag{2.3}$$

The result of this section states the concentration properties of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ in the local vicinity $\mathcal{A}_G$ of $\boldsymbol{\upsilon}_G^*$ of the form

$$\mathcal{A}_G \stackrel{\text{def}}{=} \boldsymbol{\upsilon}_G^* + \mathcal{U}_G = \bigl\{ \boldsymbol{\upsilon} = \boldsymbol{\upsilon}_G^* + \boldsymbol{u} \colon \boldsymbol{u} \in \mathcal{U}_G \bigr\} \subseteq \Upsilon^{\circ}.$$

Local Gateaux-regularity of $f(\boldsymbol{\upsilon}) = I\!\!E L(\boldsymbol{\upsilon})$ within $\mathcal{A}_G$ will be measured by the error of the second-order Taylor approximation

$$\begin{aligned} \delta_3(\boldsymbol{\upsilon}, \boldsymbol{u}) &= f(\boldsymbol{\upsilon} + \boldsymbol{u}) - f(\boldsymbol{\upsilon}) - \langle \nabla f(\boldsymbol{\upsilon}), \boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle, \\ \delta_3'(\boldsymbol{\upsilon}, \boldsymbol{u}) &= \langle \nabla f(\boldsymbol{\upsilon} + \boldsymbol{u}), \boldsymbol{u} \rangle - \langle \nabla f(\boldsymbol{\upsilon}), \boldsymbol{u} \rangle - \langle \nabla^2 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle . \end{aligned}$$

More precisely, define

$$\omega_G \stackrel{\text{def}}{=} \sup_{\boldsymbol{u} \in \mathcal{U}_G} \frac{2 |\delta_3(\boldsymbol{\upsilon}_G^*, \boldsymbol{u})|}{\| D_G \boldsymbol{u} \|^2} \, , \qquad \omega_G' \stackrel{\text{def}}{=} \sup_{\boldsymbol{u} \in \mathcal{U}_G} \frac{|\delta_3'(\boldsymbol{\upsilon}_G^*, \boldsymbol{u})|}{\| D_G \boldsymbol{u} \|^2} \, . \tag{2.4}$$

The quantities ω_G and ω_G' can be effectively bounded under smoothness conditions **$(\mathcal{T}_3)$** or **$(\mathcal{S}_3)$** given in Section A. Under **$(\mathcal{T}_3)$** at $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_G^*$ with $D^2(\boldsymbol{\upsilon}_G^*) = D_G^2$ and $\mathtt{r} = \mathtt{r}_G$, by Lemma A.1, it holds for a small constant τ_3

$$\omega_G' \le \tau_3 \, \nu^{-1} \mathtt{r}_G \, , \qquad \omega_G \le \tau_3 \, \nu^{-1} \mathtt{r}_G / 3.$$

Also under **$(\mathcal{S}_3)$**, the same bounds apply with $\tau_3 = \mathtt{c}_3 n^{-1/2}$; see Lemma A.3.

2.2.2 Concentration bound

Now we are prepared to state a very important concentration result for pMLE $\widetilde{\boldsymbol{\upsilon}}_G$.

Proposition 2.1. *Suppose* **(ζ)**, **$(\nabla\zeta)$**, *and* **$(\mathcal{C}_G)$**. *Let also*

$$1 - \nu - \omega_G' > 0; \tag{2.5}$$

see (2.4) *and* (2.3). *Then* $\widetilde{\boldsymbol{\upsilon}}_G \in \mathcal{A}_G$ *on a set* $\Omega(\mathtt{x})$ *with* $\mathbb{P}\bigl(\Omega(\mathtt{x})\bigr) \geq 1 - 3\mathrm{e}^{-\mathtt{x}}$ *, i.e.*

$$\|D_G(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*)\| \leq \nu^{-1}\mathtt{r}_G\,. \tag{2.6}$$

Proof. By $(\boldsymbol{\nabla\zeta})$, on a the random set $\Omega(\mathtt{x})$ with $\mathbb{P}(\Omega(\mathtt{x})) \geq 1 - 3\mathrm{e}^{-\mathtt{x}}$, it holds $\|D_G^{-1}\nabla\zeta\| \leq \mathtt{r}_G$. Now the result follows from Proposition A.6 with $f(\boldsymbol{\upsilon}) = \mathbb{E}L_G(\boldsymbol{\upsilon})$, $g(\boldsymbol{\upsilon}) = L_G(\boldsymbol{\upsilon})$, $\mathtt{r} = \nu^{-1}\mathtt{r}_G$, and $\boldsymbol{A} = \nabla\zeta$. □

Remark 2.1. The result (2.6) continues to apply with any matrix $\mathbb{D} \leq D_G$ in place of D_G provided that $(\boldsymbol{\nabla\zeta})$ as well as (2.4), (2.5) hold after this change.

2.2.3 Fisher and Wilks expansions. 2G bounds

Here we show how the concentration of $\widetilde{\boldsymbol{\upsilon}}_G$ around $\boldsymbol{\upsilon}_G^*$ can be used to establish a version of the Fisher expansion for the estimation error $\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*$ and the Wilks expansion for the excess $L_G(\widetilde{\boldsymbol{\upsilon}}_G) - L_G(\boldsymbol{\upsilon}_G^*)$. The result follows by Proposition A.7.

Theorem 2.2. *Assume the conditions of Proposition 2.1. Then on* $\Omega(\mathtt{x})$

$$\begin{aligned}
2L_G(\widetilde{\boldsymbol{\upsilon}}_G) - 2L_G(\boldsymbol{\upsilon}_G^*) - \bigl\|D_G^{-1}\nabla\zeta\bigr\|^2 &\leq \frac{\omega_G}{1-\omega_G}\bigl\|D_G^{-1}\nabla\zeta\bigr\|^2,\\
2L_G(\widetilde{\boldsymbol{\upsilon}}_G) - 2L_G(\boldsymbol{\upsilon}_G^*) - \bigl\|D_G^{-1}\nabla\zeta\bigr\|^2 &\geq -\omega_G\bigl\|D_G^{-1}\nabla\zeta\bigr\|^2.
\end{aligned}$$

Also

$$\begin{aligned}
\bigl\|D_G\bigl(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*\bigr) - D_G^{-1}\nabla\zeta\bigr\|^2 &\leq \frac{3\omega_G}{(1-\omega_G)^2}\,\bigl\|D_G^{-1}\nabla\zeta\bigr\|^2,\\
\bigl\|D_G\bigl(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*\bigr)\bigr\| &\leq \frac{1+\sqrt{2\omega_G}}{1-\omega_G}\,\bigl\|D_G^{-1}\nabla\zeta\bigr\|.
\end{aligned}$$

2.2.4 Effective sample size and critical dimension in pMLE

This section discusses the important question of the critical parameter dimension still ensuring the validity of the presented results. An essential feature of our results is their dimension-free and coordinate-free form. The true parametric dimension p can be very large, it does not appear in the error terms. Also, we do not use any spectral decomposition or sequence space structure, in particular, we do not require that the Fisher information matrix $\mathbb{F}$ and the penalty matrix G^2 are diagonal or can be jointly diagonalized. The results are stated for the general data $\boldsymbol{Y}$ and a quasi log-likelihood function. In particular, we do not assume independent or progressively dependent observations and additive structure of the log-likelihood. The *effective sample size* n can be defined via the smallest eigenvalue of the matrix $\mathbb{F}_G = D_G^2 = -\nabla^2 \mathbb{E} L_G(\boldsymbol{\upsilon}_G^*)$:

$$n^{-1} \stackrel{\text{def}}{=} \|\mathbb{F}_G^{-1}\|.$$

Our results apply as long as this value is sufficiently small. Alternatively, the error terms τ_3, τ_4 scale with n so that in typical examples, $\tau_3 \asymp n^{-1/2}$, $\tau_4 \asymp n^{-1}$. In typical examples like regression or density modeling such defined value n is closely related to the sample size of the data.

For the concentration result of Proposition 2.1 we need the basic conditions $(\boldsymbol{\zeta})$ and $(\boldsymbol{\mathcal{C}_G})$. Further, $(\boldsymbol{\nabla\zeta})$ identifies the radius $\mathtt{r}_G$ of the local vicinity $\mathcal{A}_G$. The final critical condition is given by (2.5). Essentially it says that the values ω_G and ω_G' are significantly smaller than 1. Under $(\boldsymbol{\mathcal{S}_3})$, $\omega_G' \leq \mathtt{c}_3 \nu^{-1} \mathtt{r}_G n^{-1/2}$; see Lemma A.3. So, (2.5) means $\mathtt{r}_G^2 \ll n$. Moreover, definition (2.1) of $\mathtt{r}_G$ yields that $\mathtt{r}_G^2 \asymp \operatorname{tr}(D_G^{-2} V^2) = \mathbb{p}_G$, where $\mathbb{p}_G$ is the *effective dimension* of the problem. We conclude that the main properties of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ are valid under the condition $\mathbb{p}_G \ll n$ meaning sufficiently many observations per effective number of parameters.

2.2.5 The use of $\widetilde{D}_G^2$ instead of D_G^2

The penalized information matrix $D_G^2 = D_G^2(\boldsymbol{\upsilon}_G^*) = -\nabla^2 \mathbb{E} L_G(\boldsymbol{\upsilon}_G^*)$ plays an important role in our results. In particular, D_G describes the shape of the concentration set $\mathcal{A}_G = \boldsymbol{\upsilon}_G^* + \mathcal{U}_G$. However, this matrix is unavailable as it involves the unknown point $\boldsymbol{\upsilon}_G^*$. If the matrix function $\mathbb{F}(\boldsymbol{\upsilon})$ is locally constant in $\mathcal{A}_G$, one can replace $\boldsymbol{\upsilon}_G^*$ with its estimate $\widetilde{\boldsymbol{\upsilon}}_G$. Variability of $\mathbb{F}(\boldsymbol{\upsilon})$, or, equivalently, $\mathbb{F}_G(\boldsymbol{\upsilon}) = \mathbb{F}(\boldsymbol{\upsilon}) + G^2$ can be measured under the Fréchet smoothness of $f(\boldsymbol{\upsilon}) = \mathbb{E} L_G(\boldsymbol{\upsilon})$ by the value ω_G^+ from (A.6) with $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_G^*$, $D(\boldsymbol{\upsilon}) = D_G$, and $\mathtt{r} = \nu^{-1} \mathtt{r}_G$. Note that $\omega_G^+ \leq \tau_3 \mathtt{r}$ under $(\boldsymbol{\mathcal{T}_3^*})$.

Proposition 2.3. *Assume the conditions of Proposition 2.1 and let $\omega_G^+ \leq 1/2$; see (A.6). The random matrix $\widetilde{D}_G^2 = \mathbb{F}_G(\widetilde{\boldsymbol{\upsilon}}_G)$ fulfills on $\Omega(\mathtt{x})$ for any $\boldsymbol{u} \in \mathbb{R}^p$*

$$\begin{gathered}
\bigl\| D_G^{-1} \widetilde{D}_G^2 D_G^{-1} - \mathbb{I}_p \bigr\| \leq \omega_G^+ , \qquad \bigl\| D_G \widetilde{D}_G^{-2} D_G - \mathbb{I}_p \bigr\| \leq \frac{\omega_G^+}{1 - \omega_G^+} , \\
(1 - \omega_G^+) \, \| D_G \boldsymbol{u} \|^2 \leq \| \widetilde{D}_G \boldsymbol{u} \|^2 \leq (1 + \omega_G^+) \, \| D_G \boldsymbol{u} \|^2 .
\end{gathered} \tag{2.7}$$

Proof. The value $\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*$ belongs to $\mathcal{U}_G$ on $\Omega(\mathtt{x})$ and (2.7) follows from (A.7). □

2.3 Expansions and risk bounds under third-order smoothness

The results of Theorem 2.2 can be refined if the second order smoothness conditions (2.4) can be strengthened to the third order. Assume for $f(\boldsymbol{\upsilon}) = \mathbb{E}L(\boldsymbol{\upsilon})$ and all $\boldsymbol{u} \in \mathcal{U}_G$

$$
\begin{aligned}
\Bigl| f(\boldsymbol{\upsilon}^* + \boldsymbol{u}) - f(\boldsymbol{\upsilon}^*) - \langle \nabla f(\boldsymbol{\upsilon}^*), \boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{\upsilon}^*), \boldsymbol{u}^{\otimes 2} \rangle \Bigr| &\le \frac{\tau_3}{6} \| D_G \boldsymbol{u} \|^3 , \\
\Bigl| \langle \nabla f(\boldsymbol{\upsilon}^* + \boldsymbol{u}), \boldsymbol{u} \rangle - \langle \nabla f(\boldsymbol{\upsilon}^*), \boldsymbol{u} \rangle - \langle \nabla^2 f(\boldsymbol{\upsilon}^*), \boldsymbol{u}^{\otimes 2} \rangle \Bigr| &\le \frac{\tau_3}{2} \| D_G \boldsymbol{u} \|^3 .
\end{aligned}
\tag{2.8}
$$

2.3.1 Fisher and Wilks expansions. 3G case

The first result substantially improves the remainder in the Fisher and Wilks expansions of Theorem 2.2. It follows from Proposition A.9.

Theorem 2.4. *Assume* $(\boldsymbol{\zeta})$, $(\boldsymbol{\nabla\zeta})$, *and* $(\boldsymbol{\mathcal{C}_G})$. *Let* $\mathcal{U}_G$ *be given by* (2.3) *with* $\nu \le 2/3$ *and* (2.8) *hold for all* $\boldsymbol{u} \in \mathcal{U}_G$ *with* $\tau_3 \, \mathtt{r}_G / 2 < 1 - \nu$. *Then on* $\Omega(\mathtt{x})$

$$
-\frac{2\tau_3}{3} \, \| D_G^{-1} \nabla \zeta \|^3 \le 2 L_G(\widetilde{\boldsymbol{\upsilon}}_G) - 2 L_G(\boldsymbol{\upsilon}_G^*) - \| D_G^{-1} \nabla \zeta \|^2 \le \tau_3 \, \| D_G^{-1} \nabla \zeta \|^3 .
$$

Moreover, under $(\boldsymbol{\mathcal{T}_3^*})$

$$
\begin{aligned}
\bigl\| D_G \bigl(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^* \bigr) - D_G^{-1} \nabla \zeta \bigr\| &\le \frac{3\tau_3}{4} \, \| D_G^{-1} \nabla \zeta \|^2 , \\
\bigl\| D_G \bigl(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^* \bigr) \bigr\| &\le \| D_G^{-1} \nabla \zeta \| + \frac{3\tau_3}{4} \, \| D_G^{-1} \nabla \zeta \|^2 .
\end{aligned}
\tag{2.9}
$$

Remark 2.2. The presented results are meaningful under the condition $\tau_3 \, \| D_G^{-1} \nabla \zeta \| \ll 1$. A sufficient condition is $\tau_3 \, \mathtt{r}_G \ll 1$ because $\| D_G^{-1} \nabla \zeta \| \le \mathtt{r}_G$ on $\Omega(\mathtt{x})$.

2.3.2 Smoothness and bias. 3G bounds

Due to Proposition 2.1, the penalized MLE $\widetilde{\boldsymbol{\upsilon}}_G$ is in fact an estimator of the vector $\boldsymbol{\upsilon}_G^*$. However, $\boldsymbol{\upsilon}_G^*$ depends on penalization which introduces some bias. This section discusses whether one can use $\widetilde{\boldsymbol{\upsilon}}_G$ for estimating the underlying truth $\boldsymbol{\upsilon}^*$ defined as the maximizer of the expected log-likelihood: $\boldsymbol{\upsilon}^* = \operatorname{argmax}_{\boldsymbol{\upsilon}} \mathbb{E}L(\boldsymbol{\upsilon})$. First, we describe the bias $\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*$ induced by penalization. It is important to mention that the previous results about the properties of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$ require strong concavity of the expected log-likelihood function $\mathbb{E}L_G(\boldsymbol{\upsilon})$ at least in a vicinity of the point $\boldsymbol{\upsilon}_G^*$. In some sense, this strong concavity is automatically forced by the penalizing term in the definition of $\boldsymbol{\upsilon}_G^*$. However, the underlying truth $\boldsymbol{\upsilon}^* = \operatorname{argmax}_{\boldsymbol{\upsilon}} \mathbb{E}L(\boldsymbol{\upsilon})$ is the maximizer of the non-penalized expected log-likelihood, and the corresponding Hessian $\mathbb{F}(\boldsymbol{\upsilon}^*) = -\nabla^2 \mathbb{E}L(\boldsymbol{\upsilon}^*)$ can degenerate. This makes the evaluation of the bias more involved. To bypass this situation, we assume later in this section that the Hessian $\nabla^2 \mathbb{E} L_G(\boldsymbol{\upsilon})$ cannot change much in a reasonably large vicinity of $\boldsymbol{\upsilon}^*$. This allows to establish an accurate quadratic approximation of $f(\boldsymbol{\upsilon})$ and to evaluate the bias $\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*$. Proposition A.13 yields the following result.

Proposition 2.5. *Let* $\mathbb{D}_G^2 = \mathbb{F}_G(\boldsymbol{\upsilon}^*)$, $\nu \le 2/3$, *and* $\mathsf{b}_G = \|\mathbb{D}_G^{-1} G^2 \boldsymbol{\upsilon}^*\|$. *Assume* $(\mathcal{T}_3^*)$ *with* $\mathtt{r} = \nu^{-1} \mathsf{b}_G$ *and let* $\tau_3 \, \mathsf{b}_G \le 1/2$. *Then the bias* $\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*$ *fulfills*

$$
\begin{aligned}
\|\mathbb{D}_G(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*)\| &\le \nu^{-1} \, \mathsf{b}_G = \nu^{-1} \, \|\mathbb{D}_G^{-1} G^2 \boldsymbol{\upsilon}^*\| , \\
\|\mathbb{D}_G(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*) + \mathbb{D}_G^{-1} G^2 \boldsymbol{\upsilon}^*\| &\le \frac{3\tau_3}{4} \, \mathsf{b}_G^2 = \frac{3\tau_3}{4} \, \|\mathbb{D}_G^{-1} G^2 \boldsymbol{\upsilon}^*\|^2.
\end{aligned}
\tag{2.10}
$$

Corollary 2.6. *Assume the conditions of Proposition 2.5. Let also* Q *be a linear operator* $Q \colon \mathbb{R}^p \to \mathbb{R}^q$. *Then*

$$
\begin{aligned}
\|Q(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^* + \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*)\| &\le \|Q \, \mathbb{D}_G^{-1}\| \, \frac{3\tau_3}{4} \, \mathsf{b}_G^2 , \\
\|Q(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*)\| &\le \|Q \, \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*\| + \|Q \, \mathbb{D}_G^{-1}\| \, \frac{3\tau_3}{4} \, \mathsf{b}_G^2 .
\end{aligned}
\tag{2.11}
$$

The same bounds apply with $D_G^2 = \mathbb{F}_G(\boldsymbol{\upsilon}^*_G)$ *in place of* $\mathbb{D}_G^2 = \mathbb{F}_G(\boldsymbol{\upsilon}^*)$.

Proof. Obviously

$$
\begin{aligned}
\|Q(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*)\| &\le \|Q \, \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*\| + \|Q(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^* + \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*)\| \\
&\le \|Q \, \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*\| + \|Q \, \mathbb{D}_G^{-1}\| \; \|\mathbb{D}_G(\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^* + \mathbb{D}_G^{-2} G^2 \boldsymbol{\upsilon}^*)\|
\end{aligned}
$$

and (2.11) follow from (2.10). The last statement is due to Remark A.2. □

2.3.3 Loss and risk of the pMLE. 3G-bounds

Now we combine the previous results about the stochastic term $\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*_G$ and the bias term $\boldsymbol{\upsilon}^*_G - \boldsymbol{\upsilon}^*$ to obtain sharp bounds on the loss and risk of the pMLE $\widetilde{\boldsymbol{\upsilon}}_G$. Everywhere in this section, we fix $\Omega(\mathtt{x})$ as the random set from $(\boldsymbol{\nabla\zeta})$ on which $\|\boldsymbol{\xi}_G\| \le \mathtt{r}_G$ with $\mathbb{P}\bigl(\Omega(\mathtt{x})\bigr) \ge 1 - 3\mathrm{e}^{-\mathtt{x}}$; see condition (2.1) of Proposition 2.1.

Theorem 2.7. *Assume the conditions of Theorem 2.4 and Proposition 2.5. With* $\mathsf{b}_G = \|D_G^{-1} G^2 \boldsymbol{\upsilon}^*\|$ *and* $\mathtt{r}_G$ *from (2.2), it holds on* $\Omega(\mathtt{x})$

$$
\begin{aligned}
\|D_G \, \{\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^* - D_G^{-2}(\nabla\zeta - G^2 \boldsymbol{\upsilon}^*)\}\| &\le \frac{3\tau_3}{4} \, \bigl(\|D_G^{-1} \nabla\zeta\|^2 + \mathsf{b}_G^2\bigr), \\
\|D_G \, (\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)\| &\le \mathtt{r}_G + \mathsf{b}_G + \frac{3\tau_3}{4} \, \bigl(\mathtt{r}_G^2 + \mathsf{b}_G^2\bigr).
\end{aligned}
\tag{2.12}
$$

Furthermore, with $\mathbb{p}_G = \mathbb{E}\|D_G^{-1}\nabla\zeta\|^2$ *, define the approximating risk* $\mathscr{R}_G$ *by*

$$\mathscr{R}_G \stackrel{\text{def}}{=} \mathbb{E}\{\|D_G^{-1}(\nabla\zeta - G^2\boldsymbol{v}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\}.$$

Then $\mathscr{R}_G \le \mathbb{p}_G + \mathsf{b}_G^2$ *and*

$$\mathbb{E}\bigl\{\|D_G(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\| \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \le \sqrt{\mathscr{R}_G} + \frac{3\tau_3}{4}\bigl(\mathbb{p}_G + \mathsf{b}_G^2\bigr). \tag{2.13}$$

Finally, introduce α_G *by*

$$\alpha_G \stackrel{\text{def}}{=} \frac{(3/4)\tau_3\,(\mathtt{r}_G\sqrt{\mathbb{p}_G} + \mathsf{b}_G^2)}{\sqrt{\mathscr{R}_G}}.$$

If $\alpha_G < 1$ *then*

$$(1-\alpha_G)^2\,\mathscr{R}_G \le \mathbb{E}\bigl\{\|D_G(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \le (1+\alpha_G)^2\,\mathscr{R}_G. \tag{2.14}$$

Proof. Define $\boldsymbol{\varepsilon}_G \stackrel{\text{def}}{=} D_G\,\{\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^* - D_G^{-2}(\nabla\zeta - G^2\boldsymbol{v}^*)\}$ and $\boldsymbol{\xi}_G = D_G^{-1}\nabla\zeta$. It follows from (2.9) of Theorem 2.4 that on $\Omega(\mathtt{x})$

$$\|\boldsymbol{\varepsilon}_G\| \le \frac{3\tau_3}{4}\,\bigl(\|\boldsymbol{\xi}_G\|^2 + \mathsf{b}_G^2\bigr).$$

This and (2.10) imply (2.12). Bound (2.13) follows from (2.12) in view of $\mathbb{E}\|\boldsymbol{\xi}_G\|^2 \le \mathbb{E}\|\boldsymbol{\xi}_G\|^2 = \mathbb{p}_G$. Now we check (2.14). As $\|\boldsymbol{\xi}_G\| \le \mathtt{r}_G$ on $\Omega(\mathtt{x})$, it holds

$$\begin{aligned}\mathbb{E}\bigl\{\|\boldsymbol{\varepsilon}_G\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} &\le \Bigl(\frac{3\tau_3}{4}\Bigr)^2\,\mathbb{E}\bigl(\mathtt{r}_G\,\|\boldsymbol{\xi}_G\| + \mathsf{b}_G^2\bigr)^2 \\ &\le \Bigl(\frac{3\tau_3}{4}\Bigr)^2\bigl(\mathtt{r}_G\sqrt{\mathbb{E}\|\boldsymbol{\xi}_G\|^2} + \mathsf{b}_G^2\bigr)^2 = \Bigl(\frac{3\tau_3}{4}\Bigr)^2\,\bigl(\mathtt{r}_G\,\sqrt{\mathbb{p}_G} + \mathsf{b}_G^2\bigr)^2 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\mathbb{E}\bigl\{\|D_G\,(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} &= \mathbb{E}\bigl\{\|D_G^{-1}(\nabla\zeta - G^2\boldsymbol{v}^*) + \boldsymbol{\varepsilon}_G\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \\ &\le \Bigl(\sqrt{\mathbb{E}\{\|D_G^{-1}(\nabla\zeta - G^2\boldsymbol{v}^*)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\}} + \sqrt{\mathbb{E}\{\|\boldsymbol{\varepsilon}_G\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\}}\Bigr)^2 \\ &\le \Bigl\{\sqrt{\mathscr{R}_G} + \frac{3\tau_3}{4}\,\bigl(\mathtt{r}_G\sqrt{\mathbb{p}_G} + \mathsf{b}_G^2\bigr)\Bigr\}^2 \le (1+\alpha_G)^2(\mathbb{p}_G + \mathsf{b}_G^2). \end{aligned} \tag{2.16}$$

The lower bound in (2.14) is proved similarly. □

Remark 2.3. The results of Theorem 2.7 implicitly assume that $\alpha_G \lesssim \tau_3\,(\mathtt{r}_G + \mathsf{b}_G) = o(1)$. Then bound (2.14) yields classical bias-variance decomposition

$$\mathbb{E}\bigl\{\|D_G\,(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} = \mathscr{R}_G\bigl\{1 + o(1)\bigr\}. \tag{2.17}$$

With $\tau_3 \asymp n^{-1/2}$ as in $(\mathcal{S}_3)$ and $\mathtt{r}_G \approx \sqrt{\mathbb{p}_G}$, the condition $\tau_3 \, \mathtt{r}_G = o(1)$ can be restated as the critical dimension condition $\mathbb{p}_G \ll n$. Our bound is sharp in the sense that for the special case of linear models, (2.17) becomes equality. Under the so-called "small bias" condition $\mathsf{b}_G^2 = \|D_G^{-1} G^2 \boldsymbol{v}^*\|^2 \ll \mathbb{p}_G$, the impact of the bias induced by penalization is negligible. The relation $\|D_G^{-1} G^2 \boldsymbol{v}^*\|^2 \asymp \mathbb{p}_G$ is usually referred to as "bias-variance trade-off".

Similarly to Corollary 2.6, the bound can be easily extended to any mapping $Q \colon \mathbb{R}^p \to \mathbb{R}^q$ in place of D_G. Define $\mathbb{B}_Q = \operatorname{Var}(Q D_G^{-2} \nabla \zeta) = Q D_G^{-2} \operatorname{Var}(\nabla \zeta) D_G^{-2} Q^\top$,

$$
\mathbb{p}_Q \stackrel{\text{def}}{=} \operatorname{tr} \mathbb{B}_Q \, , \qquad \mathtt{r}_Q \stackrel{\text{def}}{=} \sqrt{\operatorname{tr} \mathbb{B}_Q} + \sqrt{2\mathtt{x} \|\mathbb{B}_Q\|} \, . \tag{2.18}
$$

Conditions ensuring $(\nabla \zeta)$ also imply

$$
\mathbb{P}\bigl(\|Q D_G^{-2} \nabla \zeta\| > \mathtt{r}_Q \bigr) \le 3 \mathrm{e}^{-\mathtt{x}}.
$$

Later we assume without significant loss of generality, that $\|Q D_G^{-2} \nabla \zeta\| \le \mathtt{r}_Q$ on the same set $\Omega(\mathtt{x})$ shown up in $(\nabla \zeta)$.

Theorem 2.8. *Assume the conditions of Theorem 2.7. For any linear mapping $Q \colon \mathbb{R}^p \to \mathbb{R}^q$, with $\mathsf{b}_Q = \|Q D_G^{-2} G^2 \boldsymbol{v}^*\|$ and $\mathbb{p}_Q, \mathtt{r}_Q$ from (2.18), it holds on $\Omega(\mathtt{x})$*

$$
\begin{aligned}
\|Q(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\| &\le \|Q D_G^{-2} (\nabla \zeta - G^2 \boldsymbol{v}^*)\| + \|Q D_G^{-1}\| \, \frac{3\tau_3}{4} \bigl(\|D_G^{-1} \nabla \zeta\|^2 + \mathsf{b}_G^2 \bigr) \\
&\le \mathtt{r}_Q + \mathsf{b}_Q + \|Q D_G^{-1}\| \, \frac{3\tau_3}{4} \bigl(\mathtt{r}_G^2 + \mathsf{b}_G^2 \bigr) \, .
\end{aligned} \tag{2.19}
$$

Also, define

$$
\mathscr{R}_Q \stackrel{\text{def}}{=} \mathbb{E}\{ \|Q D_G^{-2} (\nabla \zeta - G^2 \boldsymbol{v}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})} \} \, .
$$

Then $\mathscr{R}_Q \le \mathbb{p}_Q + \mathsf{b}_Q^2$ and it holds

$$
\mathbb{E}\bigl\{ \|Q(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\| \, \mathbb{1}_{\Omega(\mathtt{x})} \bigr\} \le \mathscr{R}_Q^{1/2} + \|Q D_G^{-1}\| \, \frac{3\tau_3}{4} \bigl(\mathbb{p}_G + \mathsf{b}_G^2 \bigr) \, \|Q D_G^{-1}\| \, . \tag{2.20}
$$

Let α_Q be given by

$$
\alpha_Q \stackrel{\text{def}}{=} \frac{\|Q D_G^{-1}\| \, (3/4) \tau_3 \, (\mathtt{r}_G \sqrt{\mathbb{p}_G} + \mathsf{b}_G^2)}{\sqrt{\mathscr{R}_Q}} \, . \tag{2.21}
$$

If $\alpha_Q < 1$ then

$$
(1 - \alpha_Q)^2 \mathscr{R}_Q \le \mathbb{E}\bigl\{ \|Q \, (\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})} \bigr\} \le (1 + \alpha_Q)^2 \mathscr{R}_Q \, . \tag{2.22}
$$

Proof. We follow the line of the proof of Theorem 2.7. It holds by (2.9) and (2.11)

$$\begin{aligned} \|Q(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^* - D_G^{-2}\nabla\zeta\| &\leq \|QD_G^{-1}\| \, \frac{3\tau_3}{4} \, \|D_G^{-1}\nabla\zeta\|^2 \, , \\ \|Q(\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^* + D_G^{-2}G^2\boldsymbol{\upsilon}^*)\| &\leq \|QD_G^{-1}\| \, \frac{3\tau_3}{4} \, \mathsf{b}_G^2 \, , \end{aligned}$$

and hence

$$\|Q(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^* - D_G^{-2}\nabla\zeta + D_G^{-2}G^2\boldsymbol{\upsilon}^*)\| \leq \|QD_G^{-1}\| \, \frac{3\tau_3}{4} \, \bigl(\|D_G^{-1}\nabla\zeta\|^2 + \mathsf{b}_G^2\bigr)$$

yielding (2.19) and (2.20). Further, define

$$\boldsymbol{\varepsilon}_G \stackrel{\text{def}}{=} Q\{\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^* - D_G^{-2}(\nabla\zeta - G^2\boldsymbol{\upsilon}^*)\}.$$

Similarly to (2.15)

$$\mathbb{E}\bigl\{\|\boldsymbol{\varepsilon}_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \leq \|QD_G^{-1}\|^2 \Bigl(\frac{3\tau_3}{4}\Bigr)^2 \bigl(\mathtt{r}_G\sqrt{\mathbb{p}_G} + \mathsf{b}_G^2\bigr)^2 \leq \alpha_Q^2 \, \mathscr{R}_Q \, ,$$

and as in (2.16)

$$\mathbb{E}\bigl\{\|Q\,(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} = \mathbb{E}\bigl\{\|QD_G^{-2}(\nabla\zeta - G^2\boldsymbol{\upsilon}^*) + \boldsymbol{\varepsilon}_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \leq (1 + \alpha_Q)^2 \, \mathscr{R}_Q \, .$$

This yields (2.22). □

2.3.4 Critical dimension. 3G case

The results of Theorem 2.8 are meaningful only if α_G in (2.21) is small. This condition crucially depends on the operator Q. For the case $Q = D_G$ as in Theorem 2.7, it follows from the relation $\mathbb{p}_G \ll n$. However, in some other situations like semiparametric estimation when Q projects onto a low dimensional target component, (2.21) requires $\mathbb{p}_G^2 \ll n$ which can be very restrictive. An interesting question about a further improvement of the error term in (2.19) will be discussed in the next section.

2.4 Risk bounds under fourth-order smoothness

This section explains how the accuracy of the expansions for pMLE can be improved and the critical dimension condition can be relaxed under fourth-order smoothness of $f_G(\boldsymbol{\upsilon}) = \mathbb{E}L_G(\boldsymbol{\upsilon})$.

2.4.1 Fisher and Wilks expansions

Consider the third-order tensor $\mathcal{T}(\boldsymbol{u}) = \frac{1}{6}\langle \nabla^3 f(\boldsymbol{v}_G^*), \boldsymbol{u}^{\otimes 3}\rangle$. Let $\nabla\mathcal{T}(\boldsymbol{u}) = \frac{1}{2}\langle \nabla^3 f(\boldsymbol{v}_G^*), \boldsymbol{u}^{\otimes 2}\rangle$ be its gradient. Define a random vector $\boldsymbol{n}_G$ by

$$\boldsymbol{n}_G = D_G^{-2}\bigl\{\nabla\zeta + \nabla\mathcal{T}(D_G^{-2}\nabla\zeta)\bigr\}. \tag{2.23}$$

The next result shows that by light modification of the term $D_G^{-2}\nabla\zeta$ to $\boldsymbol{n}_G$ we can substantially improve the accuracy of the Fisher expansion (2.9).

Proposition 2.9. *Suppose* $(\boldsymbol{\zeta})$, $(\boldsymbol{\mathcal{C}_G})$, *and* $(\boldsymbol{\nabla\zeta})$. *Let* $(\boldsymbol{\mathcal{T}_3^*})$ *and* $(\boldsymbol{\mathcal{T}_4^*})$ *hold at* $\boldsymbol{v}_G^*$ *with* $D^2 = \mathbb{F}_G$ *and* $\mathtt{r} = \nu^{-1}\mathtt{r}_G$ *for* $\mathtt{r}_G$ *from* (2.2) *and* $\nu = 2/3$. *Finally, let* $\tau_3\,\nu^{-1}\mathtt{r}_G \le 1-\nu$. *With* $\boldsymbol{n}_G$ *from* (2.23), *it holds on* $\Omega(\mathtt{x})$

$$\|D_G(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}_G^* - \boldsymbol{n}_G)\| \le \Bigl(\frac{\tau_4}{3} + \tau_3^2\Bigr)\,\|D_G^{-1}\nabla\zeta\|^3, \tag{2.24}$$

$$\|D_G\,\boldsymbol{n}_G - D_G^{-1}\nabla\zeta\| = \|D_G^{-1}\nabla\mathcal{T}(D_G^{-2}\nabla\zeta)\| \le \frac{\tau_3}{2}\,\|D_G^{-1}\nabla\zeta\|^2, \tag{2.25}$$

and

$$\begin{aligned}&\bigl|L_G(\widetilde{\boldsymbol{v}}_G) - L_G(\boldsymbol{v}_G^*) - \frac{1}{2}\|D_G^{-1}\nabla\zeta\|^2 - \mathcal{T}(D_G^{-2}\nabla\zeta)\bigr|\\ &\quad\le \frac{\tau_4 + 7\tau_3^2}{16}\|D_G^{-1}\nabla\zeta\|^4 + \frac{(\tau_4+3\tau_3^2)^2}{5}\,\|D_G^{-1}\nabla\zeta\|^6.\end{aligned}$$

Proof. See Proposition A.11 with $\boldsymbol{A} = \nabla\zeta$, $\mathbb{F} = D_G^2$, and $\boldsymbol{a} = D_G^{-1}\nabla\zeta$. □

Expansion similar to (2.24) applies to the bias term due to Proposition A.14. Define

$$\boldsymbol{m}_G = D_G^{-2}G^2\boldsymbol{v}^* + D_G^{-2}\nabla\mathcal{T}(D_G^{-2}G^2\boldsymbol{v}^*). \tag{2.26}$$

Proposition 2.10. *Let* $\mathsf{b}_G = \|D_G^{-1}\,G^2\boldsymbol{v}^*\|$. *Assume* $(\boldsymbol{\mathcal{T}_3^*})$ *and* $(\boldsymbol{\mathcal{T}_4^*})$ *at* $\boldsymbol{v}_G^*$ *with* $\mathtt{r} = \nu^{-1}\mathsf{b}_G$, *and let* τ_3 *satisfy* $\tau_3\,\mathsf{b}_G \le 1/2$. *With* $\boldsymbol{m}_G$ *from* (2.26), *it holds*

$$\|D_G(\boldsymbol{v}_G^* - \boldsymbol{v}^* + \boldsymbol{m}_G)\| \le \Bigl(\frac{\tau_4}{3} + \tau_3^2\Bigr)\,\mathsf{b}_G^3.$$

$$\|D_G\,\boldsymbol{m}_G - D_G^{-1}G^2\boldsymbol{v}^*\| \le \frac{\tau_3}{2}\mathsf{b}_G^2. \tag{2.27}$$

2.4.2 4G risk bounds

Putting together the results on the stochastic component $\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}_G^*$ and on the bias $\boldsymbol{v}_G^* - \boldsymbol{v}^*$ yields the bound on the loss and risk of the estimator $\widetilde{\boldsymbol{v}}_G$. First, we consider the approximation $\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^* \approx \boldsymbol{n}_G - \boldsymbol{m}_G$.

Theorem 2.11. *Under the conditions of Propositions 2.9 and 2.10, it holds on* $\Omega(\mathtt{x})$ *for any linear mapping* Q

$$\|Q\,(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^* - \boldsymbol{n}_G + \boldsymbol{m}_G)\| \le \|QD_G^{-1}\| \Bigl(\frac{\tau_4}{3} + \tau_3^2\Bigr) \bigl(\|D_G^{-1}\nabla\zeta\|^3 + \mathsf{b}_G^3\bigr), \tag{2.28}$$

$$\|Q\{\boldsymbol{n}_G - \boldsymbol{m}_G - D_G^{-2}(\nabla\zeta - G^2\boldsymbol{\upsilon}^*)\}\| \le \frac{\tau_3}{2}\,\|QD_G^{-1}\|\,\bigl(\|D_G^{-1}\nabla\zeta\|^2 + \mathsf{b}_G^2\bigr).$$

Also

$$\begin{aligned}\boldsymbol{E}\bigl\{\|Q\,(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)\|\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} &\le \boldsymbol{E}\bigl\{\|Q(\boldsymbol{n}_G - \boldsymbol{m}_G)\|\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \\ &\quad + \|QD_G^{-1}\| \Bigl(\frac{\tau_4}{3} + \tau_3^2\Bigr) \bigl(\mathtt{r}_G\,\mathbb{p}_G + \mathsf{b}_G^3\bigr)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}&\Bigl|\boldsymbol{E}\bigl\{\|Q(\boldsymbol{n}_G - \boldsymbol{m}_G)\|\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} - \boldsymbol{E}\bigl\{\|QD_G^{-2}\,\nabla\zeta - QD_G^{-2}\,G^2\boldsymbol{\upsilon}^*\|\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\}\Bigr| \\ &\quad \le \|QD_G^{-1}\|\,\frac{\tau_3}{2}\,\bigl(\mathbb{p}_G + \mathsf{b}_G^2\bigr).\end{aligned}$$

Proof. All the expansions for the loss $Q\,(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)$ follow directly from Propositions 2.9 and 2.10. Also

$$\boldsymbol{E}\bigl\{\|D_G^{-1}\nabla\zeta\|^3\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \le \mathtt{r}_G\boldsymbol{E}\|D_G^{-1}\nabla\zeta\|^2 = \mathtt{r}_G\,\mathbb{p}_G$$

yielding the ℓ_1-risk bound (2.29). □

Now we study the quadratic risk of $\widetilde{\boldsymbol{\upsilon}}_G$. Define

$$\mathscr{R}_Q \stackrel{\text{def}}{=} \boldsymbol{E}\bigl\{\|QD_G^{-2}(\nabla\zeta - G^2\boldsymbol{\upsilon}^*)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\}. \tag{2.30}$$

$$\mathscr{R}_{Q,2} \stackrel{\text{def}}{=} \boldsymbol{E}\bigl\{\|Q(\boldsymbol{n}_G - \boldsymbol{m}_G)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\}. \tag{2.31}$$

Theorem 2.12. *Assume the conditions of Propositions 2.9 and 2.10. For a linear mapping* Q *and* $\mathscr{R}_{Q,2}$ *from (2.31), it holds*

$$\bigl(1 - \alpha_{Q,2}\bigr)^2\mathscr{R}_{Q,2} \le \boldsymbol{E}\bigl\{\|Q\,(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)\|^2\,\mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \le \bigl(1 + \alpha_{Q,2}\bigr)^2\mathscr{R}_{Q,2} \tag{2.32}$$

provided that

$$\alpha_{Q,2} \stackrel{\text{def}}{=} \frac{\|QD_G^{-1}\|\,(\tau_4/3 + \tau_3^2)\,(\mathtt{r}_G^3 + \mathsf{b}_G^3)}{\sqrt{\mathscr{R}_{Q,2}}} < 1.$$

If another constant $\alpha_{Q,1} < 1$ *ensures*

$$\|QD_G^{-1}\|\,\frac{\tau_3}{2}\,\bigl(\mathtt{r}_G\sqrt{\mathbb{p}_G} + \mathsf{b}_G^2\bigr) \le \alpha_{Q,1}\sqrt{\mathscr{R}_Q} \tag{2.33}$$

with $\mathscr{R}_Q$ *from* (2.30) *then*

$$\mathscr{R}_Q (1 - \alpha_{Q,1})^2 \le \mathscr{R}_{Q,2} \le \mathscr{R}_Q (1 + \alpha_{Q,1})^2 . \tag{2.34}$$

Proof. Define $\boldsymbol{\varepsilon}_Q = Q(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^* - \boldsymbol{n}_G + \boldsymbol{m}_G)$. It can be bounded by (2.28): $\|\boldsymbol{\varepsilon}_G\| \, \mathbb{1}_{\Omega(\mathtt{x})} \le \alpha_{Q,2}$. Therefore,

$$\mathbb{E}\bigl\{\|Q\,(\widetilde{\boldsymbol{v}}_G - \boldsymbol{v}^*)\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} = \mathbb{E}\bigl\{\|Q(\boldsymbol{n}_G - \boldsymbol{m}_G) + \boldsymbol{\varepsilon}_G\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} \le \bigl(\sqrt{\mathscr{R}_{Q,2}} + \alpha_{Q,2}\bigr)^2 ,$$

and (2.32) follows. Further, denote

$$\begin{aligned} \wp_Q &\stackrel{\text{def}}{=} Q D_G^{-2} (\nabla \zeta - G^2 \boldsymbol{v}^*), \\ \delta_Q &\stackrel{\text{def}}{=} Q(D_G^{-2} \nabla \zeta - \boldsymbol{n}_G) - Q(D_G^{-2} G^2 \boldsymbol{v}^* - \boldsymbol{m}_G). \end{aligned}$$

By definition, $\mathscr{R}_Q = \mathbb{E}\bigl\{\|\wp_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\}$, $\mathscr{R}_{Q,2} = \mathbb{E}\bigl\{\|\wp_Q + \delta_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\}$, and

$$\mathscr{R}_{Q,2} - \mathscr{R}_Q \, = \, \mathbb{E}\bigl\{\|\delta_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\} + 2\mathbb{E}\bigl\{\langle \wp_Q, \delta_Q \rangle \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\}.$$

Also (2.25) and (2.27) imply

$$\|\delta_Q\| \, \mathbb{1}_{\Omega(\mathtt{x})} \le \|Q D_G^{-1}\| \, \frac{\tau_3}{2} \bigl(\|D_G^{-1} \nabla \zeta\|^2 + \mathsf{b}_G^2\bigr) \, \mathbb{1}_{\Omega(\mathtt{x})} \le \|Q D_G^{-1}\| \, \frac{\tau_3}{2} \bigl(\mathtt{r}_G \|D_G^{-1} \nabla \zeta\| + \mathsf{b}_G^2\bigr) ,$$

and by (2.33)

$$\begin{aligned} \sqrt{\mathbb{E}\bigl(\|\delta_Q\|^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr)} &\le \|Q D_G^{-1}\| \, \frac{\tau_3}{2} \sqrt{\mathbb{E}\bigl\{\bigl(\mathtt{r}_G \|D_G^{-1} \nabla \zeta\| + \mathsf{b}_G^2\bigr)^2 \, \mathbb{1}_{\Omega(\mathtt{x})}\bigr\}} \\ &\le \|Q D_G^{-1}\| \, \frac{\tau_3}{2} \bigl(\mathtt{r}_G \sqrt{\mathbb{p}_G} + \mathsf{b}_G^2\bigr) \le \alpha_{Q,1} \sqrt{\mathscr{R}_Q} \, . \end{aligned}$$

This easily yields (2.34). □

2.4.3 Critical dimension. 4G case

The results of Theorem 2.11 and 2.12 enable us to improve the issue of *critical dimension*. For simplicity, let $\|Q D_G^{-1}\| = 1$. Then the derived bounds are meaningful if

$$\Bigl(\frac{\tau_4}{3} + \tau_3^2\Bigr) \bigl(\mathtt{r}_G \, \mathbb{p}_G + \mathsf{b}_G^3\bigr) \, = \, o(1).$$

Assume $\tau_4 \asymp 1/n$ and $\tau_3^2 \asymp 1/n$. As $\mathtt{r}_G^2 \approx \mathbb{p}_G$, we obtain the critical dimension condition $\mathbb{p}_G^{3/2} \ll n$ which is much weaker than $\mathbb{p}_G^3 \ll n$. Condition (2.33) ensuring equivalence of $\mathscr{R}_{Q,2}$ and $\mathscr{R}_Q$ requires $\tau_3 \, \mathbb{p}_G \ll \mathscr{R}_Q$. The target risk $\mathscr{R}_Q$ corresponds to the target effective dimension after the mapping Q and it can be a fixed value. Therefore, the mentioned condition requires $\mathbb{p}_G \ll 1/\tau_3$ or $\mathbb{p}_G \ll n^{1/2}$ as in 3G case.

3 Examples

This section illustrates the general notions on the particular examples including logistic regression, log-density estimation, and precision matrix estimation. We mainly check the general conditions. This enables us to apply the results of Section 2.

3.1 Anisotropic logistic regression

This section considers a popular logistic regression model. It is widely used e.g. in binary classification in machine learning for binary classification or in binary response models in econometrics. The results presented here can be viewed as an extension of Spokoiny and Panov (2021) and Spokoiny (2023a).

Suppose we are given a vector of independent observations/labels $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^\top$ and a set of the corresponding feature vectors $\boldsymbol{\Psi}_i \in I\!R^p$. Each binary label Y_i is modelled as a Bernoulli random variable with the parameter $\theta_i^* = I\!P(Y_i = 1)$. Logistic regression reduces this model to linear regression for the canonical parameter $\upsilon_i^* = \log \frac{\theta_i^*}{1-\theta_i^*}$ in the form $\upsilon_i^* = \langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}^* \rangle$, where $\boldsymbol{\upsilon}$ is the parameter vector in $I\!R^p$. The corresponding log-likelihood reads

$$L(\boldsymbol{\upsilon}) = \sum_{i=1}^{n} \Bigl\{ Y_i \, \langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon} \rangle - \phi\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon} \rangle \bigr) \Bigr\}$$

with $\phi(\upsilon) = \log\bigl(1 + \mathrm{e}^{\upsilon}\bigr)$. For simplicity we assume that the $\boldsymbol{\Psi}_i$'s are deterministic, otherwise, we condition on the design.

A penalized MLE $\widetilde{\boldsymbol{\upsilon}}_G$ is defined by maximization of the penalized log-likelihood $L_G(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \| G \boldsymbol{\upsilon} \|^2/2$ for the quadratic penalty $\| G \boldsymbol{\upsilon} \|^2/2$:

$$\widetilde{\boldsymbol{\upsilon}}_G = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in I\!R^p} L_G(\boldsymbol{\upsilon}).$$

The truth and the penalized truth are defined via the expected log-likelihood

$$\begin{aligned} \boldsymbol{\upsilon}^* &= \operatorname*{argmax}_{\boldsymbol{\upsilon} \in I\!R^p} I\!E L(\boldsymbol{\upsilon}), \\ \boldsymbol{\upsilon}_G^* &= \operatorname*{argmax}_{\boldsymbol{\upsilon} \in I\!R^p} I\!E L_G(\boldsymbol{\upsilon}). \end{aligned}$$

The Fisher matrix $I\!F(\boldsymbol{\upsilon})$ at $\boldsymbol{\upsilon}$ is given by

$$I\!F(\boldsymbol{\upsilon}) = \sum_{i=1}^{n} \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon} \rangle \bigr) \, \boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^\top, \qquad \phi''(\upsilon) = \frac{\mathrm{e}^{\upsilon}}{(1 + \mathrm{e}^{\upsilon})^2} \, . \tag{3.1}$$

We also write $D_G^2(\boldsymbol{\upsilon}) = I\!F_G(\boldsymbol{\upsilon}) = I\!F(\boldsymbol{\upsilon}) + G^2$ and denote $D_G^2 = D_G^2(\boldsymbol{\upsilon}_G^*) = I\!F(\boldsymbol{\upsilon}_G^*) + G^2$.

Now we check of general conditions from Section 2. Convexity of $\phi(\cdot)$ yields concavity of $L(\boldsymbol{\upsilon})$ and thus, $(\mathcal{C}_G)$. Let

$$\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - I\!E L(\boldsymbol{\upsilon}) = \sum_{i=1}^{n} \bigl(Y_i - I\!E Y_i\bigr) \langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon} \rangle$$

be the stochastic component of $L(\boldsymbol{\upsilon})$. It is obviously linear in $\boldsymbol{\upsilon}$ with

$$\nabla\zeta = \sum_{i=1}^{n} \bigl(Y_i - I\!E Y_i\bigr) \boldsymbol{\Psi}_i \, .$$

Therefore, (ζ) is granted by construction. Further, with $Y_i \sim \text{Bernoulli}(\theta_i^*)$

$$\operatorname{Var}(\nabla\zeta) = \sum_{i=1}^{n} \operatorname{Var}(Y_i) \, \boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^{\top} = \sum_{i=1}^{n} \theta_i^* (1 - \theta_i^*) \, \boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^{\top} . \tag{3.2}$$

If $\theta_i^* = \mathrm{e}^{\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}^* \rangle} / (1 + \mathrm{e}^{\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}^* \rangle})$, then $\operatorname{Var}(\nabla\zeta) = I\!\!F(\boldsymbol{\upsilon}^*)$; see (3.1).

Lemma 3.1. *Let $V^2 \geq 2 \operatorname{Var}(\nabla\zeta)$; see (3.2), and $D_G^2 = I\!\!F_G(\boldsymbol{\upsilon}_G^*)$; see (3.1). Define*

$$\mathbb{p}_G = \operatorname{tr}\bigl(D_G^{-2} V^2\bigr), \qquad \lambda_G = \| D_G^{-2} V^2 \| . \tag{3.3}$$

Let $\varkappa_0 > 0$ be such that

$$\max_{i \leq n} \| V^{-1} \boldsymbol{\Psi}_i \| \leq \varkappa_0 \leq 0.5 \sqrt{\lambda_G / \mathbb{p}_G} \, . \tag{3.4}$$

Then $(\boldsymbol{\nabla\zeta})$ is fulfilled with for all $\mathtt{x}$ such that $\sqrt{2\mathtt{x}} \leq 1/(2\varkappa_0) - \sqrt{\mathbb{p}_G / \lambda_G}$.

Proof. Let $\mathtt{g} = \log(2)/\varkappa_0$ and $\mathtt{x}_c$ be given by (B.50). By Theorem B.46,

$$I\!\!P\bigl(\| D_G^{-1} \nabla\zeta \| \geq \sqrt{\mathbb{p}_G} + \sqrt{2\mathtt{x} \, \lambda_G}\bigr) \leq 3\mathrm{e}^{-\mathtt{x}}$$

for all $\mathtt{x} \leq \mathtt{x}_c$, and by (B.52), $\sqrt{\mathtt{x}_c} \geq \mathtt{g}/2 - \sqrt{\mathbb{p}_G / (2\lambda_G)}$. This yields the assertion. $\square$

For checking the smoothness conditions $(\mathcal{T}_3^*)$ and $(\mathcal{T}_4^*)$, we need one more condition on regularity of the design $\boldsymbol{\Psi}_1, \ldots, \boldsymbol{\Psi}_n$. It assumes a point $\boldsymbol{\upsilon} \in \varUpsilon$ to be fixed. For any vector $\boldsymbol{u}$, it holds with $w_i(\boldsymbol{\upsilon}) = \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon} \rangle\bigr)$

$$\sum_{i=1}^{n} \langle \boldsymbol{\Psi}_i, \boldsymbol{u} \rangle^2 w_i(\boldsymbol{\upsilon}) = \| D(\boldsymbol{\upsilon}) \boldsymbol{u} \|^2;$$

see (3.1). Later we assume the following condition.

($\boldsymbol{\Psi}$) For some constants $\varkappa$ and $\varkappa_0$ and a matrix $\mathbb{D}^2 = \mathbb{D}^2(\boldsymbol{\upsilon})$ with $\mathbb{D}^2(\boldsymbol{\upsilon}) \le D_G^2(\boldsymbol{\upsilon})$

$$\max_{i \le n} \|\mathbb{D}^{-1}\boldsymbol{\Psi}_i\| \le \varkappa_0 \, , \tag{3.5}$$

$$\sum_{i=1}^{n} \langle \boldsymbol{\Psi}_i, \boldsymbol{z} \rangle^4 \, w_i(\boldsymbol{\upsilon}) \le \varkappa^2 \, \|\mathbb{D}\boldsymbol{z}\|^4, \qquad \boldsymbol{z} \in I\!R^p. \tag{3.6}$$

By (3.5)

$$\begin{aligned}
\sum_{i=1}^{n} \langle \boldsymbol{\Psi}_i, \boldsymbol{z} \rangle^4 \, w_i(\boldsymbol{\upsilon}) &= \sum_{i=1}^{n} \langle \boldsymbol{\Psi}_i, \boldsymbol{z} \rangle^2 \, \langle \mathbb{D}^{-1}\boldsymbol{\Psi}_i, \mathbb{D}\boldsymbol{z} \rangle^2 \, w_i(\boldsymbol{\upsilon}) \\
&\le \varkappa_0^2 \|\mathbb{D}\boldsymbol{z}\|^2 \sum_{i=1}^{n} \langle \boldsymbol{\Psi}_i, \boldsymbol{z} \rangle^2 \, w_i(\boldsymbol{\upsilon}) = \varkappa_0^2 \|\mathbb{D}\boldsymbol{z}\|^4.
\end{aligned}$$

Therefore, $\varkappa \le \varkappa_0$. However, it might be that $\varkappa \ll \varkappa_0$. As the final accuracy bound is given in terms of $\varkappa$, we keep the definition (3.6). The constant $\varkappa_0$ in (3.5) and in (3.4) can be different. However, we expect that $V^2 \asymp \mathbb{D}^2(\boldsymbol{\upsilon})$ for $\boldsymbol{\upsilon}$ close to $\boldsymbol{\upsilon}^*$ and use the same notation.

Lemma 3.2. *Assume* **($\boldsymbol{\Psi}$)** *for* $\boldsymbol{\upsilon} \in \Upsilon$. *Let a radius* $\mathtt{r} = \mathtt{r}(\boldsymbol{\upsilon})$ *satisfies*

$$\varkappa_0 \, \mathtt{r} \le 1/2 \, . \tag{3.7}$$

Then **($\mathcal{T}_3^*$)** *and* **($\mathcal{T}_4^*$)** *hold at* $\boldsymbol{\upsilon}$ *with* $\tau_3 = \sqrt{\mathrm{e}}\, \varkappa$, *and* $\tau_4 = \sqrt{\mathrm{e}}\, \varkappa^2$.

