Colored Stochastic Multiplicative Processes with Additive Noise Unveil a Third-Order PDE, Defying Conventional FPE and Fick-Law Paradigms.

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Research on stochastic differential equations (SDE) involving both additive and multiplicative noise has been extensive. In situations where the primary process is driven by a multiplicative stochastic process, additive white noise typically represents an intrinsic and unavoidable fast factor, including phenomena like thermal fluctuations, inherent uncertainties in measurement processes, or rapid wind forcing in ocean dynamics. This work focuses on a significant class of such systems, particularly those characterized by linear drift and multiplicative noise, extensively explored in the literature. Conventionally, multiplicative stochastic processes are also treated as white noise in existing studies. However, when considering colored multiplicative noise, the emphasis has been on characterizing the far tails of the probability density function (PDF), regardless of the spectral properties of the noise. In the absence of additive noise and with a general colored multiplicative SDE, standard perturbation approaches lead to a second-order PDE known as the Fokker-Planck Equation (FPE), consistent with Fick's law. This investigation unveils a notable departure from this standard behavior when introducing additive white noise. At the leading order of the stochastic process strength, perturbation approaches yield a *third-order PDE*, irrespective of the white noise intensity. The breakdown of the FPE further signifies the breakdown of Fick's law. Additionally, we derive the explicit solution for the equilibrium PDF corresponding to this third-order PDE Master Equation. Through numerical simulations, we demonstrate significant deviations from outcomes derived using the FPE obtained through the application of Fick's law

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I. INTRODUCTION

Linear equations forced by both additive and multiplicative noises are prevalent in almost every scientific discipline. In the general N-dimensional case (N-D), these equations reads

$$\dot{\boldsymbol{x}} = -\mathbb{E} \cdot \boldsymbol{x} + \boldsymbol{f}(t) - \Xi(t) \cdot \boldsymbol{x} \tag{1}$$

where $\boldsymbol{x} := (x_1, \ldots, x_N)$, $\mathbb{E}$ and $\Xi(t)$ are $N \times N$ matrices with constant and stochastic components, respectively; $\boldsymbol{f}(t)$ is a multidimensional white noise with correlation, or diffusion matrix given by $\mathbb{D}$. As shown in[1], the extension to an infinite (or continuous) vector space of (1), leads to a general model that describes a large class of important physical phenomena in fluid dynamics and in quantum mechanics. More in general, the model (1) represents a random multiplicative process (RMP), a well-known mechanism that gives rise to power-law behaviors. Widely employed as a model in various systems with both discrete and continuous time, the RMP has been applied to phenomena such as on-off intermittency[2–6] and general intermittency (see Fig. 1) with power law statistics[7,8], lasers[9,10], economic activity[11,12], fluctuations in biological populations within changing environments[13], and the advection of passive scalar fields by fluids[1,14]. It is clearly a paradigmatic model for theories on large fluctuations (e.g.,[15] and references therein).

Therefore, the significance of the model (1) cannot be overstated.

For simplicity, this work focuses on the 1-D version of the model (1):

$$\dot{x} = -\gamma x + f(t) - \epsilon\, x\xi(t), \tag{2}$$

which is the primary focus of most of the literature cited above. The extension to the N-D case (1) of the formal results is straightforward yet somewhat intricate, as detailed in Appendix A.

Thus, in (2) $f(t)$ is a white noise with diffusion coefficient D_f, $\xi(t)$ is a Gaussian stochastic process with zero average, finite correlation time $\bar{\tau}$[16] and normalized autocorrelation function $\varphi(t) = \langle \xi(t)\xi(0)\rangle_\xi / \langle \xi^2 \rangle_\xi$. We use the notation $\langle ... \rangle_\xi$ to indicate the average over the realizations of the random process $\xi(t)$, which is assumed at equilibrium. We also define $\tau := \int_0^\infty \varphi(u) \mathrm{d}u$. The value of this integral can be much smaller than the decorrelation time $\bar{\tau}$, in those cases when the function $\varphi(u)$ decays oscillating with time. Without loss of generality, we assume that $\langle \xi^2 \rangle_\xi = 1$. Thus, the intensity of the fluctuations in the stochastic perturbation is governed by ϵ. However, as demonstrated in Appendix B, the effective adimensional perturbation strength is measured by the parameter $\delta := \epsilon\tau$. Strictly speaking, a more appropriate definition of δ should be $\delta = \epsilon\bar{\tau}$. This is because the real parameter that measures the perturbation strength in the approach illustrated in Appendix B, resulting in a series of cumulants, involves the correlation parameter $\bar{\tau}$ (as properly defined in note[16]) instead of τ. Thus, for a more rigorous treatment, we should replace δ with $\delta\tau/\bar{\tau}$ in all the analytical results presented hereafter. However, to avoid complicating the formal expressions, we have chosen to stick with the current definition of δ.