Proof. We start with some technical statements. First, observe that the function $\phi(\upsilon) = \log(1 + \mathrm{e}^{\upsilon})$ satisfies for all $\upsilon \in I\!R$

$$|\phi^{(k)}(\upsilon)| \le \phi''(\upsilon), \qquad k = 3, 4. \tag{3.8}$$

Indeed, it holds

$$\begin{aligned}
\phi'(\upsilon) &= \frac{\mathrm{e}^{\upsilon}}{1 + \mathrm{e}^{\upsilon}} \, , \\
\phi''(\upsilon) &= \frac{\mathrm{e}^{\upsilon}}{(1 + \mathrm{e}^{\upsilon})^2} \, , \\
\phi^{(3)}(\upsilon) &= \frac{\mathrm{e}^{\upsilon}}{(1 + \mathrm{e}^{\upsilon})^2} - \frac{2\mathrm{e}^{2\upsilon}}{(1 + \mathrm{e}^{\upsilon})^3} \, , \\
\phi^{(4)}(\upsilon) &= \frac{\mathrm{e}^{\upsilon}}{(1 + \mathrm{e}^{\upsilon})^2} - \frac{6\mathrm{e}^{2\upsilon}}{(1 + \mathrm{e}^{\upsilon})^3} + \frac{6\mathrm{e}^{3\upsilon}}{(1 + \mathrm{e}^{\upsilon})^4} \, .
\end{aligned}$$

It is straightforward to see that $|\phi^{(k)}(\upsilon)| \le \phi''(\upsilon)$ for $k = 3, 4$ and any υ.

Next, we check local variability of $\phi''(\upsilon)$. Fix $\upsilon^\circ < 0$. As the function $\phi''(\upsilon)$ is monotonously increasing in υ, it holds

$$\sup_{|\upsilon-\upsilon^\circ|\le b} \frac{\phi''(\upsilon)}{\phi''(\upsilon^\circ)} = \frac{\phi''(\upsilon^\circ + b)}{\phi''(\upsilon^\circ)} \le \mathrm{e}^b . \tag{3.9}$$

Putting together (3.8) and (3.9) leads to a bound on variability of $D(\boldsymbol{\upsilon}+\boldsymbol{u})$. Let us fix any $\boldsymbol{u}$ with $\|\mathbb{D}\boldsymbol{u}\| \le \mathtt{r}$. By definition

$$D^2(\boldsymbol{\upsilon}+\boldsymbol{u}) = \sum_{i=1}^n \boldsymbol{\Psi}_i \boldsymbol{\Psi}_i^\top \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}+\boldsymbol{u}\rangle\bigr).$$

By (3.7) and **($\boldsymbol{\Psi}$)**, for each $i \le n$, it holds $\bigl|\langle \boldsymbol{\Psi}_i, \boldsymbol{u}\rangle\bigr| \le \|\mathbb{D}^{-1}\boldsymbol{\Psi}_i\| \, \|\mathbb{D}\boldsymbol{u}\| \le \varkappa_0 \, \mathtt{r} \le 1/2$ and by (3.9)

$$\phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}+\boldsymbol{u}\rangle\bigr) \le \sqrt{\mathrm{e}} \, \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}\rangle\bigr).$$

This yields

$$D^2(\boldsymbol{\upsilon}+\boldsymbol{u}) \le \sqrt{\mathrm{e}} \, D^2(\boldsymbol{\upsilon}). \tag{3.10}$$

As the next step we evaluate the derivative $\nabla^k f(\boldsymbol{\upsilon}+\boldsymbol{u})$ for $f(\boldsymbol{\upsilon}) = \mathbb{E} L(\boldsymbol{\upsilon})$. For any $\boldsymbol{u} \in \mathcal{U}$ with $\|\mathbb{D}\boldsymbol{u}\| \le \mathtt{r}$ and any $\boldsymbol{z} \in \mathbb{R}^p$, it holds

$$\bigl\langle \nabla^k f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}^{\otimes k} \bigr\rangle = -\sum_{i=1}^n \langle \boldsymbol{\Psi}_i, \boldsymbol{z}\rangle^k \; \phi^{(k)}\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}+\boldsymbol{u}\rangle\bigr).$$

With $w_i(\boldsymbol{\upsilon}) = \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}\rangle\bigr)$, we derive by **($\boldsymbol{\Psi}$)**, (3.8), and (3.10)

$$\begin{aligned}
\bigl|\bigl\langle \nabla^3 f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}^{\otimes 3} \bigr\rangle\bigr| &\le \sum_{i=1}^n \bigl|\langle \boldsymbol{\Psi}_i, \boldsymbol{z}\rangle\bigr|^3 \phi''\bigl(\langle \boldsymbol{\Psi}_i, \boldsymbol{\upsilon}+\boldsymbol{u}\rangle\bigr) \\
&\le \sqrt{\mathrm{e}} \sum_{i=1}^n \bigl|\langle \boldsymbol{\Psi}_i, \boldsymbol{z}\rangle\bigr|^3 w_i(\boldsymbol{\upsilon}) \le \sqrt{\mathrm{e}} \biggl(\sum_{i=1}^n \langle \boldsymbol{\Psi}_i, \boldsymbol{z}\rangle^2 w_i(\boldsymbol{\upsilon}) \biggr)^{1/2} \biggl(\sum_{i=1}^n \langle \boldsymbol{\Psi}_i, \boldsymbol{z}\rangle^4 w_i(\boldsymbol{\upsilon}) \biggr)^{1/2} \\
&\le \sqrt{\mathrm{e}} \, \varkappa \, \|\mathbb{D}\boldsymbol{z}\| \, \|\mathbb{D}\boldsymbol{z}\|^2.
\end{aligned}$$

This yields **($\mathcal{T}_3^*$)** with $\tau_3 = \sqrt{\mathrm{e}} \, \varkappa$. Similarly **($\mathcal{T}_4^*$)** holds with $\tau_4 = \sqrt{\mathrm{e}} \, \varkappa^2$. □

Now we can summarize.

Proposition 3.3. *Let $V^2 \ge 2 \operatorname{Var}(\nabla\zeta)$; see (3.2). Let also $\mathbb{p}_G$ and λ_G be given in (3.3) while $\mathtt{r}_G = \sqrt{\mathbb{p}_G} + \sqrt{2\mathtt{x}\lambda_G}$. Assume* **($\boldsymbol{\Psi}$)***, (3.4), and $\varkappa_0 \, \mathtt{r}_G \le 1/2$. Then* **($\mathcal{T}_3^*$)** *and* **($\mathcal{T}_4^*$)** *hold at $\boldsymbol{\upsilon}$ with $\tau_3 = \sqrt{\mathrm{e}} \, \varkappa$ and $\tau_4 = \sqrt{\mathrm{e}} \, \varkappa^2$.* **($\boldsymbol{\nabla\zeta}$)** *applies with $V^2 = \operatorname{Var}(\nabla\zeta)$ from (3.2) for $\mathtt{x}$ with $\sqrt{2\mathtt{x}} \le 1/(2\varkappa_0) - \sqrt{\mathbb{p}_G/\lambda_G}$.*

3.2 Log-density estimation

Suppose we are given a random sample $X_1, \ldots, X_n$ in $\mathbb{R}^d$. The density model assumes that all these random variables are independent identically distributed from some measure P with a density $f(\boldsymbol{x})$ with respect to a σ-finite measure μ_0 in $\mathbb{R}^d$. This density function is the target of estimation. By definition, the function f is non-negative, measurable, and integrates to one: $\int f(\boldsymbol{x})\, d\mu_0(\boldsymbol{x}) = 1$. Here and in what follows, the integral $\int$ without limits means the integral over the whole space $\mathbb{R}^d$. If $f(\cdot)$ has a smaller support $\mathscr{X}$, one can restrict integration to this set. Below we parametrize the model by a linear decomposition of the log-density function. Let $\bigl\{\psi_j(\boldsymbol{x}),\, j = 1, \ldots, p\bigr\}$ with $p \leq \infty$ be a collection of functions in $\mathbb{R}^d$ (a dictionary). For each $\boldsymbol{\upsilon} = (\upsilon_j) \in \mathbb{R}^p$, define

$$\ell(\boldsymbol{x}, \boldsymbol{\upsilon}) \stackrel{\text{def}}{=} \upsilon_1 \psi_1(\boldsymbol{x}) + \ldots + \upsilon_p \psi_p(\boldsymbol{x}) - \phi(\boldsymbol{\upsilon}) = \bigl\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon} \bigr\rangle - \phi(\boldsymbol{\upsilon}),$$

where $\boldsymbol{\Psi}(\boldsymbol{x})$ is a vector with components $\psi_j(\boldsymbol{x})$. Let $\phi(\boldsymbol{\upsilon})$ be given by

$$\phi(\boldsymbol{\upsilon}) \stackrel{\text{def}}{=} \log \int \mathrm{e}^{\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon} \rangle}\, d\mu_0(\boldsymbol{x}). \tag{3.11}$$

It is worth stressing that the data point $\boldsymbol{x}$ only enters in the linear term $\bigl\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon} \bigr\rangle$ of the log-likelihood $\ell(\boldsymbol{x}, \boldsymbol{\upsilon})$. The function $\phi(\boldsymbol{\upsilon})$ is entirely model-driven. Below we restrict $\boldsymbol{\upsilon}$ to a subset $\varUpsilon$ in $\mathbb{R}^p$ such that $\phi(\boldsymbol{\upsilon})$ is well defined and the integral $\int \mathrm{e}^{\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon} \rangle}\, d\mu_0(\boldsymbol{x})$ is finite. Linear log-density modeling assumes

$$\log f(\boldsymbol{x}) = \ell(\boldsymbol{x}, \boldsymbol{\upsilon}^*) = \bigl\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon}^* \bigr\rangle - \phi(\boldsymbol{\upsilon}^*) \tag{3.12}$$

for some $\boldsymbol{\upsilon}^* \in \varUpsilon$. A nice feature of such representation is that the function $\log f(\boldsymbol{x})$, in contrary to the density itself, does not need to be non-negative. Another important benefit of using the log-density is that the stochastic part of the corresponding log-likelihood is *linear* w.r.t. the parameter $\boldsymbol{\upsilon}$. With $S = \sum_{i=1}^{n} \boldsymbol{\Psi}(X_i)$, the log-likelihood $L(\boldsymbol{\upsilon})$ reads as

$$L(\boldsymbol{\upsilon}) = \sum_{i=1}^{n} \bigl\langle \boldsymbol{\Psi}(X_i), \boldsymbol{\upsilon} \bigr\rangle - n\phi(\boldsymbol{\upsilon}) = \langle S, \boldsymbol{\upsilon} \rangle - n\phi(\boldsymbol{\upsilon}).$$

The truth can be defined as its population counterpart:

$$\boldsymbol{\upsilon}^* = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \varUpsilon} \mathbb{E} L(\boldsymbol{\upsilon}) = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \varUpsilon} \bigl\{ \langle \mathbb{E} S, \boldsymbol{\upsilon} \rangle - n\phi(\boldsymbol{\upsilon}) \bigr\} = \operatorname*{argmax}_{\boldsymbol{\upsilon} \in \varUpsilon} \bigl\{ \langle \bar{\varPsi}, \boldsymbol{\upsilon} \rangle - \phi(\boldsymbol{\upsilon}) \bigr\}, \tag{3.13}$$

where $\bar{\varPsi} = n^{-1} \mathbb{E} S$. This yields the identity

$$\nabla \phi(\boldsymbol{\upsilon}^*) = \bar{\varPsi}.$$

For a given penalty operator G^2, the penalized log-likelihood $L_G(\boldsymbol{\upsilon})$ reads as

$$L_G(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \frac{1}{2}\|G\boldsymbol{\upsilon}\|^2 = \langle S, \boldsymbol{\upsilon}\rangle - n\phi(\boldsymbol{\upsilon}) - \frac{1}{2}\|G\boldsymbol{\upsilon}\|^2.$$

The penalized MLE $\widetilde{\boldsymbol{\upsilon}}_G$ and its population counterpart $\boldsymbol{\upsilon}_G^*$ are defined as

$$\widetilde{\boldsymbol{\upsilon}}_G = \operatorname*{argmax}_{\boldsymbol{\upsilon}\in\Upsilon} L_G(\boldsymbol{\upsilon}), \qquad \boldsymbol{\upsilon}_G^* = \operatorname*{argmax}_{\boldsymbol{\upsilon}\in\Upsilon} \mathbb{E} L_G(\boldsymbol{\upsilon}).$$

We are interested in sufficient conditions on the model which enables us to apply the general results of Section 2 for quantifying the error terms $\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}_G^*$, $\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*$, and the corresponding risk $\mathbb{E}\|Q(\widetilde{\boldsymbol{\upsilon}}_G - \boldsymbol{\upsilon}^*)\|^2$.

Assumptions

First note that the generalized linear structure of the model automatically yields conditions **$(\mathcal{C}_G)$** and **(ζ)**. Indeed, convexity of $\phi(\cdot)$ implies that $\mathbb{E}L(\boldsymbol{\upsilon}) = \langle \mathbb{E}S, \boldsymbol{\upsilon}\rangle - n\phi(\boldsymbol{\upsilon})$ is concave. Further, for the stochastic component $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \mathbb{E}L(\boldsymbol{\upsilon})$, it holds

$$\nabla\zeta(\boldsymbol{\upsilon}) = \nabla\zeta = S - \mathbb{E}S = \sum_{i=1}^{n}\bigl[\boldsymbol{\Psi}(X_i) - \mathbb{E}\boldsymbol{\Psi}(X_i)\bigr],$$

and **(ζ)** follows. Further, the representation $\mathbb{E}L(\boldsymbol{\upsilon}) = \langle \mathbb{E}S, \boldsymbol{\upsilon}\rangle - n\phi(\boldsymbol{\upsilon})$ implies

$$\mathbb{F}(\boldsymbol{\upsilon}) = -\nabla^2\mathbb{E}L(\boldsymbol{\upsilon}) = -\nabla^2 L(\boldsymbol{\upsilon}) = n\nabla^2\phi(\boldsymbol{\upsilon}).$$

To simplify our presentation, we assume that $X_1, \ldots, X_n$ are indeed i.i.d. and the density $f(\boldsymbol{x})$ can be represented in the form (3.12) for some parameter vector $\boldsymbol{\upsilon}^*$. This can be easily extended to a non i.i.d. case at the cost of more complicated notations. Then $\bar{\Psi} = n^{-1}\mathbb{E}S = \mathbb{E}\boldsymbol{\Psi}(X_1)$. Moreover, by (3.11), $\nabla^2\phi(\boldsymbol{\upsilon}^*) = \operatorname{Var}\bigl\{\boldsymbol{\Psi}(X_1)\bigr\}$ and

$$\operatorname{Var}(\nabla\zeta) = n\,\nabla^2\phi(\boldsymbol{\upsilon}^*) = \mathbb{F}(\boldsymbol{\upsilon}^*). \tag{3.14}$$

For any $\boldsymbol{\upsilon} \in \Upsilon$ and $\varrho > 0$, consider the elliptic set $\mathscr{B}_\varrho(\boldsymbol{\upsilon}) \subset \mathbb{R}^p$ with

$$\mathscr{B}_\varrho(\boldsymbol{\upsilon}) \stackrel{\text{def}}{=} \bigl\{\boldsymbol{u} \in \mathbb{R}^p \colon \langle \nabla^2\phi(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2}\rangle \le \varrho^2\bigr\}.$$

Assume the following conditions.

(f) $X_1, \ldots, X_n$ are i.i.d. from a density f satisfying $\log f(\boldsymbol{x}) = \boldsymbol{\Psi}(\boldsymbol{x})^\top\boldsymbol{\upsilon}^* - \phi(\boldsymbol{\upsilon}^*)$.

(Υ) The set Υ is open and convex, the value $\phi(\boldsymbol{\upsilon})$ from (3.11) is finite for all $\boldsymbol{\upsilon} \in \Upsilon$, $\boldsymbol{\upsilon}^*$ from (3.13) is an internal point in Υ such that $\mathscr{B}_{2\varrho}(\boldsymbol{\upsilon}^*) \subset \Upsilon$ for a fixed $\varrho > 0$.

(ϕ) For the Bregman divergence $\phi(\boldsymbol{\upsilon};\boldsymbol{u}) \stackrel{\text{def}}{=} \phi(\boldsymbol{\upsilon}+\boldsymbol{u}) - \phi(\boldsymbol{\upsilon}) - \langle \nabla\phi(\boldsymbol{\upsilon}), \boldsymbol{u}\rangle$, it holds

$$\sup_{\boldsymbol{\upsilon}\in\mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^{*})} \sup_{\boldsymbol{u}\in\mathscr{B}_{\varrho}(\boldsymbol{\upsilon})} \exp\phi(\boldsymbol{\upsilon};\boldsymbol{u}) \le \mathtt{C}_{\varrho}. \tag{3.15}$$

Introduce a measure $P_{\boldsymbol{\upsilon}}$ by the relation:

$$\frac{dP_{\boldsymbol{\upsilon}}}{d\mu_0}(\boldsymbol{x}) = \exp\bigl\{\bigl\langle \boldsymbol{\Psi}(\boldsymbol{x}), \boldsymbol{\upsilon}\bigr\rangle - \phi(\boldsymbol{\upsilon})\bigr\}. \tag{3.16}$$

Identity (3.11) ensures that $P_{\boldsymbol{\upsilon}}$ is a probabilistic measure. Moreover, under (3.12), the data generating measure $\mathbb{P}$ coincides with $P_{\boldsymbol{\upsilon}^{*}}^{\otimes n}$.

$(\boldsymbol{\Psi}_{4})$ *There are* $\mathtt{C}_{\Psi,3} \ge 0$ *and* $\mathtt{C}_{\Psi,4} \ge 3$ *such that for all* $\boldsymbol{\upsilon} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^{*})$ *and* $\boldsymbol{z} \in \mathbb{R}^{p}$

$$\begin{aligned} \bigl|E_{\boldsymbol{\upsilon}}\bigl\langle \boldsymbol{\Psi}(X_1) - E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1), \boldsymbol{z}\bigr\rangle^{3}\bigr| &\le \mathtt{C}_{\Psi,3}\, E_{\boldsymbol{\upsilon}}^{3/2}\bigl\langle \boldsymbol{\Psi}(X_1) - E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1), \boldsymbol{z}\bigr\rangle^{2}, \\ E_{\boldsymbol{\upsilon}}\bigl\langle \boldsymbol{\Psi}(X_1) - E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1), \boldsymbol{z}\bigr\rangle^{4} &\le \mathtt{C}_{\Psi,4}\, E_{\boldsymbol{\upsilon}}^{2}\bigl\langle \boldsymbol{\Psi}(X_1) - E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1), \boldsymbol{z}\bigr\rangle^{2}. \end{aligned}$$

In fact, conditions **(ϕ)** and **$(\boldsymbol{\Psi}_{4})$** follow from **$(\boldsymbol{\Upsilon})$** and can be considered as a kind of definition of important quantities $\mathtt{C}_{\varrho}$, $\mathtt{C}_{\Psi,3}$, and $\mathtt{C}_{\Psi,4}$ which will be used for describing the smoothness properties of $\phi(\boldsymbol{\upsilon})$.

For a penalty operator G^2, define $\mathbb{F} = \mathbb{F}(\boldsymbol{\upsilon}^{*})$, $\mathbb{F}_{G} = \mathbb{F} + G^2$, and

$$\mathbb{p}_{G} \stackrel{\text{def}}{=} \operatorname{tr}(\mathbb{F}_{G}^{-1}\mathbb{F}), \qquad \mathtt{r}_{G} = \sqrt{\mathbb{p}_{G}} + \sqrt{2\mathtt{x}}\,.$$

Proposition 3.4. *Assume* **(f)**, **$(\boldsymbol{\Upsilon})$**, **(ϕ)**, *and* **$(\boldsymbol{\Psi}_{4})$**, *and let* $\mathtt{r}_{G} \le \varrho\sqrt{n/2}$. *Then, for any* $\boldsymbol{\upsilon} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^{*})$, *the function* $f(\boldsymbol{\upsilon}) = \mathbb{E}_{\boldsymbol{\upsilon}^{*}}L(\boldsymbol{\upsilon})$ *satisfies* **$(\mathcal{S}_{3})$** *and* **$(\mathcal{S}_{4})$** *with* $h(\boldsymbol{\upsilon}) = \langle \nabla\phi(\boldsymbol{\upsilon}^{*}), \boldsymbol{\upsilon}\rangle - \phi(\boldsymbol{\upsilon})$, $\mathtt{m}^2(\boldsymbol{\upsilon}) = \nabla^2\phi(\boldsymbol{\upsilon})$, *and constants* $\mathtt{c}_3$ *and* $\mathtt{c}_4$ *satisfying*

$$\mathtt{c}_3 = \mathtt{C}_{\Psi,3}\,\mathtt{c}_{\phi}^{3/2}, \qquad \mathtt{c}_4 = (\mathtt{C}_{\Psi,4} - 3)\,\mathtt{c}_{\phi}^{2}, \qquad \mathtt{c}_{\phi} = \sqrt{\mathtt{C}_{\Psi,4}\,\mathtt{C}_{\varrho}}\,.$$

Moreover, **$(\nabla\zeta)$** *holds with* $V^2 = 2n\nabla^2\phi(\boldsymbol{\upsilon}^{*})$ *for* $\mathtt{x} \le (\varrho\sqrt{n/2} - \sqrt{\mathbb{p}_{G}})^2/4$.

Proof. Let $P_{\boldsymbol{\upsilon}}$ be defined by (3.16). It is straightforward to check that $E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1) = \nabla\phi(\boldsymbol{\upsilon})$ and $\operatorname{Var}_{\boldsymbol{\upsilon}}(\boldsymbol{\Psi}(X_1)) = \nabla^2\phi(\boldsymbol{\upsilon})$. Further, if $\boldsymbol{u} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon})$ and $\boldsymbol{\upsilon}+\boldsymbol{u} \in \Upsilon$, then

$$\phi(\boldsymbol{\upsilon}+\boldsymbol{u}) = \log E_0 \exp\{\langle \boldsymbol{\Psi}(X_1), \boldsymbol{\upsilon}+\boldsymbol{u}\rangle\} = \log E_{\boldsymbol{\upsilon}} \exp\bigl\{\bigl\langle \boldsymbol{\Psi}(X_1), \boldsymbol{u}\bigr\rangle + \phi(\boldsymbol{\upsilon})\bigr\}.$$

This yields in view of $E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1) = \nabla\phi(\boldsymbol{\upsilon})$ that $\boldsymbol{\varepsilon} = \boldsymbol{\Psi}(X_1) - E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1)$ fulfills

$$\begin{aligned} \log E_{\boldsymbol{\upsilon}} \exp(\langle \boldsymbol{\varepsilon}, \boldsymbol{u}\rangle) &= \phi(\boldsymbol{\upsilon}+\boldsymbol{u}) - \phi(\boldsymbol{\upsilon}) - \langle E_{\boldsymbol{\upsilon}}\boldsymbol{\Psi}(X_1), \boldsymbol{u}\rangle \\ &= \phi(\boldsymbol{\upsilon}+\boldsymbol{u}) - \phi(\boldsymbol{\upsilon}) - \langle \nabla\phi(\boldsymbol{\upsilon}), \boldsymbol{u}\rangle. \end{aligned} \tag{3.17}$$

Lemma 3.5. *The function* $\phi(\boldsymbol{\upsilon})$ *satisfies for any* $\boldsymbol{\upsilon} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^*)$ *and* $\boldsymbol{z} \in \mathbb{R}^p$

$$|\langle \nabla^3 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 3} \rangle| \leq \mathtt{C}_{\Psi,3} \, \langle \nabla^2 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 2} \rangle^{3/2} , \tag{3.18}$$

$$|\langle \nabla^4 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 4} \rangle| \leq (\mathtt{C}_{\Psi,4} - 3) \, \langle \nabla^2 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 2} \rangle^{2}. \tag{3.19}$$

Proof. Denote $\boldsymbol{\varepsilon} = X_1 - E_{\boldsymbol{\upsilon}} X_1$. By (3.17) with $\boldsymbol{u} = t\boldsymbol{z}$ for t sufficiently small

$$\chi(t) \stackrel{\text{def}}{=} \log E_{\boldsymbol{\upsilon}} \exp(t\langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle) = \phi(\boldsymbol{\upsilon} + t\boldsymbol{z}) - \phi(\boldsymbol{\upsilon}) - \langle \nabla \phi(\boldsymbol{\upsilon}), t\boldsymbol{z} \rangle,$$

and by **($\boldsymbol{\Psi_4}$)** with $\mathtt{C}_{\Psi,4} \geq 3$

$$\begin{aligned}
\bigl|\chi^{(3)}(0)\bigr| &= \bigl|E_{\boldsymbol{\upsilon}} \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^3 \bigr| \leq \mathtt{C}_{\Psi,3} \, E_{\boldsymbol{\upsilon}}^{3/2} \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^2 , \\
\bigl|\chi^{(4)}(0)\bigr| &= \bigl|E_{\boldsymbol{\upsilon}} \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^4 - 3E_{\boldsymbol{\upsilon}}^2 \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^2 \bigr| \leq (\mathtt{C}_{\Psi,4} - 3) E_{\boldsymbol{\upsilon}}^2 \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^2.
\end{aligned}$$

This implies the assertion. □

Lemma 3.6. *If* $\boldsymbol{\upsilon} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^*)$ *then with* $\mathtt{c}_{\phi} = \sqrt{\mathtt{C}_{\Psi,4} \, \mathtt{C}_{\varrho}}$

$$\sup_{\boldsymbol{u} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon})} \sup_{\boldsymbol{z} \in \mathbb{R}^p} \frac{\langle \nabla^2 \phi(\boldsymbol{\upsilon} + \boldsymbol{u}), \boldsymbol{z}^{\otimes 2} \rangle}{\langle \nabla^2 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 2} \rangle} \leq \mathtt{c}_{\phi} \, . \tag{3.20}$$

Proof. Let $\langle \nabla^2 \phi(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle \leq \varrho^2$. By (3.17) with $\boldsymbol{\varepsilon} = X_1 - E_{\boldsymbol{\upsilon}} X_1$

$$\nabla^2 \phi(\boldsymbol{\upsilon} + \boldsymbol{u}) = \nabla^2 \log E_{\boldsymbol{\upsilon}} \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} = \frac{E_{\boldsymbol{\upsilon}} \{ \boldsymbol{\varepsilon} \boldsymbol{\varepsilon}^{\top} \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \}}{(E_{\boldsymbol{\upsilon}} \, \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle})^2} - \frac{E_{\boldsymbol{\upsilon}} \{ \boldsymbol{\varepsilon} \, \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \} \, E_{\boldsymbol{\upsilon}} \{ \boldsymbol{\varepsilon} \, \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \}^{\top}}{(E_{\boldsymbol{\upsilon}} \, \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle})^2}$$

and by (3.19) and (3.15) in view of $E_{\boldsymbol{\upsilon}} \, \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \geq 1$

$$\begin{aligned}
\bigl\langle \nabla^2 \phi(\boldsymbol{\upsilon} + \boldsymbol{u}), \boldsymbol{z}^{\otimes 2} \bigr\rangle &\leq E_{\boldsymbol{\upsilon}} \bigl\{ \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^2 \mathrm{e}^{\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \bigr\} \\
&\leq E_{\boldsymbol{\upsilon}}^{1/2} \langle \boldsymbol{\varepsilon}, \boldsymbol{z} \rangle^4 \, E_{\boldsymbol{\upsilon}}^{1/2} \mathrm{e}^{2\langle \boldsymbol{\varepsilon}, \boldsymbol{u} \rangle} \leq \sqrt{\mathtt{C}_{\Psi,4} \, \mathtt{C}_{\varrho}} \, \bigl\langle \nabla^2 \phi(\boldsymbol{\upsilon}), \boldsymbol{z}^{\otimes 2} \bigr\rangle
\end{aligned}$$

and the assertion follows. □

Now we are prepared to finalize the check of **($\boldsymbol{\mathcal{S}_3}$)** and **($\boldsymbol{\mathcal{S}_4}$)**. Let $\boldsymbol{\upsilon} \in \mathscr{B}_{\varrho}(\boldsymbol{\upsilon}^*)$. For any $\boldsymbol{u}$ with $\|\mathbb{m}(\boldsymbol{\upsilon})\boldsymbol{u}\| \leq \mathtt{r}_G/\sqrt{n} \leq \varrho$, by (3.18) and (3.20)

$$\frac{|\langle \nabla^3 \phi(\boldsymbol{\upsilon} + t\boldsymbol{u}), \boldsymbol{z}^{\otimes 3} \rangle|}{\|\mathbb{m}(\boldsymbol{\upsilon})\boldsymbol{z}\|^3} \leq \frac{\mathtt{C}_{\Psi,3} \, \|\mathbb{m}(\boldsymbol{\upsilon} + t\boldsymbol{u})\boldsymbol{z}\|^3}{\|\mathbb{m}(\boldsymbol{\upsilon})\boldsymbol{z}\|^3} \leq \mathtt{C}_{\Psi,3} \, \mathtt{c}_{\phi}^{3/2} ,$$

and **($\boldsymbol{\mathcal{S}_3}$)** follows with $\mathtt{c}_3 = \mathtt{C}_{\Psi,3} \, \mathtt{c}_{\phi}^{3/2}$. The proof of **($\boldsymbol{\mathcal{S}_4}$)** is similar.

Now we check the deviation bound for $\nabla\zeta = \boldsymbol{S} - \mathbb{E}\boldsymbol{S}$. I.i.d. structure of $\boldsymbol{S} = \sum_i X_i$ and (3.14) yield $\operatorname{Var}(\boldsymbol{S}) = n\nabla^2\phi(\boldsymbol{\upsilon}^*)$. Further, for any $\boldsymbol{u} \in \mathscr{B}_\varrho(\boldsymbol{\upsilon}^*)$, again by the i.i.d. assumption and by (3.17)

$$n^{-1}\log \mathbb{E}_{\boldsymbol{\upsilon}^*}\exp\bigl\{\langle\nabla\zeta,\boldsymbol{u}\rangle\bigr\} = \log \mathbb{E}_{\boldsymbol{\upsilon}^*}\mathrm{e}^{\langle\boldsymbol{\varepsilon},\boldsymbol{u}\rangle} = \phi(\boldsymbol{\upsilon}^*+\boldsymbol{u}) - \phi(\boldsymbol{\upsilon}^*) - \langle\nabla\phi(\boldsymbol{\upsilon}^*),\boldsymbol{u}\rangle.$$

For $\mathtt{r}_G \le \varrho n^{1/2}$, consider all $\boldsymbol{u}$ with $n\langle\nabla^2\phi(\boldsymbol{\upsilon}^*),\boldsymbol{u}^{\otimes 2}\rangle \le \mathtt{r}_G^2$. If $\mathsf{c}_3\,\mathtt{r}_G \le 3n^{1/2}$, then by **$(\mathcal{S}_3)$** and (A.5) of Lemma A.3

$$\phi(\boldsymbol{\upsilon}^*+\boldsymbol{u}) - \phi(\boldsymbol{\upsilon}^*) - \langle\nabla\phi(\boldsymbol{\upsilon}^*),\boldsymbol{u}\rangle \le \frac{1+\mathsf{c}_3\,\mathtt{r}_G\,n^{-1/2}/3}{2}\langle\nabla^2\phi(\boldsymbol{\upsilon}^*),\boldsymbol{u}^{\otimes 2}\rangle \le \langle\nabla^2\phi(\boldsymbol{\upsilon}^*),\boldsymbol{u}^{\otimes 2}\rangle.$$

This implies (B.46) with $V^2 = 2n\phi(\boldsymbol{\upsilon}^*)$, $\mathtt{g} = \varrho\sqrt{n/2}$ and thus, the deviation bound (B.51) of Theorem B.15 implies **$(\nabla\zeta)$** for $\mathtt{x} \le \mathtt{x}_c \le (\varrho\sqrt{n/2} - \sqrt{\mathbb{p}_G})^2/4$. □

3.3 Precision matrix estimation

Let $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ be i.i.d. zero mean Gaussian vector in $\mathbb{R}^p$ with a covariance Σ: $\boldsymbol{X}_i \sim \mathcal{N}(0,\Sigma)$. Our goal is to estimate the corresponding precision matrix $\boldsymbol{\upsilon} = \Sigma^{-1}$. Later we identify the matrix $\boldsymbol{\upsilon}$ with the point in the linear subspace Υ of $\mathbb{R}^{p\times p}$ composed by symmetric matrices. The ML approach leads to the log-likelihood

$$L(\boldsymbol{\upsilon}) = -\frac{1}{2}\sum_{i=1}^{n}\langle\widehat{A}_i,\boldsymbol{\upsilon}\rangle + \frac{n}{2}\log\det(\boldsymbol{\upsilon}) \tag{3.21}$$

with $\widehat{A}_i = \boldsymbol{X}_i\boldsymbol{X}_i^\top$. Here and later $\langle A,B\rangle$ means $\operatorname{tr}(AB)$ for $A,B\in\Upsilon$. The corresponding MLE $\widetilde{\boldsymbol{\upsilon}}$ maximizes $L(\boldsymbol{\upsilon})$:

$$\widetilde{\boldsymbol{\upsilon}} = \operatorname*{argmax}_{\boldsymbol{\upsilon}\in\Upsilon} L(\boldsymbol{\upsilon}).$$

The target of estimation $\boldsymbol{\upsilon}^*$ can be defined as its population counterpart:

$$\boldsymbol{\upsilon}^* = \operatorname*{argmax}_{\boldsymbol{\upsilon}\in\Upsilon} \mathbb{E}L(\boldsymbol{\upsilon}).$$

Now introduce a quadratic penalization on $\boldsymbol{\upsilon}$ in the form $\|\mathcal{K}\boldsymbol{\upsilon}\|_{\mathrm{Fr}}^2/2$ for a linear operator $\mathcal{K}$ on the space Υ. One typical example corresponds to the case with

$$\|\mathcal{K}\boldsymbol{\upsilon}\|_{\mathrm{Fr}}^2 = \sum_{m=1}^{p}\|\mathcal{K}_m\boldsymbol{\upsilon}_m\|^2, \qquad \boldsymbol{\upsilon} = (\boldsymbol{\upsilon}_1,\ldots,\boldsymbol{\upsilon}_p),$$

for a family of linear mappings $\mathcal{K}_1, \ldots, \mathcal{K}_p$ in $\mathbb{R}^p$. The corresponding penalized MLE $\widetilde{\boldsymbol{v}}_{\mathcal{K}}$ is defined by maximizing the penalized log-likelihood $L_{\mathcal{K}}(\boldsymbol{v}) = L(\boldsymbol{v}) - \|\mathcal{K}\boldsymbol{v}\|_{\mathrm{Fr}}^2$:

$$
\begin{aligned}
\widetilde{\boldsymbol{v}}_{\mathcal{K}} &= \operatorname*{argmax}_{\boldsymbol{v}} L_{\mathcal{K}}(\boldsymbol{v}) = \operatorname*{argmax}_{\boldsymbol{v}} \Bigl\{ L(\boldsymbol{v}) - \frac{1}{2}\|\mathcal{K}\boldsymbol{v}\|_{\mathrm{Fr}}^2 \Bigr\} \\
&= \operatorname*{argmin}_{\boldsymbol{v}} \Bigl\{ \sum_{i=1}^{n} \langle \widehat{A}_i, \boldsymbol{v} \rangle - n \log\det(\boldsymbol{v}) + \|\mathcal{K}\boldsymbol{v}\|_{\mathrm{Fr}}^2 \Bigr\}.
\end{aligned}
$$

Define also the penalized target $\boldsymbol{v}_{\mathcal{K}}^*$ as

$$
\boldsymbol{v}_{\mathcal{K}}^* = \operatorname*{argmax}_{\boldsymbol{v}} \Bigl\{ \mathbb{E}L(\boldsymbol{v}) - \frac{1}{2}\|\mathcal{K}\boldsymbol{v}\|_{\mathrm{Fr}}^2 \Bigr\} = \operatorname*{argmin}_{\boldsymbol{v}} \Bigl\{ n\langle \Sigma, \boldsymbol{v} \rangle - n \log\det(\boldsymbol{v}) + \|\mathcal{K}\boldsymbol{v}\|_{\mathrm{Fr}}^2 \Bigr\}.
$$

We intend to state some sharp bounds on the loss and risk of $\widetilde{\boldsymbol{v}}_{\mathcal{K}}$ by applying the general results of Section 2. Model (3.21) is a special case of an exponential family. Therefore, the basic assumptions $(\boldsymbol{\zeta})$ and $(\boldsymbol{\mathcal{C}_G})$ are fulfilled automatically. Next, we check the smoothness properties of $\mathbb{E}L(\boldsymbol{v})$ in terms of the Gatoux derivatives.

Lemma 3.7. *Let $\boldsymbol{v} \in \Upsilon$ be positive definite. For any $\boldsymbol{z} \in \Upsilon$ and $\boldsymbol{U} = \boldsymbol{v}^{-1/2}\,\boldsymbol{z}\,\boldsymbol{v}^{-1/2}$, it holds*

$$
-\frac{d^2}{dt^2}\,\mathbb{E}L(\boldsymbol{v} + t\,\boldsymbol{z})\Big|_{t=0} = \frac{n}{2}\,\operatorname{tr}\boldsymbol{U}^2 = \frac{n}{2}\,\operatorname{tr}\{(\boldsymbol{v}^{-1}\boldsymbol{z})^2\}. \tag{3.22}
$$

Similarly

$$
\frac{d^3}{dt^3}\mathbb{E}L(\boldsymbol{v} + t\boldsymbol{z})\Big|_{t=0} = n\operatorname{tr}\boldsymbol{U}^3\,, \qquad \frac{d^4}{dt^4}\mathbb{E}L(\boldsymbol{v} + t\boldsymbol{z})\Big|_{t=0} = -3n\operatorname{tr}\boldsymbol{U}^4\,. \tag{3.23}
$$

Proof. Fix some $\boldsymbol{z} \in \Upsilon$. It holds by (3.21) with $\boldsymbol{U} = \boldsymbol{v}^{-1/2}\,\boldsymbol{z}\,\boldsymbol{v}^{-1/2}$

$$
\begin{aligned}
-\frac{d^2}{dt^2}\mathbb{E}L(\boldsymbol{v} + t\,\boldsymbol{z})\Big|_{t=0} &= -\frac{n}{2}\,\frac{d^2}{dt^2}\,\log\det(\boldsymbol{v} + t\,\boldsymbol{z})\Big|_{t=0} \\
= -\frac{n}{2}\,\frac{d^2}{dt^2}\,\log\det(\mathbb{I}_p + t\,\boldsymbol{U})\Big|_{t=0} &= \frac{n}{2}\,\operatorname{tr}\boldsymbol{U}^2 = \frac{n}{2}\,\|\boldsymbol{v}^{-1/2}\,\boldsymbol{z}\,\boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}}^2\,.
\end{aligned}
$$

This formula can easily be checked when $\boldsymbol{U}$ is diagonal, the general case is reduced to this one by an orthogonal transform. (3.23) can be checked similarly. $\square$

Bounds (3.23) help to check condition $(\boldsymbol{\mathcal{T}_3^*})$ and $(\boldsymbol{\mathcal{T}_4^*})$.

Lemma 3.8. *For $\boldsymbol{v} \in \Upsilon$ positive definite, define $\mathbb{D}^2(\boldsymbol{v})$ by*

$$
\|\mathbb{D}(\boldsymbol{v})\boldsymbol{z}\|_{\mathrm{Fr}}^2 = \frac{n}{2}\|\boldsymbol{v}^{-1/2}\,\boldsymbol{z}\,\boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}}^2\,, \qquad \boldsymbol{z} \in \Upsilon\,.
$$

Then $(\mathcal{T}_3^*)$ *and* $(\mathcal{T}_4^*)$ *are fulfilled with*

$$\tau_3 = \sqrt{8}\bigl(1 - \sqrt{2\mathtt{r}^2/n}\bigr)^{-3} n^{-1/2}, \qquad \tau_4 = 12\bigl(1 - \sqrt{2\mathtt{r}^2/n}\bigr)^{-4} n^{-1}.$$

Proof. Consider $\boldsymbol{u} \in \varUpsilon$ such that $\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|_{\mathrm{Fr}} \le \mathtt{r}$. Fix $\boldsymbol{z} \in \varUpsilon$ and define $\boldsymbol{U} = (\boldsymbol{v} + \boldsymbol{u})^{-1/2} \boldsymbol{z}\, (\boldsymbol{v} + \boldsymbol{u})^{-1/2}$. Then by (3.23) and by $\bigl|\operatorname{tr} \boldsymbol{U}^3\bigr| \le \bigl(\operatorname{tr} \boldsymbol{U}^2\bigr)^{3/2}$, the function $f(\boldsymbol{v})$ satisfies

$$\bigl|\langle \nabla^3 f(\boldsymbol{v} + \boldsymbol{u}), \boldsymbol{z}^{\otimes 3} \rangle\bigr| \le n\bigl|\operatorname{tr} \mathcal{U}^3\bigr| \le n\bigl(\operatorname{tr} \boldsymbol{U}^2\bigr)^{3/2} \le n\|(\boldsymbol{v} + \boldsymbol{u})^{-1/2} \boldsymbol{z}\, (\boldsymbol{v} + \boldsymbol{u})^{-1/2}\|_{\mathrm{Fr}}^3$$

Further, $\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|_{\mathrm{Fr}} \le \mathtt{r}$ implies

$$\|\boldsymbol{v}^{-1/2}(\boldsymbol{v} + \boldsymbol{u})\boldsymbol{v}^{-1/2} - I\!\!I\|^2 = \|\boldsymbol{v}^{-1/2} \boldsymbol{u}\, \boldsymbol{v}^{-1/2}\|^2 \le \|\boldsymbol{v}^{-1/2} \boldsymbol{u}\, \boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}}^2 \le 2\mathtt{r}^2/n$$

yielding

$$\|(\boldsymbol{v} + \boldsymbol{u})^{-1/2} \boldsymbol{v}^{1/2}\|^2 = \|(I\!\!I + \boldsymbol{v}^{-1/2} \boldsymbol{u}\, \boldsymbol{v}^{-1/2})^{-1}\| \le \frac{1}{1 - \sqrt{2\mathtt{r}^2/n}}.$$

Therefore, for any $\boldsymbol{z} \in \varUpsilon$

$$\begin{aligned} \|(\boldsymbol{v} + \boldsymbol{u})^{-1/2} \boldsymbol{z}\, (\boldsymbol{v} + \boldsymbol{u})^{-1/2}\|_{\mathrm{Fr}} &\le \|(\boldsymbol{v} + \boldsymbol{u})^{-1/2} \boldsymbol{v}\|^2 \;\|\boldsymbol{v}^{-1/2} \boldsymbol{z}\, \boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}} \\ &\le \frac{1}{1 - \sqrt{2\mathtt{r}^2/n}}\, \|\boldsymbol{v}^{-1/2} \boldsymbol{z}\, \boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}}. \end{aligned}$$

Therefore,

$$\bigl|\langle \nabla^3 f(\boldsymbol{v} + \boldsymbol{u}), \boldsymbol{z}^{\otimes 3} \rangle\bigr| \le n\bigl(1 - \sqrt{2\mathtt{r}^2/n}\bigr)^{-3} \|\boldsymbol{v}^{-1/2} \boldsymbol{z}\, \boldsymbol{v}^{-1/2}\|_{\mathrm{Fr}}^3$$

and condition $(\mathcal{T}_3^*)$ is fulfilled with $\tau_3 = \sqrt{8}\bigl(1 - \sqrt{2\mathtt{r}^2/n}\bigr)^{-3} n^{-1/2}$. Similarly, one can check $(\mathcal{T}_4^*)$. □

By (3.22), the Fisher matrix $I\!\!F(\boldsymbol{v}) \stackrel{\mathrm{def}}{=} -\nabla^2 I\!\!E L(\boldsymbol{v})$ is a linear operator in $\varUpsilon$ with

$$\langle I\!\!F(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2} \rangle = \frac{n}{2} \operatorname{tr} \boldsymbol{U}^2 = \frac{n}{2} \operatorname{tr}\{(\boldsymbol{v}^{-1}\boldsymbol{u})^2\}, \qquad \boldsymbol{U} = \boldsymbol{v}^{-1/2}\, \boldsymbol{z}\, \boldsymbol{v}^{-1/2}.$$

The penalized Fisher information operator $I\!\!D_{\mathcal{K}}^2 = I\!\!F(\boldsymbol{v}^*) + \mathcal{K}^2$ is given by

$$\|I\!\!D_{\mathcal{K}} \boldsymbol{u}\|_{\mathrm{Fr}}^2 = \frac{n}{2} \operatorname{tr}\{(\varSigma \boldsymbol{u})^2\} + \|\mathcal{K}\boldsymbol{u}\|_{\mathrm{Fr}}^2; \qquad \boldsymbol{u} \in \varUpsilon. \tag{3.24}$$

The next condition to be verified is $(\boldsymbol{\nabla\zeta})$.

Lemma 3.9. *Let* $\mathbb{D}_{\mathcal{K}}$ *be given by* (3.24)*. Define also*

$$B_{\mathcal{K}}^2 \stackrel{\text{def}}{=} \mathbb{D}_{\mathcal{K}}^{-1} \varSigma^2 \mathbb{D}_{\mathcal{K}}^{-1}. \tag{3.25}$$

If $\mathbb{p}_{\mathcal{K}} < n/8$*, then* $(\boldsymbol{\nabla\zeta})$ *is fulfilled with*

$$\mathtt{r}_{\mathcal{K}}(\mathtt{x}) \stackrel{\text{def}}{=} \sqrt{\mathbb{p}_{\mathcal{K}}} + \sqrt{2\mathtt{x}}\,, \qquad \mathbb{p}_{\mathcal{K}} \stackrel{\text{def}}{=} (\operatorname{tr} B_{\mathcal{K}})^2 + \operatorname{tr} B_{\mathcal{K}}^2\,.$$

Proof. The stochastic component $\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \mathbb{E} L(\boldsymbol{\upsilon})$ reads

$$\zeta(\boldsymbol{\upsilon}) = L(\boldsymbol{\upsilon}) - \mathbb{E} L(\boldsymbol{\upsilon}) = -\frac{1}{2} \sum_{i=1}^{n} \langle \mathcal{E}_i, \boldsymbol{\upsilon} \rangle$$

with $\mathcal{E}_i = \widehat{A}_i - \mathbb{E}\widehat{A}_i = \boldsymbol{X}_i \boldsymbol{X}_i^\top - \varSigma$. Clearly $\zeta(\boldsymbol{\upsilon})$ is linear in $\boldsymbol{\upsilon}$ and condition $(\boldsymbol{\zeta})$ is fulfilled. Moreover, for any direction $\boldsymbol{u}$ in the parameter space $\varUpsilon$,

$$\langle \nabla\zeta, \boldsymbol{u} \rangle = -\frac{1}{2} \sum_{i=1}^{n} \langle \mathcal{E}_i, \boldsymbol{u} \rangle = -\frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{X}_i^\top \boldsymbol{u} \boldsymbol{X}_i - \langle \varSigma, \boldsymbol{u} \rangle) = -\frac{1}{2} \sum_{i=1}^{n} \bigl\{ \boldsymbol{\gamma}_i^\top \boldsymbol{U} \boldsymbol{\gamma}_i - \operatorname{tr}(\boldsymbol{U}) \bigr\}$$

with $\boldsymbol{\gamma}_i = \varSigma^{-1/2} \boldsymbol{X}_i$ standard Gaussian and $\boldsymbol{U} = \varSigma^{1/2} \boldsymbol{u} \varSigma^{1/2}$. By Lemma B.1

$$\operatorname{Var}\langle \nabla\zeta, \boldsymbol{u} \rangle = \frac{n}{4} \operatorname{Var}\bigl(\boldsymbol{\gamma}_1^\top \boldsymbol{U} \boldsymbol{\gamma}_1 \bigr) = \frac{n}{2} \operatorname{tr}(\boldsymbol{U}^2) = \frac{n}{2} \operatorname{tr}(\varSigma \boldsymbol{u})^2.$$

In particular, for $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}^* = \varSigma^{-1}$, the operator $\mathbb{D}^2 = \mathbb{F}(\boldsymbol{\upsilon}^*)$ coincides with $\operatorname{Var}(\nabla\zeta)$:

$$\| \mathbb{D} \boldsymbol{u} \|_{\mathrm{Fr}}^2 = \frac{n}{2} \operatorname{tr}(\varSigma \boldsymbol{u})^2.$$

This is in agreement with the fact that under the true parametric assumption, it holds $-\nabla^2 \mathbb{E} L(\boldsymbol{\upsilon}^*) = \operatorname{Var}\bigl(\nabla\zeta\bigr)$. One can easily check for any $\boldsymbol{u} \in \varUpsilon$

$$\operatorname{Var}\langle \mathbb{D}_{\mathcal{K}}^{-1} \nabla\zeta, \boldsymbol{u} \rangle = \operatorname{Var}\langle \nabla\zeta, \mathbb{D}_{\mathcal{K}}^{-1} \boldsymbol{u} \rangle = \frac{n}{2} \operatorname{tr}(\boldsymbol{u} \mathbb{D}_{\mathcal{K}}^{-1} \varSigma^2 \mathbb{D}_{\mathcal{K}}^{-1} \boldsymbol{u}) = \frac{n}{2} \operatorname{tr}(\boldsymbol{u}\, B_{\mathcal{K}}^2\, \boldsymbol{u})$$

with $B_{\mathcal{K}}$ from (3.25). Moreover, Theorem B.26 yields under $\mathbb{p}_{\mathcal{K}} < n/8$

$$\mathbb{P}\bigl(\| \mathbb{D}_{\mathcal{K}}^{-1} \nabla\zeta \|_{\mathrm{Fr}} \geq \mathtt{r}_{\mathcal{K}}(\mathtt{x}) \bigr) \leq 3 \mathrm{e}^{-\mathtt{x}};$$

see (B.81), and $(\boldsymbol{\nabla\zeta})$ is verified. $\square$

A Local smoothness and a linearly perturbed optimization

This section discusses the problem of linearly and quadratically perturbed optimization of a smooth and concave function $f(\boldsymbol{\upsilon})$, $\boldsymbol{\upsilon} \in I\!\!R^p$.

A.1 Smoothness and self-concordance in Gateaux sense

Below we assume the function $f(\boldsymbol{\upsilon})$ to be strongly concave with the negative Hessian $\mathbb{F}(\boldsymbol{\upsilon}) \stackrel{\text{def}}{=} -\nabla^2 f(\boldsymbol{\upsilon}) \in \mathfrak{M}_p$ positive definite. Also, assume $f(\boldsymbol{\upsilon})$ three or sometimes even four times Gateaux differentiable in $\boldsymbol{\upsilon} \in \varUpsilon$. For any particular direction $\boldsymbol{u} \in I\!\!R^p$, we consider the univariate function $f(\boldsymbol{\upsilon} + t\boldsymbol{u})$ and measure its smoothness in t. Local smoothness of f will be described by the relative error of the Taylor expansion of the third or fourth order. Namely, define

$$\begin{aligned}
\delta_3(\boldsymbol{\upsilon}, \boldsymbol{u}) &= f(\boldsymbol{\upsilon} + \boldsymbol{u}) - f(\boldsymbol{\upsilon}) - \langle \nabla f(\boldsymbol{\upsilon}), \boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle, \\
\delta_3'(\boldsymbol{\upsilon}, \boldsymbol{u}) &= \langle \nabla f(\boldsymbol{\upsilon} + \boldsymbol{u}), \boldsymbol{u} \rangle - \langle \nabla f(\boldsymbol{\upsilon}), \boldsymbol{u} \rangle - \langle \nabla^2 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle ,
\end{aligned}$$

and

$$\delta_4(\boldsymbol{\upsilon}, \boldsymbol{u}) \stackrel{\text{def}}{=} f(\boldsymbol{\upsilon} + \boldsymbol{u}) - f(\boldsymbol{\upsilon}) - \langle \nabla f(\boldsymbol{\upsilon}), \boldsymbol{u} \rangle - \frac{1}{2} \langle \nabla^2 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 2} \rangle - \frac{1}{6} \langle \nabla^3 f(\boldsymbol{\upsilon}), \boldsymbol{u}^{\otimes 3} \rangle .$$

Now, for each $\boldsymbol{\upsilon}$, suppose to be given a positive symmetric operator $\mathbb{D}(\boldsymbol{\upsilon}) \in \mathfrak{M}_p$ with $\mathbb{D}^2(\boldsymbol{\upsilon}) \le \mathbb{F}(\boldsymbol{\upsilon}) = -\nabla^2 f(\boldsymbol{\upsilon})$ defining a local metric and a local vicinity around $\boldsymbol{\upsilon}$:

$$\mathcal{U}(\boldsymbol{\upsilon}) = \bigl\{ \boldsymbol{u} \in I\!\!R^p \colon \| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \| \le \mathtt{r} \bigr\}$$

for some radius $\mathtt{r}$.

Local smoothness properties of f are given via the quantities

$$\omega(\boldsymbol{\upsilon}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{u} \colon \| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \| \le \mathtt{r}} \frac{2 |\delta_3(\boldsymbol{\upsilon}, \boldsymbol{u})|}{\| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \|^2} , \quad \omega'(\boldsymbol{\upsilon}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{u} \colon \| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \| \le \mathtt{r}} \frac{|\delta_3'(\boldsymbol{\upsilon}, \boldsymbol{u})|}{\| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \|^2} . \tag{A.1}$$

The definition yields for any $\boldsymbol{u}$ with $\| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \| \le \mathtt{r}$

$$\bigl| \delta_3(\boldsymbol{\upsilon}, \boldsymbol{u}) \rangle \bigr| \le \frac{\omega(\boldsymbol{\upsilon})}{2} \| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \|^2 , \qquad \bigl| \delta_3'(\boldsymbol{\upsilon}, \boldsymbol{u}) \bigr| \le \omega'(\boldsymbol{\upsilon}) \| \mathbb{D}(\boldsymbol{\upsilon}) \boldsymbol{u} \|^2 . \tag{A.2}$$

The introduced quantities $\omega(\boldsymbol{\upsilon})$, $\omega'(\boldsymbol{\upsilon})$ strongly depend on the radius $\mathtt{r}$ of the local vicinity $\mathcal{U}(\boldsymbol{\upsilon})$. The results about Laplace approximation can be improved provided a homogeneous upper bound on the error of Taylor expansion. Assume a subset $\varUpsilon^{\circ}$ of $\varUpsilon$ to be fixed.

$(\mathcal{T}_3)$ *There exists* τ_3 *such that for all* $\boldsymbol{\upsilon} \in \Upsilon^{\circ}$

$$\bigl|\delta_3(\boldsymbol{\upsilon},\boldsymbol{u})\bigr| \le \frac{\tau_3}{6}\|\mathbb{D}(\boldsymbol{\upsilon})\,\boldsymbol{u}\|^3 , \quad \bigl|\delta_3'(\boldsymbol{\upsilon},\boldsymbol{u})\bigr| \le \frac{\tau_3}{2}\|\mathbb{D}(\boldsymbol{\upsilon})\,\boldsymbol{u}\|^3 , \quad \boldsymbol{u} \in \mathcal{U}(\boldsymbol{\upsilon}).$$

$(\mathcal{T}_4)$ *There exists* τ_4 *such that for all* $\boldsymbol{\upsilon} \in \Upsilon^{\circ}$

$$\bigl|\delta_4(\boldsymbol{\upsilon},\boldsymbol{u})\bigr| \le \frac{\tau_4}{24}\|\mathbb{D}(\boldsymbol{\upsilon})\,\boldsymbol{u}\|^4 , \qquad \boldsymbol{u} \in \mathcal{U}(\boldsymbol{\upsilon}).$$

We also present a version of **$(\mathcal{T}_3)$** resp. **$(\mathcal{T}_4)$** in terms of the third (resp. fourth) derivative of f.