The drift field $-\gamma x$ in the SDE (2) can also be interpreted as originating from the same multiplicative stochastic process, when its average is equal to $-\gamma/\epsilon$. If $x(t)$ is intended as the velocity of a Brownian particle, the SDE (2) has the important characteristic that it can be considered as a continuous process realization of Lévy flights[17] for some parameter range.

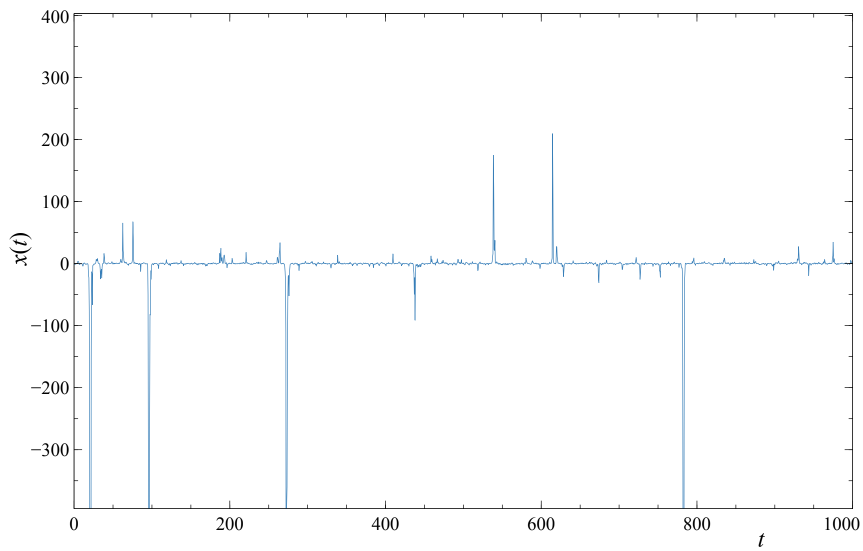

FIG. 1. A representative example, illustrating the intermittent behavior, is depicted in the time evolution of the amplitude $x(t)$ for the SDE (2) with parameters $\tau = 0.5$, $\epsilon = 5.0$, $\gamma = 2.0$, $D_f = 0.5$.

We will refer to $f(t)$ in (2) as the internal or intrinsic noise. This terminology is apt as it typically originates from intrinsic and unavoidable factors such as thermal fluctuations, inherent uncertainty in measurement processes, or rapid wind forcing in the context of ocean dynamics, among other possibilities.

As mentioned, the stochastic differential equation (SDE) (2) has been extensively studied in the scientific literature. However, in almost all of these works, besides the additive noise

$f(t)$, the stochastic process $\xi(t)$ has been considered as white noise. In cases where a colored stochastic process $\xi(t)$ has been considered, as in[1], the focus of the work has been on characterizing the far tail of the probability density function (PDF) of x, which, as we will see hereafter, does not depend on the spectral properties of the multiplicative noise.

To the best of our knowledge, there are no papers that obtain a simple closed simple form (i.e., not a formal result with infinite series of operators) for the equation of the PDF in all its support range and the corresponding equilibrium solution. We will remedy this gap, focusing on the case in which the δ parameter is small, and we will find surprisingly simple results, but not fitting either the Fokker-Planck equation (FPE) structure or Fick's law.

For an easy start, let us first assume that the intrinsic noise is absent.(i.e., $f(t) = 0$). In this case it is easy to show that, *regardless of the values of τ and ϵ*, the Master Equation (ME) for the PDF of x in (2) coincides with the following FPE (see Appendix B for simplicity, we use the shorthand $\partial_y := \partial/\partial y$):

$$\partial_t P(x;t) = \left\{ \gamma \partial_x x + \frac{\delta^2}{\tau} \partial_x x \partial_x \, x \right\} P(x;t). \tag{3}$$

This fact indicates that the process (2), with $D_f = 0$, does not depend on the spectrum (or color) features of the stochastic process $\xi(t)$. Consistently, in the white noise limit, i.e., for $\tau \to 0$ and $\delta^2/\tau = \epsilon^2 \tau$ held constant, the FPE (3) remains unaltered and corresponds to the standard FPE for SDEs with multiplicative white noise, under the Stratonovich interpretation of Wiener process differentials.

The FPE (3) can also be rewritten as a conservative equation as

$$\partial_t P(x;t) = \partial_x J(x) \tag{4}$$

with

$$J(x) := \left\{ (\gamma\tau + \delta^2) x/\tau + D_\xi(x) \partial_x \right\} P(x;t). \tag{5}$$

where we have introduced the inhomogeneous diffusion coefficient,

$$D_\xi(x) := \delta\epsilon x^2 = \delta^2 x^2/\tau, \tag{6}$$

characteristic of multiplicative noise. The interpretation of the terms appearing in Eq. (5) is straightforward: in absence of internal noise ($D_f = 0$ in (2)), the multiplicative stochastic process generates an additional friction/drift term proportional to the intensity of the stochastic perturbation and an inhomogeneous diffusion process, proportional to the gradient of the PDF (thus, following Fick's law). The equilibrium PDF of (4) is obtained by setting $J(x) = 0$ in (5), yielding $P(x)_{eq} \propto x^{-\left(1+\frac{\gamma\tau}{\delta^2}\right)}$, showcasing a singular behaviour (it is non-integrable) around $x = 0$[16].

The introduction of an internal diffusion source effectively addresses this issue and is physically plausible for many realistic models. In fact, by leveraging Fick's law and including the standard, constant diffusion coefficient D_f in the current (5), we have

$$J(x) := \left\{(\gamma\tau + \delta^2)x/\tau + (D_\xi(x) + D_f)\partial_x\right\} P(x;t). \tag{7}$$

The equilibrium PDF, obtained by setting $J(x) = 0$, is $P(x)_{eq} \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$. which no longer displays a singular behavior around $x = 0$.

Note that for $D_\xi(x) \gg D_f$, i.e., for $x \gg \sqrt{D_f\tau/\delta^2}$, it behaves similarly to the previous noiseless case. Thus, if there are no boundary conditions that constrain the variable x to a finite range, the existence condition for the moments of x remains unaltered by the introduction of this diffusive term.

We can say that the white noise $f(t)$, corresponding to a diffusion process with a diffusion coefficient D_f, introduces a repulsion from the origin, preventing any path from getting trapped at the $x = 0$ point once reached. While the introduction of such intrinsic noise in multiplicative processes has been acknowledged by many researchers (see the same works already cited above), however it has been not highlighted the fact that even though the two fluctuating processes are assumed independent of each other, in effect their contributions to the current $J(x)$ of (4) *don't simply add up*, unless the multiplicative process is a white noise too. More precisely, in this work we will show that we have

$$J(x) = \left\{(\gamma\tau + \delta^2)x/\tau + (D_\xi(x) + D_f)\partial_x + D_f D_\xi(x)\vartheta\partial_x^2\right\} P(x;t) \tag{8}$$

with ϑ, given in (20), having the dimension of time and coinciding with 2τ for $\gamma\tau \ll 1$ and with γ^{-1} for $\gamma\tau \gg 1$. Thus, in this case *the Fick's law and the corresponding FPE structure break down.*

The last term on the right-hand side of (8), which invalidates Fick's law, arises from two factors: the finite time scale of the external stochastic perturbation (colored noise) and the non-commutativity of its Liouvillian with the Liouvillian associated with the internal noise (the standard diffusion operator). Specifically, when averaging over the external stochastic process, the expansion in cumulants of the PDF coincides with a power series of the adimensional parameter δ (see Appendix B). In the second order (which is the leading one for weak perturbations), these two factors yield, in the ME of the PDF of x, a correction to the FPE obtained by applying Fick's law. This correction is proportional to both δ and D_f and turns out to be a *third-order partial differential operator on x*.