$(\mathcal{T}_3^*)$ $f(\boldsymbol{\upsilon})$ *is strongly concave,* $\mathbb{D}^2(\boldsymbol{\upsilon}) \le -\nabla^2 f(\boldsymbol{\upsilon})$ *, and*

$$\sup_{\boldsymbol{u}\colon \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\| \le \mathtt{r}} \; \sup_{\boldsymbol{z} \in I\!\!R^p} \frac{\bigl|\langle \nabla^3 f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}^{\otimes 3}\rangle\bigr|}{\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}\|^3} \le \tau_3 .$$

$(\mathcal{T}_4^*)$ $f(\boldsymbol{\upsilon})$ *is strongly concave,* $\mathbb{D}^2(\boldsymbol{\upsilon}) \le -\nabla^2 f(\boldsymbol{\upsilon})$ *, and*

$$\sup_{\boldsymbol{u}\colon \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\| \le \mathtt{r}} \; \sup_{\boldsymbol{z} \in I\!\!R^p} \frac{\bigl|\langle \nabla^4 f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}^{\otimes 4}\rangle\bigr|}{\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}\|^4} \le \tau_4 .$$

By Banach's characterization Banach (1938), **$(\mathcal{T}_3^*)$** implies

$$\bigl|\langle \nabla^3 f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}_1 \otimes \boldsymbol{z}_2 \otimes \boldsymbol{z}_3\rangle\bigr| \le \tau_3 \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}_1\| \, \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}_2\| \, \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}_3\| \tag{A.3}$$

for any $\boldsymbol{u}$ with $\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\| \le \mathtt{r}$ and all $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \in I\!\!R^p$. Similarly under **$(\mathcal{T}_4^*)$**

$$\bigl|\langle \nabla^4 f(\boldsymbol{\upsilon}+\boldsymbol{u}), \boldsymbol{z}_1 \otimes \boldsymbol{z}_2 \otimes \boldsymbol{z}_3 \otimes \boldsymbol{z}_4\rangle\bigr| \le \tau_4 \prod_{k=1}^{4} \|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{z}_k\| , \quad \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4 \in I\!\!R^p . \tag{A.4}$$

Lemma A.1. *Under* **$(\mathcal{T}_3)$** *or* **$(\mathcal{T}_3^*)$***, the values* $\omega(\boldsymbol{\upsilon})$ *and* $\omega'(\boldsymbol{\upsilon})$ *from* (A.1) *satisfy*

$$\omega(\boldsymbol{\upsilon}) \le \frac{\tau_3\,\mathtt{r}}{3} , \qquad \omega'(\boldsymbol{\upsilon}) \le \tau_3\,\mathtt{r} , \qquad \boldsymbol{\upsilon} \in \Upsilon^{\circ}.$$

Proof. For any $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{\upsilon})$ with $\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\| \le \mathtt{r}$

$$\bigl|\delta_3(\boldsymbol{\upsilon},\boldsymbol{u})\bigr| \le \frac{\tau_3}{6}\,\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\|^3 \le \frac{\tau_3\,\mathtt{r}}{6}\,\|\mathbb{D}(\boldsymbol{\upsilon})\boldsymbol{u}\|^2,$$

and the bound for $\omega(\boldsymbol{\upsilon})$ follows. The proof for $\omega'(\boldsymbol{\upsilon})$ is similar. $\square$

Lemma A.2. *Assume* $(\mathcal{T}_3^*)$. *Then*

$$\bigl\|\mathbb{D}(\boldsymbol{v})^{-1}\bigl\{\nabla f(\boldsymbol{v}+\boldsymbol{u}) - \nabla f(\boldsymbol{v}) - \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}\rangle\bigr\| \le \frac{\tau_3}{2}\, \|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|^2 .$$

Moreover, under $(\mathcal{T}_4^*)$

$$\bigl\|\mathbb{D}(\boldsymbol{v})^{-1}\bigl\{\nabla f(\boldsymbol{v}+\boldsymbol{u}) - \nabla f(\boldsymbol{v}) - \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}\rangle - \frac{1}{2}\langle \nabla^3 f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2}\rangle\bigr\}\bigr\| \le \frac{\tau_4}{6}\, \|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|^3 .$$

Proof. We write $\mathbb{D}$ in place of $\mathbb{D}(\boldsymbol{v})$. Denote

$$\boldsymbol{A} \stackrel{\text{def}}{=} \nabla f(\boldsymbol{v}+\boldsymbol{u}) - \nabla f(\boldsymbol{v}) - \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}\rangle .$$

For any vector $\boldsymbol{w}$, $(\mathcal{T}_3^*)$ and (A.3) imply

$$\bigl|\langle \boldsymbol{A}, \boldsymbol{w}\rangle\bigr| \le \frac{\tau_3}{2}\, \|\mathbb{D}\boldsymbol{u}\|^2\, \|\mathbb{D}\boldsymbol{w}\|.$$

Therefore,

$$\|\mathbb{D}^{-1}\boldsymbol{A}\| = \sup_{\|\boldsymbol{w}\|=1} \bigl|\langle \mathbb{D}^{-1}\boldsymbol{A}, \boldsymbol{w}\rangle\bigr| = \sup_{\|\boldsymbol{w}\|=1} \bigl|\langle \boldsymbol{A}, \mathbb{D}^{-1}\boldsymbol{w}\rangle\bigr| \le \frac{\tau_3}{2}\, \|\mathbb{D}\boldsymbol{u}\|^2$$

which yields the first statement. For the second one, apply $\boldsymbol{A} \stackrel{\text{def}}{=} \nabla f(\boldsymbol{v}+\boldsymbol{u}) - \nabla f(\boldsymbol{v}) - \langle \nabla^2 f(\boldsymbol{v}), \boldsymbol{u}\rangle - \frac{1}{2}\langle \nabla^3 f(\boldsymbol{v}), \boldsymbol{u}^{\otimes 2}\rangle$ and use $(\mathcal{T}_4^*)$ and (A.4) instead of $(\mathcal{T}_3^*)$ and (A.3). $\square$

The values τ_3 and τ_4 are usually very small. Some quantitative bounds are given later in this section under the assumption that the function $f(\boldsymbol{v}) = \mathbb{E}L_G(\boldsymbol{v})$ can be written in the form $-f(\boldsymbol{v}) = nh(\boldsymbol{v})$ for a fixed smooth function $h(\boldsymbol{v})$ with the Hessian $\nabla^2 h(\boldsymbol{v})$. The factor n has meaning of the sample size.

$(\mathcal{S}_3)$ $\quad$ $-f(\boldsymbol{v}) = nh(\boldsymbol{v})$ *for* $h(\boldsymbol{v})$ *convex with* $\nabla^2 h(\boldsymbol{v}) \ge \mathbb{m}^2(\boldsymbol{v}) = \mathbb{D}^2(\boldsymbol{v})/n$ *and*

$$\sup_{\boldsymbol{u}\colon \|\mathbb{m}(\boldsymbol{v})\boldsymbol{u}\| \le \mathtt{r}/\sqrt{n}} \frac{\bigl|\langle \nabla^3 h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\rangle\bigr|}{\|\mathbb{m}(\boldsymbol{v})\boldsymbol{u}\|^3} \le \mathsf{c}_3 .$$

$(\mathcal{S}_4)$ $\quad$ *the function* $h(\cdot)$ *satisfies* $(\mathcal{S}_3)$ *and*

$$\sup_{\boldsymbol{u}\colon \|\mathbb{m}(\boldsymbol{v})\boldsymbol{u}\| \le \mathtt{r}/\sqrt{n}} \frac{\bigl|\langle \nabla^4 h(\boldsymbol{v}+\boldsymbol{u}), \boldsymbol{u}^{\otimes 4}\rangle\bigr|}{\|\mathbb{m}(\boldsymbol{v})\boldsymbol{u}\|^4} \le \mathsf{c}_4 .$$

$(\mathcal{S}_3)$ and $(\mathcal{S}_4)$ are local versions of the so-called self-concordance condition; see Nesterov (1988). It is also referred to as L_4–L_2 norm equivalence; see e.g. Mendelson and Zhivotovskiy (2020). In fact, they require that each univariate function $h(\boldsymbol{v}+t\boldsymbol{u})$ of $t \in \mathbb{R}$ is self-concordant with some universal constants c_3 and c_4 . Under $(\mathcal{S}_3)$ and $(\mathcal{S}_4)$, we can use $\mathbb{D}^2(\boldsymbol{v}) = n\,\mathbb{m}^2(\boldsymbol{v})$ and easily bound the values $\delta_3(\boldsymbol{v},\boldsymbol{u})$, $\delta_4(\boldsymbol{v},\boldsymbol{u})$, and $\omega(\boldsymbol{v})$, $\omega'(\boldsymbol{v})$.

Lemma A.3. *Suppose* $(\mathcal{S}_3)$. *Then* $(\mathcal{T}_3)$ *follows with* $\tau_3 = \mathtt{c}_3 n^{-1/2}$. *Moreover, for* $\omega(\boldsymbol{v})$ *and* $\omega'(\boldsymbol{v})$ *from* (A.1), *it holds*

$$\omega(\boldsymbol{v}) \le \frac{\mathtt{c}_3\, \mathtt{r}}{3n^{1/2}}, \qquad \omega'(\boldsymbol{v}) \le \frac{\mathtt{c}_3\, \mathtt{r}}{n^{1/2}}. \tag{A.5}$$

Also $(\mathcal{T}_4)$ *follows from* $(\mathcal{S}_4)$ *with* $\tau_4 = \mathtt{c}_4 n^{-1}$.

Proof. For any $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{v})$ and $t \in [0,1]$, by the Taylor expansion of the third order

$$\begin{aligned}
|\delta(\boldsymbol{v},\boldsymbol{u})| &\le \frac{1}{6}\bigl|\langle \nabla^3 f(\boldsymbol{v}+t\boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\rangle\bigr| = \frac{n}{6}\bigl|\langle \nabla^3 h(\boldsymbol{v}+t\boldsymbol{u}), \boldsymbol{u}^{\otimes 3}\rangle\bigr| \le \frac{n\,\mathtt{c}_3}{6}\,\|\mathbb{m}(\boldsymbol{v})\boldsymbol{u}\|^3 \\
&= \frac{n^{-1/2}\,\mathtt{c}_3}{6}\,\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|^3 \le \frac{n^{-1/2}\,\mathtt{c}_3\,\mathtt{r}}{6}\,\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\|^2 .
\end{aligned}$$

This implies $(\mathcal{T}_3)$ as well as (A.5); see (A.2). The statement about $(\mathcal{T}_4)$ is similar. □

A.2 Smoothness of the Hessian

For evaluation of the bias, we also need stronger smoothness conditions. Let f be a strongly concave function. Essentially we need some continuity of the negative Hessian $\mathbb{F}(\boldsymbol{v}) = -\nabla^2 f(\boldsymbol{v})$. Let us fix $\mathtt{r}$ and for any $\boldsymbol{v} \in \varUpsilon$, some $\mathbb{D}(\boldsymbol{v}) \le \mathbb{F}^{1/2}(\boldsymbol{v})$, and define

$$\omega^{+}(\boldsymbol{v}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{u}\colon \|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\| \le \mathtt{r}} \; \sup_{\boldsymbol{z} \in I\!\!R^p} \frac{|\langle \mathbb{F}(\boldsymbol{v}+\boldsymbol{u}) - \mathbb{F}(\boldsymbol{v}), \boldsymbol{z}^{\otimes 2}\rangle|}{\|\mathbb{D}(\boldsymbol{v})\boldsymbol{z}\|^2}. \tag{A.6}$$

This definition of $\omega^{+}(\boldsymbol{v})$ is, of course, stronger than the one-directional definition of $\omega(\boldsymbol{v})$ in (A.1). However, in typical examples these quantities $\omega(\boldsymbol{v})$ and $\omega^{+}(\boldsymbol{v})$ are similar.

Lemma A.4. *Condition* $(\mathcal{T}_3^*)$ *yields* (A.6) *with* $\omega^{+}(\boldsymbol{v}) \le \tau_3\,\mathtt{r}$ *and*

$$\|\mathbb{F}^{-1/2}(\boldsymbol{v})\,\mathbb{F}(\boldsymbol{v}+\boldsymbol{u})\,\mathbb{F}^{-1/2}(\boldsymbol{v}) - I\!\!I_p\| \le \tau_3\,\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\| \le \tau_3\,\mathtt{r}, \quad \|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\| \le \mathtt{r}. \tag{A.7}$$

Proof. Let $\|\mathbb{D}(\boldsymbol{v})\boldsymbol{u}\| \le \mathtt{r}$. By $(\mathcal{T}_3^*)$, for any $\boldsymbol{z} \in I\!\!R^p$, it holds with $\boldsymbol{\delta} = \mathbb{F}^{-1/2}(\boldsymbol{v})\boldsymbol{z}$

$$\begin{aligned}
\bigl|\bigl\langle \mathbb{F}^{-1/2}(\boldsymbol{v})\,\mathbb{F}(\boldsymbol{v}+\boldsymbol{u})\,\mathbb{F}^{-1/2}(\boldsymbol{v}) - I\!\!I_p, \boldsymbol{z}^{\otimes 2}\bigr\rangle\bigr| &\le \sup_{t\in[0,1]}\bigl|\langle \nabla^3 f(\boldsymbol{v}+t\boldsymbol{u}), \boldsymbol{u}\otimes\boldsymbol{\delta}^{\otimes 2}\rangle\bigr| \\
&\le \tau_3\,\|\mathbb{D}(\boldsymbol{v})\,\boldsymbol{u}\|\,\|\mathbb{D}(\boldsymbol{v})\,\boldsymbol{\delta}\|^2 \le \tau_3\,\mathtt{r}\,\|\mathbb{D}(\boldsymbol{v})\,\mathbb{F}^{-1/2}(\boldsymbol{v})\boldsymbol{z}\|^2 \le \tau_3\,\mathtt{r}\,\|\boldsymbol{z}\|^2.
\end{aligned}$$

This yields (A.7). □

A.3 Optimization after linear perturbation. A basic lemma

Let $f(\boldsymbol{v})$ be a smooth concave function,

$$\boldsymbol{v}^* = \operatorname*{argmax}_{\boldsymbol{v}} f(\boldsymbol{v}),$$

and $\mathbb{F} = -\nabla^2 f(\boldsymbol{v}^*)$. Later we study the question of how the point of maximum and the value of maximum of f change if we add a linear or quadratic component to f.

A.3.1 A linear perturbation

This section studies the case of a linear change of f. More precisely, let another function $g(\boldsymbol{v})$ satisfy for some vector $\boldsymbol{A}$

$$g(\boldsymbol{v}) - g(\boldsymbol{v}^*) = \bigl\langle \boldsymbol{v} - \boldsymbol{v}^*, \boldsymbol{A} \bigr\rangle + f(\boldsymbol{v}) - f(\boldsymbol{v}^*). \tag{A.8}$$

A typical example corresponds to $f(\boldsymbol{v}) = \mathbb{E}L(\boldsymbol{v})$ and $g(\boldsymbol{v}) = L(\boldsymbol{v})$ for a random function $L(\boldsymbol{v})$ with a linear stochastic component $\zeta(\boldsymbol{v}) = L(\boldsymbol{v}) - \mathbb{E}L(\boldsymbol{v})$. Then (A.8) is satisfied with $\boldsymbol{A} = \nabla\zeta$. Define

$$\boldsymbol{v}^\circ \stackrel{\text{def}}{=} \operatorname*{argmax}_{\boldsymbol{v}} g(\boldsymbol{v}), \qquad g(\boldsymbol{v}^\circ) = \max_{\boldsymbol{v}} g(\boldsymbol{v}). \tag{A.9}$$

The aim of the analysis is to evaluate the quantities $\boldsymbol{v}^\circ - \boldsymbol{v}^*$ and $g(\boldsymbol{v}^\circ) - g(\boldsymbol{v}^*)$. The results will be stated in terms of the norm $\|\mathbb{F}^{-1/2}\boldsymbol{A}\|$. First, we consider the case of a quadratic function f.

Lemma A.5. *Let $f(\boldsymbol{v})$ be quadratic with $\nabla^2 f(\boldsymbol{v}) \equiv -\mathbb{F}$. If $g(\boldsymbol{v})$ satisfy (A.8), then*

$$\boldsymbol{v}^\circ - \boldsymbol{v}^* = \mathbb{F}^{-1}\boldsymbol{A}, \qquad g(\boldsymbol{v}^\circ) - g(\boldsymbol{v}^*) = \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2. \tag{A.10}$$

Proof. If $f(\boldsymbol{v})$ is quadratic, then, of course, under (A.8), $g(\boldsymbol{v})$ is quadratic as well with $-\nabla^2 g(\boldsymbol{v}) \equiv \mathbb{F}$. This implies

$$\nabla g(\boldsymbol{v}^*) - \nabla g(\boldsymbol{v}^\circ) = \mathbb{F}(\boldsymbol{v}^\circ - \boldsymbol{v}^*).$$

Further, (A.8) and $\nabla f(\boldsymbol{v}^*) = 0$ yield $\nabla g(\boldsymbol{v}^*) = \boldsymbol{A}$. Together with $\nabla g(\boldsymbol{v}^\circ) = 0$, this implies $\boldsymbol{v}^\circ - \boldsymbol{v}^* = \mathbb{F}^{-1}\boldsymbol{A}$. The Taylor expansion of g at $\boldsymbol{v}^\circ$ yields by $\nabla g(\boldsymbol{v}^\circ) = 0$

$$g(\boldsymbol{v}^*) - g(\boldsymbol{v}^\circ) = -\frac{1}{2}\|\mathbb{F}^{1/2}(\boldsymbol{v}^\circ - \boldsymbol{v}^*)\|^2 = -\frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2$$

and the assertion follows. $\square$

The next result describes the concentration properties of $\boldsymbol{\upsilon}^{\circ}$ from (A.9) in a local elliptic set of the form

$$\mathcal{A}(\mathtt{r}) \stackrel{\text{def}}{=} \{\boldsymbol{\upsilon}\colon \|\mathbb{F}^{1/2}(\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{*})\| \le \mathtt{r}\}, \tag{A.11}$$

where $\mathtt{r}$ is slightly larger than $\|\mathbb{F}^{-1/2}\boldsymbol{A}\|$.

Proposition A.6. *Let $f(\boldsymbol{\upsilon})$ be a strongly concave function with $f(\boldsymbol{\upsilon}^{*}) = \max_{\boldsymbol{\upsilon}} f(\boldsymbol{\upsilon})$ and $\mathbb{F} = -\nabla^2 f(\boldsymbol{\upsilon}^{*})$. Let further $g(\boldsymbol{\upsilon})$ and $f(\boldsymbol{\upsilon})$ be related by (A.8) with some vector $\boldsymbol{A}$. Fix $\nu < 1$ and $\mathtt{r}$ such that $\|\mathbb{F}^{-1/2}\boldsymbol{A}\| \le \nu\,\mathtt{r}$. Suppose now that $f(\boldsymbol{\upsilon})$ satisfy (A.1) for $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}^{*}$, $\mathbb{D}(\boldsymbol{\upsilon}^{*}) = \mathbb{F}^{1/2} = \mathbb{D}$, and ω' such that*

$$1 - \nu - \omega' > 0. \tag{A.12}$$

Then for $\boldsymbol{\upsilon}^{\circ}$ from (A.9), it holds

$$\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^{\circ} - \boldsymbol{\upsilon}^{*})\| \le \mathtt{r}\,.$$

Proof. With $\mathbb{D} = \mathbb{F}^{1/2}$, the bound $\|\mathbb{D}^{-1}\boldsymbol{A}\| \le \nu\,\mathtt{r}$ implies for any $\boldsymbol{u}$

$$\bigl|\langle \boldsymbol{A}, \boldsymbol{u}\rangle\bigr| = \bigl|\langle \mathbb{D}^{-1}\boldsymbol{A}, \mathbb{D}\boldsymbol{u}\rangle\bigr| \le \nu\,\mathtt{r}\|\mathbb{D}\boldsymbol{u}\|\,.$$

Let $\boldsymbol{\upsilon}$ be a point on the boundary of the set $\mathcal{A}(\mathtt{r})$ from (A.11). We also write $\boldsymbol{u} = \boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{*}$. The idea is to show that the derivative $\frac{d}{dt} g(\boldsymbol{\upsilon}^{*} + t\boldsymbol{u}) < 0$ is negative for $t > 1$. Then all the extreme points of $g(\boldsymbol{\upsilon})$ are within $\mathcal{A}(\mathtt{r})$. We use the decomposition

$$g(\boldsymbol{\upsilon}^{*} + t\boldsymbol{u}) - g(\boldsymbol{\upsilon}^{*}) = \langle \boldsymbol{A}, \boldsymbol{u}\rangle\, t + f(\boldsymbol{\upsilon}^{*} + t\boldsymbol{u}) - f(\boldsymbol{\upsilon}^{*}).$$

With $h(t) = f(\boldsymbol{\upsilon}^{*} + t\boldsymbol{u}) - f(\boldsymbol{\upsilon}^{*}) + \langle \boldsymbol{A}, \boldsymbol{u}\rangle\, t$, it holds

$$\frac{d}{dt} f(\boldsymbol{\upsilon}^{*} + t\boldsymbol{u}) = -\langle \boldsymbol{A}, \boldsymbol{u}\rangle + h'(t). \tag{A.13}$$

By definition of $\boldsymbol{\upsilon}^{*}$, it also holds $h'(0) = \langle \boldsymbol{A}, \boldsymbol{u}\rangle$. The identity $\nabla^2 f(\boldsymbol{\upsilon}^{*}) = -\mathbb{D}^2$ yields $h''(0) = -\|\mathbb{D}\boldsymbol{u}\|^2$. Bound (A.2) implies for $|t| \le 1$

$$\bigl|h'(t) - h'(0) - t h''(0)\bigr| \le t\,\bigl|h''(0)\bigr|\,\omega'\,.$$

For $t = 1$, we obtain by (A.12)

$$h'(1) \le -\langle \boldsymbol{A}, \boldsymbol{u}\rangle + h''(0) - h''(0)\,\omega' \le -\bigl|h''(0)\bigr|(1 - \omega' - \nu) < 0.$$

Moreover, concavity of $h(t)$ imply that $h'(t) - h'(0)$ decreases in t for $t > 1$. Further, summing up the above derivation yields

$$\frac{d}{dt} g(\boldsymbol{v}^* + t\boldsymbol{u})\Big|_{t=1} \le -\|\mathbb{D}\boldsymbol{u}\|^2 (1 - \nu - \omega') < 0.$$

As $\frac{d}{dt} g(\boldsymbol{v}^* + t\boldsymbol{u})$ decreases with $t \ge 1$ together with $h'(t)$ due to (A.13), the same applies to all such t. This implies the assertion. $\square$

The result of Proposition A.6 allows to localize the point $\boldsymbol{v}^\circ = \operatorname{argmax}_{\boldsymbol{v}} g(\boldsymbol{v})$ in the local vicinity $\mathcal{A}(\mathtt{r})$ of $\boldsymbol{v}^*$. The use of smoothness properties of g or, equivalently, of f, in this vicinity helps to obtain rather sharp expansions for $\boldsymbol{v}^\circ - \boldsymbol{v}^*$ and for $g(\boldsymbol{v}^\circ) - g(\boldsymbol{v}^*)$; cf. (A.10).

Proposition A.7. *Under the conditions of Proposition A.6*

$$-\frac{\omega}{1+\omega} \|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \le 2g(\boldsymbol{v}^\circ) - 2g(\boldsymbol{v}^*) - \|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \le \frac{\omega}{1-\omega} \|\mathbb{D}^{-1}\boldsymbol{A}\|^2 . \tag{A.14}$$

Also

$$\begin{aligned} \|\mathbb{D}(\boldsymbol{v}^\circ - \boldsymbol{v}^*) - \mathbb{D}^{-1}\boldsymbol{A}\|^2 &\le \frac{3\omega}{(1-\omega)^2} \|\mathbb{D}^{-1}\boldsymbol{A}\|^2 , \\ \|\mathbb{D}(\boldsymbol{v}^\circ - \boldsymbol{v}^*)\| &\le \frac{1+\sqrt{2\omega}}{1-\omega} \|\mathbb{D}^{-1}\boldsymbol{A}\| . \end{aligned} \tag{A.15}$$

Proof. By (A.1), for any $\boldsymbol{v} \in \mathcal{A}(\mathtt{r})$

$$\Big| f(\boldsymbol{v}^*) - f(\boldsymbol{v}) - \frac{1}{2}\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*)\|^2 \Big| \le \frac{\omega}{2}\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*)\|^2. \tag{A.16}$$

Further,

$$\begin{aligned} & g(\boldsymbol{v}) - g(\boldsymbol{v}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \\ & \quad = \langle \boldsymbol{v} - \boldsymbol{v}^*, \boldsymbol{A} \rangle + f(\boldsymbol{v}) - f(\boldsymbol{v}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \\ & \quad = -\frac{1}{2}\big\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*) - \mathbb{D}^{-1}\boldsymbol{A}\big\|^2 + f(\boldsymbol{v}) - f(\boldsymbol{v}^*) + \frac{1}{2}\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*)\|^2. \end{aligned} \tag{A.17}$$

As $\boldsymbol{v}^\circ \in \mathcal{A}(\mathtt{r})$ and it maximizes $g(\boldsymbol{v})$, we derive by (A.16)

$$\begin{aligned} g(\boldsymbol{v}^\circ) - g(\boldsymbol{v}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 &= \max_{\boldsymbol{v} \in \mathcal{A}(\mathtt{r})} \Big\{ g(\boldsymbol{v}) - g(\boldsymbol{v}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \Big\} \\ &\le \max_{\boldsymbol{v} \in \mathcal{A}(\mathtt{r})} \Big\{ -\frac{1}{2}\big\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*) - \mathbb{D}^{-1}\boldsymbol{A}\big\|^2 + \frac{\omega}{2}\|\mathbb{D}(\boldsymbol{v} - \boldsymbol{v}^*)\|^2 \Big\}. \end{aligned}$$

Further, $\max_{\boldsymbol{u}}\bigl\{\omega\|\boldsymbol{u}\|^2 - \|\boldsymbol{u}-\boldsymbol{\xi}\|^2\bigr\} = \frac{\omega}{1-\omega}\|\boldsymbol{\xi}\|^2$ for $\omega \in [0,1)$ and $\boldsymbol{\xi} \in I\!R^p$, yielding

$$g(\boldsymbol{\upsilon}^\circ) - g(\boldsymbol{\upsilon}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \le \frac{\omega}{2(1-\omega)}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2.$$

Similarly

$$g(\boldsymbol{\upsilon}^\circ) - g(\boldsymbol{\upsilon}^*) - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \ge \max_{\boldsymbol{\upsilon}\in\mathcal{A}(\mathtt{r})}\Bigl\{-\frac{1}{2}\bigl\|\mathbb{D}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^*) - \mathbb{D}^{-1}\boldsymbol{A}\bigr\|^2 - \frac{\omega}{2}\|\mathbb{D}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^*)\|^2\Bigr\}$$
$$= -\frac{\omega}{2(1+\omega)}\,\|\mathbb{D}^{-1}\boldsymbol{A}\|^2. \tag{A.18}$$

These bounds imply (A.14).

Now we derive similarly to (A.17) that for $\boldsymbol{\upsilon} \in \mathcal{A}(\mathtt{r})$

$$g(\boldsymbol{\upsilon}) - g(\boldsymbol{\upsilon}^*) \le \bigl\langle \boldsymbol{\upsilon}-\boldsymbol{\upsilon}^*, \boldsymbol{A}\bigr\rangle - \frac{1-\omega}{2}\|\mathbb{D}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^*)\|^2.$$

A particular choice $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}^\circ$ yields

$$g(\boldsymbol{\upsilon}^\circ) - g(\boldsymbol{\upsilon}^*) \le \bigl\langle \boldsymbol{\upsilon}^\circ-\boldsymbol{\upsilon}^*, \boldsymbol{A}\bigr\rangle - \frac{1-\omega}{2}\|\mathbb{D}(\boldsymbol{\upsilon}^\circ-\boldsymbol{\upsilon}^*)\|^2.$$

Combining this result with (A.18) allows to bound

$$\bigl\langle \boldsymbol{\upsilon}^\circ-\boldsymbol{\upsilon}^*, \boldsymbol{A}\bigr\rangle - \frac{1-\omega}{2}\|\mathbb{D}(\boldsymbol{\upsilon}^\circ-\boldsymbol{\upsilon}^*)\|^2 - \frac{1}{2}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2 \ge -\frac{\omega}{2(1+\omega)}\|\mathbb{D}^{-1}\boldsymbol{A}\|^2.$$

Further, for $\boldsymbol{\xi} = \mathbb{D}^{-1}\boldsymbol{A}$, $\boldsymbol{u} = \mathbb{D}(\boldsymbol{\upsilon}^\circ - \boldsymbol{\upsilon}^*)$, and $\omega \in [0, 1/3]$, the inequality

$$2\bigl\langle \boldsymbol{u}, \boldsymbol{\xi}\bigr\rangle - (1-\omega)\|\boldsymbol{u}\|^2 - \|\boldsymbol{\xi}\|^2 \ge -\frac{\omega}{1+\omega}\|\boldsymbol{\xi}\|^2$$

implies

$$\Bigl\|\boldsymbol{u} - \frac{1}{1-\omega}\boldsymbol{\xi}\Bigr\|^2 \le \frac{2\omega}{(1+\omega)(1-\omega)^2}\|\boldsymbol{\xi}\|^2$$

yielding for $\omega \le 1/3$

$$\|\boldsymbol{u} - \boldsymbol{\xi}\| \le \biggl(\omega + \sqrt{\frac{2\omega}{1+\omega}}\biggr)\frac{\|\boldsymbol{\xi}\|}{1-\omega} \le \frac{\sqrt{3\omega}\|\boldsymbol{\xi}\|}{1-\omega},$$
$$\|\boldsymbol{u}\| \le \biggl(1 + \sqrt{\frac{2\omega}{1+\omega}}\biggr)\frac{\|\boldsymbol{\xi}\|}{1-\omega} \le \frac{1+\sqrt{2\omega}\|\boldsymbol{\xi}\|}{1-\omega},$$

and (A.15) follows. □

Remark A.1. The roles of the functions f and g are exchangeable. In particular, the results from (A.15) apply with $\mathbb{D}^2 = -\nabla^2 g(\boldsymbol{\upsilon}^\circ) = -\nabla^2 f(\boldsymbol{\upsilon}^\circ)$ provided that (A.1) is fulfilled at $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}^\circ$.

A.3.2 Basic lemma under third order smoothness

In this section, we assume that f satisfies the local smoothness conditions **($\mathcal{T}_3$)** with some $\mathbb{D}^2 \leq \mathbb{F} = -\nabla^2 f(\boldsymbol{\upsilon}^*)$, a small constant τ_3, and some radius $\mathtt{r}$ to be specified below.

Proposition A.8. *Let $f(\boldsymbol{\upsilon})$ be a strongly concave function with $f(\boldsymbol{\upsilon}^*) = \max_{\boldsymbol{\upsilon}} f(\boldsymbol{\upsilon})$ and $\mathbb{F} = -\nabla^2 f(\boldsymbol{\upsilon}^*)$. Let $g(\boldsymbol{\upsilon})$ fulfill (A.8) with some vector $\boldsymbol{A}$. Suppose that*

- $\|\mathbb{F}^{-1/2}\boldsymbol{A}\| \leq \nu\,\mathtt{r}$ *for* $\nu \leq 2/3$ *and some* $\mathtt{r}$*;*
- $f(\boldsymbol{\upsilon})$ *follows* **($\mathcal{T}_3$)** *with this* $\mathtt{r}$ *and some* $\mathbb{D}^2 \leq \mathbb{F}$ *and* $\tau_3 \geq 0$*;*
- $\tau_3\,\mathtt{r}/2 < 1 - \nu$.

Then $\boldsymbol{\upsilon}^\circ = \operatorname{argmax}_{\boldsymbol{\upsilon}} g(\boldsymbol{\upsilon})$ satisfies

$$\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^\circ - \boldsymbol{\upsilon}^*)\| \leq \mathtt{r}. \tag{A.19}$$

Proof. Let $\boldsymbol{\upsilon}$ be a point on the boundary of the set $\mathcal{A}(\mathtt{r})$ from (A.11). We also write $\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^* = \mathbb{F}^{-1/2}\boldsymbol{u}$ for $\|\boldsymbol{u}\| = \mathtt{r}$. The idea is to show that the derivative $\frac{d}{dt} g(\boldsymbol{\upsilon}^* + t\,\mathbb{F}^{-1/2}\boldsymbol{u}) < 0$ is negative for $t > 1$. Then all the extreme points of $g(\boldsymbol{\upsilon})$ are within $\mathcal{A}(\mathtt{r})$. We use the decomposition

$$g(\boldsymbol{\upsilon}^* + t\,\mathbb{F}^{-1/2}\boldsymbol{u}) - g(\boldsymbol{\upsilon}^*) = \langle \boldsymbol{A}, \mathbb{F}^{-1/2}\boldsymbol{u}\rangle\, t + f(\boldsymbol{\upsilon}^* + t\,\mathbb{F}^{-1/2}\boldsymbol{u}) - f(\boldsymbol{\upsilon}^*).$$

With $h(t) = f(\boldsymbol{\upsilon}^* + t\,\mathbb{F}^{-1/2}\boldsymbol{u}) - f(\boldsymbol{\upsilon}^*) - \langle \boldsymbol{A}, \mathbb{F}^{-1/2}\boldsymbol{u}\rangle\, t$, it holds

$$\frac{d}{dt} f(\boldsymbol{\upsilon}^* + t\,\mathbb{F}^{-1/2}\boldsymbol{u}) = \langle \boldsymbol{A}, \mathbb{F}^{-1/2}\boldsymbol{u}\rangle + h'(t).$$

By definition of $\boldsymbol{\upsilon}^*$, it also holds $\nabla f(\boldsymbol{\upsilon}^*) = 0$ and

$$|h'(0)| = \bigl|\langle \boldsymbol{A}, \mathbb{F}^{-1/2}\boldsymbol{u}\rangle\bigr| \leq \bigl|\langle \mathbb{F}^{-1/2}\boldsymbol{A}, \boldsymbol{u}\rangle\bigr| \leq \|\mathbb{F}^{-1/2}\boldsymbol{A}\|\;\|\boldsymbol{u}\| \leq \nu\,\mathtt{r}^2.$$

By $\nabla^2 f(\boldsymbol{\upsilon}^*) = -\mathbb{F}$, it holds $h''(0) = -\|\boldsymbol{u}\|^2 = -\mathtt{r}^2$, and **($\mathcal{T}_3$)** implies by $\mathbb{D}^2 \leq \mathbb{F}$

$$\bigl|h'(t) - h'(0) - t\,h''(0)\bigr| \leq \tau_3\, t^3\,\|\boldsymbol{u}\|^3/2 = \tau_3\, t^3\,\mathtt{r}^3/2, \qquad |t| \leq 1.$$

For $t = 1$, we obtain by $\tau_3\,\mathtt{r}/2 \leq 1 - \nu$

$$h'(1) \leq h'(0) + h''(0) + \tau_3\,\mathtt{r}^3/2 \leq \nu\,\mathtt{r}^2 - \mathtt{r}^2 + \tau_3\,\mathtt{r}^3/2 < 0.$$

Moreover, concavity of $h(t)$ implies that $h'(t) - h'(0)$ decreases in t for $t > 1$ and hence, $h'(t) \leq h'(1) < 0$ for $|t| \geq 1$. This implies the assertion. $\square$

The result of Proposition A.8 allows to localize the point $\boldsymbol{\upsilon}^{\circ} = \operatorname{argmax}_{\boldsymbol{\upsilon}} g(\boldsymbol{\upsilon})$ in the local vicinity $\mathcal{A}(\mathtt{r})$ of $\boldsymbol{\upsilon}^{*}$. The use of smoothness properties of g or, equivalently, of f, in this vicinity helps to obtain rather sharp expansions for $\boldsymbol{\upsilon}^{\circ}-\boldsymbol{\upsilon}^{*}$ and for $g(\boldsymbol{\upsilon}^{\circ})-g(\boldsymbol{\upsilon}^{*})$.

Proposition A.9. *Under the conditions of Proposition A.8*

$$
-\frac{2\tau_3}{3}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3 \le 2g(\boldsymbol{\upsilon}^{\circ}) - 2g(\boldsymbol{\upsilon}^{*}) - \|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \le \tau_3\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3 . \tag{A.20}
$$

Moreover, under $(\mathcal{T}_3^{*})$

$$
\begin{aligned}
\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^{\circ}-\boldsymbol{\upsilon}^{*}) - \mathbb{F}^{-1/2}\boldsymbol{A}\| &\le \frac{3\tau_3}{4}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 ,\\
\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^{\circ}-\boldsymbol{\upsilon}^{*})\| &\le \|\mathbb{F}^{-1/2}\boldsymbol{A}\| + \frac{3\tau_3}{4}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 .
\end{aligned} \tag{A.21}
$$

Proof. By $(\mathcal{T}_3)$ and $\nabla f(\boldsymbol{\upsilon}^{*}) = 0$, for any $\boldsymbol{\upsilon} \in \mathcal{A}(\mathtt{r})$

$$
\Bigl| f(\boldsymbol{\upsilon}^{*}) - f(\boldsymbol{\upsilon}) - \frac{1}{2}\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^2 \Bigr| \le \frac{\tau_3}{6}\|\mathbb{D}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^3 \le \frac{\tau_3}{6}\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^3. \tag{A.22}
$$

Further,

$$
\begin{aligned}
& g(\boldsymbol{\upsilon}) - g(\boldsymbol{\upsilon}^{*}) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \\
&\quad = \bigl\langle \boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*}, \boldsymbol{A} \bigr\rangle + f(\boldsymbol{\upsilon}) - f(\boldsymbol{\upsilon}^{*}) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \\
&\quad = -\frac{1}{2}\bigl\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*}) - \mathbb{F}^{-1/2}\boldsymbol{A}\bigr\|^2 + f(\boldsymbol{\upsilon}) - f(\boldsymbol{\upsilon}^{*}) + \frac{1}{2}\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^2.
\end{aligned}
$$

As $\boldsymbol{\upsilon}^{\circ} \in \mathcal{A}(\mathtt{r})$ and it maximizes $g(\boldsymbol{\upsilon})$, we derive by (A.22) and Lemma A.10

$$
\begin{aligned}
& g(\boldsymbol{\upsilon}^{\circ}) - g(\boldsymbol{\upsilon}^{*}) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 = \max_{\boldsymbol{\upsilon}\in\mathcal{A}(\mathtt{r})}\Bigl\{ g(\boldsymbol{\upsilon}) - g(\boldsymbol{\upsilon}^{*}) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \Bigr\} \\
&\quad \le \max_{\boldsymbol{\upsilon}\in\mathcal{A}(\mathtt{r})}\Bigl\{ -\frac{1}{2}\bigl\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*}) - \mathbb{F}^{-1/2}\boldsymbol{A}\bigr\|^2 + \frac{\tau_3}{6}\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^3 \Bigr\} \le \frac{\tau_3}{2}\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3 .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& g(\boldsymbol{\upsilon}^{\circ}) - g(\boldsymbol{\upsilon}^{*}) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \\
&\quad \ge \max_{\boldsymbol{\upsilon}\in\mathcal{A}(\mathtt{r})}\Bigl\{ -\frac{1}{2}\bigl\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*}) - \mathbb{F}^{-1/2}\boldsymbol{A}\bigr\|^2 - \frac{\tau_3}{6}\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{*})\|^3 \Bigr\} \ge -\frac{\tau_3}{3}\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3 .
\end{aligned}
$$

These bounds imply (A.20). For proving (A.21) use that $\nabla f(\boldsymbol{\upsilon}^{*}) = 0$, $\nabla g(\boldsymbol{\upsilon}^{\circ}) = 0$, $\nabla f(\boldsymbol{\upsilon}^{\circ}) = \nabla g(\boldsymbol{\upsilon}^{\circ}) - \boldsymbol{A} = -\boldsymbol{A}$, and $-\nabla^2 f(\boldsymbol{\upsilon}^{*}) = \mathbb{F}$. By Lemma A.2 with $\boldsymbol{u} = \mathbb{F}^{-1}\boldsymbol{A}$

$$
\bigl\|\mathbb{F}^{-1/2}\{\nabla f(\boldsymbol{\upsilon}^{*} + \mathbb{F}^{-1}\boldsymbol{A}) + \boldsymbol{A}\}\bigr\| \le \frac{\tau_3}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 .
$$

Further, by (A.8)

$$\begin{aligned} \big\| \mathbb{F}^{-1/2} \nabla g(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) \big\| &= \big\| \mathbb{F}^{-1/2} \{ \nabla g(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) - \boldsymbol{A} + \boldsymbol{A} \} \big\| \\ &\le \big\| \mathbb{F}^{-1/2} \{ \nabla f(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) + \boldsymbol{A} \} \big\| \le \frac{\tau_3}{2} \| \mathbb{F}^{-1/2}\boldsymbol{A} \|^2. \end{aligned}$$

By definition $\nabla g(\boldsymbol{v}^\circ) = 0$. This yields

$$\| \mathbb{F}^{-1/2} \{ \nabla g(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) - \nabla g(\boldsymbol{v}^\circ) \} \| \le \frac{\tau_3}{2} \| \mathbb{F}^{-1/2}\boldsymbol{A} \|^2 . \tag{A.23}$$

Now we can use with $\varDelta = \boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A} - \boldsymbol{v}^\circ$

$$\mathbb{F}^{-1/2} \{ \nabla g(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) - \nabla g(\boldsymbol{v}^\circ) \} = \biggl(\int_0^1 \mathbb{F}^{-1/2} \, \nabla^2 g(\boldsymbol{v}^\circ + t\varDelta) \, \mathbb{F}^{-1/2} \, dt \biggr) \mathbb{F}^{1/2} \varDelta .$$

By (A.8) $\nabla^2 g(\boldsymbol{v}) = \nabla^2 f(\boldsymbol{v})$ for all $\boldsymbol{v}$. If $\| \mathbb{F}^{1/2}(\boldsymbol{v} - \boldsymbol{v}^*) \| \le \mathtt{r}$, then $(\boldsymbol{\mathcal{T}_3^*})$ implies $\| \mathbb{F}^{-1/2} \, \nabla^2 f(\boldsymbol{v}) \, \mathbb{F}^{-1/2} + I\!\!I_p \| \le \omega^+$ with $\omega^+ \le \tau_3 \, \mathtt{r} \le 1/3$. Hence,

$$\| \mathbb{F}^{-1/2} \{ \nabla g(\boldsymbol{v}^* + \mathbb{F}^{-1}\boldsymbol{A}) - \nabla g(\boldsymbol{v}^\circ) \} \| \ge (1 - \omega^+) \| \mathbb{F}^{1/2} (\boldsymbol{v}^\circ - \boldsymbol{v}^* - \mathbb{F}^{-1}\boldsymbol{A}) \| .$$

This and (A.23) yield

$$\| \mathbb{F}^{1/2} (\boldsymbol{v}^\circ - \boldsymbol{v}^* - \mathbb{F}^{-1}\boldsymbol{A}) \| \le \frac{\tau_3}{2(1 - \omega^+)} \| \mathbb{F}^{-1/2}\boldsymbol{A} \|^2 \le \frac{3\tau_3}{4} \| \mathbb{F}^{-1/2}\boldsymbol{A} \|^2 ,$$

and (A.21) follows. □

Lemma A.10. *For any* $\boldsymbol{\xi} \in I\!\!R^p$ *with* $\| \boldsymbol{\xi} \| \le 2\mathtt{r}/3$ *and* τ *with* $\tau \, \mathtt{r} \le 1/2$, *it holds*

$$\max_{\|\boldsymbol{u}\| \le \mathtt{r}} \Bigl(\frac{\tau}{3} \| \boldsymbol{u} \|^3 - \| \boldsymbol{u} - \boldsymbol{\xi} \|^2 \Bigr) \le \frac{\tau}{2} \, \| \boldsymbol{\xi} \|^3 , \tag{A.24}$$

$$\min_{\|\boldsymbol{u}\| \le \mathtt{r}} \Bigl(\frac{\tau}{3} \| \boldsymbol{u} \|^3 + \| \boldsymbol{u} - \boldsymbol{\xi} \|^2 \Bigr) \le \frac{\tau}{3} \, \| \boldsymbol{\xi} \|^3 . \tag{A.25}$$

Proof. Any maximizer $\boldsymbol{u}$ of the left-hand side of (A.24) satisfies

$$\tau \| \boldsymbol{u} \|^{1/2} \boldsymbol{u} - 2(\boldsymbol{u} - \boldsymbol{\xi}) = 0 .$$

Therefore, $\boldsymbol{u} = \rho \boldsymbol{\xi}$ for some ρ, reducing the problem to the univariate case:

$$\max_{\|\boldsymbol{u}\| \le \mathtt{r}} \Bigl(\frac{\tau}{3} \| \boldsymbol{u} \|^3 - \| \boldsymbol{u} - \boldsymbol{\xi} \|^2 \Bigr) = \| \boldsymbol{\xi} \|^2 \max_{\rho \colon \, \|\rho \boldsymbol{\xi}\| \le \mathtt{r}} \Bigl(\frac{\tau \| \boldsymbol{\xi} \|}{3} \rho^3 - (\rho - 1)^2 \Bigr).$$

Define $a = \tau \| \boldsymbol{\xi} \|$. The conditions $\| \boldsymbol{\xi} \| \le 2\mathtt{r}/3$ and $\tau \, \mathtt{r} \le 1/2$ imply $a \le 1/3$ and $\| \rho \boldsymbol{\xi} \| \le \mathtt{r}$ implies $|\rho| \le 3/2$. The function $a\rho^3/3 - (\rho - 1)^2$ is concave on the interval

$|\rho| \le 3/2$ and hence, the maximizer ρ fulfills $a\rho^2 - 2\rho + 2 = 0$ yielding

$$\rho = \frac{1 \pm \sqrt{1-2a}}{a}, \qquad |\rho| \le 3/2.$$

As $a \in [0,1/3]$, we can only use

$$\rho_a = \frac{1-\sqrt{1-2a}}{a} = \frac{2}{1+\sqrt{1-2a}}, \quad \rho_a - 1 = \frac{1-\sqrt{1-2a}}{1+\sqrt{1-2a}} = \frac{2a}{(1+\sqrt{1-2a})^2}.$$

Therefore,

$$\max_{\|\boldsymbol{u}\|\le \mathtt{r}} \Bigl(\frac{\tau}{3}\|\boldsymbol{u}\|^3 - \|\boldsymbol{u}-\boldsymbol{\xi}\|^2 \Bigr) = \frac{\tau\|\boldsymbol{\xi}\|^3\rho_a^3}{3} - \|\boldsymbol{\xi}\|^2(\rho_a-1)^2$$
$$= \frac{\tau\|\boldsymbol{\xi}\|^3}{3}\,\frac{8(1+\sqrt{1-2a})-12a}{(1+\sqrt{1-2a})^4} \le \frac{\tau\|\boldsymbol{\xi}\|^3}{3}\max_{a\in[0,1/3]}\frac{8(1+\sqrt{1-2a})-12a}{(1+\sqrt{1-2a})^4} \le \frac{\tau\|\boldsymbol{\xi}\|^3}{2}.$$

The function $\phi(a) \stackrel{\mathrm{def}}{=} \frac{8(1+\sqrt{1-2a})-12a}{(1+\sqrt{1-2a})^4}$ increases with $a \in [0,1/3]$. To see this, represent with $y = 1+\sqrt{1-2a}$ or $-2a = (y-1)^2 - 1 = y^2 - 2y$,

$$\phi(a) = \frac{8(1+\sqrt{1-2a})-12a}{(1+\sqrt{1-2a})^4} = \frac{8y+6y^2-12y}{y^4} = \frac{6y-4}{y^3},$$

and the latter decreases with $y \ge 1$. Moreover, $\phi(1/3) \le 3/2$, and (A.24) follows. The proof of (A.25) is similar. The general case can be reduced to the univariate one by using $\boldsymbol{u} = \rho\boldsymbol{\xi}$. With $a = \tau\|\boldsymbol{\xi}\|$, the minimizer ρ_a reads as

$$\rho_a = \frac{\sqrt{1+2a}-1}{a} = \frac{2}{1+\sqrt{1+2a}}, \quad 1-\rho_a = \frac{\sqrt{1+2a}-1}{\sqrt{1+2a}+1} = \frac{2a}{(\sqrt{1+2a}+1)^2},$$

yielding for $a \in [0,1/3]$

$$\min_{\|\boldsymbol{u}\|\le \mathtt{r}} \Bigl(\frac{\tau}{3}\|\boldsymbol{u}\|^3 + \|\boldsymbol{u}-\boldsymbol{\xi}\|^2 \Bigr) = \frac{\tau\|\boldsymbol{\xi}\|^3\rho_a^3}{3} + \|\boldsymbol{\xi}\|^2(\rho_a-1)^2$$
$$\le \frac{\tau\|\boldsymbol{\xi}\|^3}{3}\max_{a\in[0,1/3]}\frac{8(1+\sqrt{1+2a})+12a}{(1+\sqrt{1+2a})^4},$$

and with $y = 1+\sqrt{1+2a}$ or $2a = y^2 - 2y$,

$$\max_{a\in[0,1/3]}\frac{8(1+\sqrt{1+2a})+12a}{(1+\sqrt{1+2a})^4} \le \max_{y\ge 2}\frac{8y+6y^2-12y}{y^4} = \max_{y\ge 2}\frac{6y-4}{y^3} = 1,$$

and (A.25) follows. □

Remark A.2. As in Remark A.1, the roles of f and g can be exchanged. In particular, (A.21) applies with $\mathbb{F} = \mathbb{F}(\boldsymbol{v}^\circ)$ provided that $(\mathcal{T}_3^*)$ is also fulfilled at $\boldsymbol{v}^\circ$.

A.3.3 Advanced approximation under fourth order smoothness

The bounds of Proposition A.9 can be made more accurate if f follows **$(\mathcal{T}_3^*)$** and **$(\mathcal{T}_4^*)$** and one can apply the Taylor expansion of the fourth order.

Proposition A.11. *Let $f(\boldsymbol{\upsilon})$ be a strongly concave function with $f(\boldsymbol{\upsilon}^*) = \max_{\boldsymbol{\upsilon}} f(\boldsymbol{\upsilon})$ and $\mathbb{F} = -\nabla^2 f(\boldsymbol{\upsilon}^*)$. Denote $\mathcal{T}(\boldsymbol{u}) = \frac{1}{6}\langle \nabla^3 f(\boldsymbol{\upsilon}^*), \boldsymbol{u}^{\otimes 3}\rangle$ for $\boldsymbol{u} \in \mathbb{R}^p$. Let $g(\boldsymbol{\upsilon})$ fulfill (A.8) with some vector $\boldsymbol{A}$. Suppose that*

- $\|\mathbb{F}^{-1/2}\boldsymbol{A}\| \le \nu\,\mathtt{r}$ *for* $\nu \le 2/3$ *and some* $\mathtt{r}$;
- $f(\boldsymbol{\upsilon})$ *follows* **$(\mathcal{T}_3^*)$** *and* **$(\mathcal{T}_4^*)$** *with this* $\mathtt{r}$ *and some* $\mathbb{D}^2 \le \mathbb{F}$;
- τ_3 *from* **$(\mathcal{T}_3^*)$** *satisfies* $\tau_3\,\mathtt{r} < 1/3$.