This result is confirmed by the numerical simulations reported in Section III. We will undertake a detailed derivation of this phenomenon in the next session. However, it's crucial to highlight that this departure from the standard FPE/Fick's law is a general observation, applicable beyond the linear drift case of (1), and always occurs when both additive noise and multiplicative colored stochastic processes are present. For simplicity, here we focus on the linear 1D case of (2), while a more in-depth exploration of these findings will be presented in future works.

II. A THIRD ORDER PDE FOR THE PDF

Given the infinitely short time correlation of the additive noise $f(t)$, to any realization $\xi(\cdot)$ of the stochastic process $\xi(u)$, $0 \leq u \leq t$, from the SDE (2) we can write the following Liouville equation for the PDF of x: $P_{\xi(\cdot)}(x,t)$:

$$\begin{aligned}&\partial_t P_{\xi(\cdot)}(x,t)\\&= \{\mathcal{L}_a + \epsilon\,\xi(t)\mathcal{L}_I\} P_{\xi(\cdot)}(x,t),\end{aligned} \tag{9}$$

in which $\mathcal{L}_a$ is the unperturbed Liouville operator given by

$$\mathcal{L}_a := \gamma\partial_x x + D_f\partial_x^2; \tag{10}$$

and $\epsilon\,\xi(t)\mathcal{L}_I$ is the Liouville perturbation operator with:

$$\mathcal{L}_I := \partial_x x. \tag{11}$$

If the perturbing process $\epsilon\xi(t)$ is weak (characterized by small values of the δ parameter), applying a perturbation projection/cumulant approach[19–29] to Eq. (9), at the leading order of δ, we formally obtain the following standard result for the reduced PDF of x (see see Appendix B for details, note that hereafter $P(x;t) := \langle P_{\xi(\cdot)}(x,t)\rangle_\xi$, where $\langle ... \rangle_\xi$ is the average over the realizations of the stochastic process $\xi(t)$; throughout this work we will consider $t \gg \tau$):

$$\partial_t P(x;t) = \mathcal{L}_a P(x;t) + \frac{\delta^2}{\tau^2}\mathcal{L}_I \int_0^\infty \mathrm{d}u\,\varphi(u)\,\tilde{\mathcal{L}}_I(-u)\; P(x;t), \tag{12}$$

where $\varphi(t)$ is the normalized autocorrelation function of $\xi(t)$, as defined in the Introduction, and

$$\tilde{\mathcal{L}}_I(t) := e^{-\mathcal{L}_a t}\mathcal{L}_I e^{\mathcal{L}_a t} \tag{13}$$

is the interaction representation of the perturbing Liouvillian $\mathcal{L}_I$. By exploiting the Hadamard's lemma for exponentials of operators we can also write

$$\tilde{\mathcal{L}}_I(t) = e^{-\mathcal{L}_a^\times t}[\mathcal{L}_I] \tag{14}$$

in which, for any couple of operators $\mathcal{A}$ and $\mathcal{B}$, we have defined $\mathcal{A}^\times[\mathcal{B}] := [\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$. In literature (e.g.[30]), $e^{\mathcal{A}^\times t}[\mathcal{B}]$ is called the Lie evolution of the operator $\mathcal{B}$ along $\mathcal{A}$, for a time t.

Because the perturbing Liouvillian $\mathcal{L}_I$ of (11) is a first order differential operator, the order of the differential operator corresponding to the second addend in the r.h.s. of (12) is obtained by adding to one the order of differential operator of $\tilde{\mathcal{L}}_I(-u)$. From Eq. (14), we see that this latter is the result of the Lie evolution of $\mathcal{L}_I$ along the unperturbed Liouvillian $\mathcal{L}_a$.