Then $\boldsymbol{\upsilon}^\circ = \operatorname{argmax}_{\boldsymbol{\upsilon}} g(\boldsymbol{\upsilon})$ satisfies (A.19) $\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^\circ - \boldsymbol{\upsilon}^*)\| \le \mathtt{r}$. Further, define

$$\boldsymbol{a} = \mathbb{F}^{-1}\{\boldsymbol{A} + \nabla\mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A})\}. \tag{A.26}$$

Then

$$\begin{aligned} \|\mathbb{F}^{1/2}\boldsymbol{a} - \mathbb{F}^{-1/2}\boldsymbol{A}\| &\le \frac{\tau_3}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \le \frac{\tau_3\,\nu\,\mathtt{r}}{2}\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|, \\ \|\mathbb{F}^{1/2}\boldsymbol{a}\| &\le \Bigl(1 + \frac{\tau_3\,\nu\,\mathtt{r}}{2}\Bigr)\|\mathbb{F}^{-1/2}\boldsymbol{A}\|, \end{aligned} \tag{A.27}$$

and

$$\|\mathbb{F}^{1/2}(\boldsymbol{\upsilon}^\circ - \boldsymbol{\upsilon}^* - \boldsymbol{a})\| \le \frac{\tau_4 + 3\tau_3^2}{6(1 - \tau_3\,\mathtt{r})}\,\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3 \le \frac{\tau_4 + 3\tau_3^2}{3}\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3. \tag{A.28}$$

Also

$$\begin{aligned} &\Bigl| g(\boldsymbol{\upsilon}^\circ) - g(\boldsymbol{\upsilon}^*) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 - \mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A}) \Bigr| \\ &\quad \le \frac{\tau_4 + 7\tau_3^2}{16}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 + \frac{(\tau_4 + 3\tau_3^2)^2}{5}\,\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^6. \end{aligned} \tag{A.29}$$

Proof. Proposition A.8 yields (A.19). W.l.o.g. assume $\boldsymbol{\upsilon}^* = 0$. It holds by **$(\mathcal{T}_3^*)$** in view of $\mathbb{D} \le \mathbb{F}^{1/2}$

$$\begin{aligned} &\|\mathbb{F}^{1/2}\boldsymbol{a} - \mathbb{F}^{-1/2}\boldsymbol{A}\| = \|\mathbb{F}^{-1/2}\nabla\mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A})\| \\ &\quad = \sup_{\|\boldsymbol{u}\|=1} 3\bigl|\langle \mathcal{T}, \mathbb{F}^{-1}\boldsymbol{A} \otimes \mathbb{F}^{-1}\boldsymbol{A} \otimes \mathbb{F}^{-1/2}\boldsymbol{u}\rangle\bigr| \le \frac{\tau_3}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \end{aligned} \tag{A.30}$$

yielding (A.27) by $\|\mathbb{F}^{-1/2}\boldsymbol{A}\| \le \nu\,\mathtt{r}$. Similarly for any $\boldsymbol{v}$

$$\|\mathbb{F}^{-1/2}\,\nabla^2\mathcal{T}(\mathbb{F}^{-1/2}\boldsymbol{v})\,\mathbb{F}^{-1/2}\| = \sup_{\|\boldsymbol{u}\|=1} 6\bigl|\langle \mathcal{T}, \mathbb{F}^{-1/2}\boldsymbol{v} \otimes (\mathbb{F}^{-1/2}\boldsymbol{u})^{\otimes 2}\rangle\bigr| \le \tau_3\|\boldsymbol{v}\|\,.$$

Furthermore, the tensor $\nabla^2\mathcal{T}(\boldsymbol{u})$ is linear in $\boldsymbol{u}$ and hence,

$$\begin{aligned}
&\sup_{t\in[0,1]} \|\mathbb{F}^{-1/2}\,\nabla^2\mathcal{T}(t\boldsymbol{a} + (1-t)\mathbb{F}^{-1}\boldsymbol{A})\,\mathbb{F}^{-1/2}\| \\
&\quad = \max\{\|\mathbb{F}^{-1/2}\,\nabla^2\mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A})\,\mathbb{F}^{-1/2}\|, \|\mathbb{F}^{-1}\nabla^2\mathcal{T}(\boldsymbol{a})\|\} \le \tau_3 \max\{\|\mathbb{F}^{-1/2}\boldsymbol{A}\|, \|\mathbb{F}^{1/2}\boldsymbol{a}\|\}\,.
\end{aligned}$$

Later we assume $\|\mathbb{F}^{1/2}\boldsymbol{a}\| \ge \|\mathbb{F}^{-1}\boldsymbol{A}\|$ in view of (A.27). This and (A.30) yield

$$\begin{aligned}
&\|\mathbb{F}^{-1/2}\nabla\mathcal{T}(\boldsymbol{a}) - \mathbb{F}^{-1/2}\nabla\mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A})\| \\
&\quad \le \sup_{t\in[0,1]} \|\mathbb{F}^{-1/2}\,\nabla^2\mathcal{T}(t\boldsymbol{a} + (1-t)\mathbb{F}^{-1}\boldsymbol{A})\,\mathbb{F}^{-1/2}\|\;\|\mathbb{F}^{1/2}(\boldsymbol{a} - \mathbb{F}^{-1}\boldsymbol{A})\| \le \frac{\tau_3^2}{2}\,\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3\,.
\end{aligned}$$

Further, by Lemma A.2 in view of $\nabla\mathcal{T}(\boldsymbol{a}) = \frac{1}{2}\langle\nabla^3 f(\boldsymbol{v}^*), \boldsymbol{a}\otimes\boldsymbol{a}\rangle$

$$\bigl\|\mathbb{F}^{-1/2}\{\nabla f(\boldsymbol{a}) + \mathbb{F}\boldsymbol{a} - \nabla\mathcal{T}(\boldsymbol{a})\}\bigr\| \le \frac{\tau_4}{6}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3\,.$$

Now we can bound the norm of $\mathbb{F}^{-1/2}\nabla g(\boldsymbol{a})$. In view of (A.8) and (A.26), it holds

$$\begin{aligned}
\bigl\|\mathbb{F}^{-1/2}\nabla g(\boldsymbol{a})\bigr\| &= \bigl\|\mathbb{F}^{-1/2}\{\nabla g(\boldsymbol{a}) + \mathbb{F}\boldsymbol{a} - \nabla\mathcal{T}(\boldsymbol{A}) - \boldsymbol{A}\}\bigr\| \\
&\le \bigl\|\mathbb{F}^{-1/2}\{\nabla f(\boldsymbol{a}) + \mathbb{F}\boldsymbol{a} - \nabla\mathcal{T}(\boldsymbol{a})\}\bigr\| + \|\mathbb{F}^{-1/2}\{\nabla\mathcal{T}(\boldsymbol{a}) - \nabla\mathcal{T}(\boldsymbol{A})\}\| \\
&\le \frac{\tau_4 + 3\tau_3^2}{6}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3.
\end{aligned}$$

By definition $\nabla g(\boldsymbol{v}^\circ) = 0$. This yields

$$\|\mathbb{F}^{-1/2}\{\nabla g(\boldsymbol{a}) - \nabla g(\boldsymbol{v}^\circ)\}\| \le \frac{\tau_4 + 3\tau_3^2}{6}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3\,. \tag{A.31}$$

Now we can use with $\varDelta = \boldsymbol{a} - \boldsymbol{v}^\circ$

$$\mathbb{F}^{-1/2}\{\nabla g(\boldsymbol{a}) - \nabla g(\boldsymbol{v}^\circ)\} = \biggl(\int_0^1 \mathbb{F}^{-1/2}\,\nabla^2 g(\boldsymbol{v}^\circ + t\varDelta)\,\mathbb{F}^{-1/2}\,dt\biggr)\,\mathbb{F}^{1/2}\varDelta\,.$$

By (A.8) $\nabla^2 g(\boldsymbol{v}) = \nabla^2 f(\boldsymbol{v})$ for all $\boldsymbol{v}$. If $\|\mathbb{F}^{1/2}(\boldsymbol{v} - \boldsymbol{v}^*)\| \le \mathtt{r}$, then **$(\mathcal{T}_3^*)$** implies $\|\mathbb{F}^{-1/2}\,\nabla^2 f(\boldsymbol{v})\,\mathbb{F}^{-1/2} + I\!\!I_p\| \le \omega^+$ with $\omega^+ \le \tau_3\,\mathtt{r} \le 1/3$. Hence,

$$\|\mathbb{F}^{-1/2}\{\nabla g(\boldsymbol{a}) - \nabla g(\boldsymbol{v}^\circ)\}\| \ge (1 - \tau_3\,\mathtt{r})\|\mathbb{F}^{1/2}(\boldsymbol{v}^\circ - \boldsymbol{a})\|\,.$$

This, (A.27), and (A.31) yield in view of $\tau_3 \mathtt{r} \le 1/3$ and $\nu = 2/3$

$$\|\mathbb{F}^{1/2}(\boldsymbol{v}^\circ - \boldsymbol{a})\| \le \frac{\tau_4 + 3\tau_3^2}{6(1 - \tau_3 \mathtt{r})}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^3 \le \frac{\tau_4 + 3\tau_3^2}{3}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^3 , \tag{A.32}$$

and (A.28) follows. It remains to bound $g(\boldsymbol{v}^\circ) - g(0)$. By (A.30)

$$\frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 - \langle \boldsymbol{A}, \boldsymbol{a} \rangle + \frac{1}{2}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^2 = \frac{1}{2}\|\mathbb{F}^{1/2}\boldsymbol{a} - \mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \le \frac{\tau_3^2}{8}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 .$$

First consider $g(\boldsymbol{a}) - g(0)$. One more use of $(\mathcal{T}_4^*)$ yields with $\boldsymbol{v}^* = 0$ and $-\nabla^2 f(0) = \mathbb{F}$

$$\begin{aligned}
&\Bigl| g(\boldsymbol{a}) - g(0) - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 - \mathcal{T}(\boldsymbol{a}) \Bigr| \\
&\quad = \Bigl| f(\boldsymbol{a}) - f(0) + \langle \boldsymbol{A}, \boldsymbol{a} \rangle - \frac{1}{2}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 - \mathcal{T}(\boldsymbol{a}) \Bigr| \\
&\quad \le \Bigl| f(\boldsymbol{a}) - f(0) + \frac{1}{2}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^2 - \mathcal{T}(\boldsymbol{a}) \Bigr| + \frac{\tau_3^2}{8}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 \\
&\quad \le \frac{\tau_4}{24}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^4 + \frac{\tau_3^2}{8}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 \le \frac{\tau_4 + 2\tau_3^2}{16}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 .
\end{aligned}$$

Also by $\nabla g(\boldsymbol{v}^\circ) = 0$ and (A.32), it holds for some $\boldsymbol{v} \in [\boldsymbol{a}, \boldsymbol{v}^\circ]$ as in (A.32)

$$\begin{aligned}
\bigl| g(\boldsymbol{a}) - g(\boldsymbol{v}^\circ) \bigr| &\le \frac{1}{2}\|\mathbb{F}^{-1/2}\, \nabla^2 g(\boldsymbol{v})\, \mathbb{F}^{-1/2}\| \; \|\mathbb{F}^{1/2}(\boldsymbol{a} - \boldsymbol{v}^\circ)\|^2 \\
&\le \frac{(\tau_4 + 3\tau_3^2)^2}{72(1 - \tau_3 \mathtt{r})^3}\, \|\mathbb{F}^{1/2}\boldsymbol{a}\|^6 < \frac{(\tau_4 + 3\tau_3^2)^2}{5}\, \|\mathbb{F}^{-1/2}\boldsymbol{A}\|^6 ,
\end{aligned}$$

Moreover, similarly to (A.30)

$$\begin{aligned}
\bigl| \mathcal{T}(\boldsymbol{a}) - \mathcal{T}(\mathbb{F}^{-1}\boldsymbol{A}) \bigr| &\le \sup_{t \in [0,1]} \bigl\| \mathbb{F}^{-1/2} \nabla \mathcal{T}(t\mathbb{F}^{-1}\boldsymbol{A} + (1-t)\boldsymbol{a}) \bigr\| \; \bigl\| \mathbb{F}^{1/2}\boldsymbol{a} - \mathbb{F}^{-1/2}\boldsymbol{A} \bigr| \\
&\le \frac{\tau_3^2}{4}\|\mathbb{F}^{1/2}\boldsymbol{a}\|^2 \, \|\mathbb{F}^{-1/2}\boldsymbol{A}\|^2 \le \frac{5\tau_3^2}{16}\|\mathbb{F}^{-1/2}\boldsymbol{A}\|^4 .
\end{aligned}$$

Summing up the obtained bounds yields (A.29). $\square$

A.3.4 Quadratic penalization

Here we discuss the case when $g(\boldsymbol{v}) - f(\boldsymbol{v})$ is quadratic. The general case can be reduced to the situation with $g(\boldsymbol{v}) = f(\boldsymbol{v}) - \|G\boldsymbol{v}\|^2/2$. To make the dependence of G more explicit, denote $f_G(\boldsymbol{v}) \stackrel{\text{def}}{=} f(\boldsymbol{v}) - \|G\boldsymbol{v}\|^2/2$,

$$\boldsymbol{v}^* = \operatorname*{argmax}_{\boldsymbol{v}} f(\boldsymbol{v}), \quad \boldsymbol{v}_G^* = \operatorname*{argmax}_{\boldsymbol{v}} f_G(\boldsymbol{v}) = \operatorname*{argmax}_{\boldsymbol{v}} \bigl\{ f(\boldsymbol{v}) - \|G\boldsymbol{v}\|^2/2 \bigr\}.$$

We study the bias $\boldsymbol{v}_G^* - \boldsymbol{v}^*$ induced by this penalization. To get some intuition, consider first the case of a quadratic function $f(\boldsymbol{v})$.

Lemma A.12. *Let* $f(\boldsymbol{\upsilon})$ *be quadratic with* $\mathbb{F} \equiv -\nabla^2 f(\boldsymbol{\upsilon})$ *and* $\mathbb{F}_G = \mathbb{F} + G^2$ *. Then*

$$\begin{aligned} \boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^* &= -\mathbb{F}_G^{-1} G^2 \boldsymbol{\upsilon}^*, \\ f_G(\boldsymbol{\upsilon}_G^*) - f_G(\boldsymbol{\upsilon}^*) &= \frac{1}{2} \| \mathbb{F}_G^{-1/2} G^2 \boldsymbol{\upsilon}^* \|^2 . \end{aligned}$$

Proof. Quadraticity of $f(\boldsymbol{\upsilon})$ implies quadraticity of $f_G(\boldsymbol{\upsilon})$ with $\nabla^2 f_G(\boldsymbol{\upsilon}) \equiv -\mathbb{F}_G$ and

$$\nabla f_G(\boldsymbol{\upsilon}_G^*) - \nabla f_G(\boldsymbol{\upsilon}^*) = -\mathbb{F}_G \, (\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*).$$

Further, $\nabla f(\boldsymbol{\upsilon}^*) = 0$ yielding $\nabla f_G(\boldsymbol{\upsilon}^*) = -G^2 \boldsymbol{\upsilon}^*$. Together with $\nabla f_G(\boldsymbol{\upsilon}_G^*) = 0$, this implies $\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^* = -\mathbb{F}_G^{-1} G^2 \boldsymbol{\upsilon}^*$. The Taylor expansion of f_G at $\boldsymbol{\upsilon}_G^*$ yields

$$f_G(\boldsymbol{\upsilon}^*) - f_G(\boldsymbol{\upsilon}_G^*) = -\frac{1}{2} \| \mathbb{F}_G^{1/2} (\boldsymbol{\upsilon}^* - \boldsymbol{\upsilon}_G^*) \|^2 = -\frac{1}{2} \| \mathbb{F}_G^{-1/2} G^2 \boldsymbol{\upsilon}^* \|^2$$

and the assertion follows. □

Now we turn to the general case with f satisfying $(\boldsymbol{\mathcal{T}_3^*})$.

Proposition A.13. *Let* f *be concave,* $\boldsymbol{\upsilon}^* = \operatorname{argmax}_{\boldsymbol{\upsilon}} f(\boldsymbol{\upsilon})$ *,* $\mathbb{F} = -\nabla^2 f(\boldsymbol{\upsilon}^*)$ *, and* $\mathbb{F}_G = -\nabla^2 f(\boldsymbol{\upsilon}^*) + G^2$ *. Define*

$$\mathsf{b}_G = \| \mathbb{F}_G^{-1/2} \, G^2 \boldsymbol{\upsilon}^* \| .$$

With $\nu = 2/3$ *, assume* $(\boldsymbol{\mathcal{T}_3^*})$ *for* $\mathtt{r} = \nu^{-1} \mathsf{b}_G$ *and* $\mathbb{D}^2 \le \mathbb{F}_G$ *. Then* $\| \mathbb{F}_G^{1/2} (\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*) \| \le \nu^{-1} \, \mathsf{b}_G$ *or, equivalently,*

$$\boldsymbol{\upsilon}_G^* \in \mathcal{A}_G \stackrel{\mathrm{def}}{=} \{ \boldsymbol{\upsilon} \colon \| \mathbb{F}_G^{1/2} (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^*) \| \le \nu^{-1} \, \mathsf{b}_G \}. \tag{A.33}$$

Moreover,

$$\begin{aligned} \bigl\| \mathbb{F}_G^{1/2} (\boldsymbol{\upsilon}_G^* - \boldsymbol{\upsilon}^*) + \mathbb{F}_G^{-1/2} G^2 \boldsymbol{\upsilon}^* \bigr\| &\le \frac{3\tau_3}{4} \, \mathsf{b}_G^2 , \\ \bigl| 2 f_G(\boldsymbol{\upsilon}_G^*) - 2 f_G(\boldsymbol{\upsilon}^*) - \mathsf{b}_G^2 \bigr| &\le \tau_3 \, \mathsf{b}_G^3 . \end{aligned}$$

Proof. Define $g_G(\boldsymbol{\upsilon})$ by

$$g_G(\boldsymbol{\upsilon}) - g_G(\boldsymbol{\upsilon}_G^*) = f_G(\boldsymbol{\upsilon}) - f_G(\boldsymbol{\upsilon}_G^*) + \langle G^2 \boldsymbol{\upsilon}^*, \boldsymbol{\upsilon} - \boldsymbol{\upsilon}_G^* \rangle. \tag{A.34}$$

The function f_G is concave, the same holds for g_G from (A.34). Now we show that $\boldsymbol{\upsilon}^* = \operatorname{argmax} g_G(\boldsymbol{\upsilon})$. It suffices to check that $\nabla g_G(\boldsymbol{\upsilon}^*) = 0$. By definition, $\nabla f(\boldsymbol{\upsilon}^*) = 0$, and hence, $\nabla f_G(\boldsymbol{\upsilon}^*) = -G^2 \boldsymbol{\upsilon}^* + G^2 \boldsymbol{\upsilon}^* = 0$. Now the results follow from Proposition A.9 applied with $f(\boldsymbol{\upsilon}) = g_G(\boldsymbol{\upsilon}) = f_G(\boldsymbol{\upsilon}) - \langle \boldsymbol{A}, \boldsymbol{\upsilon} \rangle$, $g(\boldsymbol{\upsilon}) = f_G(\boldsymbol{\upsilon})$, and $\boldsymbol{A} = G^2 \boldsymbol{\upsilon}^*$. □

The bound on the bias can be further improved under fourth-order smoothness of f using the results of Proposition A.11.

Proposition A.14. *Let f be concave and $\boldsymbol{\upsilon}^{*} = \operatorname{argmax}_{\boldsymbol{\upsilon}} f(\boldsymbol{\upsilon})$. With $\mathbb{F}_{G} = -\nabla^{2} f(\boldsymbol{\upsilon}^{*}) + G^{2}$ and $\nu = 2/3$, assume $(\mathcal{T}_{3}^{*})$ and $(\mathcal{T}_{4}^{*})$ for $\mathbb{D}^{2} \leq \mathbb{F}_{G}$ and $\mathtt{r} = \mathtt{r}_{G} \stackrel{\text{def}}{=} \nu^{-1} \mathsf{b}_{G}$, where $\mathsf{b}_{G} = \|\mathbb{F}_{G}^{-1/2} G^{2} \boldsymbol{\upsilon}^{*}\|$. Then (A.33) holds. Furthermore, define*

$$
\boldsymbol{m}_{G} = \mathbb{F}_{G}^{-1} \{ G^{2} \boldsymbol{\upsilon}^{*} + \nabla \mathcal{T}(\mathbb{F}_{G}^{-1} G^{2} \boldsymbol{\upsilon}^{*}) \}
$$

with $\mathcal{T}(\boldsymbol{u}) = \frac{1}{6} \langle \nabla^{3} f(\boldsymbol{\upsilon}^{*}), \boldsymbol{u}^{\otimes 3} \rangle$. Then $\|\mathbb{F}_{G}^{1/2} \boldsymbol{m}_{G}\| \leq \mathtt{r}_{G}$,

$$
\begin{aligned}
\|\mathbb{F}^{1/2} \boldsymbol{m}_{G} - \mathbb{F}_{G}^{-1/2} G^{2} \boldsymbol{\upsilon}^{*}\| &\leq \frac{\tau_{3}}{2} \mathsf{b}_{G}^{2} \leq \frac{\tau_{3} \, \nu \, \mathtt{r}_{G}}{2} \, \mathsf{b}_{G}, \\
\|\mathbb{F}^{1/2} \boldsymbol{m}_{G}\| &\leq \Bigl(1 + \frac{\tau_{3} \, \nu \, \mathtt{r}_{G}}{2} \Bigr) \mathsf{b}_{G} \, ,
\end{aligned}
$$

and

$$
\|\mathbb{F}_{G}^{1/2} (\boldsymbol{\upsilon}^{*} - \boldsymbol{\upsilon}_{G}^{*} - \boldsymbol{m}_{G})\| \leq \frac{\tau_{4} + 3\tau_{3}^{2}}{6(1 - \tau_{3} \, \mathtt{r})} \, \|\mathbb{F}^{1/2} \boldsymbol{m}_{G}\|^{3} \leq \frac{\tau_{4} + 3\tau_{3}^{2}}{3} \, \mathsf{b}_{G}^{3} \, .
$$

Also

$$
\Bigl| f_{G}(\boldsymbol{\upsilon}_{G}^{*}) - f_{G}(\boldsymbol{\upsilon}^{*}) - \frac{1}{2} \mathsf{b}_{G}^{2} - \mathcal{T}(\boldsymbol{m}_{G}) \Bigr| \leq \frac{\tau_{4} + 2\tau_{3}^{2}}{16} \mathsf{b}_{G}^{4} + \frac{(\tau_{4} + 3\tau_{3}^{2})^{2}}{5} \, \mathsf{b}_{G}^{6} \, .
$$

B Deviation bounds for quadratic forms

Here we collect some useful results from probability theory mainly concerning Gaussian and non-Gaussian quadratic forms.

B.1 Moments of a Gaussian quadratic form

Let Z be standard normal in $\mathbb{R}^{p}$ for $p \leq \infty$. Given a self-adjoint trace operator B, consider a quadratic form $\bigl\langle BZ, Z \bigr\rangle$.

Lemma B.1. *It holds*

$$
\mathbb{E} \bigl\langle BZ, Z \bigr\rangle = \operatorname{tr} B.
$$

Moreover,

$$
\begin{aligned}
\mathbb{E}\big(\langle BZ, Z\rangle - \operatorname{tr} B\big)^2 &= 2\operatorname{tr} B^2, \\
\mathbb{E}\big(\langle BZ, Z\rangle - \operatorname{tr} B\big)^3 &= 8\operatorname{tr} B^3, \\
\mathbb{E}\big(\langle BZ, Z\rangle - \operatorname{tr} B\big)^4 &= 48\operatorname{tr} B^4 + 12(\operatorname{tr} B^2)^2, \\
\mathbb{E}\big(\langle BZ, Z\rangle - \operatorname{tr} B\big)^5 &= 512\operatorname{tr} B^5 + 32\operatorname{tr} B^2 \, \operatorname{tr} B^3,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\big\langle BZ, Z\big\rangle^2 &= (\operatorname{tr} B)^2 + 2\operatorname{tr} B^2, \\
\mathbb{E}\big\langle BZ, Z\big\rangle^3 &= (\operatorname{tr} B)^3 + 6\operatorname{tr} B \, \operatorname{tr} B^2 + 8\operatorname{tr} B^3, \\
\mathbb{E}\big\langle BZ, Z\big\rangle^4 &= (\operatorname{tr} B)^4 + 12(\operatorname{tr} B)^2 \operatorname{tr} B^2 + 32(\operatorname{tr} B)\operatorname{tr} B^3 + 48\operatorname{tr} B^4 + 12(\operatorname{tr} B^2)^2, \\
\operatorname{Var}\big\langle BZ, Z\big\rangle^2 &= 8(\operatorname{tr} B)^2 \operatorname{tr} B^2 + 32(\operatorname{tr} B)\operatorname{tr} B^3 + 48\operatorname{tr} B^4 + 8(\operatorname{tr} B^2)^2.
\end{aligned}
$$

Moreover, if $B \le I\!I_p$ *and* $\mathbb{p} = \operatorname{tr} B$ *, then* $\operatorname{tr} B^m \le \mathbb{p}\|B\|^{m-1}$ *for* $m \ge 1$ *and*

$$
\begin{aligned}
\mathbb{E}\big\langle BZ, Z\big\rangle^2 &\le \mathbb{p}^2 + 2\mathbb{p}\|B\| && \le (\mathbb{p} + \|B\|)^2, \\
\mathbb{E}\big\langle BZ, Z\big\rangle^3 &\le \mathbb{p}^3 + 6\mathbb{p}^2\|B\| + 8\mathbb{p}\|B\|^2 && \le (\mathbb{p} + 2\|B\|)^3, \\
\mathbb{E}\big\langle BZ, Z\big\rangle^4 &\le \mathbb{p}^4 + 12\mathbb{p}^3\|B\| + 44\mathbb{p}^2\|B\|^2 + 48\mathbb{p}\|B\|^3 && \le (\mathbb{p} + 3\|B\|)^4, \\
\mathbb{E}\big\langle BZ, Z\big\rangle^5 &\le \mathbb{p}^5 + 20\mathbb{p}^4\|B\| + 140\mathbb{p}^3\|B\|^2 + 272\mathbb{p}^2\|B\|^3 + 512\mathbb{p}\|B\|^4 && \le (\mathbb{p} + 4\|B\|)^5 . \\
\operatorname{Var}\big\langle BZ, Z\big\rangle^2 &\le 8\mathbb{p}^3 + 40\mathbb{p}^2\|B\| + 48\mathbb{p}\|B\|^2 .
\end{aligned}
$$

Finally,

$$
\mathbb{E}(ZZ^\top - I\!I_p)B(ZZ^\top - I\!I_p) = B + \operatorname{tr}(B) I\!I_p
$$

yielding

$$
\mathbb{E}\|B(ZZ^\top - I\!I_p)\|_{\mathrm{Fr}}^2 = (\operatorname{tr} B)^2 + \operatorname{tr} B^2. \tag{B.1}
$$

Proof. Let $\chi = \gamma^2 - 1$ for γ standard normal. Then $\mathbb{E}\chi = 0$, $\mathbb{E}\chi^2 = 2$, $\mathbb{E}\chi^3 = 8$, $\mathbb{E}\chi^4 = 60$. Without loss of generality assume B diagonal: $B = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Then

$$
\xi \stackrel{\text{def}}{=} \langle BZ, Z\rangle - \operatorname{tr} B = \sum_{j=1}^{p} \lambda_j(\gamma_j^2 - 1),
$$

where γ_j are i.i.d. standard normal. This easily yields with $\mathbb{p}_m = \operatorname{tr}(B^m)$

$$
\begin{aligned}
\mathbb{E}\xi^2 &= \sum_{j=1}^{p} \lambda_j^2 \mathbb{E}(\gamma_j^2 - 1)^2 = \mathbb{E}\chi^2 \operatorname{tr} B^2 = 2\mathbb{p}_2\,, \\
\mathbb{E}\xi^3 &= \sum_{j=1}^{p} \lambda_j^3 \mathbb{E}(\gamma_j^2 - 1)^3 = \mathbb{E}\chi^3 \operatorname{tr} B^3 = 8\mathbb{p}_3\,, \\
\mathbb{E}\xi^4 &= \sum_{j=1}^{p} \lambda_j^4 (\gamma_j^2 - 1)^4 + \sum_{i \neq j} \lambda_i^2 \lambda_j^2 \mathbb{E}(\gamma_i^2 - 1)^2 \mathbb{E}(\gamma_j^2 - 1)^2 \\
&= \bigl(\mathbb{E}\chi^4 - 3(\mathbb{E}\chi^2)^2\bigr) \operatorname{tr} B^4 + 3(\mathbb{E}\chi^2 \operatorname{tr} B^2)^2 = 48\mathbb{p}_4 + 12\mathbb{p}_2^2, \\
\mathbb{E}\xi^5 &= \sum_{j=1}^{p} \lambda_j^5 (\gamma_j^2 - 1)^5 + \sum_{i \neq j} \lambda_i^2 \lambda_j^3 \mathbb{E}(\gamma_i^2 - 1)^2 \mathbb{E}(\gamma_j^2 - 1)^3 \\
&= \bigl\{\mathbb{E}(\gamma^2 - 1)^5 - \mathbb{E}(\gamma^2 - 1)^2 \mathbb{E}(\gamma^2 - 1)^3\bigr\} \operatorname{tr} B^5 + \mathbb{E}(\gamma^2 - 1)^2 \mathbb{E}(\gamma^2 - 1)^3 \operatorname{tr} B^2 \operatorname{tr} B^3 \\
&= 512\mathbb{p}_5 + 32\mathbb{p}_2\,\mathbb{p}_3\,.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\bigl\langle BZ, Z\bigr\rangle^2 &= \bigl(\mathbb{E}\bigl\langle BZ, Z\bigr\rangle\bigr)^2 + \mathbb{E}\xi^2 = \mathbb{p}^2 + 2\mathbb{p}_2\,, \\
\mathbb{E}\bigl\langle BZ, Z\bigr\rangle^3 &= \mathbb{E}(\xi + \mathbb{p})^3 = \mathbb{p}^3 + \mathbb{E}\xi^3 + 3\mathbb{p}\;\mathbb{E}\xi^2 = \mathbb{p}^3 + 6\mathbb{p}\;\mathbb{p}_2 + 8\mathbb{p}_3, \\
\mathbb{E}\bigl\langle BZ, Z\bigr\rangle^4 &= \mathbb{E}\bigl(\xi + \mathbb{p}\bigr)^4 = \mathbb{p}^4 + 6\mathbb{p}^2\mathbb{E}\xi^2 + 4\mathbb{p}\,\mathbb{E}\xi^3 + \mathbb{E}\xi^4 \\
&= \mathbb{p}^4 + 12\mathbb{p}^2\,\mathbb{p}_2 + 32\mathbb{p}\,\mathbb{p}_3 + 48\mathbb{p}_4 + 12\mathbb{p}_2^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Var}\bigl\langle BZ, Z\bigr\rangle^2 = \mathbb{E}(\xi + \mathbb{p})^4 - \bigl(\mathbb{p}^2 + 2\mathbb{p}_2\bigr)^2 \\
&\quad = \mathbb{p}^4 + 6\mathbb{p}^2\mathbb{E}\xi^2 + 4\mathbb{p}\,\mathbb{E}\xi^3 + \mathbb{E}\xi^4 - \bigl(\mathbb{p}^2 + 2\mathbb{p}_2\bigr)^2 = 8\mathbb{p}^2\,\mathbb{p}_2 + 32\mathbb{p}\,\mathbb{p}_3 + 48\mathbb{p}_4 + 8\mathbb{p}_2^2\,.
\end{aligned}
$$

Also

$$
\begin{aligned}
\mathbb{E}\bigl\langle BZ, Z\bigr\rangle^5 &= \mathbb{E}\bigl(\xi + \mathbb{p}\bigr)^5 = \mathbb{p}^5 + 10\mathbb{p}^3\mathbb{E}\xi^2 + 10\mathbb{p}^2\,\mathbb{E}\xi^3 + 5\mathbb{p}\mathbb{E}\xi^4 + \mathbb{E}\xi^5 \\
&= \mathbb{p}^5 + 20\mathbb{p}^3\,\mathbb{p}_2 + 80\mathbb{p}^2\mathbb{p}_3 + 5\mathbb{p}(48\mathbb{p}_4 + 12\mathbb{p}_2^2) + 512\mathbb{p}_5 + 32\mathbb{p}_2\,\mathbb{p}_3\,.
\end{aligned}
$$

Assume $\|B\| = 1$ yielding $\mathtt{p}_m \le \mathtt{p}$. Then

$$\begin{aligned}
\mathbb{E}\bigl\langle BZ, Z \bigr\rangle^2 &\le \mathtt{p}^2 + 2\mathtt{p} \le (\mathtt{p}+1)^2 , \\
\mathbb{E}\bigl\langle BZ, Z \bigr\rangle^3 &\le \mathtt{p}^3 + 6\mathtt{p}^2 + 8\mathtt{p} \le (\mathtt{p}+2)^3, \\
\mathbb{E}\bigl\langle BZ, Z \bigr\rangle^4 &\le \mathtt{p}^4 + 12\mathtt{p}^3 + 44\mathtt{p}^2 + 48\mathtt{p} \le (\mathtt{p}+3)^4, \\
\mathbb{E}\bigl\langle BZ, Z \bigr\rangle^5 &\le \mathtt{p}^5 + 20\mathtt{p}^4 + 140\mathtt{p}^3 + 272\mathtt{p}^2 + 512\mathtt{p} \le (\mathtt{p}+4)^5 .
\end{aligned}$$

For the last result of the lemma, observe that with $B = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, the matrix $\mathbb{E}(ZZ^\top - I\!\!I_p)B(ZZ^\top - I\!\!I_p)$ is diagonal with the ith diagonal entry

$$\sum_{j=1}^{p} \lambda_i \lambda_j \mathbb{E}(\gamma_i \gamma_j - \delta_{i,j})^2 = \sum_{j=1}^{p} \lambda_i \lambda_j + \lambda_i^2$$

yielding

$$\mathbb{E}\|B^{1/2}(ZZ^\top - I\!\!I_p)B^{1/2}\|_{\mathrm{Fr}}^2 = \sum_{i,j=1}^{p} \lambda_i \lambda_j \mathbb{E}(\gamma_i \gamma_j - \delta_{i,j})^2 = \biggl(\sum_{i=1}^{p} \lambda_i \biggr)^2 + \sum_{i=1}^{p} \lambda_i^2$$

and assertion (B.1) follows. □

Now we compute the exponential moments of centered and non-centered quadratic forms.

Lemma B.2. *Let $\|B\|_{\mathrm{op}} = \lambda$ and $Z \sim \mathcal{N}(0, I\!\!I_p)$. Then for any $\mu \in (0, \lambda^{-1})$,*

$$\mathbb{E} \exp\Bigl\{ \frac{\mu}{2} \langle BZ, Z \rangle \Bigr\} = \det(I\!\!I_p - \mu B)^{-1/2} .$$

Moreover, with $\mathtt{p} = \operatorname{tr} B$ and $\mathtt{v}^2 = \operatorname{tr} B^2$

$$\log \mathbb{E} \exp\Bigl\{ \frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathtt{p} \bigr) \Bigr\} \le \frac{\mu^2 \mathtt{v}^2}{4(1 - \lambda\mu)} . \tag{B.2}$$

If B is positive semidefinite, $\lambda_j \ge 0$, then

$$\log \mathbb{E} \exp\Bigl\{ -\frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathtt{p} \bigr) \Bigr\} \le \frac{\mu^2 \mathtt{v}^2}{4} . \tag{B.3}$$

For any complex valued μ with $|\lambda\mu| < 1$,

$$\biggl| \log \mathbb{E} \exp\Bigl\{ \frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathtt{p} \bigr) - \frac{\mu^2 \operatorname{tr} B^2}{4} \Bigr\} \biggr| \le \frac{\lambda |\mu|^3 \mathtt{v}^2}{6(1 - \lambda|\mu|)} . \tag{B.4}$$

Proof. W.l.o.g. assume $\lambda = 1$. Let λ_j be the eigenvalues of B, $|\lambda_j| \le 1$. By an orthogonal transform, one can reduce the statement to the case of a diagonal matrix $B = \operatorname{diag}\bigl(\lambda_j\bigr)$. Then $\langle BZ, Z \rangle = \sum_{j=1}^{p} \lambda_j \gamma_j^2$ and by independence of the γ_j 's

$$\mathbb{E}\Bigl\{\frac{\mu}{2}\langle BZ, Z\rangle\Bigr\} = \prod_{j=1}^{p} \mathbb{E}\exp\Bigl(\frac{\mu}{2}\lambda_j \varepsilon_j^2\Bigr) = \prod_{j=1}^{p} \frac{1}{\sqrt{1-\mu\lambda_j}} = \det\bigl(\mathbb{I}_p - \mu B\bigr)^{-1/2}.$$

Below we use the simple bounds:

$$\begin{aligned} -\log(1-u) - u &= \sum_{k=2}^{\infty} \frac{u^k}{k} \le \frac{u^2}{2}\sum_{k=0}^{\infty} u^k = \frac{u^2}{2(1-u)}, \qquad u \in (0,1), \\ -\log(1-u) + u &= \sum_{k=2}^{\infty} \frac{u^k}{k} \le \frac{u^2}{2}, \qquad u \in (-1,0). \end{aligned}$$

Now it holds for $\mu > 0$

$$\begin{aligned} &\log \mathbb{E}\Bigl\{\frac{\mu}{2}\bigl(\langle BZ, Z\rangle - \mathbb{p}\bigr)\Bigr\} = \log\det(\mathbb{I}_p - \mu B)^{-1/2} - \frac{\mu\,\mathbb{p}}{2} \\ &\quad = -\frac{1}{2}\sum_{j=1}^{p}\bigl\{\log(1-\mu\lambda_j) + \mu\lambda_j\bigr\} \le \sum_{j=1}^{p} \frac{\mu^2\lambda_j^2}{4(1-\mu\lambda_j)} \le \frac{\mu^2 \mathrm{v}^2}{4(1-\mu\lambda)}. \end{aligned}$$

Similarly for any complex μ with $|\mu|\lambda < 1$

$$\begin{aligned} &\biggl|\log \mathbb{E}\Bigl\{\frac{\mu}{2}\bigl(\langle BZ, Z\rangle - \mathbb{p}\bigr) - \frac{\mu^2 \operatorname{tr} B^2}{4}\Bigr\}\biggr| = \biggl|\log\det(\mathbb{I}_p - \mu B)^{-1/2} - \frac{\mu\,\mathbb{p}}{2} - \frac{\mu^2 \operatorname{tr} B^2}{4}\biggr| \\ &\quad = \frac{1}{2}\biggl|\sum_{j=1}^{p}\Bigl\{\log(1-\mu\lambda_j) - \mu\lambda_j - \frac{\mu^2\lambda_j^2}{2}\Bigr\}\biggr| \le \sum_{j=1}^{p} \frac{|\mu\lambda_j|^3}{6(1-|\mu|)} = \frac{|\mu|^3 \lambda \mathrm{v}^2}{6(1-|\mu|)}. \end{aligned}$$

Statement (B.3) can be proved similarly. □

Now we consider the case of a non-centered quadratic form $\langle BZ, Z\rangle/2 + \langle \boldsymbol{A}, Z\rangle$ for a fixed vector $\boldsymbol{A}$.

Lemma B.3. *Let $\|B\| = \lambda_{\max}(B) < 1$. Then for any $\boldsymbol{A}$*

$$\mathbb{E}\exp\Bigl\{\frac{1}{2}\langle BZ, Z\rangle + \langle \boldsymbol{A}, Z\rangle\Bigr\} = \exp\Bigl\{\frac{\|(\mathbb{I}_p - B)^{-1/2}\boldsymbol{A}\|^2}{2}\Bigr\} \det(\mathbb{I}_p - B)^{-1/2}.$$

Moreover, for any $\mu \in (0,1)$

$$
\begin{aligned}
& \log I\!\!E \exp\Bigl\{ \frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathbb{p} \bigr) + \langle \boldsymbol{A}, Z \rangle \Bigr\} \\
& \qquad = \frac{\| (I\!\!I_p - \mu B)^{-1/2} \boldsymbol{A} \|^2}{2} + \log \det (I\!\!I_p - \mu B)^{-1/2} - \mu \, \mathbb{p} \\
& \qquad \le \frac{\| (I\!\!I_p - \mu B)^{-1/2} \boldsymbol{A} \|^2}{2} + \frac{\mu^2 \mathtt{v}^2}{4(1 - \mu \| B \|)} \, .
\end{aligned}
\tag{B.5}
$$

Proof. Denote $\boldsymbol{a} = (I\!\!I_p - B)^{-1/2} \boldsymbol{A}$. It holds by change of variables $(I\!\!I_p - B)^{1/2} \boldsymbol{x} = \boldsymbol{u}$ for $\mathtt{C}_p = (2\pi)^{-p/2}$

$$
\begin{aligned}
& I\!\!E \exp\Bigl\{ \frac{1}{2} \langle BZ, Z \rangle + \langle \boldsymbol{A}, Z \rangle \Bigr\} = \mathtt{C}_p \int \exp\Bigl\{ -\frac{1}{2} \langle (I\!\!I_p - B) \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{A}, \boldsymbol{x} \rangle \Bigr\} d\boldsymbol{x} \\
& \quad = \mathtt{C}_p \det (I\!\!I_p - B)^{-1/2} \int \exp\Bigl\{ -\frac{1}{2} \| \boldsymbol{u} \|^2 + \langle \boldsymbol{a}, \boldsymbol{u} \rangle \Bigr\} d\boldsymbol{u} = \det (I\!\!I_p - B)^{-1/2} \, \mathrm{e}^{\| \boldsymbol{a} \|^2 / 2}.
\end{aligned}
$$

The last inequality (B.5) follows by (B.2). ☐

B.2 Deviation bounds for Gaussian quadratic forms

The next result explains the concentration effect of $\| Q \boldsymbol{\xi} \|^2$ for a centered Gaussian vector $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{V}^2)$ and a linear operator $Q \colon I\!\!R^p \to I\!\!R^q$, $p, q \le \infty$. We use a version from Laurent and Massart (2000). For completeness, we present a simple proof.

Theorem B.4. *Let* $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbb{V}^2)$ *be a Gaussian element in* $I\!\!R^p$ *and let* $Q \colon I\!\!R^p \to I\!\!R^q$ *be such that* $B = Q \mathbb{V}^2 Q^\top$ *is a trace operator in* $I\!\!R^q$. *Then with* $\mathbb{p} = \operatorname{tr}(B)$, $\mathtt{v}^2 = \operatorname{tr}(B^2)$, *and* $\lambda = \| B \|$, *it holds for each* $\mathtt{x} \ge 0$

$$
I\!\!P\Bigl(\| Q \boldsymbol{\xi} \|^2 - \mathbb{p} > 2 \mathtt{v} \sqrt{\mathtt{x}} + 2 \lambda \mathtt{x} \Bigr) \le \mathrm{e}^{-\mathtt{x}}, \tag{B.6}
$$

$$
I\!\!P\Bigl(\| Q \boldsymbol{\xi} \|^2 - \mathbb{p} \le -2 \mathtt{v} \sqrt{\mathtt{x}} \Bigr) \le \mathrm{e}^{-\mathtt{x}}. \tag{B.7}
$$

It also implies

$$
I\!\!P\bigl(\bigl| \| Q \boldsymbol{\xi} \|^2 - \mathbb{p} \bigr| > z_2(B, \mathtt{x}) \bigr) \le 2 \mathrm{e}^{-\mathtt{x}},
$$

with

$$
z_2(B, \mathtt{x}) \stackrel{\mathrm{def}}{=} 2 \mathtt{v} \sqrt{\mathtt{x}} + 2 \lambda \mathtt{x} \, . \tag{B.8}
$$

Proof. W.l.o.g. assume that $\lambda = \|B\| = 1$. We use the identity $\|Q\boldsymbol{\xi}\|^2 = \langle BZ, Z \rangle$ with $Z \sim \mathcal{N}(0, I\!\!I_q)$. We apply Markov's inequality: with $\mu > 0$

$$I\!\!P\Bigl(\langle BZ, Z \rangle - \mathbb{p} > z_2(B, \mathtt{x}) \Bigr) \leq I\!\!E \exp\Bigl(\frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathbb{p} \bigr) - \frac{\mu \, z_2(B, \mathtt{x})}{2} \Bigr) .$$

Given $\mathtt{x} > 0$, fix $\mu < 1$ by the equation

$$\frac{\mu}{1 - \mu} = \frac{2\sqrt{\mathtt{x}}}{\mathtt{v}} \qquad \text{or} \qquad \mu^{-1} = 1 + \frac{\mathtt{v}}{2\sqrt{\mathtt{x}}} . \tag{B.9}$$

Let λ_j be the eigenvalues of B, $|\lambda_j| \leq 1$. It holds with $\mathbb{p} = \operatorname{tr} B$ in view of (B.2)

$$\log I\!\!E \Bigl\{ \frac{\mu}{2} \bigl(\langle BZ, Z \rangle - \mathbb{p} \bigr) \Bigr\} \leq \frac{\mu^2 \mathtt{v}^2}{4(1 - \mu)} . \tag{B.10}$$

For (B.6), it remains to check that the choice μ by (B.9) yields

$$\frac{\mu^2 \mathtt{v}^2}{4(1 - \mu)} - \frac{\mu \, z_2(B, \mathtt{x})}{2} = \frac{\mu^2 \mathtt{v}^2}{4(1 - \mu)} - \mu \bigl(\mathtt{v}\sqrt{\mathtt{x}} + \mathtt{x} \bigr) = \mu \Bigl(\frac{\mathtt{v}\sqrt{\mathtt{x}}}{2} - \mathtt{v}\sqrt{\mathtt{x}} - \mathtt{x} \Bigr) = -\mathtt{x}.$$

The bound (B.7) is obtained similarly from Markov's inequality applied to $-\langle BZ, Z \rangle + \mathbb{p}$ with $\mu = 2\mathtt{v}^{-1}\sqrt{\mathtt{x}}$. The use of (B.3) yields

$$\begin{aligned} I\!\!P\Bigl(\langle BZ, Z \rangle - \mathbb{p} < -2\mathtt{v}\sqrt{\mathtt{x}} \Bigr) &\leq I\!\!E \exp\Bigl\{ \frac{\mu}{2} \bigl(-\langle BZ, Z \rangle + \mathbb{p} \bigr) - \mu \, \mathtt{v}\sqrt{\mathtt{x}} \Bigr\} \\ &\leq \exp\Bigl(\frac{\mu^2 \mathtt{v}^2}{4} - \mu \, \mathtt{v}\sqrt{\mathtt{x}} \Bigr) = \mathrm{e}^{-\mathtt{x}} \end{aligned}$$

as required. $\square$

Corollary B.5. *Assume the conditions of Theorem B.4. Then for* $z > \mathtt{v}$

$$I\!\!P\bigl(\bigl| \|Q\boldsymbol{\xi}\|^2 - \mathbb{p} \bigr| \geq z \bigr) \leq 2 \exp\Bigl\{ - \frac{z^2}{\bigl(\mathtt{v} + \sqrt{\mathtt{v}^2 + 2\lambda z} \bigr)^2} \Bigr\} \leq 2 \exp\Bigl(- \frac{z^2}{4\mathtt{v}^2 + 4\lambda z} \Bigr). \tag{B.11}$$

Proof. Given z, define $\mathtt{x}$ by $2\mathtt{v}\sqrt{\mathtt{x}} + 2\lambda \mathtt{x} = z$ or $2\lambda\sqrt{\mathtt{x}} = \sqrt{\mathtt{v}^2 + 2\lambda z} - \mathtt{v}$. Then

$$I\!\!P\bigl(\|Q\boldsymbol{\xi}\|^2 - \mathbb{p} \geq z \bigr) \leq \mathrm{e}^{-\mathtt{x}} = \exp\Bigl\{ - \frac{\bigl(\sqrt{\mathtt{v}^2 + 2\lambda z} - \mathtt{v} \bigr)^2}{4\lambda^2} \Bigr\} = \exp\Bigl\{ - \frac{z^2}{\bigl(\mathtt{v} + \sqrt{\mathtt{v}^2 + 2\lambda z} \bigr)^2} \Bigr\}.$$

This yields (B.11) by direct calculus. $\square$

Of course, bound (B.11) is sensible only if $z \gg \mathtt{v}$.

Corollary B.6. *Assume the conditions of Theorem B.4. If also* $B \geq 0$ *, then*

$$\mathbb{P}\Bigl(\|Q\boldsymbol{\xi}\|^2 \geq z^2(B,\mathtt{x}) \Bigr) \leq \mathrm{e}^{-\mathtt{x}}$$

with

$$z^2(B,\mathtt{x}) \stackrel{\mathrm{def}}{=} \mathbb{p} + 2\mathtt{v}\sqrt{\mathtt{x}} + 2\lambda\mathtt{x} \leq \bigl(\sqrt{\mathbb{p}} + \sqrt{2\lambda\mathtt{x}} \bigr)^2 .$$

Also

$$\mathbb{P}\Bigl(\|Q\boldsymbol{\xi}\|^2 - \mathbb{p} < -2\mathtt{v}\sqrt{\mathtt{x}} \Bigr) \leq \mathrm{e}^{-\mathtt{x}}.$$

Proof. The definition implies $\mathtt{v}^2 \leq \mathbb{p}\lambda$. One can use a sub-optimal choice of the value $\mu(\mathtt{x}) = \bigl\{ 1 + 2\sqrt{\lambda\mathbb{p}/\mathtt{x}} \bigr\}^{-1}$ yielding the statement of the corollary. □

As a special case, we present a bound for the chi-squared distribution corresponding to $Q = \mathbb{V}^2 = \mathbb{I}_p$, $p < \infty$. Then $B = \mathbb{I}_p$, $\operatorname{tr}(B) = p$, $\operatorname{tr}(B^2) = p$ and $\lambda(B) = 1$.

Corollary B.7. *Let* Z *be a standard normal vector in* $\mathbb{R}^p$. *Then for any* $\mathtt{x} > 0$

$$\begin{aligned}
\mathbb{P}\bigl(\|Z\|^2 \geq p + 2\sqrt{p\,\mathtt{x}} + 2\mathtt{x} \bigr) &\leq \mathrm{e}^{-\mathtt{x}}, \\
\mathbb{P}\bigl(\|Z\| \;\geq \sqrt{p} + \sqrt{2\mathtt{x}} \bigr) \quad &\leq \mathrm{e}^{-\mathtt{x}}, \\
\mathbb{P}\bigl(\|Z\|^2 \leq p - 2\sqrt{p\,\mathtt{x}} \bigr) \quad &\leq \mathrm{e}^{-\mathtt{x}}.
\end{aligned}$$

The bound of Theorem B.4 can be represented as a usual deviation bound.

Theorem B.8. *Assume the conditions of Theorem B.4. For* $\mathtt{y} > 0$ *, define*

$$\mathtt{x}(\mathtt{y}) \stackrel{\mathrm{def}}{=} \frac{(\sqrt{\mathtt{y} + \mathbb{p}} - \sqrt{\mathbb{p}})^2}{4\lambda} .$$

Then

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\|^2 \geq \mathbb{p} + \mathtt{y} \bigr) \leq \mathrm{e}^{-\mathtt{x}(\mathtt{y})}, \tag{B.12}$$

$$\mathbb{E}\bigl\{ (\|Q\boldsymbol{\xi}\|^2 - \mathbb{p})\, \mathbb{1}\bigl(\|Q\boldsymbol{\xi}\|^2 \geq \mathbb{p} + \mathtt{y} \bigr) \bigr\} \leq 2\Bigl(\frac{\mathtt{y} + \mathbb{p}}{\lambda\,\mathtt{x}(\mathtt{y})} \Bigr)^{1/2} \mathrm{e}^{-\mathtt{x}(\mathtt{y})} . \tag{B.13}$$

Moreover, let $\mu > 0$ *fulfill* $\epsilon = \mu\lambda + \mu\sqrt{\lambda\mathbb{p}/\mathtt{x}(\mathtt{y})} < 1$. *Then*

$$\mathbb{E}\bigl\{ \mathrm{e}^{\mu(\|Q\boldsymbol{\xi}\|^2 - \mathbb{p})/2}\, \mathbb{1}(\|Q\boldsymbol{\xi}\|^2 \geq \mathbb{p} + \mathtt{y}) \bigr\} \leq \frac{1}{1-\epsilon} \exp\{-(1-\epsilon)\mathtt{x}(\mathtt{y})\} . \tag{B.14}$$

Proof. Normalizing by λ reduces the statements to the case with $\lambda = 1$. Define $\eta = \|Q\boldsymbol{\xi}\|^2 - \mathbb{p}$ and

$$z(\mathtt{x}) = 2\sqrt{\mathbb{p}\,\mathtt{x}} + 2\mathtt{x}. \tag{B.15}$$

Then by (B.6) $\mathbb{P}(\eta \ge z(\mathtt{x})) \le \mathrm{e}^{-\mathtt{x}}$. Inverting the relation (B.15) yields

$$\mathtt{x}(z) = \frac{1}{4}\bigl(\sqrt{z+\mathbb{p}} - \sqrt{\mathbb{p}}\bigr)^2$$

and (B.12) follows by applying $z = \mathtt{y}$. Further,

$$\mathbb{E}\bigl\{\eta\, \mathbb{I}(\eta \ge \mathtt{y})\bigr\} = \int_{\mathtt{y}}^{\infty} \mathbb{P}(\eta \ge z)\,dz \le \int_{\mathtt{y}}^{\infty} \mathrm{e}^{-\mathtt{x}(z)}\,dz = \int_{\mathtt{x}(\mathtt{y})}^{\infty} \mathrm{e}^{-\mathtt{x}}\, z'(\mathtt{x})\,d\mathtt{x}\,.$$

As $z'(\mathtt{x}) = 2 + \sqrt{\mathbb{p}/\mathtt{x}}$ monotonously decreases with $\mathtt{x}$, we derive

$$\mathbb{E}\bigl\{\eta\, \mathbb{I}(\eta \ge \mathtt{y})\bigr\} \le z'(\mathtt{x}(\mathtt{y}))\mathrm{e}^{-\mathtt{x}(\mathtt{y})} = \frac{1}{\mathtt{x}'(\mathtt{y})}\,\mathrm{e}^{-\mathtt{x}(\mathtt{y})} = \frac{4\sqrt{\mathtt{y}+\mathbb{p}}}{\sqrt{\mathtt{y}+\mathbb{p}} - \sqrt{\mathbb{p}}}\,\mathrm{e}^{-\mathtt{x}(\mathtt{y})}$$

and (B.13) follows.

In a similar way, define $\mathsf{z}(\mathtt{x})$ from the relation $\mu^{-1}\log \mathsf{z}(\mathtt{x}) = \sqrt{\mathbb{p}\,\mathtt{x}} + \mathtt{x}$ yielding

$$\mathsf{z}(\mathtt{x}) = \exp\bigl(\mu\sqrt{\mathbb{p}\,\mathtt{x}} + \mu\,\mathtt{x}\bigr).$$

The inverse relation reads

$$\mathtt{x}_{\mathrm{e}}(\mathsf{z}) = \bigl(\sqrt{\mu^{-1}\log \mathsf{z} + \mathbb{p}/4} - \sqrt{\mathbb{p}/4}\bigr)^2.$$

Then with $\mathtt{x}(\mathtt{y}) = \mathtt{x}_{\mathrm{e}}(\mathrm{e}^{\mu\mathtt{y}/2}) = \bigl(\sqrt{\mathtt{y}+\mathbb{p}} - \sqrt{\mathbb{p}}\bigr)^2/4$

$$\begin{aligned}
\mathbb{E}\bigl\{\mathrm{e}^{\mu\eta/2}\, \mathbb{I}(\eta \ge \mathtt{y})\bigr\} &= \int_{\mathrm{e}^{\mu\mathtt{y}/2}}^{\infty} \mathbb{P}(\mathrm{e}^{\mu\eta/2} \ge \mathsf{z})\,d\mathsf{z} = \int_{\mathrm{e}^{\mu\mathtt{y}/2}}^{\infty} \mathbb{P}(\eta \ge 2\mu^{-1}\log\mathsf{z})\,d\mathsf{z} \\
&\le \int_{\mathrm{e}^{\mu\mathtt{y}/2}}^{\infty} \mathrm{e}^{-\mathtt{x}_{\mathrm{e}}(\mathsf{z})}\,d\mathsf{z} = \int_{\mathtt{x}(\mathtt{y})}^{\infty} \mathrm{e}^{-\mathtt{x}}\,\mathsf{z}'(\mathtt{x})\,d\mathtt{x}.
\end{aligned}$$

Further, in view of $\mu + 0.5\,\mu\sqrt{\mathbb{p}/\mathtt{x}} < \mu + \mu\sqrt{\mathbb{p}/\mathtt{x}(\mathtt{y})} = \epsilon < 1$ for $\mathtt{x} \ge \mathtt{x}(\mathtt{y})$, it holds

$$\mathsf{z}'(\mathtt{x}) = \bigl(\mu + 0.5\,\mu\sqrt{\mathbb{p}/\mathtt{x}}\bigr)\exp\bigl(\mu\sqrt{\mathbb{p}\,\mathtt{x}} + \mu\,\mathtt{x}\bigr) \le \exp\bigl(\mu\,\mathtt{x}\sqrt{\mathbb{p}/\mathtt{x}(\mathtt{y})} + \mu\,\mathtt{x}\bigr) = \exp(\epsilon\,\mathtt{x})$$

and

$$\mathbb{E}\bigl\{\mathrm{e}^{\mu\eta/2}\, \mathbb{I}(\eta \ge \mathtt{y})\bigr\} \le \int_{\mathtt{x}(\mathtt{y})}^{\infty} \mathrm{e}^{-(1-\epsilon)\mathtt{x}}\,d\mathtt{x} = \frac{1}{1-\epsilon}\,\mathrm{e}^{-(1-\epsilon)\mathtt{x}(\mathtt{y})}$$

and (B.14) follows. $\square$

B.3 Deviation bounds for sub-gaussian quadratic forms

This section collects some probability bounds for sub-gaussian quadratic forms.

B.3.1 A rough upper bound

Let $\boldsymbol{\xi}$ be a random vector in $\mathbb{R}^p$, $p \le \infty$, with $\mathbb{E}\boldsymbol{\xi} = 0$. We suppose that there exists an operator $\mathbb{V}$ in $\mathbb{R}^p$ such that

$$\log \mathbb{E} \exp\bigl(\langle \boldsymbol{u}, \mathbb{V}^{-1}\boldsymbol{\xi}\rangle\bigr) \le \frac{\|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p. \tag{B.16}$$

In the Gaussian case, one can take $\mathbb{V}^2 = \operatorname{Var}(\boldsymbol{\xi})$. In general, $\mathbb{V}^2 \ge \operatorname{Var}(\boldsymbol{\xi})$. We consider a quadratic form $\|Q\boldsymbol{\xi}\|^2$, where $\boldsymbol{\xi}$ satisfies (B.16) and Q is a given linear operator $\mathbb{R}^p \to \mathbb{R}^q$. Denote

$$B \stackrel{\text{def}}{=} Q\mathbb{V}^2 Q^\top, \qquad \mathbb{p} \stackrel{\text{def}}{=} \operatorname{tr}(B), \qquad \mathtt{v}^2 \stackrel{\text{def}}{=} \operatorname{tr}(B^2). \tag{B.17}$$

We show that under (B.16), the quadratic form $\|Q\boldsymbol{\xi}\|^2$ follows the same upper deviation bound $\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\|^2 - \operatorname{tr} B \ge z_2(B,\mathtt{x})\bigr) \le \mathrm{e}^{-\mathtt{x}}$ with $z_2(B,\mathtt{x})$ from (B.8) as in the Gaussian case. Similar results can be found e.g. in Hsu et al. (2012). We present an independent proof for reference convenience.