If the decay time of $\varphi(u)$ is significantly shorter than $1/\gamma$, we can safely assume the approximation $\tilde{\mathcal{L}}_I(-u) \approx \mathcal{L}_I$ inside the integral on the r.h.s. of (12). Consequently, the ME (12) effectively reduces to a FPE. However, when this is not the case, we must address the challenge of evaluating the full Lie evolution of $\mathcal{L}_I$ along $\mathcal{L}_a$. In-depth exploration of this topic, from a formal and general perspective, can be found in[30]. Specifically, Proposition 1 in[30] is of particular relevance. For the simple 1-D case with linear drift, corresponding to the present SDE (2), we can derive the Lie evolution of $\mathcal{L}_I$ along the unperturbed Liouvillian $\mathcal{L}_a$ in (14) as follows. From (14) we have

$$\frac{d}{dt}\tilde{\mathcal{L}}_I(t) = -\mathcal{L}_a^\times\left[e^{-\mathcal{L}_a^\times t}[\mathcal{L}_I]\right] = -e^{-\mathcal{L}_a^\times t}\left[[\mathcal{L}_a,\mathcal{L}_I]\right]. \tag{15}$$

By using (10) and (11), we get

$$[\mathcal{L}_a,\mathcal{L}_I] = [\gamma\partial_x x + D_f\partial_x^2\,,\,\partial_x x] = 2D_f\partial_x^2 \tag{16}$$

thus, Eq. (15) can be written as

$$
\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{L}}_I(t) =& -2e^{-\mathcal{L}_a^{\times}t}\left[D_f\partial_x^2\right] \\
=& -2e^{-\mathcal{L}_a^{\times}t}\left[\gamma\partial_x x + D_f\partial_x^2 - \gamma\partial_x x\right] = -2(\gamma\partial_x x + D_f\partial_x^2) + 2\gamma\tilde{\mathcal{L}}_I(t)
\end{aligned}
\tag{17}
$$

of which the solution is

$$
\tilde{\mathcal{L}}_I(t) = \partial_x\, x + D_f\frac{1-e^{2\gamma t}}{\gamma}\partial_x^2. \tag{18}
$$

By using Eq. (18) into the ME (12), and exploiting again (10) and (11), we finally obtain

$$
\begin{aligned}
\partial_t P(x;t) =& \left\{\gamma\partial_x\, x + D_f\partial_x^2 + \frac{\delta^2}{\tau}\partial_x x\partial_x x + D_f\delta^2\frac{\vartheta}{\tau}\partial_x\, x\partial_x^2\right\}P(x;t) \\
=& \,\partial_x J(x)
\end{aligned}
\tag{19}
$$

with $J(x)$ is given in (8) and the time ϑ defined as

$$
\vartheta := \frac{1}{\gamma\tau}\left(\tau - \hat{\varphi}(2\gamma)\right). \tag{20}
$$

The hat over a function indicates its Laplace transform: $\hat{\varphi}(s) := \int_0^\infty \mathrm{d}u\, \varphi(u)e^{-su}$. The third order PDE (19) with (20) is the main result of the present work. At the leading order in powers of the δ parameter, Eq. (19) is exact, irrespective of the value of the diffusion coefficient D_f.

Thus, upon introducing the internal noise, alongside the standard diffusion process, *an additional mutual contribution is activated.* As we can observe from (19), this mutual contribution of the white internal noise and the external multiplicative stochastic process takes on an odd nature in terms of partial derivatives. As previously emphasized in the Introduction, we reiterate that the time parameter ϑ of (20) coincides with 2τ for $\gamma\tau \ll 1$ and with γ^{-1} for $\gamma\tau \gg 1$. Consequently, the adimensional parameter $r := \epsilon\vartheta$ is akin the δ parameter, but is rescaled based on the time scale relationship between the stochastic process and the unperturbed dynamics.