Theorem B.9. *Suppose* (B.16). *With notation from* (B.17), *it holds for any* $\mu < 1/\|B\|$

$$\mathbb{E} \exp\Bigl(\frac{\mu}{2}\|Q\boldsymbol{\xi}\|^2\Bigr) \le \exp\Bigl(\frac{\mu^2 \mathtt{v}^2}{4(1-\|B\|\mu)} + \frac{\mu\,\mathbb{p}}{2}\Bigr)$$

and for any $\mathtt{x} > 0$

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\|^2 > \mathbb{p} + 2\mathtt{v}\sqrt{\mathtt{x}} + 2\mathtt{x}\bigr) \le \mathrm{e}^{-\mathtt{x}}. \tag{B.18}$$

Proof. Normalization by $\|B\|$ reduces the proof to $\|B\| = 1$. For $\mu \in (0,1)$,

$$\mathbb{E} \exp\bigl(\mu\|Q\boldsymbol{\xi}\|^2/2\bigr) = \mathbb{E}\,\mathbb{E}_{\gamma} \exp\bigl(\mu^{1/2}\langle \mathbb{V}Q^\top\boldsymbol{\gamma}, \mathbb{V}^{-1}\boldsymbol{\xi}\rangle\bigr), \tag{B.19}$$

where $\boldsymbol{\gamma}$ is standard Gaussian in $\mathbb{R}^q$ under $\mathbb{E}_{\gamma}$ independent on $\boldsymbol{\xi}$. Application of Fubini's theorem, (B.16), and (B.10) yields

$$\mathbb{E} \exp\Bigl(\frac{\mu}{2}\|Q\boldsymbol{\xi}\|^2\Bigr) \le \mathbb{E}_{\gamma} \exp\Bigl(\frac{\mu}{2}\|\mathbb{V}Q^\top\boldsymbol{\gamma}\|^2\Bigr) \le \exp\Bigl(\frac{\mu^2 \operatorname{tr}(B^2)}{4(1-\mu)} + \frac{\mu \operatorname{tr}(B)}{2}\Bigr).$$

Further, we proceed as in the Gaussian case; see the proof of Theorem B.4. □

The bound (B.18) looks identical to the Gaussian case, however, there is an essential difference: $\mathbb{p} = \operatorname{tr}(B)$ can be much larger than $\mathbb{E}\|Q\boldsymbol{\xi}\|^2 = \operatorname{tr}\bigl\{Q \operatorname{Var}(\boldsymbol{\xi}) Q^\top\bigr\}$. The result from (B.18) is not accurate enough for supporting the concentration phenomenon that $\|Q\boldsymbol{\xi}\|^2$ concentrates around its expectation $\mathbb{E}\|Q\boldsymbol{\xi}\|^2$. The next section presents some sufficient conditions for obtaining sharp Gaussian-like deviation bounds.

B.3.2 Concentration of the squared norm of a sub-gaussian vector

Let $\boldsymbol{\xi}$ be a centered random vector in $\mathbb{R}^p$ with sub-gaussian tails. We study concentration effect of the squared norm $\|Q\boldsymbol{X}\|^2$ for a linear mapping Q and for $\boldsymbol{X} = \mathbb{V}^{-1}\boldsymbol{\xi}$ being the standardized version of $\boldsymbol{\xi}$, where $\mathbb{V}^2 = \operatorname{Var}(\boldsymbol{\xi})$. More generally, we allow $\mathbb{V}^2 \geq \operatorname{Var}(\boldsymbol{\xi})$ yielding $\operatorname{Var}(\boldsymbol{X}) \leq \mathbb{I}_p$ to incorporate the case when $\operatorname{Var}(\boldsymbol{\xi})$ is ill-posed. The aim is to establish the results similar to (B.6) with $B = Q\mathbb{V}^2 Q^\top$ as in Gaussian case. Later we assume the following condition.

$(\boldsymbol{X})$ *A random vector* $\boldsymbol{X} \in \mathbb{R}^p$ *satisfies* $\mathbb{E}\boldsymbol{X} = 0$, $\operatorname{Var}(\boldsymbol{X}) \leq \mathbb{I}_p$ *for the identity matrix* $\mathbb{I}_p$. *The function* $\phi(\boldsymbol{u}) \stackrel{\mathrm{def}}{=} \log \mathbb{E}\mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X}\rangle}$ *is finite and fulfills for some* $\mathtt{C}_\phi$

$$\phi(\boldsymbol{u}) \stackrel{\mathrm{def}}{=} \log \mathbb{E}\mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X}\rangle} \leq \frac{\mathtt{C}_\phi \|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p. \tag{B.20}$$

The constant $\mathtt{C}_\phi$ can be quite large, it does not show up in the leading term of the obtained bound. Also, we will only use this condition for $\|\boldsymbol{u}\| \geq \mathtt{g}$ for some sufficiently large $\mathtt{g}$. For $\|\boldsymbol{u}\| \leq \mathtt{g}$, we use smoothness properties of $\phi(\boldsymbol{u})$ in terms of its third and fourth derivatives.

The bounds in (B.6) and in (B.18) are uniform in the sense that they apply for all $\mathtt{x}$ and all B. The results of this section are limited to a high dimensional situation with $\operatorname{tr}(B) \gg \|B\|$ and apply only for $\mathtt{x} \ll \operatorname{tr}(B)/\|B\|$. This corresponds to high dimensional concentration of $\|Q\boldsymbol{X}\|^2$ for $\boldsymbol{X}$ Gaussian. As compensation for this local behavior, the bounds are surprisingly sharp. In fact, they perfectly replicate bounds (B.6) from the Gaussian case, the upper and lower quantiles are exactly as in (B.6) and the deviation probability is increased from $\mathrm{e}^{-\mathtt{x}}$ to $(1 + \varDelta_\mu)\mathrm{e}^{-\mathtt{x}}$ for a small value $\varDelta_\mu$. For larger $\mathtt{x}$, one can still apply rough upper bound (B.18) involving $\mathtt{C}_\phi$.

With $\boldsymbol{\gamma}$ standard normal in $\mathbb{R}^q$, define

$$\begin{aligned} \mathbb{p} &\stackrel{\mathrm{def}}{=} \mathbb{E}\|Q\boldsymbol{X}\|^2 = \operatorname{tr}\{Q \operatorname{Var}(\boldsymbol{X}) Q^\top\} = \operatorname{tr}(B), \\ \mathbb{p}_Q &\stackrel{\mathrm{def}}{=} \mathbb{E}\|Q^\top \boldsymbol{\gamma}\|^2 = \operatorname{tr}(QQ^\top). \end{aligned} \tag{B.21}$$

The presented results apply to a high-dimensional situation when $\mathbb{p}$ and hence $\mathbb{p}_Q$ is a large number. Define for $\boldsymbol{u} \in I\!\!R^p$, define a measure $I\!\!E_{\boldsymbol{u}}$ by

$$I\!\!E_{\boldsymbol{u}} \eta \stackrel{\text{def}}{=} \frac{I\!\!E(\eta \, \mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X} \rangle})}{I\!\!E \mathrm{e}^{\langle \boldsymbol{u}, \boldsymbol{X} \rangle}} . \tag{B.22}$$

Also fix some $\mathtt{g} > 0$ and introduce

$$\tau_3 \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \le \mathtt{g}} \frac{1}{\|\boldsymbol{u}\|^3} \bigl| I\!\!E_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - I\!\!E_{\boldsymbol{u}} \boldsymbol{X} \rangle^3 \bigr| , \tag{B.23}$$

$$\tau_4 \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \le \mathtt{g}} \frac{1}{\|\boldsymbol{u}\|^4} \bigl| I\!\!E_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - I\!\!E_{\boldsymbol{u}} \boldsymbol{X} \rangle^4 - 3 \bigl\{ I\!\!E_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - I\!\!E_{\boldsymbol{u}} \boldsymbol{X} \rangle^2 \bigr\}^2 \bigr| . \tag{B.24}$$

The quantities τ_3 and τ_4 are typically not only finite but also very small. Indeed, for $\boldsymbol{X}$ Gaussian they just vanish. If $\boldsymbol{X}$ is a normalized sum of independent centred random vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ then $\tau_3 \asymp n^{-1/2}$ and $\tau_4 \asymp n^{-1}$; see Section B.3.3.

First, we present an upper bound which nicely replicates (B.6) under some restrictions; see Section B.3.4 for a further discussion.

Theorem B.10. *Let* $\boldsymbol{X}$ *satisfy* $I\!\!E \boldsymbol{X} = 0$, $\operatorname{Var}(\boldsymbol{X}) \le I\!\!I_p$, *and* $\boldsymbol{(X)}$. *For any linear mapping* Q *with* $\|Q\| = 1$, *set* $B = Q \operatorname{Var}(\boldsymbol{X}) Q^\top$ *and define* $\mathbb{p}$, $\mathbb{p}_Q$ *by* (B.21). *Fix* $\mathtt{g}$ *such that* $\mathtt{g}^2 \ge 3\mathbb{p}_Q$ *and* $\mathtt{g}\tau_3 \le 2/3$ *for* τ_3 *from* (B.23). *For any* $\mathtt{x} > 0$ *with* $\sqrt{4\mathtt{x}} \le \sqrt{\operatorname{tr}(B^2)}/(3\mathtt{C}_\phi)$, *it holds*

$$I\!\!P\bigl(\|Q\boldsymbol{X}\|^2 > \operatorname{tr}(B) + 2\sqrt{\mathtt{x} \operatorname{tr}(B^2)} + 2\mathtt{x} \bigr) \le (1 + \Delta_\mu) \mathrm{e}^{-\mathtt{x}} , \tag{B.25}$$

where $\mu = \mu(\mathtt{x})$ *is given by* $\mu^{-1} = 1 + \sqrt{\operatorname{tr}(B^2)/(4\mathtt{x})}$ *and* Δ_μ *depends on* τ_3, τ_4, $\mathbb{p}$, $\mathbb{p}_Q$ *only and will be given explicitly in the proof. Moreover,* $\Delta_\mu \ll 1$ *under* $\mathbb{p}_Q \gg 1$, $(\tau_3^2 + \tau_4)\, \mathbb{p}_Q^2 \ll 1$.

The key step of the proof is the following statement.

Proposition B.11. *Assume the conditions of Theorem B.10. If* $\mu > 0$ *satisfies*

$$\mathtt{C}_\phi \, \mu \le 1/3 , \tag{B.26}$$

then it holds

$$\bigl| I\!\!E \exp(\mu \|Q\boldsymbol{X}\|^2/2) - \det(I\!\!I_q - \mu B)^{-1/2} \bigr| \le \Delta_\mu \det(I\!\!I_q - \mu B)^{-1/2} \tag{B.27}$$

for some constant Δ_μ *such that* $\Delta_\mu \ll 1$ *under* $\mathbb{p}_Q \gg 1$, $(\tau_3^2 + \tau_4)\, \mathbb{p}_Q^2 \ll 1$; *see the proof for a closed-form representation.*

Proof. We use (B.19) and Fubini theorem: with $\mathbb{E}_{\gamma} = \mathbb{E}_{\gamma \sim \mathcal{N}(0,\mathbb{I}_q)}$

$$\mathbb{E}\exp\bigl(\mu\|Q\boldsymbol{X}\|^2/2\bigr) = \mathbb{E}\,\mathbb{E}_{\boldsymbol{\gamma}}\exp\bigl(\mu^{1/2}\langle Q^\top\boldsymbol{\gamma},\boldsymbol{X}\rangle\bigr) = \mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^\top\boldsymbol{\gamma}). \tag{B.28}$$

Further, with $\mathtt{g}^2 = 3\mathbb{p}_Q$,

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^\top\boldsymbol{\gamma}) &= \mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^\top\boldsymbol{\gamma})\,\mathbb{1}(\|\mu^{1/2}Q^\top\boldsymbol{\gamma}\|\le\mathtt{g})\\&\quad+\mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^\top\boldsymbol{\gamma})\,\mathbb{1}(\|\mu^{1/2}Q^\top\boldsymbol{\gamma}\|>\mathtt{g}).\end{aligned} \tag{B.29}$$

Each summand here will be bounded separately starting from the second one. Define

$$\mathfrak{z}_\mu \stackrel{\text{def}}{=} \frac{1}{4}\Bigl(\sqrt{\mathtt{C}_\phi^{-1}\mu^{-1}\mathtt{g}^2} - \sqrt{\mathbb{p}_Q}\Bigr)^2, \qquad \omega_\mu \stackrel{\text{def}}{=} \mathtt{C}_\phi\,\mu + \mathtt{C}_\phi\,\mu\sqrt{\mathbb{p}_Q/\mathfrak{z}_\mu}\,.$$

Then (B.26) ensures that $\mathfrak{z}_\mu \ge \bigl(\sqrt{9\mathbb{p}_Q} - \sqrt{\mathbb{p}_Q}\bigr)^2/4 = \mathbb{p}_Q$ and $\omega_\mu \le 2/3$. By (B.20) and (B.14) of Theorem B.8, it holds under the condition $\omega_\mu \le 2/3$

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{\gamma}}\exp\phi(\mu^{1/2}Q^\top\boldsymbol{\gamma})\,\mathbb{1}(\|\mu^{1/2}Q^\top\boldsymbol{\gamma}\|>\mathtt{g})\\&\quad\le\mathbb{E}_{\boldsymbol{\gamma}}\exp\bigl(\mathtt{C}_\phi\,\mu\|Q^\top\boldsymbol{\gamma}\|^2/2\bigr)\,\mathbb{1}(\|Q^\top\boldsymbol{\gamma}\|^2>\mu^{-1}\mathtt{g}^2)\\&\quad\le\exp\bigl(\mathtt{C}_\phi\,\mu\,\mathbb{p}_Q/2\bigr)\,\mathbb{E}_{\boldsymbol{\gamma}}\exp\bigl(\mathtt{C}_\phi\,\mu(\|Q^\top\boldsymbol{\gamma}\|^2-\mathbb{p}_Q)/2\bigr)\,\mathbb{1}(\|Q^\top\boldsymbol{\gamma}\|^2>\mu^{-1}\mathtt{g}^2)\\&\quad\le\frac{1}{1-\omega_\mu}\exp\{\mathtt{C}_\phi\,\mu\,\mathbb{p}_Q/2-(1-\omega_\mu)\mathfrak{z}_\mu\}\,.\end{aligned} \tag{B.30}$$

Note that $\omega_\mu \le 2/3$, $\mathfrak{z}_\mu \ge \mathbb{p}_Q$, and $\mathtt{C}_\phi\,\mu \le 1/3$ imply

$$\frac{1}{1-\omega_\mu}\exp\{\mathtt{C}_\phi\,\mu\,\mathbb{p}_Q/2-(1-\omega_\mu)\mathfrak{z}_\mu\} \le 3\mathrm{e}^{-\mathbb{p}_Q/6}\,. \tag{B.31}$$

Now we check that $\phi(\boldsymbol{u})$ satisfies conditions $(\mathbb{D}_3^*)$ and $(\mathbb{D}_4)$:

$$|\langle\nabla^3\phi(\boldsymbol{x}),\boldsymbol{u}^{\otimes 3}\rangle| \le \tau_3\|\boldsymbol{u}\|^3, \quad \boldsymbol{u}\in\mathbb{R}^p, \tag{B.32}$$

and

$$|\delta_4(\boldsymbol{u})| \stackrel{\text{def}}{=} \Bigl|\phi(\boldsymbol{u}) - \frac{1}{2}\langle\phi''(0),\boldsymbol{u}^{\otimes 2}\rangle - \frac{1}{6}\langle\phi'''(0),\boldsymbol{u}^{\otimes 3}\rangle\Bigr| \le \frac{\tau_4}{24}\|\boldsymbol{u}\|^4, \quad \|\boldsymbol{u}\|\le\mathtt{g}\,. \tag{B.33}$$

Consider first the univariate case. Let X satisfy $\mathbb{E}X = 0$ and $\mathbb{E}X^2 \le \sigma^2$. Define for any $t\in[0,\mathtt{g}]$ a measure $\mathbb{P}_t$ such that for any random variable η

$$\mathbb{E}_t\,\eta \stackrel{\text{def}}{=} \frac{\mathbb{E}(\eta\,\mathrm{e}^{tX})}{\mathbb{E}\mathrm{e}^{tX}}\,.$$

Consider $\phi(t) \stackrel{\text{def}}{=} \log \mathbb{E} e^{tX}$ as a function of $t \in [0, \lambda]$. It is well defined and satisfies $\phi(0) = \phi'(0) = 0$, $\phi''(0) = \mathbb{E} X^2 \le \sigma^2$, and

$$\begin{aligned} \phi'''(t) &= \mathbb{E}_t (X - \mathbb{E}_t X)^3 , \\ \phi^{(4)}(t) &= \mathbb{E}_t (X - \mathbb{E}_t X)^4 - 3 \bigl\{ \mathbb{E}_t (X - \mathbb{E}_t X)^2 \bigr\}^2 . \end{aligned}$$

Therefore, conditions $(\mathbb{D}_3^*)$ and $(\mathbb{D}_4)$ follow from (B.23) and (B.24). The multivariate case can be reduced to the univariate one by fixing a direction $\boldsymbol{u} \in \mathbb{R}^p$ and considering the function $\phi(t\boldsymbol{u})$ of t.

Next, consider the first term on the right-hand side of (B.29). Define $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}^q : \|\mu^{1/2} Q^\top \boldsymbol{w}\| \le \mathtt{g}\}$. Then with $\mathtt{C}_q = (2\pi)^{-q/2}$ and $\boldsymbol{\gamma} \sim \mathcal{N}(0, \mathbb{I}_p)$

$$\mathbb{E}_{\boldsymbol{\gamma}} \exp \phi(\mu^{1/2} Q^\top \boldsymbol{\gamma}) \, \mathbb{1}(\|\mu^{1/2} Q^\top \boldsymbol{\gamma}\| \le \mathtt{g}) = \mathtt{C}_q \int_{\mathcal{W}} e^{f_\mu(\boldsymbol{w})} \, d\boldsymbol{w} ,$$

where for $\boldsymbol{w} \in \mathbb{R}^q$

$$f_\mu(\boldsymbol{w}) = \phi(\mu^{1/2} Q^\top \boldsymbol{w}) - \|\boldsymbol{w}\|^2/2$$

so that $f_\mu(0) = 0$, $\nabla f_\mu(0) = 0$. Also, define

$$\begin{aligned} D_\mu^2 &\stackrel{\text{def}}{=} -\nabla^2 f_\mu(0) = -\mu Q \operatorname{Var}(\boldsymbol{X}) Q^\top + \mathbb{I}_q = \mathbb{I}_q - \mu B, \\ \mathbb{p}_\mu &\stackrel{\text{def}}{=} \operatorname{tr}\bigl\{ D_\mu^{-1} (\mu Q Q^\top) D_\mu^{-1} \bigr\}, \\ \alpha_\mu &\stackrel{\text{def}}{=} \| D_\mu^{-1} (\mu Q Q^\top) D_\mu^{-1} \|. \end{aligned}$$

Note that $\|B\| \le 1$ implies $(1-\mu)\mathbb{I}_p \le D_\mu^2 \le \mathbb{I}_p$ and with $\mathbb{p}_Q = \operatorname{tr}(QQ^\top)$

$$\mathbb{p}_\mu \le \bar{\mu} \, \mathbb{p}_Q, \qquad \alpha_\mu \le \bar{\mu} , \qquad \bar{\mu} \stackrel{\text{def}}{=} \frac{\mu}{1-\mu}.$$

The function $f_\mu(\boldsymbol{w})$ inherits smoothness properties of $\phi(\mu^{1/2} Q^\top \boldsymbol{w})$. In particular,

$$\bigl| \langle \nabla^3 f_\mu(0), \boldsymbol{u}^{\otimes 3} \rangle \bigr| \le \tau_3 \|\mu^{1/2} Q^\top \boldsymbol{u}\|^3 \le \tau_3 \, \bar{\mu}^{3/2} \|D_\mu \boldsymbol{u}\|^3 = \tau_{3,\mu} \|D_\mu \boldsymbol{u}\|^3,$$

and for any $\boldsymbol{w}$ with $\|\mu^{1/2} Q^\top \boldsymbol{w}\| \le \mathtt{g}$

$$\bigl| \langle \nabla^4 f_\mu(\boldsymbol{w}), \boldsymbol{u}^{\otimes 4} \rangle \bigr| \le \tau_4 \|\mu^{1/2} Q^\top \boldsymbol{u}\|^4 \le \tau_4 \, \bar{\mu}^2 \, \|D_\mu \boldsymbol{u}\|^4 = \tau_{4,\mu} \|D_\mu \boldsymbol{u}\|^4 .$$

Also $\boldsymbol{w} \in \mathcal{W}$ implies $\bar{\mu} \|D_\mu \boldsymbol{w}\|^2 \le \mathtt{g}^2$. We apply Proposition B.43 to $f_\mu(\boldsymbol{w})$ yielding

$$\left| \frac{\int_{\mathcal{W}} e^{f_\mu(\boldsymbol{w})} \, d\boldsymbol{w} - \int_{\mathcal{W}} e^{-\|D_\mu \boldsymbol{w}\|^2/2} \, d\boldsymbol{w}}{\int e^{-\|D_\mu \boldsymbol{w}\|^2/2} \, d\boldsymbol{w}} \right| \le \diamondsuit . \tag{B.34}$$

The quantity $\diamondsuit$ here is computed as follows. Let $\mathcal{T}(\boldsymbol{u}) = \langle \nabla^3 f_\mu(0), \boldsymbol{u}^{\otimes 3} \rangle$, $\boldsymbol{\gamma}_\mu \sim \mathcal{N}(0, D_\mu^{-2})$. Define

$$
\begin{aligned}
\epsilon_\mu &= \frac{\tau_{3,\mu}\, \mathtt{g}^2 \sqrt{\alpha_\mu}}{2\bar{\mu}} \le \frac{1}{2}\, \tau_3\, \mathtt{g}^2 , \\
\sigma_\mu^2 &= \mathbb{E} \mathcal{T}^2(\boldsymbol{\gamma}_\mu) \le \sqrt{5/12}\, \tau_{3,\mu}\, \mathbb{p}_\mu \le \sqrt{5/12}\, \tau_3\, \mathbb{p}_Q , \\
\delta_{4,\mu} &= \mathbb{E}_{\mathcal{U}}\, \delta_4^2(\boldsymbol{\gamma}_\mu) \le \frac{1}{24}\, \tau_{4,\mu} (\mathbb{p}_\mu + 3\alpha_\mu)^2 \le \frac{1}{24}\, \tau_4 (\mathbb{p}_Q + 3)^2 .
\end{aligned}
$$

Then

$$
\Bigl| \diamondsuit - \frac{\sigma_\mu^2}{2} \Bigr| \le \sigma_\mu\, \delta_{4,\mu} + \frac{\delta_{4,\mu}^2}{2} + \frac{5}{3} \epsilon_\mu^3 \exp(\epsilon_\mu^2) , \quad \diamondsuit \le \frac{1}{2} (\sigma_\mu + \delta_{4,\mu})^2 + \frac{5}{3} \epsilon_\mu^3 \exp(\epsilon_\mu^2) . \tag{B.35}
$$

Furthermore, it holds

$$
\begin{aligned}
\rho_\mu &\stackrel{\text{def}}{=} 1 - \frac{\int_{\mathcal{W}} \mathrm{e}^{-\|D_\mu \boldsymbol{w}\|^2/2}\, d\boldsymbol{w}}{\int \mathrm{e}^{-\|D_\mu \boldsymbol{w}\|^2/2}\, d\boldsymbol{w}} = \mathbb{P}\bigl(\|\mu^{1/2} Q^\top D_\mu^{-1} \boldsymbol{\gamma}\| > \mathtt{g} \bigr) \\
&\le \mathbb{P}\bigl(\|Q^\top \boldsymbol{\gamma}\|^2 > (1-\mu) \mu^{-1} \mathtt{g}^2 \bigr).
\end{aligned} \tag{B.36}
$$

By (B.26), it holds $\mu^{-1} \mathtt{g}^2 \ge 9 \mathtt{C}_\phi\, \mathbb{p}_Q \ge 9 \mathbb{p}_Q$ and $\mu \le 1/(3\mathtt{C}_\phi) \le 1/3$, and hence

$$
\rho_\mu \le \mathbb{P}\bigl(\|Q^\top \boldsymbol{\gamma}\|^2 > 6 \mathbb{p}_Q \bigr) \le \mathrm{e}^{-\mathbb{p}_Q} . \tag{B.37}
$$

By (B.34) and (B.36)

$$
\biggl| \frac{\int_{\mathcal{W}} \mathrm{e}^{f_\mu(\boldsymbol{w})}\, d\boldsymbol{w}}{\int \mathrm{e}^{-\|D_\mu \boldsymbol{w}\|^2/2}\, d\boldsymbol{w}} - 1 \biggr| \le \diamondsuit + \rho_\mu . \tag{B.38}
$$

It remains to be noted that

$$
\mathtt{C}_q \int \mathrm{e}^{-\|D_\mu \boldsymbol{w}\|^2/2}\, d\boldsymbol{w} = \frac{1}{\det D_\mu} = \det(\mathbb{I}_q - \mu B)^{-1/2} \le 1
$$

and (B.27) follows from (B.30) and (B.38) with

$$
\varDelta_\mu \le \diamondsuit + \rho_\mu + \frac{1}{1-\omega_\mu} \exp\{ \mathtt{C}_\phi\, \mu\, \mathbb{p}_Q/2 - (1-\omega_\mu) \mathfrak{z}_\mu \} . \tag{B.39}
$$

Moreover, the last two quantities on the right-hand side of (B.39) are small for $\mathbb{p}_Q$ large in view of (B.31) and (B.37) while $\diamondsuit$ is small provided that $(\tau_3^2 + \tau_4)\, \mathbb{p}_Q^2$ is small. $\square$

Proof of Theorem B.10. Upper deviation bounds for $\|Q\boldsymbol{X}\|^2$ can now be derived as in the Gaussian case by applying (B.27) with a proper choice of μ. Let $\mathtt{x}$ satisfy $\sqrt{4\mathtt{x}} \le \sqrt{\operatorname{tr}(B^2)}/(3\mathtt{C}_\phi)$. We check (B.26) for $\mu = \mu(\mathtt{x})$. Indeed, the definition $\mu^{-1} =$

$1 + \sqrt{\operatorname{tr}(B^2)/(4\mathtt{x})}$ implies $\mu \le \sqrt{4\mathtt{x}/\operatorname{tr}(B^2)}$. Therefore, $\sqrt{4\mathtt{x}} \le \sqrt{\operatorname{tr}(B^2)}/(3\mathtt{C}_\phi)$ yields $\mu \le 1/(3\mathtt{C}_\phi)$ and (B.26) is fulfilled for $\mathtt{g}^2 = 3\operatorname{tr}(B)$. The bound (B.25) follows from (B.27) as in the Gaussian case of Theorem B.4. $\square$

For getting the bound on the lower deviation probability, we need an analog of (B.27) for μ negative. Representation (B.28) reads as

$$\mathbb{E}\, \mathrm{e}^{-\mu\|Q\boldsymbol{X}\|^2/2} = \mathbb{E}\, \mathbb{E}_{\boldsymbol{\gamma}}\, \mathrm{e}^{\mathtt{i}\sqrt{\mu}\langle Q^\top \boldsymbol{\gamma}, \boldsymbol{X}\rangle} = \mathbb{E}_{\boldsymbol{\gamma}} \mathbb{E}\, \mathrm{e}^{\mathtt{i}\sqrt{\mu}\langle Q^\top \boldsymbol{\gamma}, \boldsymbol{X}\rangle} \tag{B.40}$$

with $\mathtt{i} = \sqrt{-1}$. For our approach, it is necessary that the characteristic function $\mathbb{E}\exp\bigl(\mathtt{i}\langle \boldsymbol{u}, \boldsymbol{X}\rangle\bigr)$ does not vanish. This allows to define

$$\mathbb{f}(\boldsymbol{u}) \stackrel{\text{def}}{=} \log \mathbb{E}\, \mathrm{e}^{\mathtt{i}\langle \boldsymbol{u}, \boldsymbol{X}\rangle}.$$

Later we assume that the function $\mathbb{f}(\boldsymbol{u})$ is bounded on the ball $\|\boldsymbol{u}\| \le \mathtt{g}$. On the contrary to $(\boldsymbol{X})$, we don't require finite exponential moments for the vector $\boldsymbol{X}$.

$(\mathtt{i}\boldsymbol{X})$ *For some fixed* $\mathtt{g}$ *and* $\mathtt{C}_{\mathbb{f}}$, *the function* $\mathbb{f}(\boldsymbol{u}) = \log \mathbb{E}\, \mathrm{e}^{\mathtt{i}\langle \boldsymbol{u}, \boldsymbol{X}\rangle}$ *satisfies*

$$|\mathbb{f}(\boldsymbol{u})| = |\log \mathbb{E}\, \mathrm{e}^{\mathtt{i}\langle \boldsymbol{u}, \boldsymbol{X}\rangle}| \le \mathtt{C}_{\mathbb{f}}, \qquad \|\boldsymbol{u}\| \le \mathtt{g}.$$

Note that this condition can easily be ensured by replacing $\boldsymbol{X}$ with $\boldsymbol{X} + \alpha\boldsymbol{\gamma}$ for any positive α and $\boldsymbol{\gamma} \sim \mathcal{N}(0, \mathbb{I}_p)$. The constant $\mathtt{C}_{\mathbb{f}}$ is unimportant, it does not show up in our results. It, however, enables us to define similarly to (B.24)

$$\tau_4 \stackrel{\text{def}}{=} \sup_{\|\boldsymbol{u}\| \le \mathtt{g}} \frac{1}{\|\boldsymbol{u}\|^4} \bigl| \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \langle \mathtt{i}\boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \boldsymbol{X}\rangle^4 - 3\bigl\{ \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \langle \mathtt{i}\boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \boldsymbol{X}\rangle^2 \bigr\}^2 \bigr|. \tag{B.41}$$

Theorem B.12. *Let* $\boldsymbol{X}$ *satisfy* $\mathbb{E}\boldsymbol{X} = 0$, $\operatorname{Var}(\boldsymbol{X}) \le \mathbb{I}_p$. *Let also* Q *be a linear mapping* Q *with* $\|Q\| = 1$, $\mathbb{p}_Q = \operatorname{tr}(QQ^\top)$, $B = Q\operatorname{Var}(\boldsymbol{X})Q^\top$, $\mathtt{v}^2 = \operatorname{tr}(B^2)$. *Assume* $(\mathtt{i}\boldsymbol{X})$ *for some* $\mathtt{g}$ *with* $\mathtt{g}^2 \ge 4\mathbb{p}_Q^2/\mathtt{v}^2$. *Let also* τ_3 *be given by* (B.23) *and* $\omega \stackrel{\text{def}}{=} \mathtt{g}\,\tau_3/2 \le 1/3$. *Then for any* $\mathtt{x} \le \mathtt{v}^2/4$, *it holds*

$$\mathbb{P}\bigl(\|Q\boldsymbol{X}\|^2 < \operatorname{tr}(B) - 2\mathtt{v}\sqrt{\mathtt{x}}\bigr) \le (2 + \diamondsuit + \rho_\mu)\mathrm{e}^{-\mathtt{x}}, \tag{B.42}$$

where $\mu \stackrel{\text{def}}{=} 2\mathtt{v}^{-1}\sqrt{\mathtt{x}}$ *and*

$$\rho_\mu \stackrel{\text{def}}{=} \mathbb{P}\biggl(\|Q^\top \boldsymbol{\gamma}\|^2 \ge \frac{4\mu^{-1}\mathbb{p}_Q^2}{\mathtt{v}^2}\biggr) \le \exp\Bigl\{-\frac{\mathbb{p}_Q^2}{4\mathtt{v}^2}(2\mu^{-1/2} - 1)^2\Bigr\} \le \exp\Bigl(-\frac{\mathbb{p}_Q^2}{4\mathtt{v}^2}\Bigr). \tag{B.43}$$

The value $\diamondsuit$ *is described in the proof of Theorem B.11 and it is small provided* $\mathbb{p}_Q \gg 1$ *and* $(\tau_3^2 + \tau_4)\,\mathbb{p}_Q \ll 1$.

This result is based on an approximation $\mathbb{E}\mathrm{e}^{-\mu\|Q\boldsymbol{X}\|^2/2} \approx \det(I\!\!I_q + \mu B)^{-1/2}$.

Proposition B.13. *Assume the conditions of Theorem B.12. For any* $\mu \in (0,1)$, *it holds with* $B = Q\,\mathrm{Var}(\boldsymbol{X})Q^\top$

$$
\begin{aligned}
&\bigl|\mathbb{E}\mathrm{e}^{-\mu\|Q\boldsymbol{X}\|^2/2} - \det(I\!\!I_q + \mu B)^{-1/2}\bigr| \le (\diamondsuit + \rho_\mu)\det(I\!\!I_q + \mu B)^{-1/2} + \rho_\mu\,,\\
&\rho_\mu \le \mathbb{P}_{\boldsymbol{\gamma}}\bigl(\|Q^\top\boldsymbol{\gamma}\|^2 \ge 4\mu^{-1}\mathbb{p}_Q\bigr) \le \exp\Bigl\{-\frac{\mathbb{p}_Q}{4}(2\mu^{-1/2}-1)^2\Bigr\}.
\end{aligned}
\tag{B.44}
$$

Proof. We follow the line of the proof of Theorem B.11 replacing everywhere $\phi(\boldsymbol{u})$ with $\mathbb{f}(\boldsymbol{u})$. In particular, we start with representation (B.40) and apply with $\mathrm{g}^2 = 4\mathbb{p}_Q$

$$
\begin{aligned}
\mathbb{E}\,\mathrm{e}^{-\mu\|Q\boldsymbol{X}\|^2/2} &= \mathbb{E}_{\boldsymbol{\gamma}}\,\mathrm{e}^{\mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{\gamma})}\\
&= \mathbb{E}_{\boldsymbol{\gamma}}\,\mathrm{e}^{\mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{\gamma})}\,\mathbb{1}(\|\sqrt{\mu}\,Q^\top\boldsymbol{\gamma}\| \le \mathrm{g}) + \mathbb{E}_{\boldsymbol{\gamma}}\,\mathrm{e}^{\mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{\gamma})}\,\mathbb{1}(\|\sqrt{\mu}\,Q^\top\boldsymbol{\gamma}\| > \mathrm{g}).
\end{aligned}
$$

It holds

$$
\mathbb{f}(0) = 0, \quad \nabla\mathbb{f}(0) = 0, \quad -\nabla^2\mathbb{f}(0) = \mathrm{Var}(\boldsymbol{X}) \le I\!\!I_p\,.
$$

Moreover, smoothness conditions (B.32), (B.33) are automatically fulfilled for $\mathbb{f}(\boldsymbol{u})$ with the same τ_3 and τ_4 from (B.41). The most important observation for the proof is that the bound (B.38) continues to apply for $\mu < 0$ and

$$
f_\mu(\boldsymbol{w}) = \mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{w}) - \|\boldsymbol{w}\|^2/2,
$$

with $\diamondsuit$ from (B.35) and

$$
\begin{aligned}
D_\mu^2 &\stackrel{\mathrm{def}}{=} -\nabla^2 f_\mu(0) = \mu Q\,\mathrm{Var}(\boldsymbol{X})Q^\top + I\!\!I_q = I\!\!I_q + \mu B,\\
\mathbb{p}_\mu &\stackrel{\mathrm{def}}{=} \mathrm{tr}\bigl\{D_\mu^{-2}(\mu QQ^\top)\bigr\} \le \frac{\mu}{1+\mu}\,\mathrm{tr}(QQ^\top) \le \mu\,\mathbb{p}_Q\,,\\
\alpha_\mu &\stackrel{\mathrm{def}}{=} \|D_\mu^{-1}(\mu QQ^\top)D_\mu^{-1}\| \le \frac{\mu}{1+\mu}\,,
\end{aligned}
$$

and $\rho_\mu \le \mathbb{P}\bigl(\|Q^\top\boldsymbol{\gamma}\|^2 \ge \mu^{-1}\mathrm{g}^2\bigr)$; cf. (B.36). This yields

$$
\left|\mathbb{E}_{\boldsymbol{\gamma}}\,\mathrm{e}^{\mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{\gamma})}\,\mathbb{1}(\|\sqrt{\mu}\,Q^\top\boldsymbol{\gamma}\| \le \mathrm{g}) - \frac{1}{\det(I\!\!I_q + \mu B)^{1/2}}\right| \le \frac{\diamondsuit + \rho_\mu}{\det(I\!\!I_q + \mu B)^{1/2}}\,.
$$

Finally we use $|\mathrm{e}^{\mathbb{f}(\boldsymbol{u})}| \le 1$ and thus,

$$
\bigl|\mathbb{E}_{\boldsymbol{\gamma}}\,\mathrm{e}^{\mathbb{f}(\sqrt{\mu}\,Q^\top\boldsymbol{\gamma})}\,\mathbb{1}(\|\sqrt{\mu}\,Q^\top\boldsymbol{\gamma}\| > \mathrm{g})\bigr| \le \mathbb{P}\bigl(\|\sqrt{\mu}\,Q^\top\boldsymbol{\gamma}\| > \mathrm{g}\bigr) = \mathbb{P}_{\boldsymbol{\gamma}}\bigl(\|Q^\top\boldsymbol{\gamma}\|^2 \ge 4\mu^{-1}\mathbb{p}_Q\bigr)
$$

and (B.44) follows in view of (B.12) of Theorem B.8. ∎

Proof of Theorem B.12. By the exponential Chebyshev inequality and (B.44)

$$\begin{aligned}\mathbb{P}\bigl(\operatorname{tr}(B) - \|Q\boldsymbol{X}\|^2 > 2\mathtt{v}\sqrt{\mathtt{x}}\bigr) &\le \exp(-\mu\,\mathtt{v}\sqrt{\mathtt{x}})\mathbb{E}\exp\bigl\{\mu\operatorname{tr}(B)/2 - \mu\|Q\boldsymbol{X}\|^2/2\bigr\} \\ &\le \exp(\mu\operatorname{tr}(B)/2 - \mu\,\mathtt{v}\sqrt{\mathtt{x}})\bigl\{(1+\diamondsuit+\rho_\mu)\det(\mathbb{I}_q+\mu B)^{-1/2} + \rho_\mu\bigr\}.\end{aligned}$$

In view of $x - \log(1+x) \le x^2/2$ and $\mu = 2\mathtt{v}^{-1}\sqrt{\mathtt{x}}$, it holds as in the proof of Lemma B.2

$$-\mu\,\mathtt{v}\sqrt{\mathtt{x}} + \mu\operatorname{tr}(B)/2 + \log\det(\mathbb{I}_q+\mu B)^{-1/2} \le -\mu\,\mathtt{v}\sqrt{\mathtt{x}} + \mu^2\mathtt{v}^2/4 = -\mathtt{x}.$$

Also $\mu\operatorname{tr}(B)/2 - \mu\,\mathtt{v}\sqrt{\mathtt{x}} = \mathtt{v}^{-1}\operatorname{tr}(B)\sqrt{\mathtt{x}} - 2\mathtt{x} \le \mathtt{v}^{-1}\mathbb{p}_Q\sqrt{\mathtt{x}} - 2\mathtt{x}$. The bound on ρ_μ in (B.43) follows from (B.12) of Theorem B.8 in view of $\mathbb{p}_Q \ge \mathtt{v}^2$ and hence, $\mathbb{p}_Q \le \mathbb{p}_Q^2/\mathtt{v}^2$. Finally, observe that

$$\begin{aligned}\rho_\mu\,\exp\bigl(\mathtt{v}^{-1}\mathbb{p}_Q\sqrt{\mathtt{x}} - 2\mathtt{x}\bigr) &\le \exp\Bigl(-\frac{\mathbb{p}_Q^2}{4\mathtt{v}^2} + \frac{\mathbb{p}_Q\sqrt{\mathtt{x}}}{\mathtt{v}} - 2\mathtt{x}\Bigr) \\ &\le \exp\Bigl\{-\Bigl(\frac{\mathbb{p}_Q}{2\mathtt{v}} - \sqrt{\mathtt{x}}\Bigr)^2 - \mathtt{x}\Bigr\} \le \mathrm{e}^{-\mathtt{x}}\end{aligned}$$

and (B.42) follows as well. □

B.3.3 Sum of i.i.d. random vectors

Here we specify the obtained results to the case when $\boldsymbol{X} = n^{-1/2}\sum_{i=1}^n \boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_i$ are i.i.d. in $\mathbb{R}^p$ with $\mathbb{E}\boldsymbol{\xi}_i = 0$ and $\operatorname{Var}(\boldsymbol{\xi}_i) = \Sigma \le \mathbb{I}_p$. In fact, only independence of the $\boldsymbol{\xi}_i$'s is used provided that all the moment conditions later on are satisfied uniformly over $i \le n$. However, the formulation is slightly simplified in the i.i.d case. Let some $Q\colon \mathbb{R}^p \to \mathbb{R}^q$ be fixed with $\|Q\| = 1$. It holds

$$\mathbb{p} = \mathbb{E}\|Q\boldsymbol{X}\|^2 = \operatorname{tr}(B), \qquad B = Q\Sigma Q^\top.$$

Also define $\mathbb{p}_Q = QQ^\top$. We study the concentration phenomenon for $\|Q\boldsymbol{X}\|^2$. The goal is to apply Theorem B.10 and Theorem B.12 claiming that $\|Q\boldsymbol{X}\|^2 - \mathbb{p}$ can be sandwiched between $-2\mathtt{v}\sqrt{\mathtt{x}}$ and $2\mathtt{v}\sqrt{\mathtt{x}} + 2\mathtt{x}$ with probability at least $1 - 2\mathrm{e}^{-\mathtt{x}}$. The major required condition is sub-gaussian behavior of $\boldsymbol{\xi}_1$. The conditions are summarized here.

$(\boldsymbol{\xi}_1)$ *A random vector* $\boldsymbol{\xi}_1 \in \mathbb{R}^p$ *satisfies* $\mathbb{E}\boldsymbol{\xi}_1 = 0$, $\operatorname{Var}(\boldsymbol{\xi}_1) = \Sigma \le \mathbb{I}_p$. *Also*

1. *The function* $\phi_1(\boldsymbol{u}) \stackrel{\text{def}}{=} \log\mathbb{E}\mathrm{e}^{\langle\boldsymbol{u},\boldsymbol{\xi}_1\rangle}$ *is finite and fulfills for some* $\mathtt{C}_\phi$

$$\phi_1(\boldsymbol{u}) \stackrel{\text{def}}{=} \log\mathbb{E}\mathrm{e}^{\langle\boldsymbol{u},\boldsymbol{\xi}_1\rangle} \le \frac{\mathtt{C}_\phi\|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p.$$

2. *For some* $\varrho > 0$ *and some constants* $\mathtt{c}_3$ *and* $\mathtt{c}_4$ *, it holds with* $\mathbb{E}_{\boldsymbol{u}}$ *from* (B.22)

$$\sup_{\|\boldsymbol{u}\| \le \varrho} \frac{1}{\|\boldsymbol{u}\|^3} \bigl| \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle^3 \bigr| \le \mathtt{c}_3 \,;$$
$$\sup_{\|\boldsymbol{u}\| \le \varrho} \frac{1}{\|\boldsymbol{u}\|^4} \bigl| \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^4 - 3 \bigl\{ \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^2 \bigr\}^2 \bigr| \le \mathtt{c}_4 \,.$$

3. *The function* $\log \mathbb{E}\, \mathrm{e}^{\mathtt{i} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle}$ *is well defined and*

$$\sup_{\|\boldsymbol{u}\| \le \varrho} \frac{1}{\|\boldsymbol{u}\|^4} \bigl| \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \langle \mathtt{i}\boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^4 - 3 \bigl\{ \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \langle \mathtt{i}\boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\mathtt{i}\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^2 \bigr\}^2 \bigr| \le \mathtt{c}_4 \,.$$

We are now well prepared to state the result for the i.i.d. case. Apart $(\boldsymbol{\xi}_1)$, we need $\operatorname{tr}(B^2)$ to be sufficiently large to ensure the condition $\operatorname{tr}(B^2) \gg \mathtt{C}_\phi^2$. Also we require n to be large enough for the relation $\mathbb{p}_Q^2 \ll n$; see Section B.3.4 for a further discussion.

Theorem B.14. *Let* $\boldsymbol{X} = n^{-1/2} \sum_{i=1}^n \boldsymbol{\xi}_i$ *, where* $\boldsymbol{\xi}_i$ *are i.i.d. in* $\mathbb{R}^p$ *satisfying* $\mathbb{E} \boldsymbol{\xi}_1 = 0$ *and* $\operatorname{Var}(\boldsymbol{\xi}_1) = \Sigma \le \mathbb{I}_p$ *, and* $(\boldsymbol{\xi}_1)$ *. For a fixed* Q *with* $\|Q\| = 1$ *, assume* $n\,\varrho^2 \ge 4 \mathbb{p}_Q$ *and* $n \gg \mathbb{p}_Q^2$ *. Then with* $B = Q \Sigma Q^\top$ *, it holds*

$$\mathbb{P}\Bigl(\|Q\boldsymbol{X}\|^2 - \operatorname{tr}(B) > 2\sqrt{\mathtt{x} \operatorname{tr}(B^2)} + 2\mathtt{x} \Bigr) \le (1 + \Delta_\mu) \mathrm{e}^{-\mathtt{x}} , \qquad \text{if} \quad \sqrt{4\mathtt{x}} \le \frac{\sqrt{\operatorname{tr}(B^2)}}{3\mathtt{C}_\phi} ,$$
$$\mathbb{P}\Bigl(\|Q\boldsymbol{X}\|^2 - \operatorname{tr}(B) < -2\sqrt{\mathtt{x} \operatorname{tr}(B^2)} \Bigr) \le (2 + \Delta_\mu) \mathrm{e}^{-\mathtt{x}} , \qquad \text{if} \quad \mathtt{x} \le \operatorname{tr}(B^2)/4,$$

where

$$\Delta_\mu \lesssim \frac{\mathbb{p}_Q^2}{n} \,. \tag{B.45}$$

Proof. The definition and i.i.d structure of the $\boldsymbol{\xi}_i$'s yield

$$\phi(\boldsymbol{u}) = \log \mathbb{E} \mathrm{e}^{\langle \boldsymbol{X}, \boldsymbol{u} \rangle} = n \phi_1(n^{-1/2} \boldsymbol{u}).$$

Moreover,

$$\mathbb{E} \langle \boldsymbol{u}, \boldsymbol{X} \rangle^2 = \mathbb{E} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle^2, \qquad \mathbb{E} \langle \boldsymbol{u}, \boldsymbol{X} \rangle^3 = n^{-1/2} \mathbb{E} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 \rangle^3,$$

and for any $\boldsymbol{u}$

$$\begin{aligned} & \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{X} \rangle^4 - 3 \bigl\{ \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{X} - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{X} \rangle^2 \bigr\}^2 \\ & \quad = n^{-1} \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^4 - 3 n^{-1} \bigl\{ \mathbb{E}_{\boldsymbol{u}} \langle \boldsymbol{u}, \boldsymbol{\xi}_1 - \mathbb{E}_{\boldsymbol{u}} \boldsymbol{\xi}_1 \rangle^2 \bigr\}^2. \end{aligned}$$

This implies (B.24) for any $\mathtt{g}$ with $\mathtt{g}/\sqrt{n} \le \varrho$ and

$$\tau_3 \le n^{-1/2}\mathtt{c}_3 , \qquad \tau_4 \le n^{-1}\mathtt{c}_4 .$$

Moreover, the quantity $\diamondsuit$ from (B.35) satisfies $\diamondsuit \lesssim \mathbb{p}_Q^2/n$ yielding (B.45). Now the upper bound follows from Theorem B.10. Similar arguments can be used for checking the lower bound by Theorem B.12. $\square$

B.3.4 Range of applicability, critical dimension

This section discusses the range of applicability of the presented results, in particular, of the concentration phenomenon. It was already mentioned earlier that concentration of the squared norm $\|Q\boldsymbol{X}\|^2$ is only possible in a high dimensional situation, even for $\boldsymbol{X}$ Gaussian. This condition can be written as $\operatorname{tr}(B)/\|B\| \gg 1$. In our results, this condition is further detailed. For instance, bound (B.25) of Theorem B.10 is only meaningful if $\operatorname{tr}(B^2) \gg \mathtt{C}_\phi^2$. This is the only place where the value $\mathtt{C}_\phi$ shows up.

Another important issue is the value $\varDelta_\mu$ which is presented in all our results. It should be small to make the results meaningful. A sufficient condition for this property are $(\tau_3^2 + \tau_4)\,\mathbb{p}_Q^2 \ll 1$. For the case of additive structure of $\boldsymbol{X}$, this condition transforms into "critical dimension" condition $\mathbb{p}_Q^2 \ll n$. Recent results from Katsevich (2023) indicate that Laplace approximation could fail if $\mathbb{p}_Q^2 \ll n$ is not fulfilled even for a simple generalized linear model. One can guess that a further relaxation of the "critical dimension" condition $\mathbb{p}_Q^2 \ll n$ is not possible and approximation $\mathbb{P}\bigl(\|Q\boldsymbol{X}\| > z(B,\mathtt{x})\bigr) \approx \mathbb{P}\bigl(\|Q\widetilde{\boldsymbol{X}}\| > z(B,\mathtt{x})\bigr)$ with $\widetilde{\boldsymbol{X}}$ gaussian can fail if $\mathbb{p}_Q^2 \gg n$.

B.4 Deviation bounds under light exponential tails

Let $\boldsymbol{\xi}$ be a zero mean random vector in $\mathbb{R}^p$ with covariance $\operatorname{Var}(\boldsymbol{\xi})$ and let $Q\colon \mathbb{R}^p \to \mathbb{R}^q$ be a linear mapping. This section presents some deviation bounds on the norm $\|Q\boldsymbol{\xi}\|$ for the case of light exponential tails of $\boldsymbol{\xi}$. Namely,

($\mathtt{g}$) *for some fixed* $\mathtt{g} > 0$ *and some self-adjoint operator* $\mathbb{V}^2$ *in* $\mathbb{R}^p$ *with* $\mathbb{V}^2 \ge \operatorname{Var}(\boldsymbol{\xi})$,

$$\phi(\boldsymbol{u}) \stackrel{\text{def}}{=} \log \mathbb{E} \exp\bigl(\langle \boldsymbol{u}, \mathbb{V}^{-1}\boldsymbol{\xi}\rangle\bigr) \le \frac{\|\boldsymbol{u}\|^2}{2}, \qquad \boldsymbol{u} \in \mathbb{R}^p,\ \|\boldsymbol{u}\| \le \mathtt{g}, \tag{B.46}$$

In fact, it is sufficient to assume that

$$\sup_{\|\boldsymbol{u}\| \le \mathtt{g}} \mathbb{E} \exp\bigl(\langle \boldsymbol{u}, \mathbb{V}^{-1}\boldsymbol{\xi}\rangle\bigr) \le \mathtt{C}. \tag{B.47}$$

The quantity $\mathtt{C}$ can be very large but it is not important. Indeed, the function $\phi(\boldsymbol{u})$ is analytic on the disk $\|\boldsymbol{u}\| \le \mathtt{g}$, and condition (B.47) implies an analog of (B.46):

$$\phi(\boldsymbol{u}) \le \frac{\|\boldsymbol{u}\|^2}{2} + \frac{\tau_3 \|\boldsymbol{u}\|^3}{6} \le \frac{\|\boldsymbol{u}\|^2}{2}\Bigl(1 + \frac{\tau_3 \mathtt{g}}{3}\Bigr), \qquad \|\boldsymbol{u}\| \le \mathtt{g},$$

for a fixed value τ_3. Moreover, reducing $\mathtt{g}$ allows to take $\mathbb{V}^2$ equal or close to $\operatorname{Var}(\boldsymbol{\xi})$ and τ_3 close to zero. The next section presents our main results under (g). The proofs are postponed until the end of the section.

B.4.1 Main results

Let a random vector $\boldsymbol{\xi}$ satisfy $\mathbb{E}\boldsymbol{\xi} = 0$ and (g). The goal is to establish possibly sharp deviation bounds on $\|Q\boldsymbol{\xi}\|^2$ for a given linear mapping $Q\colon \mathbb{R}^p \to \mathbb{R}^q$. Define

$$\begin{gathered} B \stackrel{\text{def}}{=} Q\mathbb{V}^2 Q^\top, \quad \mathbb{p} \stackrel{\text{def}}{=} \operatorname{tr}(B), \quad \mathtt{v}^2 \stackrel{\text{def}}{=} \operatorname{tr}(B^2), \quad \lambda \stackrel{\text{def}}{=} \|B\|, \\ z^2(B,\mathtt{x}) \stackrel{\text{def}}{=} \operatorname{tr} B + 2\sqrt{\mathtt{x}\operatorname{tr}(B^2)} + 2\mathtt{x}\|B\| = \mathbb{p} + 2\mathtt{v}\sqrt{\mathtt{x}} + 2\mathtt{x}\lambda. \end{gathered} \tag{B.48}$$

Also fix some $\rho < 1$, a standard choice is $\rho = 1/2$. Our main result applies for all $\mathtt{x}$ satisfying the condition

$$z^2(B,\mathtt{x}) \le \rho \left(\frac{\mathtt{g}\sqrt{\lambda}}{\mu(\mathtt{x})} - \sqrt{\frac{\mathbb{p}}{\mu(\mathtt{x})}} \right)^2 \tag{B.49}$$

with $z(B,\mathtt{x})$ from (B.48) and $\mu(\mathtt{x})$ defined by $\mu^{-1}(\mathtt{x}) = 1 + \frac{\mathtt{v}}{2\lambda\sqrt{\mathtt{x}}}$; see (B.9). One can see that the left hand-side of (B.49) increases with $\mathtt{x}$ while the right hand-side decreases. Therefore, there exists a unique root $\mathtt{x}_c$ such that with $\mu_c = \mu(\mathtt{x}_c)$

$$z^2(B,\mathtt{x}_c) = \rho \left(\frac{\mathtt{g}\sqrt{\lambda}}{\mu_c} - \sqrt{\frac{\mathbb{p}}{\mu_c}} \right)^2. \tag{B.50}$$

The value $\mathtt{x}_c$ is important, it describes the *phase transition* effect: the upper quantile function of $\|Q\boldsymbol{\xi}\|$ exhibits the Gaussian-like behavior for $\mathtt{x} \le \mathtt{x}_c$, while it grows linearly with $\mathtt{x}/\mathtt{g}$ for $\mathtt{x} > \mathtt{x}_c$ as in a sub-exponential case.

Theorem B.15. *Assume* (g). *Fix* $\mathtt{x}_c$ *by* (B.50) *for some* $\rho \le 1/2$. *It holds*

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| \ge z(B,\mathtt{x})\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, \qquad \mathtt{x} \le \mathtt{x}_c. \tag{B.51}$$

For $\rho = 1/2$, *the value* $\mathtt{x}_c$ *from* (B.50) *fulfills*

$$\frac{1}{4}\left(\mathtt{g} - \sqrt{\frac{2\mathbb{p}}{\lambda}}\right)_+^2 \le \mathtt{x}_c \le \frac{\mathtt{g}^2}{4}. \tag{B.52}$$

If $\mathtt{g} > \sqrt{2\mathbb{p}/\lambda}$ *then* $z_c = z(B, \mathtt{x}_c)$ *follows*

$$\mathtt{g}\sqrt{\lambda/2} - (1 - 2^{-1/2})\sqrt{\mathbb{p}} \le z_c \le \mathtt{g}\sqrt{\lambda/2} + \sqrt{\mathbb{p}}\,. \tag{B.53}$$

The results of Theorem B.15 state nearly Gaussian deviation bounds for the norm of the vector $Q\boldsymbol{\xi}$ satisfying (g). Namely, the Gaussian deviation bound $\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| \ge z(B,\mathtt{x})\bigr) \le \mathrm{e}^{-\mathtt{x}}$ from Theorem B.4 applies with the additional factor 3 for all $\mathtt{x} \le \mathtt{x}_c$. Condition $\mathtt{g} \gg \sqrt{\mathbb{p}/\lambda}$ is important. Otherwise, the value $\mathtt{x}_c$ is not significantly large and the zone $\mathtt{x} \le \mathtt{x}_c$ with Gaussian-like quantiles is too narrow. It turns out that out of this range, the norm $\|Q\boldsymbol{\xi}\|$ exhibits a sub-exponential behavior.

Theorem B.16. *Assume* (g)*. With* $\mathtt{x}_c$ *from* (B.50) *and* $z_c = z(B,\mathtt{x}_c)$*, set* $\varkappa = \frac{\sqrt{\rho}\,\mathtt{g}}{(2+\sqrt{\rho})\sqrt{\lambda}}$*. It holds*

$$\begin{aligned} &\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| > z_c + \varkappa^{-1}(\mathtt{x} - \mathtt{x}_c)\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, && \mathtt{x} \ge \mathtt{x}_c\,, \\ &\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| > z\bigr) \le 3\exp\{-\mathtt{x}_c - \varkappa(z - z_c)\}, && z \ge z_c\,. \end{aligned} \tag{B.54}$$

The obtained deviation bounds of Theorem B.15 and Theorem B.16 can be fused into one. To be more specific, we fix $\rho = 1/2$.

Corollary B.17. *Assume* (g)*. Let* $\mathtt{x}_c$ *be defined by* (B.50) *with* $\rho = 1/2$*. For all* $\mathtt{x} > 0$

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| > z_c(B,\mathtt{x})\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, \tag{B.55}$$

where with $\varkappa \stackrel{\text{def}}{=} \frac{\mathtt{g}}{(\sqrt{8}+1)\sqrt{\lambda}}$ *and* $\mathtt{x} \wedge \mathtt{x}_c \stackrel{\text{def}}{=} \min\{\mathtt{x}, \mathtt{x}_c\}$

$$z_c(B,\mathtt{x}) \stackrel{\text{def}}{=} z(B, \mathtt{x} \wedge \mathtt{x}_c) + \varkappa^{-1}(\mathtt{x} - \mathtt{x}_c)_+ = \begin{cases} z(B,\mathtt{x}), & \mathtt{x} \le \mathtt{x}_c\,, \\ z(B,\mathtt{x}_c) + \dfrac{\mathtt{x} - \mathtt{x}_c}{\varkappa}, & \mathtt{x} > \mathtt{x}_c\,. \end{cases} \tag{B.56}$$

Moreover, $\mathtt{x}_c$ *follows* (B.52) *and* $z_c = z(B,\mathtt{x}_c)$ *satisfies* (B.53) *provided* $\mathtt{g} \ge \sqrt{2\mathbb{p}/\lambda}$*.*

If $\mathtt{g} \gg \sqrt{\mathbb{p}/\lambda}$ then $\mathtt{x}_c$ is large and $z_c(B,\mathtt{x}) = z(B,\mathtt{x}) \le \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}\lambda}$ for all reasonable $\mathtt{x}$. For $\mathtt{g} < \sqrt{2\mathbb{p}/\lambda}$, the accurate bound (B.56) can be simplified by a linear majorant which does not involve $\mathtt{x}_c$.

Theorem B.18. *Assume* (g)*. Fix* $\varkappa = \frac{\mathtt{g}}{(\sqrt{8}+1)\sqrt{\lambda}}$*. Then* (B.55) *applies with*

$$z_c(B,\mathtt{x}) \le \sqrt{\mathbb{p}} + \frac{\varkappa}{\sqrt{2}} + \varkappa^{-1}\mathtt{x}\,.$$

The next result provides some upper bounds on the exponential moments of $\|Q\boldsymbol{\xi}\|$. We distinguish between zones $z \le z_c$ and $z > z_c$ with $z_c = z(B, \mathtt{x}_c)$; see (B.50).

Theorem B.19. *Assume* (g)*. Let* $\mathtt{x}_c$ *fulfill* (B.50) *and* $z_c = z(B, \mathtt{x}_c)$*. For any* $z \in [\sqrt{\mathbb{p}}, z_c]$ *and any* $\nu \le \frac{z-\sqrt{\mathbb{p}}}{2\sqrt{\lambda}}$*, it holds*

$$\mathbb{E} \mathrm{e}^{\nu\|Q\boldsymbol{\xi}\|} \, \mathbb{1}(\|Q\boldsymbol{\xi}\| \ge z) \le 6 \exp\Bigl\{ \nu z - \frac{(z-\sqrt{\mathbb{p}})^2}{2\lambda} \Bigr\}. \tag{B.57}$$

Further, for any $\nu < \varkappa \stackrel{\mathrm{def}}{=} \frac{\mathtt{g}\sqrt{\rho}}{\sqrt{\lambda}\,(2+\sqrt{\rho})}$

$$\mathbb{E} \mathrm{e}^{\nu\|Q\boldsymbol{\xi}\|} \, \mathbb{1}(\|Q\boldsymbol{\xi}\| > z_c) \le \frac{3\varkappa}{\varkappa-\nu} \exp\biggl\{ \nu z_c - \frac{(z_c-\sqrt{\mathbb{p}})^2}{2\lambda} \biggr\}. \tag{B.58}$$

Moreover, for $z \ge z_c$

$$\mathbb{E} \mathrm{e}^{\nu\|Q\boldsymbol{\xi}\|} \, \mathbb{1}(\|Q\boldsymbol{\xi}\| > z) \le \frac{3\varkappa}{\varkappa-\nu} \exp\Bigl\{ \nu z_c - \frac{(z_c-\sqrt{\mathbb{p}})^2}{2\lambda} - (\varkappa-\nu)(z-z_c) \Bigr\}. \tag{B.59}$$

B.4.2 Proof of Theorem B.15

By normalization, one can easily reduce the study to the case $\|B\| = 1$. Moreover, replacing $\boldsymbol{\xi}$ with $\mathbb{V}^{-1}\boldsymbol{\xi}$ and Q with $Q\mathbb{V}$ reduces the proof to the situation with $\mathbb{V} = \mathbb{I}_p$. This will be assumed later on. For $\mu \in (0,1)$ and $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu} > 0$, define trimming $t_\mu(\boldsymbol{u})$ of $\boldsymbol{u} \in \mathbb{R}^p$ as

$$t_\mu(\boldsymbol{u}) \stackrel{\mathrm{def}}{=} \begin{cases} \boldsymbol{u}, & \text{if } \|\boldsymbol{u}\| \le \mathfrak{z}(\mu), \\ \frac{\mathfrak{z}(\mu)}{\|\boldsymbol{u}\|}\,\boldsymbol{u}, & \text{otherwise.} \end{cases} \tag{B.60}$$

By construction $\|t_\mu(\boldsymbol{u})\| \le \mathfrak{z}(\mu)$ for all $\boldsymbol{u} \in \mathbb{R}^p$.

Lemma B.20. *Assume* (g) *and let* $\|B\| = 1$*. Fix* $\mu \in (0,1)$ *s.t.* $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu} > 0$*. Then with* $t_\mu(\cdot)$ *from* (B.60)

$$\mathbb{E} \exp\Bigl\{ \frac{\mu}{2}\, t_\mu^2(Q\boldsymbol{\xi}) \Bigr\} \le 2 \exp\{\varPhi(\mu)\}, \tag{B.61}$$

where

$$\varPhi(\mu) \stackrel{\mathrm{def}}{=} \frac{\mu^2 \mathtt{v}^2}{4(1-\mu)} + \frac{\mu\,\mathbb{p}}{2}. \tag{B.62}$$

Furthermore, for any $\mathfrak{z} < \mathfrak{z}(\mu)$

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| > \mathfrak{z}, \|Q\boldsymbol{\xi}\| \le \mathfrak{z}(\mu) \bigr) \le 2 \exp\Bigl\{ -\frac{\mu\,\mathfrak{z}^2}{2} + \varPhi(\mu) \Bigr\}. \tag{B.63}$$

Proof. Let us fix any value of $\boldsymbol{\xi}$. We intend to show that

$$\exp\Bigl\{\frac{\mu}{2}\,\|t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr\} \le 2\mathbb{E}_{\boldsymbol{\gamma}}\,\exp\{\mu^{1/2}\boldsymbol{\gamma}^{\top}t_{\mu}(Q\boldsymbol{\xi})\}. \tag{B.64}$$

Here $\mathbb{E}_{\boldsymbol{\gamma}}$ means conditional expectation w.r.t. $\boldsymbol{\gamma}\sim\mathcal{N}(0,I\!\!I_p)$ given $\boldsymbol{\xi}$. Obviously, with $A=\{\boldsymbol{u}\colon \mu^{1/2}\|Q^{\top}\boldsymbol{u}\|\le \mathtt{g}\}$, it suffices to check that

$$\mathcal{I}_{\mu}(\boldsymbol{\xi}) \stackrel{\text{def}}{=} \mathbb{E}_{\boldsymbol{\gamma}}\exp\Bigl\{\mu^{1/2}\boldsymbol{\gamma}^{\top}t_{\mu}(Q\boldsymbol{\xi})-\frac{\mu}{2}\|t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr\}\,\mathbb{I}(\boldsymbol{\gamma}\in A)\ge 1/2. \tag{B.65}$$

With $\mathtt{C}_p=(2\pi)^{-p/2}$, it holds

$$\begin{aligned}
\mathcal{I}_{\mu}(\boldsymbol{\xi}) &= \mathtt{C}_p\int_A \exp\Bigl(\mu^{1/2}\boldsymbol{u}^{\top}t_{\mu}(Q\boldsymbol{\xi})-\frac{\mu}{2}\|t_{\mu}(Q\boldsymbol{\xi})\|^2-\frac{1}{2}\|\boldsymbol{u}\|^2\Bigr)d\boldsymbol{u}\\
&= \mathtt{C}_p\int_A \exp\Bigl(-\frac{1}{2}\|\boldsymbol{u}-\mu^{1/2}t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr)d\boldsymbol{u} = \mathbb{P}_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}-\mu^{1/2}t_{\mu}(Q\boldsymbol{\xi})\in A).
\end{aligned}$$

The definition of A and the condition $\|t_{\mu}(Q\boldsymbol{\xi})\|\le\mathfrak{z}(\mu)$ imply in view of $\|Q\|\le 1$

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}-\mu^{1/2}t_{\mu}(Q\boldsymbol{\xi})\in A) &= \mathbb{P}_{\boldsymbol{\gamma}}\bigl(\|Q^{\top}(\boldsymbol{\gamma}-\mu^{1/2}t_{\mu}(Q\boldsymbol{\xi}))\|\le \mathtt{g}/\mu^{1/2}\bigr)\\
&\ge \mathbb{P}_{\boldsymbol{\gamma}}\bigl(\|Q^{\top}\boldsymbol{\gamma}\|\le \mathtt{g}/\mu^{1/2}-\mu^{1/2}\mathfrak{z}(\mu)\bigr)\ge \mathbb{P}_{\boldsymbol{\gamma}}\bigl(\|Q^{\top}\boldsymbol{\gamma}\|\le\sqrt{\mathbb{p}}\bigr)\ge 1/2
\end{aligned}$$

and (B.65) follows. Taking expectation for both sides of (B.64) and the use of Fubini's theorem yield

$$\mathbb{E}\exp\Bigl\{\frac{\mu}{2}\,\|t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr\}\le 2\mathbb{E}_{\boldsymbol{\gamma}}\bigl\{\mathbb{E}\exp\{\mu^{1/2}\boldsymbol{\gamma}^{\top}t_{\mu}(Q\boldsymbol{\xi})\}\;\mathbb{I}(\mu^{1/2}\|Q^{\top}\boldsymbol{\gamma}\|\le\mathtt{g})\bigr\}.$$

Obviously, for any $\boldsymbol{u}\in\mathbb{R}^p$

$$\exp\{\boldsymbol{u}^{\top}t_{\mu}(Q\boldsymbol{\xi})\}+\exp\{-\boldsymbol{u}^{\top}t_{\mu}(Q\boldsymbol{\xi})\}\le\exp\{\boldsymbol{u}^{\top}Q\boldsymbol{\xi}\}+\exp\{-\boldsymbol{u}^{\top}Q\boldsymbol{\xi}\}$$

and by (B.46)

$$\begin{aligned}
\mathbb{E}\exp\Bigl\{\frac{\mu}{2}\,\|t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr\} &\le 2\mathbb{E}_{\boldsymbol{\gamma}}\Bigl\{\exp\Bigl(\frac{1}{2}\,\|\mu^{1/2}\boldsymbol{\gamma}^{\top}Q\|^2\Bigr)\;\mathbb{I}(\mu^{1/2}\|Q^{\top}\boldsymbol{\gamma}\|\le\mathtt{g})\Bigr\}\\
&\le 2\mathbb{E}_{\boldsymbol{\gamma}}\,\exp\Bigl(\frac{1}{2}\,\|\mu^{1/2}\boldsymbol{\gamma}^{\top}Q\|^2\Bigr) = 2\det(I\!\!I_p-\mu Q^{\top}Q)^{-1/2}.
\end{aligned}$$

We also use that for any $\mu>0$ by (B.10),

$$\log\det\bigl(I\!\!I-\mu B\bigr)^{-1/2}\le\frac{\mu\,\mathrm{tr}(B)}{2}+\frac{\mu^2\,\mathrm{tr}(B^2)}{4(1-\mu)}=\varPhi(\mu)\,,$$

and the first statement follows. Moreover, by Markov's inequality

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\|>\mathfrak{z},\|Q\boldsymbol{\xi}\|\le\mathfrak{z}(\mu)\bigr)\le \mathrm{e}^{-\mu\mathfrak{z}^2/2}\mathbb{E}\exp\Bigl\{\frac{\mu}{2}\,\|t_{\mu}(Q\boldsymbol{\xi})\|^2\Bigr\}\le 2\exp\Bigl\{-\frac{\mu\mathfrak{z}^2}{2}+\varPhi(\mu)\Bigr\},$$

and (B.63) follows as well. $\square$

The use of $\mu = \mu(\mathtt{x})$ from (B.9) in (B.61) yields

$$-\frac{\mu z^2(B,\mathtt{x})}{2} + \varPhi(\mu) \, = \, -\mathtt{x}\,, \tag{B.66}$$

and similarly to the proof of Theorem B.4

$$I\!\!P\Bigl(\|Q\boldsymbol{\xi}\|^2 > z^2(B,\mathtt{x}),\; \|Q\boldsymbol{\xi}\| \le \mathfrak{z}(\mu)\Bigr) \, \le \, 2\mathrm{e}^{-\mathtt{x}}. \tag{B.67}$$

It remains to consider the probability of large deviation $I\!\!P\bigl(\|Q\boldsymbol{\xi}\| > \mathfrak{z}(\mu)\bigr)$.

Lemma B.21. *Assume* $\|B\| = 1$. *Given* $\mathtt{x} > 0$, *fix* $\mu = \mu(\mathtt{x})$ *and* $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu}$. *Assume* (B.49) *for some* $\rho \le 1/2$. *Then*

$$I\!\!P\bigl(\|Q\boldsymbol{\xi}\| > \mathfrak{z}(\mu)\bigr) \, \le \, \mathrm{e}^{-\mathtt{x}}. \tag{B.68}$$

Proof. Denote $\eta = \|Q\boldsymbol{\xi}\|$. By (B.67)

$$I\!\!P\Bigl(\eta > z(B,\mathtt{x}),\; \eta \le \mathfrak{z}(\mu)\Bigr) \, \le \, 2\mathrm{e}^{-\mathtt{x}}, \tag{B.69}$$

For $\mu = \mu(\mathtt{x})$, it holds (B.66) with $\varPhi(\mu)$ given by (B.62). Bounding the tails of η in the region $\eta > \mathfrak{z}(\mu)$ requires another choice of μ. Namely, we apply (B.63) with $\rho\mu$ instead of μ yielding

$$I\!\!P\bigl(\eta > \mathfrak{z}(\mu), \eta \le \mathfrak{z}(\rho\mu)\bigr) \, \le \, 2\exp\Bigl\{-\frac{\rho\mu\,\mathfrak{z}^2(\mu)}{2} + \varPhi(\rho\mu)\Bigr\}.$$

In a similar way, applying (B.69) with $\rho^2\mu$ in place of μ and using that

$$\rho\,\mathfrak{z}(\rho\mu) \, = \, \mathtt{g}/\mu - \sqrt{\rho\,\mathbb{p}/\mu} \le \mathfrak{z}(\mu) \tag{B.70}$$

yields

$$\begin{aligned} I\!\!P\bigl(\eta > \mathfrak{z}(\rho\mu), \eta \le \mathfrak{z}(\rho^2\mu)\bigr) &\le 2\exp\Bigl\{-\frac{\rho^2\mu\,\mathfrak{z}^2(\rho\mu)}{2} + \varPhi(\rho^2\mu)\Bigr\} \\ &\le 2\exp\Bigl\{-\frac{\mu\,\mathfrak{z}^2(\mu)}{2} + \varPhi(\rho^2\mu)\Bigr\}. \end{aligned}$$

This trick can be applied again and again yielding in view of (B.70)

$$
\begin{aligned}
\mathbb{P}\bigl(\eta > \mathfrak{z}(\mu)\bigr) &\leq \sum_{k=0}^{\infty} \mathbb{P}\bigl(\eta > \mathfrak{z}(\rho^{k}\mu), \eta \leq \mathfrak{z}(\rho^{k+1}\mu)\bigr) \\
&\leq \sum_{k=0}^{\infty} 2\exp\bigl\{-\rho^{k+1}\mu\,\mathfrak{z}^{2}(\rho^{k}\mu)/2 + \varPhi(\rho^{k+1}\mu)\bigr\} \\
&\leq \sum_{k=0}^{\infty} 2\exp\bigl\{-\rho^{-k+1}\mu\,\mathfrak{z}^{2}(\mu)/2 + \varPhi(\rho^{k+1}\mu)\bigr\}.
\end{aligned}
$$

Condition $\rho\,\mathfrak{z}^{2}(\mu) \geq z^{2}(B,\mu)/2$ and (B.66) ensure for $\rho \leq 1/2$

$$
\begin{aligned}
\mathbb{P}\bigl(\eta > \mathfrak{z}(\mu)\bigr) &\leq \sum_{k=0}^{\infty} 2\exp\bigl\{-\rho^{-k}\mu\,z^{2}(B,\mu)/2 + \varPhi(\rho^{k+1}\mu)\bigr\} \\
&\leq 2\sum_{k=0}^{\infty} \exp\bigl\{\varPhi(\rho^{k+1}\mu) - \rho^{-k}\varPhi(\mu) - \rho^{-k}\mathtt{x}\bigr\} \leq \mathrm{e}^{-\mathtt{x}}.
\end{aligned}
$$

This yields (B.68). □

Putting together (B.67) and (B.68) yields (B.51).

Now we check (B.52). Normalization by λ reduces the proof to the case $\|B\| = \|Q\,\mathbb{V}^{2}Q^{\top}\| = 1$. We use the simplified bounds $z(B,\mathtt{x}) \leq \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}$ and $\mu^{-1} = 1 + \sqrt{\mathbb{p}/(4\mathtt{x})}$. Now (B.49) with $\rho = 1/2$ can be rewritten as

$$
\mathtt{g} \geq \sqrt{\mu\,\mathbb{p}} + \mu\sqrt{2}\bigl(\sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}\bigr). \tag{B.71}
$$

The use of $\mu = \sqrt{4\mathtt{x}}/(\sqrt{4\mathtt{x}} + \sqrt{\mathbb{p}})$ yields

$$
\mu\sqrt{2}\bigl(\sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}\bigr) = \sqrt{8\mathtt{x}}\,\frac{\sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}}{\sqrt{\mathbb{p}} + \sqrt{4\mathtt{x}}} \geq \sqrt{4\mathtt{x}}\,,
$$

and (B.71) is not possible for $\mathtt{x} > \mathtt{g}^{2}/4$. Further, with $\mathtt{y} = \sqrt{4\mathtt{x}}/\mathtt{g}$ and $\alpha = \sqrt{\mathbb{p}}/\mathtt{g}$

$$
\frac{\sqrt{\mu\,\mathbb{p}} + \mu\sqrt{2}\bigl(\sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}\bigr)}{\mathtt{g}} = \sqrt{\frac{\mathtt{y}\alpha^{2}}{\alpha + \mathtt{y}}} + \frac{\mathtt{y}(\sqrt{2}\alpha + \mathtt{y})}{\alpha + \mathtt{y}} \leq \alpha + \mathtt{y} + \frac{\mathtt{y}(\sqrt{2} - 1)\alpha}{\alpha + \mathtt{y}} \leq \mathtt{y} + \sqrt{2}\alpha.
$$

Together with (B.71), this yields $\mathtt{y} \geq 1 - \sqrt{2}\alpha$ and (B.52) follows. For (B.53) we use $z_{c} \leq \sqrt{\mathbb{p}} + \sqrt{2\lambda\mathtt{x}_{c}}$ and $z_{c} \geq \sqrt{\mathbb{p}/2} + \sqrt{2\lambda\mathtt{x}_{c}}$.

B.4.3 Proof of Theorem B.16

Assume w.l.o.g. $\lambda = 1$. First we present an accurate deviation bound, which, however, does not provide a closed form quantile function for $\|Q\boldsymbol{\xi}\|$. Then we show how it implies a rough linear upper bound on this quantile function. For $\mathtt{x}_c$ from (B.50) and $\mathtt{x} > \mathtt{x}_c$, fix μ by the relation

$$\frac{\rho\,\mu\,\mathfrak{z}^2(\mu)}{2} = \mathtt{x} + \varPhi(\mu) = \mathtt{x} + \frac{\mu\,\mathbb{p}}{2} + \frac{\mu^2 \mathtt{v}^2}{4(1-\mu)}\,, \tag{B.72}$$

where $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu}$; cf. (B.66). It is easy to see that the solution μ exists and unique. Moreover, if $\mathtt{x} = \mathtt{x}_c$ then $\mu = \mu_c$ and $\mathfrak{z}^2(\mu_c) = z^2(B, \mathtt{x}_c)$; see (B.50). If $\mathtt{x} > \mathtt{x}_c$, then $\mu < \mu_c$ and $\mathfrak{z}^2(\mu) > z^2(B, \mathtt{x})$.

Lemma B.22. *For* $\mathtt{x} > \mathtt{x}_c$, *define* μ *by* (B.72). *Then with* $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu}$

$$I\!\!P\bigl(\|Q\boldsymbol{\xi}\|^2 > \rho\,\mathfrak{z}^2(\mu)\bigr) \le 3\mathrm{e}^{-\mathtt{x}}\,. \tag{B.73}$$

Proof. We again apply Lemma B.20, however, the choice $\mu = \mu(\mathtt{x})$ from (B.9) is not possible anymore in view of $z(B, \mathtt{x}) > \mathfrak{z}(\mu)$. More precisely, for $\mathtt{x}$ large, the value $\mu(\mathtt{x})$ approaches one and this choice of μ yields the value $\mathfrak{z}(\mu)$ smaller than we need. To cope with this problem, we apply (B.63) of Lemma B.20 with a sub-optimal μ from (B.72) ensuring $\rho\mu\,\mathfrak{z}^2(\mu) - \varPhi(\mu) = \mathtt{x}$. By (B.63) of Lemma B.20

$$I\!\!P\bigl(\|Q\boldsymbol{\xi}\| > \sqrt{\rho}\,\mathfrak{z}(\mu), \|Q\boldsymbol{\xi}\| \le \mathfrak{z}(\mu)\bigr) \le 2\exp\Bigl\{-\frac{\rho\mu\,\mathfrak{z}^2(\mu)}{2} + \varPhi(\mu)\Bigr\} = 2\mathrm{e}^{-\mathtt{x}}.$$

Repeating the arguments from the proof of Lemma B.21 implies

$$\begin{aligned} I\!\!P\bigl(\|Q\boldsymbol{\xi}\|^2 > \rho\,\mathfrak{z}^2(\mu)\bigr) &\le \sum_{k=0}^{\infty} 2\exp\Bigl\{-\frac{1}{2}\rho^{k+1}\mu\,\mathfrak{z}^2(\rho^k\mu) + \varPhi(\rho^k\mu)\Bigr\} \\ &\le \sum_{k=0}^{\infty} 2\exp\Bigl\{-\frac{1}{2}\rho^{-k+1}\mu\,\mathfrak{z}^2(\mu) + \varPhi(\rho^k\mu)\Bigr\} \\ &\le 2\mathrm{e}^{-\mathtt{x}} + 2\mathrm{e}^{-\mathtt{x}}\sum_{k=1}^{\infty}\exp\Bigl\{-\frac{1}{2}(\rho^{-k}-1)\rho\mu\,\mathfrak{z}^2(\mu) + \varPhi(\rho^k\mu) - \varPhi(\mu)\Bigr\} \le 3\mathrm{e}^{-\mathtt{x}}. \end{aligned}$$

as stated in (B.73). □

It remains to evaluate $\rho\,\mathfrak{z}^2(\mu)$ with μ from (B.72) and $\mathfrak{z}(\mu) = \mathtt{g}/\mu - \sqrt{\mathbb{p}/\mu}$. For $\mu \le \mu_c$

$$\frac{\rho}{2}\Bigl(\frac{\mathtt{g}}{\sqrt{\mu}} - \sqrt{\mathbb{p}}\Bigr)^2 = \mathtt{x} + \varPhi(\mu)$$

and

$$\frac{\sqrt{\rho}\,\mathtt{g}}{\sqrt{\mu}} = \sqrt{2\mathtt{x} + 2\varPhi(\mu)} + \sqrt{\rho\,\mathbb{p}}\,.$$

This results in

$$\begin{aligned}\sqrt{\rho}\,\mathfrak{z}(\mu) &= \frac{\sqrt{\rho}}{\sqrt{\mu}}\Bigl(\frac{\mathtt{g}}{\sqrt{\mu}} - \sqrt{\mathbb{p}}\Bigr) \le \frac{1}{\sqrt{\rho}\,\mathtt{g}}\bigl(\sqrt{2\mathtt{x} + 2\varPhi(\mu)} + \sqrt{\rho\,\mathbb{p}}\bigr)\sqrt{2\mathtt{x} + 2\varPhi(\mu)} \\ &\le \frac{1}{\sqrt{\rho}\,\mathtt{g}}\bigl(2\mathtt{x} + 2\varPhi(\mu_c) + \sqrt{\rho\,\mathbb{p}(2\mathtt{x} + 2\varPhi(\mu_c))}\bigr) \stackrel{\text{def}}{=} \bar{z}(\mathtt{x})\,.\end{aligned}$$

By (B.50), this inequality becomes equality for $\mathtt{x} = \mathtt{x}_c$ and $\mu = \mu_c$ with $\sqrt{\rho}\,\mathfrak{z}(\mu_c) = \bar{z}(\mathtt{x}_c) = z(B, \mathtt{x}_c)$. Furthermore, the derivative of $\bar{z}(\mathtt{x})$ w.r.t. $\mathtt{x}$ satisfies

$$\frac{d}{d\mathtt{x}}\,\bar{z}(\mathtt{x}) = \frac{1}{\sqrt{\rho}\,\mathtt{g}}\biggl(2 + \frac{\sqrt{\rho\,\mathbb{p}}}{\sqrt{2\mathtt{x} + 2\varPhi(\mu_c)}}\biggr) \le \frac{1}{\sqrt{\rho}\,\mathtt{g}}\biggl(2 + \frac{\sqrt{\rho\,\mathbb{p}}}{\sqrt{2\mathtt{x}_c + 2\varPhi(\mu_c)}}\biggr).$$

Moreover, $2\mathtt{x}_c + 2\varPhi(\mu_c) = z^2(B, \mathtt{x}_c)$ and

$$\frac{d}{d\mathtt{x}}\,\bar{z}(\mathtt{x}) \le \frac{1}{\sqrt{\rho}\,\mathtt{g}}\biggl(2 + \frac{\sqrt{\rho\,\mathbb{p}}}{z(B, \mathtt{x}_c)}\biggr) \le \frac{2 + \sqrt{\rho}}{\sqrt{\rho}\,\mathtt{g}}$$

yielding

$$\bar{z}(\mathtt{x}) \le \bar{z}(\mathtt{x}_c) + \frac{2 + \sqrt{\rho}}{\sqrt{\rho}\,\mathtt{g}}(\mathtt{x} - \mathtt{x}_c) = z(B, \mathtt{x}_c) + \frac{2 + \sqrt{\rho}}{\sqrt{\rho}\,\mathtt{g}}(\mathtt{x} - \mathtt{x}_c)$$

and hence,

$$\sqrt{\rho}\,\mathfrak{z}(\mu) \le z(B, \mathtt{x}_c) + \frac{2 + \sqrt{\rho}}{\sqrt{\rho}\,\mathtt{g}}(\mathtt{x} - \mathtt{x}_c) = z_c + \frac{\mathtt{x} - \mathtt{x}_c}{\varkappa}\,. \tag{B.74}$$

This implies (B.54).

B.4.4 Proof of Theorem B.18

As previously, assume $\lambda = 1$. We use $z(B, \mathtt{x}_c) \le \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}_c}$. Further, $\varkappa^{-1}\mathtt{x}_c - \sqrt{2\mathtt{x}_c} + \varkappa/\sqrt{2} \ge 0$ and thus,

$$\sqrt{2\mathtt{x}_c} - \varkappa^{-1}\mathtt{x}_c \le \varkappa/\sqrt{2}\,.$$

Therefore, for $\mathtt{x} \ge \mathtt{x}_c$, it holds

$$z_c(B, \mathtt{x}) = z(B, \mathtt{x}_c) + \frac{\mathtt{x} - \mathtt{x}_c}{\varkappa} \le \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}_c} - \frac{\mathtt{x}_c}{\varkappa} + \frac{\mathtt{x}}{\varkappa} \le \sqrt{\mathbb{p}} + \frac{\varkappa}{\sqrt{2}} + \frac{\mathtt{x}}{\varkappa}\,.$$

In the zone $\mathtt{x} \le \mathtt{x}_c$, it holds $z_c(B, \mathtt{x}) = z(B, \mathtt{x}) \le \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}$ and it remains to note that $\sqrt{2\mathtt{x}} \le \varkappa/\sqrt{2} + \varkappa^{-1}\mathtt{x}$.

B.4.5 Proof of Theorem B.19

Assume w.o.l.g. $\lambda = 1$. First consider $z \geq z_c$. By (B.74) of Theorem B.16, it holds with $\varkappa = \mathtt{g}\sqrt{\rho}/(2+\sqrt{\rho})$ and $\mathtt{x}_c = (z_c - \sqrt{\mathbb{p}})^2/2$

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| \geq z\bigr) = \mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| \geq z_c + z - z_c\bigr) \leq 3\mathrm{e}^{-\mathtt{x}_c - \varkappa(z-z_c)}.$$

In particular, $\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z_c) \leq 3\mathrm{e}^{-\mathtt{x}_c}$. Integration by parts yields for $\nu < \varkappa$

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{\nu(\|Q\boldsymbol{\xi}\|-z_c)}\,\mathbb{I}(\|Q\boldsymbol{\xi}\| > z_c) &= -\int_{z_c}^{\infty}\mathrm{e}^{\nu(z-z_c)}d\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z) \\
&= \mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z_c) + \nu\int_{z_c}^{\infty}\mathrm{e}^{\nu(z-z_c)}\,\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z)\,dz \\
&\leq 3\mathrm{e}^{-\mathtt{x}_c} + \nu\int_{z_c}^{\infty}\mathrm{e}^{\nu(z-z_c)-\mathtt{x}_c-\varkappa(z-z_c)}\,dz = \left(3 + \frac{3\nu}{\varkappa-\nu}\right)\mathrm{e}^{-\mathtt{x}_c}
\end{aligned} \tag{B.75}$$

and (B.58) follows. Similarly, for $z \geq z_c$, we derive (B.59) as follows

$$\begin{aligned}
&\mathbb{E}\mathrm{e}^{\nu\|Q\boldsymbol{\xi}\|}\,\mathbb{I}(\|Q\boldsymbol{\xi}\| > z) = -\int_{z}^{\infty}\mathrm{e}^{\nu t}d\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq t) \\
&\quad\leq 3\mathrm{e}^{\nu z_c-\mathtt{x}_c-\varkappa(z-z_c)} + \frac{3\nu}{\varkappa-\nu}\mathrm{e}^{\nu z_c-\mathtt{x}_c-\varkappa(z-z_c)} = \frac{3\varkappa}{\varkappa-\nu}\mathrm{e}^{\nu z_c-\mathtt{x}_c-(\varkappa-\nu)(z-z_c)}.
\end{aligned}$$

Now fix $z_\circ$ with $z_\circ - \sqrt{\mathbb{p}} \geq 2\nu$ but $z_\circ \leq z_c$. Then

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{\nu\|Q\boldsymbol{\xi}\|}\,\mathbb{I}(\|Q\boldsymbol{\xi}\| > z_\circ) &= -\int_{z_\circ}^{\infty}\mathrm{e}^{\nu z}d\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z) \\
&= \mathrm{e}^{\nu z_\circ}\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z_\circ) + \nu\left(\int_{z_\circ}^{z_c} + \int_{z_c}^{\infty}\right)\mathrm{e}^{\nu z}\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z)dz.
\end{aligned}$$

By (B.51), for any $z \in [z_\circ, z_c]$, it holds in view of $z(B,\mathtt{x}) \leq \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}}$

$$\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z) \leq 3\mathrm{e}^{-(z-\sqrt{\mathbb{p}})^2/2}.$$

As $(\nu z - (z-\sqrt{\mathbb{p}})^2/2)' = \nu - z + \sqrt{\mathbb{p}} \leq -\nu$ for $z - \sqrt{\mathbb{p}} \geq 2\nu$, it holds

$$\nu\int_{z_\circ}^{z_c}\mathrm{e}^{\nu z-(z-\sqrt{\mathbb{p}})^2/2}dz \leq \mathrm{e}^{\nu z_\circ-(z_\circ-\sqrt{\mathbb{p}})^2/2}\,\nu\int_{z_\circ}^{z_c}\mathrm{e}^{-\nu(z-z_\circ)}dz \leq \mathrm{e}^{\nu z_\circ-(z_\circ-\sqrt{\mathbb{p}})^2/2}$$

and also $\nu z_\circ - (z_\circ - \sqrt{\mathbb{p}})^2/2 > \nu z_c - (z_c - \sqrt{\mathbb{p}})^2/2$. Putting this together with the above bound on $\int_{z_c}^{\infty}\mathrm{e}^{\nu z}\mathbb{P}(\|Q\boldsymbol{\xi}\| \geq z)dz$ as in (B.75) completes the proof of (B.57).

B.5 Frobenius norm losses for empirical covariance

Let $\boldsymbol{X}_i \sim \mathcal{N}(0, \varSigma)$ be i.i.d. zero mean Gaussian vectors in $\mathbb{R}^p$ with a covariance matrix $\varSigma \in \mathfrak{M}_p$. By $\widehat{\varSigma}$ we denote the empirical covariance

$$\widehat{\varSigma} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{X}_i^{\top}.$$

Our goal is to establish sharp dimension free deviation bounds on the squared Frobenius norm $\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$:

$$\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2 = \operatorname{tr}(\widehat{\varSigma} - \varSigma)^2.$$

We demonstrate how the general results of Section B.4 can be used for obtaining accurate deviation bounds for $\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$ and for supporting the concentration phenomenon.

B.5.1 Upper bounds

First we establish a tight upper bound on $\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$. We identify the matrix $\widehat{\varSigma}$ with the vector in the linear subspace of $\mathbb{R}^{p \times p}$ composed by symmetric matrices. Our aim is in showing that the quantiles of $\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$ mimic well similar quantiles of $\|\widetilde{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$ for a Gaussian matrix $\widetilde{\varSigma}$ with the same covariance structure as $\widehat{\varSigma}$. Define

$$\mathbb{p}(\varSigma) = (\operatorname{tr} \varSigma)^2 + \operatorname{tr} \varSigma^2, \qquad \mathtt{v}^2(\varSigma) = \bigl(\operatorname{tr} \varSigma^2\bigr)^2 + \operatorname{tr} \varSigma^4. \tag{B.76}$$

Later we show that $\mathbb{p}(\varSigma) = \mathbb{E}\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2 = \operatorname{tr} \operatorname{Var}(\widetilde{\varSigma})$ and $\mathtt{v}^2(\varSigma) = \operatorname{tr}\{\operatorname{Var}(\widetilde{\varSigma})\}^2$ while $\lambda(\varSigma) = \|\operatorname{Var}(\widetilde{\varSigma})\| = 2\|\varSigma\|^2$. In our results we implicitly assume a high dimensional situation with $\mathbb{p}(\varSigma)$ large. The presented bounds also require that $n \gg \mathbb{p}(\varSigma)$.

Theorem B.23. *Assume* $\|\varSigma\| = 1$ *and* $\mathbb{p}(\varSigma) < n/8$. *Given* $\mathtt{x}$ *with* $4\sqrt{\mathtt{x}} < \sqrt{n/8} - \sqrt{\mathbb{p}(\varSigma)}$, *fix* $\rho < 1$ *by*

$$\rho(1 - \rho)\sqrt{n/8} = \sqrt{\mathbb{p}(\varSigma)} + 4\sqrt{\mathtt{x}}. \tag{B.77}$$

Then

$$\mathbb{P}\Bigl(n\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2 > \frac{1}{1 - \rho}\bigl\{\mathbb{p}(\varSigma) + 2\mathtt{v}(\varSigma)\sqrt{\mathtt{x}} + 4\mathtt{x}\bigr\}\Bigr) \le 3e^{-\mathtt{x}}. \tag{B.78}$$

B.5.2 Lower bounds

This section presents a lower bound on the Frobenius norm of $\widehat{\varSigma} - \varSigma$. Later in Section B.5.3 we state the concentration phenomenon for $\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2$.

Theorem B.24. *Let* $\|\Sigma\| = 1$ *and* $\mathbb{p}(\Sigma)$ *and* $\mathtt{v}(\Sigma)$ *be defined by* (B.76)*. For* $\mathtt{x} > 0$ *with* $2\sqrt{\mathtt{x}} \le \mathbb{p}(\Sigma)/\mathtt{v}(\Sigma)$*, define* $\mu = \mu(\mathtt{x}) = 2\sqrt{\mathtt{x}}/\mathtt{v}(\Sigma)$ *and assume that there is* $\alpha < 1/2$ *satisfying*

$$\alpha\sqrt{\frac{1-2\alpha}{1-\alpha}} \ge \sqrt{\frac{\mu(\mathtt{x})}{n}}\biggl(\sqrt{2\mathbb{p}(\Sigma)} + \frac{\sqrt{2}\,\mathbb{p}(\Sigma)}{\mathtt{v}(\Sigma)}\biggr). \tag{B.79}$$

Then

$$I\!\!P\Bigl(n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 < \frac{1-2\alpha}{1-\alpha}\,\mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}}\Bigr) \le 2\mathrm{e}^{-\mathtt{x}}.$$

B.5.3 Concentration of the Frobenius loss

Putting together Theorem B.23 and Theorem B.24 yields the following corollary.

Corollary B.25. *Under conditions of Theorem B.23 and Theorem B.24, it holds for any* $\mathtt{x}$ *resolving* (B.77) *and* (B.79) *on a random set* $\Omega(\mathtt{x})$ *with* $I\!\!P\bigl(\Omega(\mathtt{x})\bigr) \ge 1 - 5\mathrm{e}^{-\mathtt{x}}$

$$\frac{1-2\alpha}{1-\alpha}\,\mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}} \le n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 \le \frac{1}{1-\rho}\bigl\{\mathbb{p}(\Sigma) + 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}} + 4\mathtt{x}\bigr\}. \tag{B.80}$$

This result mimics similar bound of Theorem B.4 for $\widehat{\Sigma}$ Gaussian and of Theorem B.15 for $\widehat{\Sigma}$ sub-Gaussian. However, the empirical covariance $\widehat{\Sigma}$ is quadratic in the $\boldsymbol{X}_i$'s and thus, only sub-exponential. We pay an additional factor $(1-\rho)^{-1}$ in the upper quantile function and the factor $\frac{1-2\alpha}{1-\alpha}$ in the lower quantile function for this extension.

Further we discuss the concentration phenomenon for the Frobenius error $n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2$ around its expectation $\mathbb{p}(\Sigma)$. Even in the Gaussian case, it meets only in high-dimensional situation with $\mathbb{p}(\Sigma)$ large. As $\mathtt{v}^2(\Sigma) \le \mathbb{p}(\Sigma)\lambda(\Sigma) = 2\mathbb{p}(\Sigma)$, this also implies $\mathtt{v}(\Sigma) \ll \mathbb{p}(\Sigma)$. Statement (B.80) can be rewritten as

$$-\frac{\alpha\,\mathbb{p}(\Sigma)}{1-\alpha} - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}} \le n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 - \mathbb{p}(\Sigma) \le \frac{\rho\,\mathbb{p}(\Sigma)}{1-\rho} + \frac{2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}} + 4\mathtt{x}}{1-\rho}.$$

Therefore, concentration effect of the loss $n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2$ requires $\mathbb{p}(\Sigma)$ large and α and ρ small. Then for $\mathtt{x} \ll \mathbb{p}(\Sigma)$, quantiles of $n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 - \mathbb{p}(\Sigma)$ are smaller in order than $\mathbb{p}(\Sigma)$. Definition (B.77) of ρ ensures $\rho \asymp \sqrt{\mathbb{p}(\Sigma)/n}$, and hence, "$\rho \ll 1$" is equivalent to "$\mathbb{p}(\Sigma) \ll n$". Condition ensuring $\alpha \ll 1$ is similar. To see this, assume $\mathtt{v}^2(\Sigma) \asymp \mathbb{p}(\Sigma)$. Then $\mathtt{x} \ll \mathbb{p}(\Sigma)$ yields $\mu(\mathtt{x}) = 2\sqrt{\mathtt{x}}/\mathtt{v}(\Sigma) \ll 1$ and definition (B.79) of α implies

$$\alpha \lesssim \sqrt{\frac{\mu}{n}}\biggl(\sqrt{2\mathbb{p}(\Sigma)} + \frac{\sqrt{2}\,\mathbb{p}(\Sigma)}{\mathtt{v}(\Sigma)}\biggr) \lesssim \sqrt{\frac{\mathbb{p}(\Sigma)}{n}}.$$

B.5.4 Weighted Frobenius norm

The result can be easily extended to the case of a weighted Frobenius norm. Consider for any linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ the value $n\|A(\widehat{\Sigma} - \Sigma)A^\top\|_{\mathrm{Fr}}^2$.

Theorem B.26. *Let* $\|\Sigma\| = 1$ *and* $A\colon \mathbb{R}^p \to \mathbb{R}^q$ *be a linear operator with* $\|A\| = \|A^\top A\| = 1$. *Define* $\Sigma_A \stackrel{\mathrm{def}}{=} A\Sigma A^\top$,

$$\mathbb{p}_A \stackrel{\mathrm{def}}{=} \mathbb{p}(\Sigma_A) = \operatorname{tr}^2(\Sigma_A) + \operatorname{tr}(\Sigma_A)^2, \quad \mathtt{v}_A^2 \stackrel{\mathrm{def}}{=} \mathtt{v}^2(\Sigma_A) = \bigl\{\operatorname{tr}(\Sigma_A^2)\bigr\}^2 + \operatorname{tr}(\Sigma_A)^4, \tag{B.81}$$

and assume $\mathbb{p}_A < n/8$. *The the statements of Theorem B.23 and Theorem B.24 apply to* $n\|A(\widehat{\Sigma} - \Sigma)A^\top\|_{\mathrm{Fr}}^2$ *after replacing* $\mathbb{p}(\Sigma)$ *and* $\mathtt{v}(\Sigma)$ *with* $\mathbb{p}_A$ *and* $\mathtt{v}_A$.

Proof. We can represented

$$\sqrt{n}\, A(\widehat{\Sigma} - \Sigma)A^\top = A\,\Sigma^{1/2}\mathcal{E}\,\Sigma^{1/2}A^\top$$

with $\mathcal{E}$ from (B.82). This reduces the result to the previous case with $\Sigma_A = A\Sigma A^\top$ in place of Σ. □

B.5.5 Proof of Theorem B.23

Each vector $\boldsymbol{\gamma}_i = \Sigma^{-1/2}\boldsymbol{X}_i$ is standard normal. Define

$$\mathcal{E} = \frac{1}{n^{1/2}} \sum_{i=1}^{n} (\boldsymbol{\gamma}_i\boldsymbol{\gamma}_i^\top - \mathbb{I}_p). \tag{B.82}$$

We will use the representation $\widehat{\Sigma} - \Sigma = n^{-1/2}\Sigma^{1/2}\,\mathcal{E}\,\Sigma^{1/2}$ and

$$n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 = \operatorname{tr}(\Sigma^{1/2}\mathcal{E}\,\Sigma\,\mathcal{E}\,\Sigma^{1/2}) = \|\Sigma^{1/2}\mathcal{E}\,\Sigma^{1/2}\|_{\mathrm{Fr}}^2 .$$

The main step is in applying Theorem B.15 to the quadratic form $\|Q\mathcal{E}\|_{\mathrm{Fr}}^2$ with $Q\mathcal{E} = \Sigma^{1/2}\,\mathcal{E}\,\Sigma^{1/2}$. First check (B.46) for $\boldsymbol{\xi} = \mathcal{E}$.

Lemma B.27. *For any symmetric* $\Gamma \in \mathfrak{M}_p$ *with* $\|\Gamma\|_{\mathrm{Fr}} \le \mathtt{g} < \sqrt{n}/2$, *it holds*

$$\mathbb{E}\langle \Gamma, \mathcal{E}\rangle^2 = 2\|\Gamma\|_{\mathrm{Fr}}^2 , \tag{B.83}$$
$$\log \mathbb{E}\exp\langle \Gamma, \mathcal{E}\rangle \le \frac{1}{1 - 2n^{-1/2}\|\Gamma\|}\,\|\Gamma\|_{\mathrm{Fr}}^2 \le \frac{1}{1 - 2n^{-1/2}\mathtt{g}}\,\|\Gamma\|_{\mathrm{Fr}}^2 .$$

Proof. Let us fix any symmetric $\Gamma \in \mathfrak{M}_p$ with $\|\Gamma\|_{\mathrm{Fr}} \le \mathtt{g}$. For the scalar product $\langle \Gamma, \mathcal{E}\rangle$, we use the representation

$$\langle \Gamma, \mathcal{E}\rangle = \operatorname{tr}(\Gamma\mathcal{E}) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \bigl\{\boldsymbol{\gamma}_i^\top \Gamma \boldsymbol{\gamma}_i - \mathbb{E}(\boldsymbol{\gamma}_i^\top \Gamma \boldsymbol{\gamma}_i)\bigr\}.$$

Then by independence of the $\boldsymbol{\gamma}_i$'s and Lemma B.1, it holds

$$\mathbb{E}\langle \varGamma, \mathcal{E} \rangle^2 = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\bigl\{ \boldsymbol{\gamma}_i^\top \varGamma \boldsymbol{\gamma}_i - \mathbb{E}(\boldsymbol{\gamma}_i^\top \varGamma \boldsymbol{\gamma}_i) \bigr\}^2 = 2 \operatorname{tr} \varGamma^2.$$

Now consider the exponential moment of $\langle \varGamma, \mathcal{E} \rangle$. Again, independence of the $\boldsymbol{\gamma}_i$'s yields

$$\begin{aligned} \log \mathbb{E} \exp \langle \varGamma, \mathcal{E} \rangle &= \sum_{i=1}^{n} \log \mathbb{E} \exp \frac{\boldsymbol{\gamma}_i^\top \varGamma \boldsymbol{\gamma}_i}{\sqrt{n}} - \sqrt{n} \operatorname{tr} \varGamma \\ &= \frac{n}{2} \log \det\bigl(I\!\!I_p - \frac{2}{\sqrt{n}} \varGamma \bigr) - \sqrt{n} \operatorname{tr} \varGamma \end{aligned}$$

provided that $2\varGamma < \sqrt{n} I\!\!I_p$. Moreover, by Lemma B.2

$$\Bigl| \frac{n}{2} \log \det(I\!\!I_p - 2n^{-1/2} \varGamma) - \sqrt{n} \operatorname{tr} \varGamma \Bigr| \le \frac{\operatorname{tr} \varGamma^2}{1 - 2n^{-1/2} \| \varGamma \|} = \frac{\| \varGamma \|_{\mathrm{Fr}}^2}{1 - 2n^{-1/2} \| \varGamma \|},$$

and the assertion follows in view of $\| \varGamma \| \le \| \varGamma \|_{\mathrm{Fr}} \le \mathtt{g}$. $\square$

We now fix $\mathtt{g} = \rho \sqrt{n}/2$. Then the random matrix $\boldsymbol{\xi} = \mathcal{E}$ follows condition (B.46) with $\mathbb{V}^2 = 2(1-\rho)^{-1} I\!\!I$. This enables us to apply Theorem B.15 to the quadratic form $\| Q\mathcal{E} \|_{\mathrm{Fr}}^2$ for $Q\mathcal{E} = \varSigma^{1/2} \, \mathcal{E} \, \varSigma^{1/2}$. By (B.83), it holds $\operatorname{Var}(\mathcal{E}) = 2 I\!\!I$. Now introduce a Gaussian element $\widetilde{\mathcal{E}}$ with the same covariance structure. One can use $\widetilde{\mathcal{E}} = (\boldsymbol{\zeta} + \boldsymbol{\zeta}^\top)/\sqrt{2}$, where $\boldsymbol{\zeta} = (\zeta_{ij})$ is a random p-matrix with i.i.d. standard normal entries ζ_{ij}. Indeed, for any symmetric p-matrix $\varGamma$,

$$\mathbb{E}\langle \widetilde{\mathcal{E}}, \varGamma \rangle^2 = 2 \mathbb{E} \langle \boldsymbol{\zeta}, \varGamma \rangle^2 = 2.$$

Statement (B.51) of Theorem B.15 yields nearly the same deviation bounds for $\| Q\mathcal{E} \|_{\mathrm{Fr}}^2$ as for $\| Q\widetilde{\mathcal{E}} \|_{\mathrm{Fr}}^2$ with $\widetilde{\mathcal{E}} \sim \mathcal{N}(0, \operatorname{Var}(\mathcal{E}))$. Theorem B.4 claims

$$\mathbb{P}\bigl(\| Q\widetilde{\mathcal{E}} \|_{\mathrm{Fr}}^2 > z^2(\widetilde{B}, \mathtt{x}) \bigr) \le \mathrm{e}^{-\mathtt{x}},$$

where $\widetilde{B} = \operatorname{Var}(Q\widetilde{\mathcal{E}})$ and the quantile $z(B, \mathtt{x})$ is defined as

$$z^2(B, \mathtt{x}) = \operatorname{tr} B + 2 \sqrt{\mathtt{x} \operatorname{tr}(B^2)} + 2\mathtt{x} \| B \|. \tag{B.84}$$

Lemma B.28. *Let* $\widetilde{\mathcal{E}} = (\boldsymbol{\zeta} + \boldsymbol{\zeta}^\top)/\sqrt{2}$, *where* $\boldsymbol{\zeta} = (\zeta_{ij})$ *is a random* p*-matrix with i.i.d. standard normal entries* ζ_{ij}. *Consider* $Q\widetilde{\mathcal{E}} = \varSigma^{1/2} \, \widetilde{\mathcal{E}} \, \varSigma^{1/2}$. *It holds for* $\widetilde{B} = \operatorname{Var}(Q\widetilde{\mathcal{E}})$

$$\operatorname{tr} \widetilde{B} = \mathbb{p}(\varSigma), \quad \operatorname{tr} \widetilde{B}^2 = \mathtt{v}^2(\varSigma), \quad \| \widetilde{B} \| = 2.$$

Proof. We may assume $\varSigma = \operatorname{diag}\{\lambda_1, \ldots, \lambda_p\}$. Then it holds by Lemma B.1

$$\|Q\widetilde{\mathcal{E}}\|_{\mathrm{Fr}}^2 = \|\varSigma^{1/2}\,\widetilde{\mathcal{E}}\,\varSigma^{1/2}\|_{\mathrm{Fr}}^2 = \frac{1}{2}\sum_{i,j=1}^{p} \lambda_i\,\lambda_j\,(\zeta_{ij}+\zeta_{ji})^2 \stackrel{\mathrm{d}}{=} 2\sum_{i\le j} \lambda_i\,\lambda_j\,\zeta_{ij}^2 \tag{B.85}$$

and thus

$$\operatorname{tr}\widetilde{B} = I\!\!E\|Q\widetilde{\mathcal{E}}\|_{\mathrm{Fr}}^2 = 2\sum_{i\le j}\lambda_i\,\lambda_j = \Bigl(\sum_{i=1}^{p}\lambda_i\Bigr)^2 + \sum_{i=1}^{p}\lambda_i^2 = \mathbb{p}(\varSigma)\,.$$

Further we compute $\mathtt{v}^2(\varSigma) = \operatorname{tr}\widetilde{B}^2$. Note that $\operatorname{Var}(\|Q\widetilde{\mathcal{E}}\|_{\mathrm{Fr}}^2) \neq \operatorname{Var}(\|Q\mathcal{E}\|_{\mathrm{Fr}}^2)$. Due to Lemma B.1, it holds $\mathtt{v}^2(\varSigma) = \operatorname{Var}(\|Q\widetilde{\mathcal{E}}\|_{\mathrm{Fr}}^2)/2$ yielding by (B.85)

$$\mathtt{v}^2(\varSigma) = 2\sum_{i\le j}\lambda_i^2\lambda_j^2\operatorname{Var}(\zeta_{ij}^2) = 2\sum_{i\ne j}\lambda_i^2\lambda_j^2 + 2\sum_{i=1}^{p}\lambda_i^4 = \bigl(\operatorname{tr}\varSigma^2\bigr)^2 + \operatorname{tr}\varSigma^4.$$

Finally, $\operatorname{Var}(\mathcal{E}) = 2I\!\!I$ and $\|\varSigma\| = 1$ implies $\lambda(\varSigma) = \|Q\operatorname{Var}(\mathcal{E})Q^\top\| = 2$. $\square$

Now we apply Theorem B.15 to $n\|\widehat{\varSigma} - \varSigma\|_{\mathrm{Fr}}^2 = \|Q\mathcal{E}\|_{\mathrm{Fr}}^2$. Following to Lemma B.27, define $B = (1-\nu)^{-1}\widetilde{B}$. Then with $z^2(B,\mathtt{x})$ from (B.84)

$$I\!\!P\Bigl(n\|\widehat{\varSigma}-\varSigma\|_{\mathrm{Fr}}^2 > z^2(B,\mathtt{x})\Bigr) = I\!\!P\bigl(\|Q\mathcal{E}\|_{\mathrm{Fr}}^2 > z^2(B,\mathtt{x})\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, \quad \mathtt{x}\le\mathtt{x}_c\,,$$

and assertion (B.78) follows in view of Lemma B.28 and $z^2(B,\mathtt{x}) = (1-\nu)^{-1}z^2(\widetilde{B},\mathtt{x})$. However, it is still necessary to check that the upper bound (B.78) applies for a given $\mathtt{x}$. (B.52) provides a sufficient condition $\mathtt{g}/\lambda \ge \sqrt{\mathbb{p}/\lambda} + \sqrt{8\mathtt{x}}$ with $\mathbb{p} = \mathbb{p}(\varSigma)/(1-\rho)$ and $\lambda = 2/(1-\rho)$ for $\mathtt{g} = \rho\sqrt{n}/2$. By (B.77)

$$\frac{\mathtt{g}}{\lambda} - \sqrt{\frac{\mathbb{p}}{\lambda}} = \frac{\rho\sqrt{n}}{2\lambda} - \sqrt{\frac{\mathbb{p}(\varSigma)}{2}} \ge \frac{\rho(1-\rho)\sqrt{n}}{4} - \sqrt{\frac{\mathbb{p}(\varSigma)}{2}} > \sqrt{8\mathtt{x}}$$

and the result follows.

B.5.6 Proof of Theorem B.24

As in the proof of the upper bound, we apply Markov's inequality

$$I\!\!P\bigl(n\|\widehat{\varSigma}-\varSigma\|_{\mathrm{Fr}}^2 < z\bigr) \le \mathrm{e}^{\mu z/2}\,I\!\!E\exp\Bigl(-\frac{\mu}{2}n\|\widehat{\varSigma}-\varSigma\|_{\mathrm{Fr}}^2\Bigr). \tag{B.86}$$

However, now we are free to choose any positive μ. Later we evaluate the exponential moments of $-n\|\widehat{\varSigma}-\varSigma\|_{\mathrm{Fr}}^2$ for all $\mu > 0$ and then, given $\mathtt{x}$, fix μ and z similarly to the Gaussian case to ensure the prescribed deviation probability $\mathrm{e}^{-\mathtt{x}}$.