Imposing the equilibrium condition to the ME (19), i.e., setting $J(x) = 0$, we obtain two different analytical solutions, both involving the Kummer confluent hypergeometric function of first kind[18]:

$$
\begin{aligned}
P_1(x) =& \,{}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2}+1\right); \frac{1}{2}\left(\frac{1}{\delta r}+1\right); -\frac{x^2}{2D_f\vartheta}\right) \\
P_2(x) =& \,2^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} r^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} D_f^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} x^{1-\frac{1}{\delta r}} \\
&\times\, {}_1F_1\left(\frac{r\left(2\delta\epsilon+\gamma\right)-1}{2\delta r}; \frac{3}{2}-\frac{1}{2\delta r}; -\frac{x^2}{2D_f\vartheta}\right).
\end{aligned}
\tag{21}
$$

This fact is due to the third order nature of the PDE (19). From a mathematical point of view any linear combination of these two functions is also a possible solution. However, it is easy to show that the second one is not physically acceptable. In fact, let us consider the behaviour of these two functions around $x = 0$. We have

$$P_1(x) \approx 1 - \frac{x^2\left[\gamma\tau + \delta^2\right]}{(2D_f\tau)(\delta r + 1)} + O\left(x^3\right) \tag{22}$$

$$P_2(x) \approx 2^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} r^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} D_f^{\frac{1}{2}\left(\frac{1}{\delta r}-1\right)} x^{1-\frac{1}{\delta r}} + O\left(x^3\right). \tag{23}$$

We see that if $R := \delta r < 1$, a condition which is typically met, the solution $P_2(x)$ is not integrable, therefore it must be discarded. The expression of the function $P_1(x)$ in (22) implies that the presence of $r > 0$ smears the equilibrium PDF around $x = 0$.

Thus, the final result is given by

$$P_{eq}(x) \propto {}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2}+1\right); \frac{1}{2}\left(\frac{1}{\delta r}+1\right); -\frac{x^2}{2D_f\vartheta}\right). \tag{24}$$

In the case in which the support of the PDF is not limited (for example, if there are not reflecting boundary conditions at some finite values of x), we can evaluate the asymptotic behavior as $x \to \pm\infty$ of the equilibrium PDF in (24), and the result is $P_{eq}(x) \sim |x|^{-\left(\frac{\gamma\tau}{\delta^2}+1\right)}$. From this expression we observe that even when considering the contribution from the third partial derivative, the far tails of the equilibrium PDF of x remain unaffected by the presence of the additive white noise $f(t)$. This implies that, for the case of infinite support of the PDF, the condition for the existence of the n-th moment of x depends only on the fraction $\gamma\tau/(\delta^2)$, i.e., it remains independent of the spectral properties of $\xi(t)$. Still in the case of an unbounded domain for PDF, from the PDE (19) it is possible to obtain the following time differential equation for the moment of order n of x:

$$\begin{aligned}\partial_t\langle x^n\rangle = &- n\gamma\langle x^n\rangle\left(1 - n\delta^2/(\gamma\tau)\right)\\ &+ n(n-1)D_f\left(1 - n\delta r\right)\langle x^{n-2}\rangle.\end{aligned} \tag{25}$$

For any fixed n, Eq. (25) is a linear relationship between the first n moments of x. The eigenvalues of the corresponding matrix are $-n\gamma\left(1 - n\delta^2/(\gamma\tau)\right)$, thus, they do not depend on D_f and r. Therefore, the relaxation behaviour of the moments is independent on D_f and r as well and they exist only if $\left(1 - n\delta^2/(\gamma\tau)\right) > 0$. On the other hand, it is clear from the same Eq. (25) that the equilibrium values of the moments (if they exist) do depend on D_f, and also on the value of $R := \delta r$. When it exists, the equilibrium solution of Eq. (25) is given by

$$\langle x^n \rangle_{eq} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \left(\frac{D_f}{\gamma}\right)^{n/2} (n-1)!! \prod_{j=1}^{n/2} \frac{(1-2j\,\delta r)}{(1-2j\,\delta^2/(\gamma\tau))} & \text{for } n \text{ even} \end{cases} \tag{26}$$

From (20) we always have $\vartheta < 1/\gamma$, from which $(1 - 2j\delta^2/(\gamma\tau)) > (1 - 2j\,\delta r)$. Therefore, if $(1 - n\delta^2/(\gamma\tau)) > 0$ (as it must be for the n-th moment do exist), then un-physical situation of an even moment smaller than zero is incompatible with Eq. (26).