Denote by $\boldsymbol{\zeta} = (\zeta_{ij})$ a random $p \times p$ matrix with i.i.d. standard Gaussian entries ζ_{ij} and $\bar{\boldsymbol{\zeta}} \stackrel{\text{def}}{=} (\boldsymbol{\zeta} + \boldsymbol{\zeta}^{\top})/2$. Then for any $\mu > 0$

$$\exp\bigl(-\mu n \|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2/2\bigr) = \mathbb{E}_{\zeta} \exp\bigl\{\mathtt{i}\sqrt{\mu n}\,\langle \widehat{\Sigma} - \Sigma, \boldsymbol{\zeta}\rangle\bigr\} = \mathbb{E}_{\zeta} \exp\bigl\{\mathtt{i}\sqrt{\mu n}\,\langle \widehat{\Sigma} - \Sigma, \bar{\boldsymbol{\zeta}}\rangle\bigr\}.$$

Therefore, by independence of the $\boldsymbol{X}_i$'s

$$\begin{aligned}
\mathbb{E} \exp\bigl(-\mu n \|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2/2\bigr) &= \mathbb{E}_{\zeta} \mathbb{E} \exp\bigl(\mathtt{i}\sqrt{\mu n}\,\langle \widehat{\Sigma} - \Sigma, \bar{\boldsymbol{\zeta}}\rangle\bigr) \\
&= \mathbb{E}_{\zeta} \Bigl\{\mathbb{E} \exp\bigl(\mathtt{i}\sqrt{\mu/n}\,\langle \boldsymbol{X}_1 \boldsymbol{X}_1^{\top} - \Sigma, \bar{\boldsymbol{\zeta}}\rangle\bigr)\Bigr\}^n \\
&= \mathbb{E}_{\zeta} \Bigl\{\mathbb{E} \exp\bigl(\mathtt{i}\sqrt{\mu/n}\,\langle \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top} - I\!\!I_p\,, \Sigma^{1/2}\,\bar{\boldsymbol{\zeta}}\,\Sigma^{1/2}\rangle\bigr)\Bigr\}^n.
\end{aligned}$$

Further, by Lemma B.2, with $\mathcal{B} = \Sigma^{1/2}\,\bar{\boldsymbol{\zeta}}\,\Sigma^{1/2}$

$$\begin{aligned}
&\Bigl\{\mathbb{E} \exp\bigl(\mathtt{i}\sqrt{\mu/n}\,\langle \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top} - I\!\!I_p\,, \mathcal{B}\rangle\bigr)\Bigr\}^n \\
&\quad = \exp\bigl\{n \log\det\bigl(I\!\!I_p - 2\mathtt{i}\sqrt{\mu/n}\,\mathcal{B}\bigr)^{-1/2} - \mathtt{i}\sqrt{\mu n}\,\operatorname{tr}(\mathcal{B})\bigr\}.
\end{aligned} \tag{B.87}$$

Let some $\mathtt{x} > 0$ and some $\alpha \in (0, 1/2)$ be fixed. Define

$$\mu \stackrel{\text{def}}{=} \frac{2\sqrt{\mathtt{x}}}{\mathtt{v}(\Sigma)}, \qquad \mu_{\alpha} \stackrel{\text{def}}{=} \frac{1-\alpha}{1-2\alpha}\,\mu = \frac{1-\alpha}{1-2\alpha}\,\frac{2\sqrt{\mathtt{x}}}{\mathtt{v}(\Sigma)}, \tag{B.88}$$

and introduce a random set $\Omega(\alpha)$ with

$$\Omega(\alpha) \stackrel{\text{def}}{=} \bigl\{\boldsymbol{\zeta} \colon 2\sqrt{\mu_{\alpha}/n}\,\|\mathcal{B}\| \le \alpha\bigr\}, \qquad \mathcal{B} = \Sigma^{1/2}\,(\boldsymbol{\zeta} + \boldsymbol{\zeta}^{\top})\,\Sigma^{1/2}/2. \tag{B.89}$$

It holds on $\Omega(\alpha)$ by (B.87) similarly to (B.4) of Lemma B.2

$$\begin{aligned}
\mathbb{E}^n \exp\bigl\{\mathtt{i}\sqrt{\mu_{\alpha}/n}\,\langle \boldsymbol{\gamma}\boldsymbol{\gamma}^{\top} - I\!\!I_p, \mathcal{B}\rangle\bigr\} &\le \exp\biggl(-\mu_{\alpha}\operatorname{tr}(\mathcal{B}^2) + \frac{\mu_{\alpha}\,\alpha\operatorname{tr}(\mathcal{B}^2)}{1-\alpha}\biggr) \\
= \exp\biggl(-\frac{1-2\alpha}{1-\alpha}\,\mu_{\alpha}\,\operatorname{tr}(\mathcal{B}^2)\biggr) &= \exp\bigl(-\mu\,\operatorname{tr}(\mathcal{B}^2)\bigr).
\end{aligned} \tag{B.90}$$

Exponential moments of $\operatorname{tr}(\mathcal{B}^2)$ from (B.91) under $I\!\!P_{\zeta}$ can be easily computed. We proceed assuming $\Sigma = \operatorname{diag}\{\lambda_j\}$ and using that $\zeta_{ij} + \zeta_{ji} \sim \mathcal{N}(0, 2)$ for $i \ne j$, and all $\zeta_{ij} + \zeta_{ji}$ are mutually independent for $i \le j$. This implies

$$\operatorname{tr}(\mathcal{B}^2) = \frac{1}{4}\sum_{i,j=1}^{p} \lambda_i\,\lambda_j\,(\zeta_{ij} + \zeta_{ji})^2 \stackrel{\mathrm{d}}{=} \sum_{i \le j} \lambda_i\,\lambda_j\,\zeta_{ij}^2 \tag{B.91}$$

and

$$\mathbb{E}_{\zeta} \operatorname{tr}(\mathcal{B}^2) = \sum_{i \le j} \lambda_i \lambda_j = \frac{\mathbb{p}(\Sigma)}{2}, \tag{B.92}$$

$$\mathbb{E}_{\zeta} \exp\{-\mu \operatorname{tr}(\mathcal{B}^2)\} = \mathbb{E}_{\zeta} \exp\biggl(-\mu \sum_{i \le j} \lambda_i \lambda_j \zeta_{ij}^2\biggr) = \exp\biggl(-\frac{1}{2} \sum_{i \le j} \log(1 + 2\mu \lambda_i \lambda_j)\biggr).$$

The latter expression can be evaluated by using (B.3) of Lemma B.2:

$$\mathbb{E}_{\zeta} \exp\{-\mu \operatorname{tr}(\mathcal{B}^2)\} \le \exp\biggl(-\mu \sum_{i \le j} \lambda_i \lambda_j + \mu^2 \sum_{i \le j} \lambda_i^2 \lambda_j^2\biggr) = \exp\biggl(-\frac{\mu\, \mathbb{p}(\Sigma)}{2} + \frac{\mu^2 \mathtt{v}^2(\Sigma)}{4}\biggr).$$

This and (B.90) yield

$$\mathbb{E} \exp\biggl(-\frac{\mu_\alpha}{2} n \|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2\biggr) \le \mathbb{P}_{\zeta}\bigl(\Omega(\alpha)^c\bigr) + \exp\biggl(-\frac{\mu\, \mathbb{p}(\Sigma)}{2} + \frac{\mu^2 \mathtt{v}^2(\Sigma)}{4}\biggr)$$

and for any z by Markov's inequality (B.86)

$$\mathbb{P}\bigl(n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 < z\bigr) \le \mathrm{e}^{\mu_\alpha z/2} \mathbb{P}_{\zeta}\bigl(\Omega(\alpha)^c\bigr) + \exp\biggl(\frac{\mu_\alpha z}{2} - \frac{\mu\, \mathbb{p}(\Sigma)}{2} + \frac{\mu^2 \mathtt{v}^2(\Sigma)}{4}\biggr).$$

With $\mu = 2\sqrt{\mathtt{x}}/\mathtt{v}(\Sigma)$, we define z by

$$\mu_\alpha z = \mu\{\mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}}\} = \frac{2\sqrt{\mathtt{x}}\, \mathbb{p}(\Sigma)}{\mathtt{v}(\Sigma)} - 4\mathtt{x} \tag{B.93}$$

yielding

$$\frac{\mu_\alpha z}{2} - \frac{\mu\, \mathbb{p}(\Sigma)}{2} + \frac{\mu^2 \mathtt{v}^2(\Sigma)}{4} = \frac{\mu}{2}\bigl\{\mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}}\bigr\} - \frac{\mu\, \mathbb{p}(\Sigma)}{2} + \frac{\mu^2 \mathtt{v}^2(\Sigma)}{4} = -\mathtt{x}$$

and

$$\mathbb{P}(n\|\widehat{\Sigma} - \Sigma\|_{\mathrm{Fr}}^2 < z) \le \mathrm{e}^{-\mathtt{x}} + \mathrm{e}^{\mu_\alpha z/2} \mathbb{P}_{\zeta}\bigl(\Omega(\alpha)^c\bigr)$$

where

$$z = \biggl(1 - \frac{\alpha}{1-\alpha}\biggr)\bigl\{\mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}}\bigr\} \ge \mathbb{p}(\Sigma) - \frac{\alpha}{1-\alpha}\, \mathbb{p}(\Sigma) - 2\mathtt{v}(\Sigma)\sqrt{\mathtt{x}}.$$

For bounding the probability of the set $\Omega(\alpha)^c$ from (B.89), one can apply the advanced results from the random matrix theory. To keep the proof self-contained, we use a simple bound $\|\mathcal{B}\|^2 \le \|\mathcal{B}\|_{\mathrm{Fr}}^2 = \operatorname{tr}(\mathcal{B}^2)$. For any matrix Γ, it holds

$$\operatorname{Var}\langle \bar{\boldsymbol{\zeta}}, \Gamma \rangle = \frac{1}{4} \mathbb{E}\biggl(\sum_{i,j=1}^{p} \Gamma_{ij}(\zeta_{ij} + \zeta_{ji})\biggr)^2 = \|\Gamma\|_{\mathrm{Fr}}^2$$

yielding $\|\operatorname{Var}(\bar{\boldsymbol{\zeta}})\| \leq 1$ and $\|\operatorname{Var}(\mathcal{B})\| \leq 1$. Also by (B.92) $I\!E\|\mathcal{B}\|_{\mathrm{Fr}}^2 = \mathbb{p}(\Sigma)/2$. Therefore, by Theorem B.4 applied to $\|\mathcal{B}\|_{\mathrm{Fr}}^2$, it holds for any $\mathtt{x}_\circ$

$$I\!P_{\zeta}\bigl(\|\mathcal{B}\|_{\mathrm{Fr}} > \sqrt{\mathbb{p}(\Sigma)/2} + \sqrt{2\mathtt{x}_\circ}\bigr) \leq \mathrm{e}^{-\mathtt{x}_\circ}.$$

By (B.88) and (B.93), it holds

$$\mathtt{x}_\circ \stackrel{\mathrm{def}}{=} \mathtt{x} + \frac{\mu_\alpha z}{2} \leq \frac{\mathbb{p}(\Sigma)\sqrt{\mathtt{x}}}{\mathtt{v}(\Sigma)} - \mathtt{x} \leq \frac{\mathbb{p}^2(\Sigma)}{4\mathtt{v}^2(\Sigma)}$$

and

$$I\!P_{\zeta}\Biggl(\|\mathcal{B}\|_{\mathrm{Fr}} > \sqrt{\frac{\mathbb{p}(\Sigma)}{2} + \frac{\mathbb{p}(\Sigma)}{\sqrt{2}\,\mathtt{v}(\Sigma)}}\Biggr) \leq \mathrm{e}^{-\mathtt{x}-\mu_\alpha z/2}.$$

Therefore, by definition (B.89) and condition (B.79)

$$\mathrm{e}^{\mu_\alpha z/2} I\!P_{\zeta}\bigl(\Omega(\alpha)^c\bigr) \leq \mathrm{e}^{\mu_\alpha z/2} I\!P\Biggl(\|\mathcal{B}\|_{\mathrm{Fr}} > \frac{\alpha\sqrt{n}}{2\sqrt{\mu_\alpha}}\Biggr) \leq \mathrm{e}^{-\mathtt{x}}$$

and the result follows.

B.6 Concentration for a family of second order tensors

Suppose to be given a family of Gaussian quadratic forms

$$\mathbb{T}_i = \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k}\,\gamma_j\,\gamma_k\,, \quad i = 1,\ldots,p,$$

with standard Gaussian r.v.'s γ_j. Without loss of generality assume that each matrix $\mathcal{T}_i = (\mathcal{T}_{i,j,k})_{j,k\leq p}$ is symmetric. The value $\mathbb{T}_i$ can be written as

$$\mathbb{T}_i = \boldsymbol{\gamma}^\top \mathcal{T}_i \boldsymbol{\gamma} = \langle \mathcal{T}_i\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle = \langle \mathcal{T}_i, \gamma^{\otimes 2}\rangle.$$

We study the concentration phenomenon of the vector $\mathbb{T}$ around its expectation in terms of its covariance matrix $S^2 = \operatorname{Var}(\mathbb{T})$. Note that the use of $S^2 = \operatorname{Var}(\mathbb{T})$ is not mandatory. All the results presented later apply with any matrix S^2 satisfying $S^2 \geq \operatorname{Var}(\mathbb{T})$. Denote

$$\|\mathcal{T}\|_{\mathrm{Fr}}^2 \stackrel{\mathrm{def}}{=} \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}^2\,.$$

Given $\boldsymbol{u} \in I\!R^p$, define

$$\mathcal{T}[\boldsymbol{u}] \stackrel{\mathrm{def}}{=} \sum_{i=1}^{p} u_i\,\mathcal{T}_i\,. \tag{B.94}$$

First, describe the covariance structure of $\mathbb{T}$.

Lemma B.29. *Denote*

$$\langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle \stackrel{\text{def}}{=} \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k} \, \mathcal{T}_{i',j,k} \, , \qquad i, i' = 1, \ldots, p.$$

Then

$$S^2 \stackrel{\text{def}}{=} \operatorname{Var}(\mathbb{T}) = \bigl(2 \langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle \bigr)_{i,i'=1,\ldots,p} \, , \tag{B.95}$$

$$\operatorname{tr} S^2 \, = \, 2 \sum_{i=1}^{p} \| \mathcal{T}_i \|_{\text{Fr}}^2 = 2 \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}^2 = 2 \| \mathcal{T} \|_{\text{Fr}}^2 \, .$$

Moreover,

$$\| S \boldsymbol{u} \|^2 \, = \, 2 \bigl\| \mathcal{T}[\boldsymbol{u}] \bigr\|_{\text{Fr}}^2 \, , \qquad \boldsymbol{u} \in \mathbb{R}^p. \tag{B.96}$$

Proof. For any i, i', it holds in view of $\mathbb{E}(\gamma_j \gamma_k - \delta_{j,k})^2 = 1 + \delta_{j,k}$ for all $j, k \le p$

$$\begin{aligned}
\mathbb{E}(\mathbb{T}_i - \mathbb{E}\mathbb{T}_i)(\mathbb{T}_{i'} - \mathbb{E}\mathbb{T}_{i'}) \, &= \, \mathbb{E} \left(\sum_{j,k=1}^{p} \mathcal{T}_{i,j,k} \, (\gamma_j \gamma_k - \delta_{j,k}) \sum_{j',k'=1}^{p} \mathcal{T}_{i',j',k'} \, (\gamma_{j'} \gamma_{k'} - \delta_{j',k'}) \right) \\
&= 2 \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k} \, \mathcal{T}_{i',j,k} = 2 \langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle.
\end{aligned}$$

This yields (B.95). Further

$$\operatorname{tr} S^2 \, = 2 \sum_{i=1}^{p} \langle \mathcal{T}_i, \mathcal{T}_i \rangle = 2 \sum_{i=1}^{p} \| \mathcal{T}_i \|_{\text{Fr}}^2 \stackrel{\text{def}}{=} 2 \| \mathcal{T} \|_{\text{Fr}}^2 \, .$$

Similarly, for any $\boldsymbol{u} = (u_i) \in \mathbb{R}^p$

$$\| S \boldsymbol{u} \|^2 \, = \boldsymbol{u}^\top S^2 \boldsymbol{u} = 2 \sum_{i,i'=1}^{p} u_i \, u_{i'} \langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle = 2 \left\| \sum_{i=1}^{p} u_i \mathcal{T}_i \right\|_{\text{Fr}}^2 = 2 \bigl\| \mathcal{T}[\boldsymbol{u}] \bigr\|_{\text{Fr}}^2 \tag{B.97}$$

completing the proof. $\square$

Given $\mathbb{V}^2 \ge S^2$ we characterize regularity of the family $(\mathcal{T}_i)$ by the value δ such that

$$\sup_{\boldsymbol{u} \colon \| \mathbb{V} \boldsymbol{u} \| \le 1} 2 \, \bigl\| \mathcal{T}[\boldsymbol{u}] \bigr\| \, \le \, \delta \, . \tag{B.98}$$

Remark B.1. With $\mathbb{V}^2 = S^2$, by (B.97), bound (B.98) follows from the condition

$$\sqrt{2}\,\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\| \le \delta \bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}, \qquad \boldsymbol{u} \in I\!\!R^p .$$

Clearly this condition meets for $\delta = \sqrt{2}$. We, however, need this condition to be fulfilled with sufficiently small δ. This can be ensured by choosing another matrix $\mathbb{V}^2 \ge S^2$. For instance, with $\mathbb{V}^2 = \mathtt{C}^2 S^2$, the inequalities $\sqrt{2}\,\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\| \le \delta \bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}$ and $\|\mathbb{V}\boldsymbol{u}\| \le 1$ imply $2\,\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\| \le \delta/\mathtt{C}$.

B.6.1 An upper bound on $\|Q(\mathbb{T} - I\!\!E\mathbb{T})\|$

This section presents an upper bound on the norm of $Q\mathcal{E}$ for $\mathcal{E} = \mathbb{T} - I\!\!E\mathbb{T}$ and a linear mapping Q. With $\mathbb{V}^2 \ge S^2$, define $B = Q\mathbb{V}^2 Q^\top$ and $z^2(B,\mathtt{x}) = \mathbb{p} + 2\mathtt{v}\sqrt{\mathtt{x}} + 2\lambda\mathtt{x}$ with

$$\begin{aligned} \mathbb{p} &\stackrel{\mathrm{def}}{=} \operatorname{tr} B &&= \operatorname{tr}(Q\mathbb{V}^2 Q^\top), \\ \mathtt{v}^2 &= \operatorname{tr} B^2 &&= \operatorname{tr}(Q\mathbb{V}^2 Q^\top)^2, \\ \lambda &\stackrel{\mathrm{def}}{=} \|B\| &&= \|Q\mathbb{V}^2 Q^\top\|. \end{aligned} \tag{B.99}$$

A "high dimensional" situation means $\mathbb{p}/\lambda$ large. As $\mathbb{p}\lambda \ge \mathtt{v}^2$, this implies $\mathbb{p} \gg \mathtt{v}$.

Theorem B.30. *Assume (B.98) and let $\mathtt{g}$ fulfill $\delta\mathtt{g} < 1$. Given Q, consider*

$$\mathcal{Z} = \sqrt{1 - \delta\mathtt{g}}\, \|Q(\mathbb{T} - I\!\!E\mathbb{T})\|.$$

Then with $B = Q\mathbb{V}^2 Q^\top$, $\mathbb{p}, \mathtt{v}, \lambda$ from (B.99), and $\mathtt{x}_c$ from (B.50), it holds

$$I\!\!P\bigl(\mathcal{Z} > z(B,\mathtt{x})\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, \qquad \mathtt{x} \le \mathtt{x}_c . \tag{B.100}$$

Moreover, with $z_c = z(B,\mathtt{x}_c)$, $\varkappa = \frac{\mathtt{g}}{\sqrt{\lambda}\,(\sqrt{8}+1)}$, it holds

$$\begin{aligned} &I\!\!P\bigl(\mathcal{Z} \ge z_c + \varkappa^{-1}(\mathtt{x} - \mathtt{x}_c)\bigr) \le 3\mathrm{e}^{-\mathtt{x}}, && \mathtt{x} \ge \mathtt{x}_c , \\ &I\!\!P\bigl(\mathcal{Z} \ge z\bigr) \le 3\exp\{-\mathtt{x}_c - \varkappa(z - z_c)\}, && z \ge z_c . \end{aligned} \tag{B.101}$$

For any $z \le z_c$ and ν with $2\nu \le \frac{z - \sqrt{\mathbb{p}}}{\sqrt{\lambda}}$, it holds

$$I\!\!E\, \mathrm{e}^{\nu\mathcal{Z}}\, \mathrm{1\!I}(\mathcal{Z} \ge z) \le 6\exp\biggl\{\nu z - \frac{(z - \sqrt{\mathbb{p}})^2}{2\lambda}\biggr\}, \tag{B.102}$$

while the condition $2\nu < \varkappa = \frac{\mathtt{g}}{\sqrt{\lambda}\,(\sqrt{8}+1)}$ ensures

$$I\!\!E\, \mathrm{e}^{\nu\mathcal{Z}}\, \mathrm{1\!I}(\mathcal{Z} > z) \le 6\,\exp\biggl\{\nu z_c - \frac{(z_c - \sqrt{\mathbb{p}})^2}{2\lambda} - (\varkappa - \nu)(z - z_c)\biggr\}, \quad z \ge z_c . \tag{B.103}$$

Proof. Let $\boldsymbol{\xi} = \mathbb{V}^{-1}\mathcal{E}$. For any $\boldsymbol{v} \in I\!\!R^p$ with $2\|\mathcal{T}[\mathbb{V}^{-1}\boldsymbol{v}]\| < 1$, we check

$$\log I\!\!E \exp\bigl(\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle\bigr) \leq \frac{\|\boldsymbol{v}\|^2}{2\bigl(1 - 2\|\mathcal{T}[\mathbb{V}^{-1}\boldsymbol{v}]\|\bigr)} . \tag{B.104}$$

Fix $\boldsymbol{v} \in I\!\!R^p$ and define $\boldsymbol{w} = 2\mathbb{V}^{-1}\boldsymbol{v}$ and $\mathcal{T}[\boldsymbol{w}]$ by (B.94). By Lemma B.2,

$$\begin{aligned}
\log I\!\!E \exp\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle &= \log I\!\!E \exp\langle \mathcal{E}, \mathbb{V}^{-1}\boldsymbol{v} \rangle \\
&= \log I\!\!E \exp\Bigl(\frac{1}{2}\langle \mathcal{T}[\boldsymbol{w}], \boldsymbol{\gamma}^{\otimes 2} \rangle - \frac{1}{2} I\!\!E \langle \mathcal{T}[\boldsymbol{w}], \boldsymbol{\gamma}^{\otimes 2} \rangle\Bigr) \\
&= \exp\Bigl\{-\frac{\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])}{2} + \log\det\bigl(I\!\!I_p - \mathcal{T}[\boldsymbol{w}]\bigr)^{-1/2}\Bigr\} \leq \frac{\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])^2}{4(1 - \|\mathcal{T}[\boldsymbol{w}]\|)} .
\end{aligned} \tag{B.105}$$

By (B.96)

$$\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])^2 = \bigl\|2\mathcal{T}[\mathbb{V}^{-1}\boldsymbol{v}]\bigr\|_{\mathrm{Fr}}^2 = 2\boldsymbol{v}^{\top}\mathbb{V}^{-1}S^2\,\mathbb{V}^{-1}\boldsymbol{v} \leq 2\|\boldsymbol{v}\|^2, \tag{B.106}$$

and (B.104) follows. If $\|\boldsymbol{v}\| \leq \mathtt{g}$, then by (B.98) $2\|\mathcal{T}[\mathbb{V}^{-1}\boldsymbol{v}]\| \leq \delta\mathtt{g}$, and condition (B.46) is fulfilled with $I\!\!V^2 = (1-\delta\mathtt{g})^{-1}I\!\!I_p$. Now Theorem B.15 applied to $\mathcal{Z} = \sqrt{1-\delta\mathtt{g}}\;\|Q\mathbb{V}\boldsymbol{\xi}\|$ implies (B.100). Furthermore, Theorem B.16 with $\rho = 1/2$ yields (B.101) while Corollary B.17 yields (B.102) and (B.103). □

B.6.2 A lower bound

We also present a lower bound on the quadratic forms $\|Q\mathcal{E}\|^2$. Here we assume $\mathbb{V}^2 = S^2$.

Theorem B.31. *Assume* (B.98) *with* $\mathbb{V}^2 = S^2$. *Fix* $\mathtt{x} \leq \mathbb{p}^2/(4\mathtt{v}^2)$, $\mu = 2\sqrt{\mathtt{x}}/\mathtt{v}$, *and* $\alpha < 1/2$ *s.t.*

$$\alpha\sqrt{\frac{1-\alpha}{1-2\alpha}} \geq \delta\sqrt{\mathbb{p}}\left(1 + \sqrt{\frac{\mathbb{p}\lambda}{2\mathtt{v}^2}}\right). \tag{B.107}$$

Then

$$I\!\!P\Bigl(\|Q\mathcal{E}\|^2 - \mathbb{p} < -\frac{\alpha\,\mathbb{p}}{1-\alpha} - 2\mathtt{v}\sqrt{\mathtt{x}}\Bigr) \leq 2\mathrm{e}^{-\mathtt{x}}. \tag{B.108}$$

Proof. The main step of the proof is a bound on negative exponential moments of $\|Q\mathcal{E}\|^2$. Then we apply Markov's inequality with a proper choice of the exponent. Namely, given $\mu = 2\sqrt{\mathtt{x}}/\mathtt{v}$ and $\alpha < 1/2$, define μ_α by

$$\frac{1-2\alpha}{1-\alpha}\,\mu_\alpha = \mu.$$

For $\boldsymbol{\zeta} \sim \mathcal{N}(0, I\!\!I_p)$ independent of $\mathcal{G}$ and $\mathtt{i} = \sqrt{-1}$

$$\begin{aligned} I\!\!E \exp\Bigl(-\frac{\mu_\alpha}{2}\|Q\mathcal{E}\|^2\Bigr) &= I\!\!E\, I\!\!E_{\boldsymbol{\zeta}} \exp\bigl\{\mathtt{i}\mu_\alpha^{1/2}\langle Q\mathcal{E}, \boldsymbol{\zeta}\rangle\bigr\} \\ &= I\!\!E_{\boldsymbol{\zeta}}\, I\!\!E\bigl[\exp\bigl\{\mathtt{i}\mu_\alpha^{1/2}\langle \mathcal{E}, Q^\top\boldsymbol{\zeta}\rangle\bigr\} \,\big|\, \boldsymbol{\zeta}\bigr], \end{aligned} \tag{B.109}$$

and similarly to (B.105), it holds by Lemma B.2 with $\boldsymbol{w} = 2\mu_\alpha^{1/2} Q^\top \boldsymbol{\zeta}$

$$I\!\!E\Bigl\{\exp\Bigl(\mathtt{i}\mu_\alpha^{1/2}\langle\mathcal{E}, Q^\top\boldsymbol{\zeta}\rangle\Bigr) \,\Big|\, \boldsymbol{\zeta}\Bigr\} = \exp\Bigl\{-\frac{\mathtt{i}\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])}{2} + \log\det\bigl(I\!\!I_p - \mathtt{i}\mathcal{T}[\boldsymbol{w}]\bigr)^{-1/2}\Bigr\}.$$

Now introduce a random set

$$\Omega(\alpha) \stackrel{\text{def}}{=} \bigl\{2\mu_\alpha^{1/2}\bigl\|\mathcal{T}[Q^\top\boldsymbol{\zeta}]\bigr\| \le \alpha\bigr\}.$$

Then $\|\mathcal{T}[\boldsymbol{w}]\| \le \alpha$ on this set and by (B.4) of Lemma B.2

$$\biggl|\log\det\bigl(I\!\!I_p - \mathtt{i}\mathcal{T}[\boldsymbol{w}]\bigr)^{-1/2} - \frac{\mathtt{i}\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])}{2} + \frac{\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])^2}{4}\biggr| \le \frac{\alpha\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])^2}{6(1-\alpha)}\,.$$

This implies on $\Omega(\alpha)$ in view of (B.106)

$$\begin{aligned} &\Bigl|I\!\!E\Bigl\{\exp\Bigl(\mathtt{i}\mu_\alpha^{1/2}\langle Q\mathcal{E}, \boldsymbol{\zeta}\rangle\Bigr) \,\Big|\, \boldsymbol{\zeta}\Bigr\}\Bigr| \le \exp\Bigl(-\frac{(1-2\alpha)\operatorname{tr}(\mathcal{T}[\boldsymbol{w}])^2}{4(1-\alpha)}\Bigr) \\ &= \exp\Bigl(-\frac{(1-2\alpha)\mu_\alpha\operatorname{tr}(2\mathcal{T}[Q^\top\boldsymbol{\zeta}])^2}{4(1-\alpha)}\Bigr) \\ &= \exp\Bigl\{-\frac{\mu_\alpha(1-2\alpha)}{1-\alpha}\,\frac{\|SQ^\top\boldsymbol{\zeta}\|^2}{2}\Bigr\} = \exp\Bigl(-\frac{\mu}{2}\|SQ^\top\boldsymbol{\zeta}\|^2\Bigr). \end{aligned}$$

Now, by (B.109) and by (B.3) of Lemma B.2

$$\begin{aligned} &I\!\!E\exp\Bigl\{-\frac{\mu_\alpha}{2}\|Q\mathcal{E}\|^2\Bigr\} \le I\!\!E_{\boldsymbol{\zeta}}\exp\Bigl\{-\frac{\mu}{2}\|SQ^\top\boldsymbol{\zeta}\|^2\Bigr\} + I\!\!P\bigl(\Omega(\alpha)\bigr) \\ &\quad= \det\bigl(I\!\!I_p + \mu B\bigr)^{-1/2} + I\!\!P\bigl(\Omega(\alpha)\bigr) \le \exp\Bigl\{-\frac{\mu\operatorname{tr}(B)}{2} + \frac{\mu^2\operatorname{tr}(B^2)}{4}\Bigr\} + I\!\!P\bigl(\Omega(\alpha)\bigr). \end{aligned}$$

For any fixed z, by Markov's inequality

$$\begin{aligned} I\!\!P\bigl(\|Q\mathcal{E}\|^2 < z\bigr) &\le \exp\Bigl(\frac{\mu_\alpha z}{2}\Bigr) I\!\!E\exp\Bigl(-\frac{\mu_\alpha}{2}\|Q\mathcal{E}\|^2\Bigr) \\ &\le \exp\Bigl\{\frac{\mu_\alpha z}{2} - \frac{\mu\operatorname{tr}(B)}{2} + \frac{\mu^2\operatorname{tr}(B^2)}{4}\Bigr\} + \exp\Bigl(\frac{\mu_\alpha z}{2}\Bigr) I\!\!P\bigl(\Omega(\alpha)\bigr). \end{aligned} \tag{B.110}$$

With $\mathbb{p} = \operatorname{tr} B$, $\mathtt{v}^2 = \operatorname{tr} B^2$, and $\mu = 2\sqrt{\mathtt{x}}/\mathtt{v}$, define z by

$$\frac{\mu_\alpha\, z}{2} = \frac{\mu}{2}\bigl(\mathbb{p} - 2\mathtt{v}\sqrt{\mathtt{x}}\bigr) = \frac{\mathbb{p}\sqrt{\mathtt{x}}}{\mathtt{v}} - 2\mathtt{x}$$

yielding

$$\frac{\mu_\alpha z}{2} - \frac{\mu\,\mathbb{p}}{2} + \frac{\mu^2 \mathtt{v}^2}{4} = \frac{\mu}{2}\bigl(\mathbb{p} - 2\mathtt{v}\sqrt{\mathtt{x}}\bigr) - \frac{\mu\,\mathbb{p}}{2} + \frac{\mu^2 \mathtt{v}^2}{4} = -\mathtt{x} \tag{B.111}$$

while

$$z = \frac{1-2\alpha}{1-\alpha}\,(\mathbb{p} - 2\mathtt{v}\sqrt{\mathtt{x}}) \ge \mathbb{p} - \frac{\alpha}{1-\alpha}\,\mathbb{p} - 2\mathtt{v}\sqrt{\mathtt{x}}\,.$$

Now we check that $\mathrm{e}^{\mu_\alpha z/2}\mathbb{P}\bigl(\Omega(\alpha)^c\bigr) \le \mathrm{e}^{-\mathtt{x}}$. By (B.98) $2\bigl\|\mathcal{T}[Q^\top\boldsymbol{\zeta}]\bigr\| \le \delta\|SQ^\top\boldsymbol{\zeta}\|$, and

$$\mathbb{P}\bigl(\Omega(\alpha)^c\bigr) \le \mathbb{P}\bigl(2\sqrt{\mu_\alpha}\bigl\|\mathcal{T}[Q^\top\boldsymbol{\zeta}]\bigr\| > \alpha\bigr) \le \mathbb{P}\bigl(\delta\sqrt{\mu_\alpha}\|SQ^\top\boldsymbol{\zeta}\| > \alpha\bigr).$$

Gaussian deviation bound (B.6) yields for any $\mathtt{x}_\circ > 0$ by $\|SQ^\top\boldsymbol{\zeta}\|^2 = \boldsymbol{\zeta}^\top B\boldsymbol{\zeta}$

$$\mathbb{P}\bigl(\|SQ^\top\boldsymbol{\zeta}\| > \sqrt{\mathbb{p}} + \sqrt{2\mathtt{x}_\circ\lambda}\bigr) \le \mathrm{e}^{-\mathtt{x}_\circ}.$$

By construction,

$$\mathtt{x} + \frac{\mu_\alpha\,z}{2} = \mathtt{x} + \frac{\mu}{2}(\mathbb{p} - 2\mathtt{v}\sqrt{\mathtt{x}}) = \frac{\mathbb{p}\sqrt{\mathtt{x}}}{\mathtt{v}} - \mathtt{x} \le \frac{\mathbb{p}^2}{4\mathtt{v}^2}\,,$$

and the use of $\mathtt{x}_\circ = \mathbb{p}^2/(4\mathtt{v}^2)$ ensures under (B.107)

$$\begin{aligned}
\mathrm{e}^{\mu_\alpha z/2}\mathbb{P}\bigl(\Omega(\alpha)^c\bigr) &= \mathrm{e}^{\mu_\alpha z/2}\mathbb{P}\biggl(\|SQ^\top\boldsymbol{\zeta}\| > \frac{\alpha}{\delta\sqrt{\mu_\alpha}}\biggr) \\
&\le \mathrm{e}^{\mu_\alpha z/2}\mathbb{P}\biggl(\|SQ^\top\boldsymbol{\zeta}\| > \sqrt{\mathbb{p}} + \frac{\mathbb{p}\,\sqrt{\lambda}}{\sqrt{2}\,\mathtt{v}}\biggr) \le \mathrm{e}^{-\mathtt{x}}.
\end{aligned}$$

Putting this together with (B.110) and (B.111) yields (B.108). □

B.7 Some bounds for a third order Gaussian tensor

Let $\mathcal{T} = \bigl(\mathcal{T}_{i,j,k}\bigr)$ be a third order symmetric tensor, that is, $\mathcal{T}_{i,j,k} = \mathcal{T}_{\pi(i,j,k)}$ for any permutation π of the triple (i,j,k). This section present a deviation bound for a Gaussian tensor sum $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}}) \stackrel{\mathrm{def}}{=} \langle \mathcal{T}, \boldsymbol{\gamma}_{\mathsf{D}}^{\otimes 3}\rangle$ for a Gaussian zero mean vector $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$ in $\mathbb{R}^p$. Much more general results for higher order tensors are available in the literature, see e.g. Götze et al. (2021) and Adamczak and Wolff (2013) and references therein. We, however, present an independent self-contained study which delivers finite sample and sharp results. Later we use notations

$$\|\mathcal{T}\| = \sup_{\|\boldsymbol{u}_1\| = \|\boldsymbol{u}_2\| = \|\boldsymbol{u}_3\| = 1} \bigl|\langle \mathcal{T}, \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3\rangle\bigr|\,.$$

Banach's characterization Banach (1938); Nie (2017) yields

$$\|\mathcal{T}\| = \sup_{\|\boldsymbol{u}\|=1} \bigl|\langle \mathcal{T}, \boldsymbol{u}^{\otimes 3} \rangle\bigr| . \tag{B.112}$$

Define

$$\mathcal{T}(\boldsymbol{u}) = \langle \mathcal{T}, \boldsymbol{u}^{\otimes 3} \rangle = \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}\, u_i\, u_j\, u_k\,, \qquad \boldsymbol{u} = (u_i) \in \mathbb{R}^p .$$

Clearly $\mathcal{T}(\boldsymbol{u})$ is a third order polynomial function on $\mathbb{R}^p$. Define also its gradient $\nabla\mathcal{T}(\boldsymbol{u}) \in \mathbb{R}^p$. Each entry of the gradient vector $\nabla\mathcal{T}(\boldsymbol{u})$ is a second order polynomial of $\boldsymbol{u}$. Symmetricity of $\mathcal{T}$ implies for any $\boldsymbol{u} \in \mathbb{R}^p$

$$\begin{aligned} \nabla\mathcal{T}(\boldsymbol{u}) &= \Biggl(3 \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k}\, u_j\, u_k \Biggr)_{i=1,\ldots,p} = 3\bigl(\boldsymbol{u}^\top \mathcal{T}_i\, \boldsymbol{u}\bigr)_{i=1,\ldots,p} \,, \\ \nabla^2\mathcal{T}(\boldsymbol{u}) &= \Biggl(6 \sum_{i=1}^{p} \mathcal{T}_{i,j,k}\, u_i \Biggr)_{j,k=1,\ldots,p} = 6\mathcal{T}[\boldsymbol{u}] \,, \end{aligned} \tag{B.113}$$

where $\mathcal{T}_i$ is the sub-tensor of order 2 with $(\mathcal{T}_i)_{j,k} = \mathcal{T}_{i,j,k}$ and

$$\mathcal{T}[\boldsymbol{u}] \stackrel{\text{def}}{=} \sum_{i=1}^{p} u_i\, \mathcal{T}_i \,.$$

Also

$$\mathcal{T}(\boldsymbol{u}) = \frac{1}{3} \langle \nabla\mathcal{T}(\boldsymbol{u}), \boldsymbol{u} \rangle = \frac{1}{6} \langle \nabla^2\mathcal{T}(\boldsymbol{u}), \boldsymbol{u}^{\otimes 2} \rangle \,.$$

Denote

$$\begin{aligned} \mathcal{T}^{(1)}(\boldsymbol{u}) &\stackrel{\text{def}}{=} \frac{1}{3}\nabla\mathcal{T}(\boldsymbol{u}) = \bigl(\boldsymbol{u}^\top \mathcal{T}_i\, \boldsymbol{u}\bigr)_{i=1,\ldots,p} \,, \\ \mathcal{T}^{(2)}(\boldsymbol{u}) &\stackrel{\text{def}}{=} \frac{1}{6}\nabla^2\mathcal{T}(\boldsymbol{u}) = \mathcal{T}[\boldsymbol{u}]. \end{aligned} \tag{B.114}$$

For the norm of the vector $\mathcal{T}^{(1)}(\boldsymbol{u})$ and of the matrix $\mathcal{T}^{(2)}(\boldsymbol{u})$, it holds by (B.112)

$$\begin{aligned} \|\mathcal{T}^{(1)}(\boldsymbol{u})\| &= \sup_{\boldsymbol{\phi} \in \mathbb{R}^p \colon \|\boldsymbol{\phi}\|=1} \langle \mathcal{T}^{(1)}(\boldsymbol{u}), \boldsymbol{\phi} \rangle = \sup_{\boldsymbol{\phi} \in \mathbb{R}^p \colon \|\boldsymbol{\phi}\|=1} \langle \mathcal{T}, \boldsymbol{u} \otimes \boldsymbol{u} \otimes \boldsymbol{\phi} \rangle = \|\mathcal{T}\|\, \|\boldsymbol{u}\|^2, \\ \|\mathcal{T}^{(2)}(\boldsymbol{u})\| &= \sup_{\boldsymbol{\phi} \in \mathbb{R}^p \colon \|\boldsymbol{\phi}\|=1} \bigl| \boldsymbol{\phi}^\top \mathcal{T}^{(2)}(\boldsymbol{u})\, \boldsymbol{\phi} \bigr| = \sup_{\boldsymbol{\phi} \in \mathbb{R}^p \colon \|\boldsymbol{\phi}\|=1} \bigl|\langle \mathcal{T}, \boldsymbol{u} \otimes \boldsymbol{\phi} \otimes \boldsymbol{\phi} \rangle\bigr| = \|\mathcal{T}\|\, \|\boldsymbol{u}\|. \end{aligned}$$

B.7.1 Moments of a Gaussian 3-tensor

Consider a Gaussian 3-tensor $\mathcal{T}(\boldsymbol{\gamma}) = \langle \mathcal{T}, \boldsymbol{\gamma}^{\otimes 3} \rangle$. Define

$$M_i = \sum_{j=1}^{p} \mathcal{T}_{i,j,j} = \operatorname{tr} \mathcal{T}_i \,, \qquad i = 1, \ldots, n \,.$$

Lemma B.32. *Let $\mathcal{T} = (\mathcal{T}_{i,j,k})$ be a 3-dimensional symmetric tensor in $\mathbb{R}^p$ and $\mathcal{T}(\boldsymbol{\gamma}) = \langle \mathcal{T}, \boldsymbol{\gamma}^{\otimes 3} \rangle$ for $\boldsymbol{\gamma} \sim \mathcal{N}(0, \mathbb{I}_p)$. With $\boldsymbol{M} = (M_i) \in \mathbb{R}^p$ and $\|\mathcal{T}\|_{\mathrm{Fr}}^2 = \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}^2$, it holds*

$$\begin{aligned} \mathbb{E}\bigl(\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle\bigr)^2 &= 6\|\mathcal{T}\|_{\mathrm{Fr}}^2 \,, \\ \mathbb{E}\, \mathcal{T}^2(\boldsymbol{\gamma}) &= 6\|\mathcal{T}\|_{\mathrm{Fr}}^2 + 9\|\boldsymbol{M}\|^2 \,. \end{aligned} \tag{B.115}$$

Proof. By definition

$$\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle = \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}\, \gamma_i \gamma_j \gamma_k - 3 \sum_{i=1}^{p} \gamma_i \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k}\, \delta_{j,k} \,. \tag{B.116}$$

It is easy to see that for each i by symmetricity of $\mathcal{T}$

$$\mathbb{E}\biggl(\gamma_i \sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}\, \gamma_i \gamma_j \gamma_k\biggr) = 3 \sum_{j \in I_i^c} \mathcal{T}_{i,j,j} \mathbb{E}(\gamma_i^2 \gamma_j^2) + \sum_{i=1}^{p} \mathcal{T}_{i,i,i} \mathbb{E}\gamma_i^4 = 3 \sum_{j=1}^{p} \mathcal{T}_{i,j,j} = 3M_i \,,$$

where the index set $I_i^c = \{1, \ldots, i-1, i+1, \ldots, p\}$ is obtained by removing the index i from $1, \ldots, p$. This implies orthogonality

$$\mathbb{E}\bigl\{\bigl(\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle\bigr)\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle\bigr\} = 0. \tag{B.117}$$

Introduce the index set $\mathcal{I} = \{(i,j,k) \colon i \neq j \neq k\}$:

$$\mathcal{I} \stackrel{\mathrm{def}}{=} \{(i,j,k) \colon\; \mathbb{1}(i=j) + \mathbb{1}(i=k) + \mathbb{1}(j=k) = 0\} \,.$$

Represent (B.116) as

$$\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle = \sum_{\mathcal{I}} \mathcal{T}_{i,j,k}\, \gamma_i\, \gamma_j\, \gamma_k + 3 \sum_{i=1}^{p} \sum_{j \in I_i^c} \mathcal{T}_{i,j,j}\, \gamma_i (\gamma_j^2 - 1) + \sum_{i=1}^{p} \mathcal{T}_{i,i,i}\, (\gamma_i^3 - 3\gamma_i) \,.$$

All terms in the right hand-side are orthogonal to each other allowing to compute $\mathbb{E}\bigl(\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle\bigr)^2$:

$$\begin{aligned} \mathbb{E}\bigl(\mathcal{T}(\boldsymbol{\gamma}) - 3\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle\bigr)^2 &= \mathbb{E}\biggl(\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}\, \gamma_i\, \gamma_j\, \gamma_k\biggr)^2 \\ &\quad + \mathbb{E}\biggl(3 \sum_{i=1}^{p} \sum_{j \in I_i^c} \mathcal{T}_{i,j,j}\, \gamma_i (\gamma_j^2 - 1)\biggr)^2 + \mathbb{E}\biggl(\sum_{i=1}^{p} \mathcal{T}_{i,i,i}\, (\gamma_i^3 - 3\gamma_i)\biggr)^2 . \end{aligned}$$

Further, by symmetricity of $\mathcal{T}$

$$\begin{aligned}\mathbb{E}\Bigl(\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}\,\gamma_i\,\gamma_j\,\gamma_k\Bigr)^2 &= \mathbb{E}\Bigl(\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}\,\gamma_i\,\gamma_j\,\gamma_k \sum_{\mathcal{I}} \mathcal{T}_{i',j',k'}\,\gamma_{i'}\,\gamma_{j'}\,\gamma_{k'}\Bigr) \\ &= \mathbb{E}\Bigl(\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}\,\gamma_i\,\gamma_j\,\gamma_k \sum_{(i',j',k')=\pi(i,j,k)} \mathcal{T}_{i',j',k'}\,\gamma_{i'}\,\gamma_{j'}\,\gamma_{k'}\Bigr) = 6\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}^2\,.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{E}\Bigl(3\sum_{i=1}^{p}\sum_{j\in I_i^c} \mathcal{T}_{i,j,j}\,\gamma_i(\gamma_j^2-1)\Bigr)^2 &= 9\sum_{i=1}^{p}\sum_{j\in I_i^c} \mathcal{T}_{i,j,j}^2\,\mathbb{E}\bigl\{\gamma_i^2(\gamma_j^2-1)^2\bigr\} = 18\sum_{i=1}^{p}\sum_{j\in I_i^c} \mathcal{T}_{i,j,j}^2\,, \\ \mathbb{E}\Bigl(\sum_{i=1}^{p} \mathcal{T}_{i,i,i}\,(\gamma_i^3-3\gamma_i)\Bigr)^2 &= \sum_{i=1}^{p} \mathcal{T}_{i,i,i}^2\,\mathbb{E}(\gamma_i^3-3\gamma_i)^2 = 6\sum_{i=1}^{p} \mathcal{T}_{i,i,i}^2\end{aligned}$$

yielding again by symmetricity of $\mathcal{T}$

$$\mathbb{E}\bigl(\mathcal{T}(\boldsymbol{\gamma})-3\langle \boldsymbol{M},\boldsymbol{\gamma}\rangle\bigr)^2 = 6\sum_{\mathcal{I}} \mathcal{T}_{i,j,k}^2 + 18\sum_{i=1}^{p}\sum_{j\in I_i^c} \mathcal{T}_{i,j,j}^2 + 6\sum_{i=1}^{p} \mathcal{T}_{i,i,i}^2 = 6\|\mathcal{T}\|_{\mathrm{Fr}}^2$$

and assertion (B.115) follows in view of orthogonality (B.117). □

Similarly we study the moments of the scaled gradient vector

$$\mathbb{T} = \mathcal{T}^{(1)}(\boldsymbol{\gamma}) = \frac{1}{3}\nabla\mathcal{T}(\boldsymbol{\gamma}).$$

The entries $\mathbb{T}_i$ of $\mathbb{T}$ can be written as $\mathbb{T}_i = \boldsymbol{\gamma}^\top \mathcal{T}_i\,\boldsymbol{\gamma}$; see (B.113).

Lemma B.33. *It holds* $\mathbb{E}\mathbb{T} = \boldsymbol{M}$,

$$\mathrm{Var}(\mathbb{T}) = S^2 = \bigl(2\langle \mathcal{T}_i, \mathcal{T}_{i'}\rangle\bigr)_{i,i'=1,\ldots,p}\,, \tag{B.118}$$

$$\mathrm{tr}\,S^2 = 2\sum_{i=1}^{p} \|\mathcal{T}_i\|_{\mathrm{Fr}}^2 = 2\sum_{i,j,k=1}^{p} \mathcal{T}_{i,j,k}^2 = 2\|\mathcal{T}\|_{\mathrm{Fr}}^2\,, \tag{B.119}$$

$$\mathbb{E}\|\mathbb{T}\|^2 = \|\boldsymbol{M}\|^2 + 2\|\mathcal{T}\|_{\mathrm{Fr}}^2 \le \frac{1}{3}\mathbb{E}\mathcal{T}^2(\boldsymbol{\gamma})\,.$$

Moreover, for any $\boldsymbol{u} \in \mathbb{R}^p$

$$\|S\boldsymbol{u}\|^2 = 2\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}^2\,. \tag{B.120}$$

Proof. The first statement follows directly from $\mathbb{E}\mathbb{T}_i = \mathbb{E}\boldsymbol{\gamma}^\top \mathcal{T}_i \boldsymbol{\gamma} = \operatorname{tr} \mathcal{T}_i$. For any i, i', it holds in view of $\mathbb{E}(\gamma_j \gamma_k - \delta_{j,k})^2 = 1 + \delta_{j,k}$ for all $j, k \le p$

$$\begin{aligned}\mathbb{E}(\mathbb{T}_i - \mathbb{E}\mathbb{T}_i)(\mathbb{T}_{i'} - \mathbb{E}\mathbb{T}_{i'}) &= \mathbb{E}\left(\sum_{j,k=1}^{p} \mathcal{T}_{i,j,k}\,(\gamma_j\gamma_k - \delta_{j,k}) \sum_{j',k'=1}^{p} \mathcal{T}_{i',j',k'}\,(\gamma_{j'}\gamma_{k'} - \delta_{j',k'})\right) \\ &= 2 \sum_{j,k=1}^{p} \mathcal{T}_{i,j,k}\,\mathcal{T}_{i',j,k} = 2\langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle.\end{aligned}$$

This yields (B.118). Further

$$\operatorname{tr} S^2 = 2\sum_{i=1}^{p} \langle \mathcal{T}_i, \mathcal{T}_i \rangle = 2\sum_{i=1}^{p} \|\mathcal{T}_i\|_{\mathrm{Fr}}^2 \stackrel{\mathrm{def}}{=} 2\|\mathcal{T}\|_{\mathrm{Fr}}^2 ,$$

which proves (B.119). Similarly, for any $\boldsymbol{u} = (u_i) \in \mathbb{R}^p$

$$\|S\boldsymbol{u}\|^2 = \boldsymbol{u}^\top S^2 \boldsymbol{u} = 2\sum_{i,i'=1}^{p} u_i\,u_{i'} \langle \mathcal{T}_i, \mathcal{T}_{i'} \rangle = 2\left\|\sum_{i=1}^{p} u_i \mathcal{T}_i\right\|_{\mathrm{Fr}}^2 = 2\big\|\mathcal{T}[\boldsymbol{u}]\big\|_{\mathrm{Fr}}^2$$

completing the proof. □

B.7.2 $\ell_3 - \ell_2$ condition

This section introduces a special $\ell_3 - \ell_2$ condition for a symmetric 3-tensor $\mathcal{T}$.

$(\boldsymbol{\mathit{\Gamma}})$ *For some symmetric p-matrix $\mathit{\Gamma}$ and $\tau > 0$, $\mathcal{T}(\boldsymbol{u}) = \langle \mathcal{T}, \boldsymbol{u}^{\otimes 3} \rangle$ fulfills*

$$|\mathcal{T}(\boldsymbol{u})| \le \tau\,\|\mathit{\Gamma}\boldsymbol{u}\|^3, \qquad \boldsymbol{u} \in \mathbb{R}^p . \tag{B.121}$$

Lemma B.34. *Suppose that the tensor $\mathcal{T}$ satisfies $(\boldsymbol{\mathit{\Gamma}})$. Then*

$$|\langle \mathcal{T}, \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3 \rangle| \le \tau\,\|\mathit{\Gamma}\boldsymbol{u}_1\|\,\|\mathit{\Gamma}\boldsymbol{u}_2\|\,\|\mathit{\Gamma}\boldsymbol{u}_3\| , \quad \boldsymbol{u}_1 , \boldsymbol{u}_2 , \boldsymbol{u}_3 \in \mathbb{R}^p , \tag{B.122}$$

and it holds for $\mathcal{T}^{(1)}(\boldsymbol{u}) = \frac{1}{3}\nabla\mathcal{T}(\boldsymbol{u})$, $\mathcal{T}^{(2)}(\boldsymbol{u}) = \mathcal{T}[\boldsymbol{u}]$ from (B.114), any $\boldsymbol{u} \in \mathbb{R}^p$

$$\|\mathcal{T}^{(1)}(\boldsymbol{u})\| \le \tau\,\|\mathit{\Gamma}\boldsymbol{u}\|^2\,\|\mathit{\Gamma}\| , \tag{B.123}$$

$$\mathcal{T}[\boldsymbol{u}] \le \tau\,\|\mathit{\Gamma}\boldsymbol{u}\|\,\mathit{\Gamma}^2 , \tag{B.124}$$

yielding

$$\begin{aligned}\|\mathcal{T}[\boldsymbol{u}]\|_{\mathrm{Fr}}^2 &\le \tau^2\,\|\mathit{\Gamma}\boldsymbol{u}\|^2\,\operatorname{tr}(\mathit{\Gamma}^4) , \qquad \boldsymbol{u} \in \mathbb{R}^p , \\ \|\mathcal{T}\|_{\mathrm{Fr}}^2 &\le \tau^2\,\operatorname{tr}(\mathit{\Gamma}^2)\,\operatorname{tr}(\mathit{\Gamma}^4) .\end{aligned} \tag{B.125}$$

Further, for $\boldsymbol{M} = (M_i) \in \mathbb{R}^p$ *with* $M_i = \operatorname{tr} \mathcal{T}_i$ *, it holds*

$$\|\boldsymbol{M}\| \leq \tau \, \|\Gamma\| \, \operatorname{tr}(\Gamma^2) \, ,$$

The matrix S^2 *from* (B.118) *fulfills*

$$S^2 \leq 2\tau^2 \operatorname{tr}(\Gamma^4) \, \Gamma^2. \tag{B.126}$$

It holds for the Gaussian tensor $\mathcal{T}(\boldsymbol{\gamma})$

$$\mathbb{E}\, \mathcal{T}^2(\boldsymbol{\gamma}) \leq 6\tau^2 \operatorname{tr}(\Gamma^2) \operatorname{tr}(\Gamma^4) + 9\tau^2 \, \|\Gamma\|^2 \operatorname{tr}^2(\Gamma^2) \leq 15\tau^2 \, \|\Gamma\|^2 \operatorname{tr}^2(\Gamma^2). \tag{B.127}$$

Proof. Define 3-tensor $\mathcal{T}_\Gamma$ by $\mathcal{T}_\Gamma(\boldsymbol{u}) = \mathcal{T}(\Gamma^{-1}\boldsymbol{u})$. Then condition (B.121) reads $|\mathcal{T}_\Gamma(\boldsymbol{u})| \leq \tau$ for all $\|\boldsymbol{u}\| \leq 1$ while (B.122) can be written as

$$|\langle \mathcal{T}_\Gamma, \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3 \rangle| \leq \tau \, , \quad \forall \|\boldsymbol{u}_j\| \leq 1, \;\; j = 1,2,3.$$

The latter holds by Banach's characterization as in (B.112). Further,

$$\begin{aligned}
\|\mathcal{T}^{(1)}(\boldsymbol{u})\| &= \sup_{\|\boldsymbol{u}_1\|=1} \bigl|\langle \mathcal{T}^{(1)}(\boldsymbol{u}), \boldsymbol{u}_1 \rangle\bigr| = \sup_{\|\boldsymbol{u}_1\|=1} \bigl|\langle \mathcal{T}, \boldsymbol{u} \otimes \boldsymbol{u} \otimes \boldsymbol{u}_1 \rangle\bigr| \\
&\leq \tau \, \|\Gamma \boldsymbol{u}\|^2 \sup_{\|\boldsymbol{u}_1\|=1} \|\Gamma \boldsymbol{u}_1\| \leq \tau \, \|\Gamma \boldsymbol{u}\|^2 \, \|\Gamma\| \, , \\
\|\mathcal{T}[\boldsymbol{u}]\| &= \sup_{\|\boldsymbol{u}_1\|=1} \langle \mathcal{T}[\boldsymbol{u}], \boldsymbol{u}_1^{\otimes 2} \rangle = \sup_{\|\boldsymbol{u}_1\|=1} \langle \mathcal{T}, \boldsymbol{u} \otimes \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 \rangle \\
&\leq \tau \, \|\Gamma \boldsymbol{u}\| \sup_{\|\boldsymbol{u}_1\|=1} \|\Gamma \boldsymbol{u}_1\|^2 \leq \tau \, \|\Gamma \boldsymbol{u}\| \, \|\Gamma^2\| \, ,
\end{aligned}$$

yielding (B.124). Further, $\langle \boldsymbol{M}, \boldsymbol{u} \rangle = \operatorname{tr} \mathcal{T}[\boldsymbol{u}]$ and by (B.124)

$$\|\boldsymbol{M}\| = \sup_{\|\boldsymbol{u}\|=1} \bigl|\langle \boldsymbol{M}, \boldsymbol{u} \rangle\bigr| = \sup_{\|\boldsymbol{u}\|=1} \bigl|\operatorname{tr} \mathcal{T}[\boldsymbol{u}]\bigr| \leq \tau \, \|\Gamma\| \, \operatorname{tr}(\Gamma^2) \, .$$

Similarly for $\boldsymbol{u} \in \mathbb{R}^p$

$$\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}^2 = \operatorname{tr}(\mathcal{T}[\boldsymbol{u}]^2) \leq \tau^2 \, \|\Gamma \boldsymbol{u}\|^2 \, \operatorname{tr}(\Gamma^4) \, .$$

Finally, the use of $\mathcal{T}_i = \mathcal{T}[\boldsymbol{e}_i]$ for the canonic basis vectors $\boldsymbol{e}_i \in \mathbb{R}^p$ yields

$$\|\mathcal{T}\|_{\mathrm{Fr}}^2 = \sum_{i=1}^p \operatorname{tr}(\mathcal{T}[\boldsymbol{e}_i]^2) \leq \tau^2 \sum_{i=1}^p \|\Gamma \boldsymbol{e}_i\|^2 \operatorname{tr}(\Gamma^4) = \tau^2 \operatorname{tr}(\Gamma^2) \operatorname{tr}(\Gamma^4) \, ,$$

and (B.125) follows. By (B.120) and (B.124), it holds for any $\boldsymbol{u} \in \mathbb{R}^p$

$$\|S\boldsymbol{u}\|^2 = 2\bigl\|\mathcal{T}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}^2 \leq 2\tau^2 \operatorname{tr}(\Gamma^4) \|\Gamma \boldsymbol{u}\|^2 \, .$$

This yields (B.126). The obtained bounds lead to (B.127) in view of (B.115). $\square$

B.7.3 Colored case

This section extends the established upper bound to the case when the standard Gaussian vector $\boldsymbol{\gamma}$ is replaced by a general zero mean Gaussian vector $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$ for a symmetric covariance matrix D^2. Then $\boldsymbol{\gamma}_{\mathsf{D}} = \mathsf{D}^{-1}\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}$ standard normal and $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}}) = \mathcal{T}(\mathsf{D}^{-1}\boldsymbol{\gamma}) = \widetilde{\mathcal{T}}(\boldsymbol{\gamma})$ with $\widetilde{\mathcal{T}}(\boldsymbol{u}) = \mathcal{T}(\mathsf{D}^{-1}\boldsymbol{u})$. If $\mathcal{T}$ satisfies $(\boldsymbol{\mathit{\Gamma}})$ then $\widetilde{\mathcal{T}}$ does as well but $\mathit{\Gamma}^2$ has to be replaced by $\mathbb{v}^2 = \mathsf{D}^{-1}\mathit{\Gamma}^2\mathsf{D}^{-1}$.

Lemma B.35. *Let $\mathcal{T}(\boldsymbol{u})$ satisfies $(\boldsymbol{\mathit{\Gamma}})$ with some $\mathit{\Gamma}$ and τ. Then $\widetilde{\mathcal{T}}(\boldsymbol{u}) = \mathcal{T}(\mathsf{D}^{-1}\boldsymbol{u})$ satisfies $(\boldsymbol{\mathit{\Gamma}})$ with $\mathbb{v}^2 = \mathsf{D}^{-1}\mathit{\Gamma}^2\mathsf{D}^{-1}$ in place of $\mathit{\Gamma}^2$ and the same τ. In particular, with $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{T}}_i)$, $\widetilde{\boldsymbol{M}} = (\operatorname{tr}\widetilde{\mathcal{T}}_i)$, and $\widetilde{S}^2 \stackrel{\text{def}}{=} \bigl(2\langle\widetilde{\mathcal{T}}_i, \widetilde{\mathcal{T}}_{i'}\rangle\bigr)_{i,i'=1,\ldots,p}$, it holds*

$$\|\widetilde{\mathcal{T}}\|_{\mathrm{Fr}}^2 \le \tau^2 \operatorname{tr}(\mathbb{v}^2) \operatorname{tr}(\mathbb{v}^4), \tag{B.128}$$

$$\|\widetilde{\boldsymbol{M}}\| \le \tau \|\mathbb{v}\| \operatorname{tr}(\mathbb{v}^2), \tag{B.129}$$

$$\widetilde{S}^2 \le 2\tau^2 \operatorname{tr}(\mathbb{v}^4)\, \mathbb{v}^2,$$

Moreover, for any $\boldsymbol{u} \in \mathbb{R}^p$

$$\begin{aligned}
\|\widetilde{\mathcal{T}}^{(1)}(\boldsymbol{u})\| &= \frac{1}{3}\|\nabla\widetilde{\mathcal{T}}(\boldsymbol{u})\| \le \tau\, \|\mathbb{v}\, \boldsymbol{u}\|^2\, \|\mathbb{v}\|. \\
\bigl\|\widetilde{\mathcal{T}}[\boldsymbol{u}]\bigr\|_{\mathrm{Fr}}^2 &\le \tau^2\, \|\mathbb{v}\, \boldsymbol{u}\|^2 \operatorname{tr}\mathbb{v}^4,
\end{aligned}$$

Proof. By definition, for any $\boldsymbol{u} \in \mathbb{R}^p$

$$\widetilde{\mathcal{T}}(\boldsymbol{u}) = \mathcal{T}(\mathsf{D}^{-1}\boldsymbol{u}) \le \tau\|\mathit{\Gamma}\mathsf{D}^{-1}\boldsymbol{u}\|^3 = \tau\|\mathbb{v}\, \boldsymbol{u}\|^3$$

yielding $(\boldsymbol{\mathit{\Gamma}})$ for $\widetilde{\mathcal{T}}$. Now Lemma B.35 enables us to apply the results of Lemma B.34 with $\mathbb{v}$ in place of $\mathit{\Gamma}$. Finally, for any $\boldsymbol{u}$ with $\|\mathbb{v}\, \boldsymbol{u}\| \le \mathtt{r}$, it holds by (B.123)

$$\|\widetilde{\mathcal{T}}^{(1)}(\boldsymbol{u})\| \le \tau\, \|\mathbb{v}\, \boldsymbol{u}\|^2\, \|\mathbb{v}\| \le \tau\, \mathtt{r}^2\, \|\mathbb{v}\|.$$

This completes the proof. □

Lemma B.36. *For the Gaussian tensor $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})$, it holds*

$$\mathbb{E}\,\mathcal{T}^2(\boldsymbol{\gamma}_{\mathsf{D}}) \le 15\tau^2\, \|\mathbb{v}\|^2 \operatorname{tr}^2(\mathbb{v}^2). \tag{B.130}$$

Moreover, with $\mathsf{c}_2 = ???$,

$$\sqrt{\mathbb{E}\,\mathcal{T}^4(\boldsymbol{\gamma}_{\mathsf{D}})} \le \mathsf{c}_2\, \tau^2\, \|\mathbb{v}\|^2 \operatorname{tr}^2(\mathbb{v}^2).$$

Proof. Apply (B.115) of Lemma B.32 to $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}}) = \widetilde{\mathcal{T}}(\boldsymbol{\gamma})$ and use (B.128) and (B.129). □

B.7.4 An exponential bound on $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})$

Let $\delta(\boldsymbol{u})$ be a smooth function on $\mathbb{R}^p$. A typical example we have in mind is $\delta(\boldsymbol{u}) = \mathcal{T}(\boldsymbol{u})$, where $\mathcal{T}$ is a symmetric 3-tensor satisfying $(\boldsymbol{\mathit{\Gamma}})$ with some Γ^2 and τ. Let also $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$. Our aim is a possibly accurate exponential bound for $\delta(\boldsymbol{\gamma}_{\mathsf{D}})$, in particular, for the Gaussian tensor $\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}}) = \langle \mathcal{T}, \boldsymbol{\gamma}_{\mathsf{D}}^{\otimes 3} \rangle$. We use $\delta(\boldsymbol{\gamma}_{\mathsf{D}}) = \delta(\mathsf{D}^{-1}\boldsymbol{\gamma}) = \widetilde{\delta}(\boldsymbol{\gamma})$ for $\widetilde{\delta}(\boldsymbol{u}) = \delta(\mathsf{D}^{-1}\boldsymbol{u})$ and $\boldsymbol{\gamma}$ standard normal. The results use a bound on the norm $\|\nabla\widetilde{\delta}(\boldsymbol{u})\|$ which is hard to verify on the whole domain $\mathbb{R}^p$. Therefore, we limit the domain of $\delta(\boldsymbol{u})$ to a subset $\mathcal{U}$ on which the Gaussian measure $\mathcal{N}(0, \mathsf{D}^{-2})$ well concentrates. Clearly, for any $\boldsymbol{u}$ with $\mathsf{D}^{-1}\boldsymbol{u} \in \mathcal{U}$

$$\nabla\widetilde{\delta}(\boldsymbol{u}) = \mathsf{D}^{-1}\nabla\delta(\mathsf{D}^{-1}\boldsymbol{u}).$$

For the rest of this section, we assume that $\|\nabla\widetilde{\delta}(\boldsymbol{u})\|$ is uniformly bounded over the set of $\boldsymbol{u}$ with $\mathsf{D}^{-1}\boldsymbol{u} \in \mathcal{U}$ for the set $\mathcal{U}$ with the elliptic shape

$$\mathcal{U} \stackrel{\text{def}}{=} \{\boldsymbol{u} \colon \|\Gamma\boldsymbol{u}\| \le \mathtt{r}\}. \tag{B.131}$$

This applies to $\delta(\boldsymbol{u}) = \mathcal{T}(\boldsymbol{u})$ for a tensor $\mathcal{T}$ satisfying $(\boldsymbol{\mathit{\Gamma}})$. We consider local behavior of $\delta(\boldsymbol{\gamma}_{\mathsf{D}})$ for $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$. With $\mathcal{U}$ fixed, introduce the notation

$$\mathbb{E}_{\mathcal{U}}\, \xi \stackrel{\text{def}}{=} \mathbb{E}\xi\, \mathbb{I}(\boldsymbol{\gamma}_{\mathsf{D}} \in \mathcal{U}).$$

Remind the definition

$$\mathbb{v}^2 \stackrel{\text{def}}{=} \mathsf{D}^{-1}\Gamma^2\mathsf{D}^{-1}.$$

The next lemma explains the choice of the radius $\mathtt{r}$ to ensure a concentration effect of $\boldsymbol{\gamma}_{\mathsf{D}}$ on $\mathcal{U}$.

Lemma B.37. *For a fixed* $\mathtt{x}$, *set* $\mathtt{r} = \mathtt{r}(\mathtt{x}) = z(\mathbb{v}^2, \mathtt{x})$ *with*

$$z^2(\mathbb{v}^2, \mathtt{x}) = \operatorname{tr}(\mathbb{v}^2) + 2\sqrt{\mathtt{x}\operatorname{tr}(\mathbb{v}^4)} + 2\mathtt{x}\|\mathbb{v}^2\|.$$

For the set $\mathcal{U}$ *from* (B.131), *suppose*

$$\sup_{\boldsymbol{v} \colon \mathsf{D}^{-1}\boldsymbol{v} \in \mathcal{U}} \|\nabla\widetilde{\delta}(\boldsymbol{v})\| = \sup_{\boldsymbol{u} \in \mathcal{U}} \|\mathsf{D}^{-1}\nabla\delta(\boldsymbol{u})\| \le \epsilon\,. \tag{B.132}$$

Then it holds for $X = \delta(\boldsymbol{\gamma}_{\mathsf{D}}) - \mathbb{E}_{\mathcal{U}}\, \delta(\boldsymbol{\gamma}_{\mathsf{D}})$, *with any* μ *and any integer* k

$$\mathbb{E}_{\mathcal{U}}\, \mathrm{e}^{\mu X} \le \exp\bigl(\mu^2\epsilon^2/2\bigr)\,, \tag{B.133}$$

$$\mathbb{E}_{\mathcal{U}}\, |X|^{2k} \le \mathtt{C}_k^2\epsilon^{2k}\,, \qquad \mathtt{C}_k^2 = 2^{k+1}k!\,. \tag{B.134}$$

Also

$$\mathbb{P}\Big(X > \epsilon\sqrt{2\mathtt{x}}\Big) \le 2\mathrm{e}^{-\mathtt{x}}.$$

Proof. With $\boldsymbol{\gamma} \sim \mathcal{N}(0, I\!\!I_p)$ and $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$, it holds

$$\mathbb{P}(\boldsymbol{\gamma}_{\mathsf{D}} \not\in \mathcal{U}) = \mathbb{P}\big(\|\varGamma \mathsf{D}^{-1}\boldsymbol{\gamma}\| > \mathtt{r}\big).$$

For $\mathtt{r} = z(\mathtt{v}^2, \mathtt{x})$, Gaussian concentration bound yields

$$\mathbb{P}(\boldsymbol{\gamma}_{\mathsf{D}} \not\in \mathcal{U}) = \mathbb{P}\big(\|\varGamma \mathsf{D}^{-1}\boldsymbol{\gamma}\| > z(\mathtt{v}^2, \mathtt{x})\big) \le \mathrm{e}^{-\mathtt{x}}.$$

Further, $\delta(\boldsymbol{\gamma}_{\mathsf{D}}) = \delta(\mathsf{D}^{-1}\boldsymbol{\gamma}) = \widetilde{\delta}(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma}$ standard normal and by (B.132), the norm of the gradient $\nabla\widetilde{\delta}(\boldsymbol{v})$ is bounded by ϵ for all $\boldsymbol{v}$ with $\mathsf{D}^{-1}\boldsymbol{v} \in \mathcal{U}$. The use of log-Sobolev inequality and Herbst's arguments yields (B.133) for $X = \delta(\mathsf{D}^{-1}\boldsymbol{\gamma}) - \mathbb{E}_{\mathcal{U}}\,\delta(\mathsf{D}^{-1}\boldsymbol{\gamma})$; see Theorem 5.5 in Boucheron et al. (2013) or Proposition 5.4.1 in Bakry et al. (2013). Result (B.133) also implies the probability bound

$$\mathbb{P}\Big(X > \sqrt{2\mathtt{x}}\,\epsilon\Big) \le \mathbb{P}(\boldsymbol{\gamma}_{\mathsf{D}} \not\in \mathcal{U}) + \mathbb{P}\Big(X > \epsilon\sqrt{2\mathtt{x}}\,,\, \boldsymbol{\gamma}_{\mathsf{D}} \in \mathcal{U}\Big) \le 2\mathrm{e}^{-\mathtt{x}};$$

see (5.4.2) in Bakry et al. (2013). Now Lemma B.38 and (B.133) imply (B.134). □

Lemma B.38 (Boucheron et al. (2013), Theorem 2.1)**.** *Let a random variable* X *satisfy* $\mathbb{E}\exp(\mu X) \le \exp(\mu^2\epsilon^2/2)$ *for all* μ *with some* $\epsilon^2 > 0$*. Then for any integer* k

$$\mathbb{E}|X|^{2k} \le \mathtt{C}_k^2\,\epsilon^{2k}, \qquad \mathtt{C}_k^2 = 2^{k+1}k!\,. \tag{B.135}$$

In particular, $\mathtt{C}_1 = 2$, $\mathtt{C}_2 = 4$, $\mathtt{C}_3 = \sqrt{96} \le 10$, $\mathtt{C}_4 = 16\sqrt{3} \le 28$.

Proof. Conditions of the lemma and Markov inequality imply for any $u > 0$ with $\mu = u$

$$\mathbb{P}\big(X/\epsilon > u\big) \le \mathrm{e}^{-\mu u}\,\mathbb{E}\exp(\mu X/\epsilon) \le \exp(-u^2/2)$$

and similarly for $\mathbb{P}\big(-X/\epsilon > u\big)$ hence,

$$\begin{aligned}\mathbb{E}|X/s|^{2k} &= \int_0^\infty \mathbb{P}\big(|X/s|^{2k} > x\big)\,dx = 2k\int_0^\infty x^{2k-1}\mathbb{P}\big(|X/s| > x\big)\,dx \\ &\le 4k\int_0^\infty x^{2k-1}\mathrm{e}^{-x^2/2}\,dx = 4k\int_0^\infty (2t)^{k-1}\mathrm{e}^{-t}\,dt = 2^{k+1}k!\end{aligned}$$

as claimed. □

Let X satisfy $\mathbb{E}\exp(\mu X) \le \exp(\mu^2\epsilon^2/2)$ for all μ with some ϵ small. One can expect that e^X can be well approximated for $k \ge 2$ by

$$\mathcal{E}_k(X) \stackrel{\text{def}}{=} 1 + X + \ldots + \frac{X^{k-1}}{(k-1)!}\,. \tag{B.136}$$

Lemma B.39. *Let a random variable X satisfy $\mathbb{E}\exp(\mu X) \le \exp(\mu^2\epsilon^2/2)$ for all μ with some $\epsilon^2 > 0$. Then for a random variable ξ such that $|\xi| \le 1$ and any integer k with $\mathtt{C}_k$ from (B.134) and $\mathcal{E}_k(X)$ from (B.136)*

$$\bigl|\mathbb{E}\bigl(\mathrm{e}^X - \mathcal{E}_k(X)\bigr)\xi\bigr| \le \frac{\mathtt{C}_k}{k!}\,\epsilon^k \mathrm{e}^{\epsilon^2}\,.$$

In particular, with $\mathtt{C}_2 = 4$ and $\mathtt{C}_3^2 = 96$

$$\bigl|\mathbb{E}\bigl(\mathrm{e}^X - 1 - X\bigr)\xi\bigr| \le 2\epsilon^2\,\mathrm{e}^{\epsilon^2}\,,$$

$$\Bigl|\mathbb{E}\Bigl(\mathrm{e}^X - 1 - X - \frac{X^2}{2}\Bigr)\xi\Bigr| \le \frac{5}{3}\epsilon^3\,\mathrm{e}^{\epsilon^2}\,.$$

If ξ is not bounded but $\mathbb{E}\xi^{2k+2} < \infty$, then with $\rho = k/(k+1)$

$$\bigl|\mathbb{E}\bigl(\mathrm{e}^X - \mathcal{E}_k(X)\bigr)\xi\bigr| \le \frac{\mathtt{C}_{k+1}^{\rho}}{k!}\,\epsilon^k\,\mathrm{e}^{\epsilon^2}\,\bigl(\mathbb{E}\xi^{2k+2}\bigr)^{\frac{1}{2k+2}}\,. \tag{B.137}$$

In particular, with $\mathtt{C}_3^{2/3} = 96^{1/3} \le 4.6$ and $\mathtt{C}_4^{3/4} \le 12$

$$\bigl|\mathbb{E}\bigl(\mathrm{e}^X - 1 - X\bigr)\xi\bigr| \le 2.3\,\epsilon^2\,\mathrm{e}^{\epsilon^2}\,\bigl(\mathbb{E}\xi^6\bigr)^{1/6}\,, \tag{B.138}$$

$$\Bigl|\mathbb{E}\Bigl(\mathrm{e}^X - 1 - X - \frac{X^2}{2}\Bigr)\xi\Bigr| \le 2\,\epsilon^3\,\mathrm{e}^{\epsilon^2}\,\bigl(\mathbb{E}\xi^8\bigr)^{1/8}\,. \tag{B.139}$$

Proof. Define

$$\mathscr{R}(t) \stackrel{\text{def}}{=} \mathbb{E}\bigl\{\bigl(\mathrm{e}^{tX} - \mathcal{E}_k(tX)\bigr)\xi\bigr\}\,.$$

Obviously $\mathscr{R}(0) = \mathscr{R}'(0) = \ldots = \mathscr{R}^{(k-1)}(0) = 0$. The Taylor expansion of order k yields

$$\bigl|\mathscr{R}(1)\bigr| \le \frac{1}{k!}\sup_{t\in[0,1]} |\mathscr{R}^{(k)}(t)|.$$

Further,

$$\mathscr{R}^{(k)}(t) = \mathbb{E}(X^k\,\xi\,\mathrm{e}^{tX})\,.$$

Consider first the case $|\xi| \le 1$ a.s. By the Cauchy-Schwarz inequality, (B.133) of Lemma B.37, and (B.135), it holds for any $t \in [0,1]$

$$|\mathscr{R}^{(k)}(t)|^2 \le \mathbb{E}|X|^{2k}\;\mathbb{E}\,\mathrm{e}^{2tX} \le \mathtt{C}_k^2\epsilon^{2k}\mathrm{e}^{2\epsilon^2}\,.$$

For a general ξ, in a similar way, it holds with $\rho = k/(k+1)$

$$
\begin{aligned}
|\mathscr{R}^{(k)}(t)|^2 &\le \mathbb{E}(|X|^{2k}\xi^2)\, \mathbb{E}\,\mathrm{e}^{2tX} \le \bigl(\mathbb{E}|X|^{2k+2}\bigr)^{\rho}\bigl(\mathbb{E}\xi^{2k+2}\bigr)^{1-\rho}\, \mathbb{E}\,\mathrm{e}^{2\epsilon^2} \\
&\le \mathtt{C}_{k+1}^{2\rho}\epsilon^{2k}\mathrm{e}^{2\epsilon^2}\left(\mathbb{E}\xi^{2k+2}\right)^{1-\rho},
\end{aligned}
$$

and (B.137) follows. □

This result with $\xi = 1$ yields an approximation $\mathbb{E}\mathrm{e}^X \approx 1 + \mathbb{E}X + \mathbb{E}X^2/2$ and with $\xi = X$ an approximation $\mathbb{E}(X\mathrm{e}^X) \approx \mathbb{E}X + \mathbb{E}X^2$.

Lemma B.40. *Let a random variable X satisfy $\mathbb{E}\exp(\mu X) \le \exp(\mu^2\epsilon^2/2)$ for all μ with some $\epsilon^2 > 0$. Then*

$$
\begin{aligned}
\bigl|\mathbb{E}\mathrm{e}^X - 1 - \mathbb{E}X - \mathbb{E}X^2/2)\bigr| &\le 2\epsilon^3\mathrm{e}^{\epsilon^2}, \\
\bigl|\mathbb{E}(X\mathrm{e}^X) - \mathbb{E}X - \mathbb{E}X^2)\bigr| &\le 5\epsilon^3\mathrm{e}^{\epsilon^2}.
\end{aligned}
$$

Proof. The first bound follows from (B.139) with $\xi \equiv 1$. Further, (B.135) for $k = 3$ implies $\mathbb{E}X^6 \le 96\epsilon^6$ and (B.138) with $\xi = X$ yields the second bound. □

Now we specify the obtained bounds for two scenarios. Let f be a function on $\mathbb{R}^p$. First we consider a symmetric 3-tensor $\mathcal{T}$ which can be viewed as third order derivative of f at some point $\boldsymbol{x}$ and define $X = \mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})$ for $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathbb{D}^{-2})$.

Lemma B.41. *Let $\mathcal{T}$ be a symmetric 3-tensor $\mathcal{T}$ satisfying* $(\boldsymbol{\mathit{\Gamma}})$ *and $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$. Consider the set $\mathcal{U}$ from Lemma B.37. Then all the statements of Lemma B.37 and Lemma B.39 continue to apply with $X = \mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})$ and $\epsilon \stackrel{\mathrm{def}}{=} 3\,\tau\,\mathtt{r}^2\|\mathbb{v}\|$. In particular, it holds for any μ and any integer k*

$$
\begin{aligned}
\mathbb{E}_{\mathcal{U}}\,\mathrm{e}^{\mu\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})} &\le \exp\bigl(\mu^2\epsilon^2/2\bigr), \\
\mathbb{E}_{\mathcal{U}}\,|\mathcal{T}(\boldsymbol{\gamma}_{\mathsf{D}})|^{2k} &\le \mathtt{C}_k^2\epsilon^{2k}, \qquad \mathtt{C}_k = 2^{k+1}k!\,.
\end{aligned}
$$

Proof. Condition $(\boldsymbol{\mathit{\Gamma}})$ ensures that the gradient of $\mathcal{T}(\mathbb{D}^{-1}\boldsymbol{u})$ is uniformly bounded by $\epsilon = 3\,\tau\,\mathtt{r}^2\|\mathbb{v}\|$ on the local set $\mathcal{U}$; see Lemma B.35. This enables the statements of Lemma B.37 and Lemma B.39. □

For the second scenario, let $\delta(\boldsymbol{u})$ be the third order remainder in the Taylor expansion of $f(\boldsymbol{x} + \boldsymbol{u})$ at a fixed point $\boldsymbol{x}$:

$$
\delta(\boldsymbol{u}) \stackrel{\mathrm{def}}{=} f(\boldsymbol{x} + \boldsymbol{u}) - f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{x}), \boldsymbol{u} \rangle - \frac{1}{2}\langle \nabla^2 f(\boldsymbol{x}), \boldsymbol{u}^{\otimes 2} \rangle \tag{B.140}
$$

and consider $\delta(\boldsymbol{\gamma}_{\mathsf{D}})$. In this case we assume that f satisfies the following condition.

$\boldsymbol{(\Gamma_3)}$ *For some* Γ *and* $\tau_3 > 0$, *it holds for any* $\boldsymbol{u} \in \mathcal{U} = \{\boldsymbol{u}\colon \|\Gamma \boldsymbol{u}\| \le \mathtt{r}\}$,

$$|\langle \nabla^3 f(\boldsymbol{x}+\boldsymbol{u}), \boldsymbol{u}_1^{\otimes 3}\rangle| \le \tau_3 \|\Gamma \boldsymbol{u}_1\|^3, \qquad \boldsymbol{u}_1 \in \mathbb{R}^p.$$

Banach's characterization Banach (1938) yields for any $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^p$

$$\bigl|\langle \nabla^3 f(\boldsymbol{x}+\boldsymbol{u}), \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3\rangle\bigr| \le \tau_3 \|\Gamma \boldsymbol{u}_1\| \, \|\Gamma \boldsymbol{u}_2\| \, \|\Gamma \boldsymbol{u}_3\| ; \tag{B.141}$$

see Lemma B.34.

Lemma B.42. *Let a function* f *satisfy* $\boldsymbol{(\Gamma_3)}$ *and* $\boldsymbol{\gamma}_{\mathsf{D}} \sim \mathcal{N}(0, \mathsf{D}^{-2})$. *Define* $\mathbb{v}$ *by* $\mathbb{v}^2 = \mathsf{D}^{-1}\Gamma^2\mathsf{D}^{-1}$. *Then all the statements of Lemma B.37 and Lemma B.39 continue to apply with* $X = \delta(\boldsymbol{\gamma}_{\mathsf{D}})$ *for* $\delta(\boldsymbol{u})$ *from* (B.140) *and* $\epsilon \stackrel{\mathrm{def}}{=} \tau_3 \mathtt{r}^2 \|\mathbb{v}\|/2$.

Proof. Define $\widetilde{\delta}(\boldsymbol{u}) = \delta(\mathsf{D}^{-1}\boldsymbol{u})$. Note that $\mathsf{D}^{-1}\boldsymbol{u} \in \mathcal{U}$ means $\|\Gamma\mathsf{D}^{-1}\boldsymbol{u}\| = \|\mathbb{v}\boldsymbol{u}\| \le \mathtt{r}$. We only have to check that condition $\boldsymbol{(\Gamma_3)}$ implies with $\epsilon = \tau_3 \mathtt{r}^2 \|\mathbb{v}\|/2$

$$\sup_{\boldsymbol{u}\colon \|\mathbb{v}\boldsymbol{u}\| \le \mathtt{r}} \|\nabla \widetilde{\delta}(\boldsymbol{u})\| = \sup_{\boldsymbol{u}\colon \|\mathbb{v}\boldsymbol{u}\| \le \mathtt{r}} \|\mathsf{D}^{-1}\nabla\delta(\boldsymbol{u})\| \le \epsilon.$$

Indeed, the Taylor expansion at $\boldsymbol{u} = 0$ yields by $\nabla\delta(0) = 0$ and $\nabla^2\delta(0) = 0$

$$\begin{aligned}
\|\mathsf{D}^{-1}\nabla\delta(\boldsymbol{u})\| &\le \sup_{\|w\|=1} \langle \nabla\delta(\boldsymbol{u}), \mathsf{D}^{-1}\boldsymbol{w}\rangle = \sup_{\|w\|=1} \langle \nabla\delta(\boldsymbol{u}) - \nabla\delta(0) - \nabla^2\delta(0)\boldsymbol{u}, \mathsf{D}^{-1}\boldsymbol{w}\rangle \\
&= \frac{1}{2} \sup_{\|w\|=1} \langle \nabla^3\delta(t\boldsymbol{u}), \boldsymbol{u} \otimes \boldsymbol{u} \otimes \mathsf{D}^{-1}\boldsymbol{w}\rangle
\end{aligned}$$

By (B.141)

$$\begin{aligned}
\|\mathsf{D}^{-1}\nabla\delta(\boldsymbol{u})\| &\le \frac{1}{2} \sup_{\|w\|=1} \langle \nabla^3\delta(t\boldsymbol{u}), \boldsymbol{u} \otimes \boldsymbol{u} \otimes \mathsf{D}^{-1}\boldsymbol{w}\rangle \le \frac{\tau_3}{2} \sup_{\|w\|=1} \|\Gamma\boldsymbol{u}\| \, \|\Gamma\boldsymbol{u}\| \, \|\Gamma\mathsf{D}^{-1}\boldsymbol{w}\| \\
&\le \frac{\tau_3}{2} \mathtt{r}^2 \sup_{\|w\|=1} \|\mathbb{v}\boldsymbol{w}\| = \frac{\tau_3}{2} \mathtt{r}^2 \, \|\mathbb{v}\|
\end{aligned}$$

as required. □

B.8 Local Laplace approximation

This section presents the bounds on the error of local Laplace approximation. Let $f(\boldsymbol{x})$ be a function in a high-dimensional Euclidean space $\mathbb{R}^p$ such that $\int \mathrm{e}^{f(\boldsymbol{x})}\, d\boldsymbol{x} = \mathtt{C} < \infty$, where the integral sign $\int$ without limits means the integral over the whole space $\mathbb{R}^p$.

Then f determines a distribution $\boldsymbol{P}_f$ with the density $\mathtt{C}^{-1}\mathrm{e}^{f(\boldsymbol{x})}$. Let $\boldsymbol{x}^*$ be a point of maximum:

$$f(\boldsymbol{x}^*) = \sup_{\boldsymbol{u}\in\mathbb{R}^p} f(\boldsymbol{x}^*+\boldsymbol{u}).$$

We also assume that $f(\cdot)$ is at least three time differentiable. Introduce the negative Hessian $\mathbb{F} = -\nabla^2 f(\boldsymbol{x}^*)$ and assume $\mathbb{F}$ strictly positive definite. Moreover, implicitly we assume that the negative Hessian $\mathbb{F} = -\nabla^2 f(\boldsymbol{x}^*)$ is sufficiently large in the sense that the Gaussian measure $\mathcal{N}(0,\mathbb{F}^{-1})$ concentrates on a small local set $\mathcal{U}$. This allows to use a local Taylor expansion for $f(\boldsymbol{x}^*;\boldsymbol{u}) \approx -\|\mathbb{F}^{1/2}\boldsymbol{u}\|^2/2$ in $\boldsymbol{u}$ on $\mathcal{U}$. For this local set $\mathcal{U}$, we evaluate the quantity

$$\diamondsuit \stackrel{\mathrm{def}}{=} \left| \frac{\int_{\mathcal{U}} \mathrm{e}^{f(\boldsymbol{x}^*+\boldsymbol{u})-f(\boldsymbol{x}^*)}\,d\boldsymbol{u} - \int_{\mathcal{U}} \mathrm{e}^{-\|\mathbb{F}^{1/2}\boldsymbol{u}\|^2/2}\,d\boldsymbol{u}}{\int \mathrm{e}^{-\|\mathbb{F}^{1/2}\boldsymbol{u}\|^2/2} d\boldsymbol{u}} \right|.$$

As $\boldsymbol{x}^* = \operatorname{argmax}_{\boldsymbol{x}} f(\boldsymbol{x})$, it holds $\nabla f(\boldsymbol{x}^*) = 0$ and

$$\diamondsuit = \left| \frac{\int_{\mathcal{U}} \mathrm{e}^{f(\boldsymbol{x}^*;\boldsymbol{u})}\,d\boldsymbol{u} - \int_{\mathcal{U}} \mathrm{e}^{-\|\mathbb{F}^{1/2}\boldsymbol{u}\|^2/2}\,d\boldsymbol{u}}{\int \mathrm{e}^{-\|\mathbb{F}^{1/2}\boldsymbol{u}\|^2/2} d\boldsymbol{u}} \right|, \tag{B.142}$$

where $f(\boldsymbol{x};\boldsymbol{u})$ is the Bregman divergence

$$f(\boldsymbol{x};\boldsymbol{u}) = f(\boldsymbol{x}+\boldsymbol{u}) - f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{x}), \boldsymbol{u} \rangle. \tag{B.143}$$

Our setup is motivated by Bayesian inference. Assume for a moment that

$$f(\boldsymbol{x}) = \ell(\boldsymbol{x}) - \|G(\boldsymbol{x}-\boldsymbol{x}_0)\|^2/2$$

for some $\boldsymbol{x}_0$ and a symmetric p-matrix $G^2 \geq 0$. Here $\ell(\cdot)$ stands for a log-likelihood function while the quadratic penalty $\|G(\boldsymbol{x}-\boldsymbol{x}_0)\|^2/2$ corresponds to a Gaussian prior $\mathcal{N}(\boldsymbol{x}_0, G^{-2})$. Let also $\mathbb{D}^2 \stackrel{\mathrm{def}}{=} -\nabla^2\ell(\boldsymbol{x}^*) > 0$. Then

$$\mathbb{F} = -\nabla^2 f(\boldsymbol{x}^*) = -\nabla^2\ell(\boldsymbol{x}^*) + G^2 = \mathbb{D}^2 + G^2. \tag{B.144}$$

With decomposition (B.144) in mind, define

$$\mathbb{D}_G^2 = \mathbb{F} = \mathbb{D}^2 + G^2, \qquad \mathbb{v}^2 \stackrel{\mathrm{def}}{=} \mathbb{D}_G^{-1}\mathbb{D}^2\mathbb{D}_G^{-1},$$

Also, given $\mathtt{r}$, define the local set $\mathcal{U}$ as

$$\mathcal{U} = \bigl\{\boldsymbol{u}\colon \|\mathbb{D}\boldsymbol{u}\| \leq \mathtt{r}\bigr\}. \tag{B.145}$$

Assume that $f(\cdot)$ be a four times continuously differentiable function on $\mathbb{R}^p$. We fix a local region around $\boldsymbol{x}^*$ given by the local set $\mathcal{U} \subset \mathbb{R}^p$ from (B.145). Consider the remainder of the second and third order Taylor approximation

$$\begin{aligned}\delta_3(\boldsymbol{u}) &= f(\boldsymbol{x}^*;\boldsymbol{u}) - \bigl\langle \nabla^2 f(\boldsymbol{x}^*), \boldsymbol{u}^{\otimes 2} \bigr\rangle/2, \\ \delta_4(\boldsymbol{u}) &= f(\boldsymbol{x}^*;\boldsymbol{u}) - \bigl\langle \nabla^2 f(\boldsymbol{x}^*), \boldsymbol{u}^{\otimes 2} \bigr\rangle/2 - \bigl\langle \nabla^3 f(\boldsymbol{x}^*), \boldsymbol{u}^{\otimes 3} \bigr\rangle/6,\end{aligned}$$

where $f(\boldsymbol{x};\boldsymbol{u})$ is given by (B.143). We will use the decomposition

$$f(\boldsymbol{x}^*;\boldsymbol{u}) = -\frac{1}{2}\|\mathbb{D}_G^{-1}\boldsymbol{u}\|^2 + \delta_3(\boldsymbol{u}) = -\frac{1}{2}\|\mathbb{D}_G^{-1}\boldsymbol{u}\|^2 + \mathcal{T}(\boldsymbol{u}) + \delta_4(\boldsymbol{u}), \tag{B.146}$$

where $\mathcal{T}(\boldsymbol{u}) = \langle \nabla^3 f(\boldsymbol{x}^*), \boldsymbol{u}^{\otimes 3} \rangle/6$ is the third order tensor corresponding to the third derivative in the fourth order Taylor expansion for $f(\boldsymbol{x}^*;\boldsymbol{u})$. For ease of notation, we skip dependence of $\mathcal{T}$, δ_3, and δ_4 on $\boldsymbol{x}^*$.

Introduce the following conditions.

$(\mathbb{D}_3^*)$ *For some $\tau_3 > 0$,*

$$\sup_{\boldsymbol{x}\colon \boldsymbol{x}-\boldsymbol{x}^* \in \mathcal{U}} |\langle \nabla^3 f(\boldsymbol{x}), \boldsymbol{u}^{\otimes 3} \rangle| \le \frac{\tau_3}{6}\,\|\mathbb{D}\boldsymbol{u}\|^3, \qquad \boldsymbol{u} \in \mathbb{R}^p .$$

$(\mathbb{D}_4)$ *For some $\tau_4 > 0$ and the local set $\mathcal{U}$ from (B.145),*

$$|\delta_4(\boldsymbol{u})| \le \frac{\tau_4}{24}\|\mathbb{D}\boldsymbol{u}\|^4, \qquad \boldsymbol{u} \in \mathcal{U}.$$

Expansion (B.146) allows to represent the error $\diamondsuit$ from (B.142) as

$$\diamondsuit = \frac{\int_{\mathcal{U}} \mathrm{e}^{f(\boldsymbol{x};\boldsymbol{u})}\,d\boldsymbol{u} - \int_{\mathcal{U}} \mathrm{e}^{-\|\mathbb{D}_G\boldsymbol{u}\|^2/2}\,d\boldsymbol{u}}{\int \mathrm{e}^{-\|\mathbb{D}_G\boldsymbol{u}\|^2/2} d\boldsymbol{u}} = \mathbb{E}_{\mathcal{U}}\Bigl[\bigl\{\exp\delta_3(\boldsymbol{\gamma}_G) - 1\bigr\}\Bigr],$$

where $\boldsymbol{\gamma}_G \sim \mathcal{N}(0, \mathbb{D}_G^{-2})$ and $\mathbb{E}_{\mathcal{U}}\,\xi$ means $\mathbb{E}\{\xi\,\mathbb{I}(\boldsymbol{\gamma}_G \in \mathcal{U})\}$.

Proposition B.43. *Assume $(\mathbb{D}_3^*)$, $(\mathbb{D}_4)$. Then with $\epsilon = \tau_3\,\mathtt{r}^2\|\mathbb{v}\|/2$, $\mathbb{v}^2 = \mathbb{D}_G^{-1}\mathbb{D}^2\,\mathbb{D}_G^{-1}$, $\sigma_G^2 = \mathbb{E}\mathcal{T}^2(\boldsymbol{\gamma}_G)$, and $\delta_{4,G} = \mathbb{E}_{\mathcal{U}}\,\delta_4^2(\boldsymbol{\gamma}_G)$, it holds*

$$\Bigl|\diamondsuit - \frac{\sigma_G^2}{2}\Bigr| \le \sigma_G\,\delta_{4,G} + \frac{\delta_{4,G}^2}{2} + \frac{5}{3}\epsilon^3\mathrm{e}^{\epsilon^2}, \quad \diamondsuit \le \frac{1}{2}(\sigma_G + \delta_{4,G})^2 + \frac{5}{3}\epsilon^3\mathrm{e}^{\epsilon^2}. \tag{B.147}$$

Moreover,

$$\delta_{4,G} \le \frac{1}{24}\,\tau_4\,\bigl\{\mathrm{tr}(\mathbb{v}^2) + 3\|\mathbb{v}^2\|\bigr\}^2, \tag{B.148}$$

$$\sigma_G \le \sqrt{\frac{5}{12}}\,\tau_3\,\|\mathbb{v}\|\;\mathrm{tr}(\mathbb{v}^2). \tag{B.149}$$

Moreover, with $\sigma_{1,G} = \mathbb{E}|\mathcal{T}(\boldsymbol{\gamma}_G)|$

$$\bigl|\mathbb{E}_{\mathcal{U}}\,\mathcal{T}(\boldsymbol{\gamma}_G)\,g(\boldsymbol{\gamma}_G)\bigr| \le \sigma_{1,G} \le \sigma_G \le \sqrt{\frac{5}{12}\,\tau_3\,\|\mathbb{v}\|\,\operatorname{tr}(\mathbb{v}^2)}\,.$$

If $g(\boldsymbol{u})$ *is centrally symmetric,* $g(\boldsymbol{u}) = g(-\boldsymbol{u})$, *then* $\mathbb{E}_{\mathcal{U}}\{\mathcal{T}(\boldsymbol{\gamma}_G)\,g(\boldsymbol{\gamma}_G)\} = 0$.

Proof. We start with a technical assertion.

Lemma B.44. *Assume* $(\mathbb{D}_3^*)$ *and* $(\mathbb{D}_4)$. *Then for any* g *with* $\sup_{\boldsymbol{u}\in\mathcal{U}}|g(\boldsymbol{u})| \le 1$,

$$\Bigl|\mathbb{E}_{\mathcal{U}}\Bigl\{\Bigl(\mathrm{e}^{\widehat{\delta}_3(\boldsymbol{\gamma}_G)} - 1 - \widehat{\delta}_3(\boldsymbol{\gamma}_G) - \frac{\widehat{\delta}_3^2(\boldsymbol{\gamma}_G)}{2}\Bigr)g(\boldsymbol{\gamma}_G)\Bigr\}\Bigr| \le \frac{5}{3}\epsilon^3\mathrm{e}^{\epsilon^2}, \tag{B.150}$$

and with $\sigma_{1,G} = \mathbb{E}_{\mathcal{U}}\,|\mathcal{T}(\boldsymbol{\gamma}_G)|$, $\sigma_G = \sqrt{\mathbb{E}_{\mathcal{U}}\,\mathcal{T}^2(\boldsymbol{\gamma}_G)}$

$$\begin{aligned}\mathbb{E}_{\mathcal{U}}\bigl|\widehat{\delta}_3(\boldsymbol{\gamma}_G) - \mathcal{T}(\boldsymbol{\gamma}_G)\bigr| &= \mathbb{E}_{\mathcal{U}}\bigl|\widehat{\delta}_4(\boldsymbol{\gamma}_G)\bigr| \le \delta_{4,G}\,,\\ \mathbb{E}_{\mathcal{U}}\bigl|\widehat{\delta}_3^2(\boldsymbol{\gamma}_G) - \mathcal{T}^2(\boldsymbol{\gamma}_G)\bigr| &\le 2\sigma_G\,\delta_{4,G} + \delta_{4,G}^2\,.\end{aligned} \tag{B.151}$$

Proof. Condition $(\mathbb{D}_3^*)$ enables us to apply Lemma B.42 with $X = \widehat{\delta}_3(\boldsymbol{\gamma}_G)$ and $k = 3$. This yields (B.150). Further, by $(\mathbb{D}_4)$ and Lemma B.1

$$\mathbb{E}_{\mathcal{U}}\,\delta_4^2(\boldsymbol{\gamma}_G) \le \frac{\tau_4^2}{24^2}\,\mathbb{E}\|\mathbb{D}\,\mathbb{D}_G^{-1}\boldsymbol{\gamma}\|^8 \le \frac{\tau_4^2}{24^2}\bigl\{\operatorname{tr}(\mathbb{v}^2) + 3\|\mathbb{v}^2\|\bigr\}^4$$

and (B.148) follows. As $\widehat{\delta}_3(\boldsymbol{\gamma}_G) = \mathcal{T}(\boldsymbol{\gamma}_G) + \widehat{\delta}_4(\boldsymbol{\gamma}_G)$, it holds

$$\mathbb{E}_{\mathcal{U}}\,|\widehat{\delta}_3(\boldsymbol{\gamma}_G) - \mathcal{T}(\boldsymbol{\gamma}_G)| = \mathbb{E}_{\mathcal{U}}\,|\widehat{\delta}_4(\boldsymbol{\gamma}_G)| \le \sqrt{\mathbb{E}_{\mathcal{U}}\,\widehat{\delta}_4^2(\boldsymbol{\gamma}_G)} \le \sqrt{\mathbb{E}_{\mathcal{U}}\,\delta_4^2(\boldsymbol{\gamma}_G)}\,,$$

and by $\widehat{\delta}_4(\boldsymbol{\gamma}_G) = \delta_4(\boldsymbol{\gamma}_G) - \mathbb{E}_{\mathcal{U}}\,\delta_4(\boldsymbol{\gamma}_G)$

$$\begin{aligned}\mathbb{E}_{\mathcal{U}}\bigl|\widehat{\delta}_3^2(\boldsymbol{\gamma}_G) - \mathcal{T}^2(\boldsymbol{\gamma}_G)\bigr| &\le 2\mathbb{E}_{\mathcal{U}}\bigl|\widehat{\delta}_4(\boldsymbol{\gamma}_G)\,\mathcal{T}(\boldsymbol{\gamma}_G)\bigr| + \mathbb{E}_{\mathcal{U}}\,\widehat{\delta}_4^2(\boldsymbol{\gamma}_G)\\ &\le 2\sqrt{\mathbb{E}\mathcal{T}^2(\boldsymbol{\gamma}_G)\,\mathbb{E}_{\mathcal{U}}\,\widehat{\delta}_4^2(\boldsymbol{\gamma}_G)} + \mathbb{E}_{\mathcal{U}}\,\widehat{\delta}_4^2(\boldsymbol{\gamma}_G) \le 2\sigma_G\,\delta_{4,G} + \delta_{4,G}^2\,,\end{aligned}$$

and (B.151) follows as well. □

Now we are prepared to finalize the proof of the proposition. As $\mathbb{E}_{\mathcal{U}}\,\widehat{\delta}_3(\boldsymbol{\gamma}_G) = 0$, (B.150) with $g(\cdot) \equiv 1$ and (B.151) imply (B.147). The use of (B.130) from Lemma B.36 with $\tau = \tau_3/6$ yields

$$\mathbb{E}_{\mathcal{U}}\,|\mathcal{T}(\boldsymbol{\gamma}_G)| \le \sqrt{\mathbb{E}_{\mathcal{U}}\,\mathcal{T}^2(\boldsymbol{\gamma}_G)} \le \frac{1}{6}\sqrt{15\tau_3^2\,\|\mathbb{v}\|^2\,\operatorname{tr}^2(\mathbb{v}^2)}$$

and (B.149) follows as well. □

B.9 Deviation bounds for Bernoulli vector sums

Let Y_i be independent $\mathrm{Bernoulli}(\theta_i^*)$, $i = 1, \ldots, n$. We denote $\boldsymbol{Y} = (Y_i) \in \mathbb{R}^n$. Weighted sums of the Y_i naturally appear in various statistical tasks including classification, binary response models, logistic regression etc. Recent applications include e.g. stochastic block modeling; see e.g. Gao et al. (2017), Abbe (2018) and references therein, or ranking from pairwise comparison Chen et al. (2022) among many others. We show how the general bounds of Section B.4 can be used for vector sums of Bernoulli r.v.s. For a linear mapping $\boldsymbol{\Psi} \colon \mathbb{R}^n \to \mathbb{R}^p$, define $\boldsymbol{\xi} = \boldsymbol{\Psi}(\boldsymbol{Y} - \mathbb{E}\boldsymbol{Y})$. Below we state some deviation bounds on the squared norm $\|\boldsymbol{\xi}\|^2$ starting from the univariate case.

B.9.1 Weighted sums of Bernoulli r.v.'s: univariate case

Given a collections of weights (w_i), define

$$
\begin{aligned}
S &= \sum_{i=1}^n Y_i w_i \,, \\
V^2 &= \mathrm{Var}(S) = \sum_{i=1}^n \theta_i^* (1 - \theta_i^*) w_i^2 \,, \\
w^* &= \max_i |w_i|.
\end{aligned}
$$

First we state a deviation bound for a centered sum $S - \mathbb{E}S$.

Proposition B.45. *Let Y_i be independent* $\mathrm{Bernoulli}(\theta_i^*)$ *and $w_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then $S = \sum_{i=1}^n Y_i w_i$ satisfies*

$$
\log \mathbb{E} \exp\Bigl\{ \frac{\lambda (S - \mathbb{E}S)}{V} \Bigr\} \le \lambda^2, \qquad \lambda \le \frac{\log(2) V}{w^*} \,. \tag{B.152}
$$

Furthermore, suppose that given $\mathtt{x} \ge 0$,

$$
V \ge \frac{3}{2} w^* \sqrt{\mathtt{x}} \,. \tag{B.153}
$$

Then

$$
\mathbb{P}\bigl(V^{-1} |S - \mathbb{E}S| \ge 2\sqrt{\mathtt{x}} \bigr) \le 2\mathrm{e}^{-\mathtt{x}}. \tag{B.154}
$$

Without (B.153), the bound (B.154) applies with V replaced by $V_{\mathtt{x}} = V \vee (3\, w^* \sqrt{\mathtt{x}} \,/2)$.

Proof. Without loss of generality assume $w^* = 1$, otherwise just rescale all the weights by the factor $1/w^*$. We use that

$$f(u) \stackrel{\text{def}}{=} \log \mathbb{E} \exp\Bigl\{u(S - \mathbb{E}S)\Bigr\} = \sum_{i=1}^{N} \Bigl[\log\bigl(\theta_i^* \mathrm{e}^{uw_i} + 1 - \theta_i^*\bigr) - uw_i\theta_i^*\Bigr].$$

This is an analytic function of u for $|u| \le \log 2$ satisfying $f(0) = 0$, $f'(0) = 0$, and, with $\upsilon_i^* = \log\theta_i^* - \log(1 - \theta_i^*)$,

$$f''(u) = \sum_{i=1}^{N} \frac{w_i^2\, \theta_i^*(1-\theta_i^*)\, \mathrm{e}^{uw_i}}{(\theta_i^* \mathrm{e}^{uw_i} + 1 - \theta_i^*)^2} = \sum_{i=1}^{N} \frac{w_i^2\, \mathrm{e}^{\upsilon_i^* + uw_i}}{(\mathrm{e}^{\upsilon_i^* + uw_i} + 1)^2} = \sum_{i=1}^{N} \theta_i(u)\bigl\{1 - \theta_i(u)\bigr\} w_i^2$$

for $\theta_i(u) = \mathrm{e}^{\upsilon_i^* + uw_i}/(\mathrm{e}^{\upsilon_i^* + uw_i} + 1)$. Clearly $\theta_i(u)$ and thus, $\theta_i(u)\bigl\{1 - \theta_i(u)\bigr\}$ monotonously increases with u and it holds for $\theta_i^* = \theta_i(0)$

$$\theta_i(u)\bigl\{1 - \theta_i(u)\bigr\} \le \mathrm{e}^{|u|}\, \theta_i^*\, (1 - \theta_i^*) \le 2\, \theta_i^*\, (1 - \theta_i^*), \qquad |u| \le \log 2.$$

This yields

$$f(u) \le V^2 u^2 \qquad |u| \le \log 2.$$

As $\mathtt{x} \le 4V^2/9$, the value $\lambda = \sqrt{\mathtt{x}}$ fulfills $\lambda/V = \sqrt{\mathtt{x}}/V \le \log 2 \le 2^{-1/2}$. Now by the exponential Chebyshev inequality

$$\begin{aligned} \mathbb{P}\Bigl(V^{-1}(S - \mathbb{E}S) \ge 2\sqrt{\mathtt{x}}\Bigr) &\le \exp\bigl\{-2\lambda\sqrt{\mathtt{x}} + f(\lambda/V)\bigr\} \\ &\le \exp\bigl(-2\lambda\sqrt{\mathtt{x}} + \lambda^2\bigr) = \mathrm{e}^{-\mathtt{x}}. \end{aligned}$$

Similarly one can bound $\mathbb{E}S - S$. □

B.9.2 Deviation bounds for Bernoulli vector sums

Now we present an upper bound on the norm of a vector $\boldsymbol{\xi} = \boldsymbol{\Psi}(\boldsymbol{Y} - \mathbb{E}\boldsymbol{Y})$, where $\boldsymbol{\Psi}$ is a linear mapping $\boldsymbol{\Psi}\colon \mathbb{R}^n \to \mathbb{R}^p$. It holds

$$\operatorname{Var}(\boldsymbol{\xi}) = \operatorname{Var}(\boldsymbol{\Psi}\boldsymbol{Y}) = \boldsymbol{\Psi} \operatorname{Var}(\boldsymbol{Y}) \boldsymbol{\Psi}^\top.$$

We aim at bounding the squared norm $\|Q\boldsymbol{\xi}\|^2$ for another linear mapping $Q\colon \mathbb{R}^p \to \mathbb{R}^q$.

Theorem B.46. *Let* $Y_i \sim \operatorname{Bernoulli}(\theta_i^*)$, $i = 1, \ldots, n$. *Consider* $\boldsymbol{\xi} = \boldsymbol{\Psi}(\boldsymbol{Y} - \mathbb{E}\boldsymbol{Y})$, *and let* $\mathbb{V}^2 \ge 2 \operatorname{Var}(\boldsymbol{\xi})$. *Define*

$$w^* = \max_{i \le n} \| \mathbb{V}^{-1} \boldsymbol{\Psi}_i \|, \qquad \mathtt{g} = \log(2)/w^*.$$

Then with $B = Q\mathbb{V}^2 Q^\top$ *and* $z_c(B,\mathtt{x})$ *from* (B.56), *it holds*

$$\mathbb{P}\bigl(\|Q\boldsymbol{\xi}\| \geq z_c(B,\mathtt{x})\bigr) \leq 3\mathrm{e}^{-\mathtt{x}}.$$

Proof. We apply the general result of Corollary B.17 under conditions (B.46). For any vector $\boldsymbol{u}$, consider the scalar product $\langle \mathbb{V}^{-1}\boldsymbol{\xi}, \boldsymbol{u}\rangle = \langle \mathbb{V}^{-1}\boldsymbol{\Psi}(\boldsymbol{Y} - \mathbb{E}\boldsymbol{Y}), \boldsymbol{u}\rangle$. It is obviously a weighted centered sum of the Bernoulli r.v.'s $Y_i - \theta_i^*$ with

$$\operatorname{Var}\langle \mathbb{V}^{-1}\boldsymbol{\xi}, \boldsymbol{u}\rangle \leq \|\boldsymbol{u}\|^2/2.$$

One can write with $\varepsilon_i = Y_i - \theta_i^*$ and $\boldsymbol{\varepsilon} = (\varepsilon_i)$

$$\langle \mathbb{V}^{-1}\boldsymbol{\xi}, \boldsymbol{u}\rangle = \bigl\langle \boldsymbol{\varepsilon}, \boldsymbol{\Psi}^\top \mathbb{V}^{-1}\boldsymbol{u}\bigr\rangle.$$

By the Cauchy-Schwarz inequality, it holds

$$\|\boldsymbol{\Psi}^\top \mathbb{V}^{-1}\boldsymbol{u}\|_\infty = \max_i \bigl|(\mathbb{V}^{-1}\boldsymbol{\Psi}_i)^\top \boldsymbol{u}\bigr| \leq w^* \|\boldsymbol{u}\|.$$

Bound (B.152) of Proposition B.45 on the exponential moments of $\langle \mathbb{V}^{-1}\boldsymbol{\xi}, \boldsymbol{u}\rangle$ implies

$$\log \mathbb{E} \exp\bigl\{\langle \mathbb{V}^{-1}\boldsymbol{\xi}, \boldsymbol{u}\rangle\bigr\} \leq \|\boldsymbol{u}\|^2/2, \qquad \|\boldsymbol{u}\| \leq \log(2)/w^* .$$

Therefore, (B.46) is fulfilled with $\mathtt{g} = \log(2)/w^*$. The deviation bound (B.55) of Corollary B.17 yields the assertion. $\square$

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