This observation appears to contradict both the earlier findings that around $x = 0$ the equilibrium PDF broadens with increasing R and that the far tails of this PDF do not depend on R. The explanation of this apparent contradiction lies in the fact as we move away from the origin, where the expansion (22) holds, but before we arrive at the asymptotic tail, the equilibrium PDF decays more quickly as function of x due to the presence of $R > 0$. This fact is easily confirmed by plotting the equilibrium PDF with $R = 0$ and $R \neq 0$ (see next section).

If there is a very large time scale separation, i.e. $\gamma\tau \ll 1$, from (20) we have

$$R = \delta r \approx \epsilon^2 \int_0^\infty \mathrm{d}u\, \varphi(u) u \approx 2(\epsilon\tau)^2 = 2\delta^2, \tag{27}$$

that does not depend on γ. The reader should note that in the white noise limit, i.e., as τ approaches zero while keeping $\epsilon^2\tau = \delta\epsilon$ fixed, R in (27) tends to zero. Consequently, the non-Fick contribution to J becomes negligible as well. Conversely, if δ, the relevant parameter for the cumulant series, is held fixed (small enough to allow truncating the series to the second cumulant), while changing the time scale of the noise, R in (27) remains constant.

In essence, with a small strength of the stochastic process $\xi(t)$ (while it diverges in the white noise limit), the breakdown of Fick's law and of the associated FPE for the model (2) persists, resulting in a ME with a third derivative term.

III. THE CASE OF ORNSTEIN UHLENBECK EXTERNAL STOCHASTIC PROCESS: ANALYTICAL AND NUMERICAL RESULTS

To explore the full range of τ and γ, we consider the specific case of exponentially decaying correlation function: $\varphi(u) = \exp(-u/\tau)$, from which, by also exploiting (20), we have:

$$\vartheta = \frac{2\tau}{(2\gamma\tau + 1)} \tag{28}$$

i.e.,

$$R := \delta r = \frac{2\delta^2}{(2\gamma\tau + 1)}. \tag{29}$$

We observe that R depends solely on δ (the small parameter in the cumulant expansion) and $\gamma\tau$ (quantifying the time scale separation between the unperturbed relaxation process and the relaxation of the correlation function of $\xi(t)$). It is evident from (29) that R decreases when the time scale separation decreases ($\gamma\tau$ increases) and increases quadratically with δ. The equilibrium PDF (24) in this case reads:

$$P_{eq}(x) = N\, {}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2} + 1\right); \frac{1}{4}\left(\frac{2\gamma\tau + 1}{\delta^2} + 2\right); -\frac{(2\gamma\tau + 1)x^2}{4D_f\tau}\right). \tag{30}$$

where N is a normalization factor. We note that, except for the quantity $D_f\tau$, which acts as a scale factor for x, also the equilibrium PDF (30) depends only on δ and $\gamma\tau$.

In figures 3-4, solid lines depict the plots of the PDF (30) for a fixed $D_f\tau = 0.5$ and different values of δ and $\gamma\tau$ corresponding to the points $(a)-(d)$ in the diagram (2). We have also included the corresponding results of the numerical simulation of the SDE (2) (circles), where $\xi(t)$ is the Ornstein-Uhlenbeck process. Additionally, to assess the relevance of the non Fick contribution to the current, we have also plotted, with dashed lines, the normalized function $P_{eq,FPE}(x) \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$ which is the solution for the vanishing "Fick" current of (7) (or the equilibrium PDF of the corresponding FPE). The excellent agreement of the analytical result (30) with numerical simulations is evident, while when relying on the $P_{eq,FPE}(x)$, the comparison with numerical simulations is not at all so good.

IV. CONCLUSIONS

The Fokker-Planck Equation (FPE) holds a central position in statistical mechanics. Initially derived as the Kramers-Moyal expansion of the Master Equation (ME), limited to

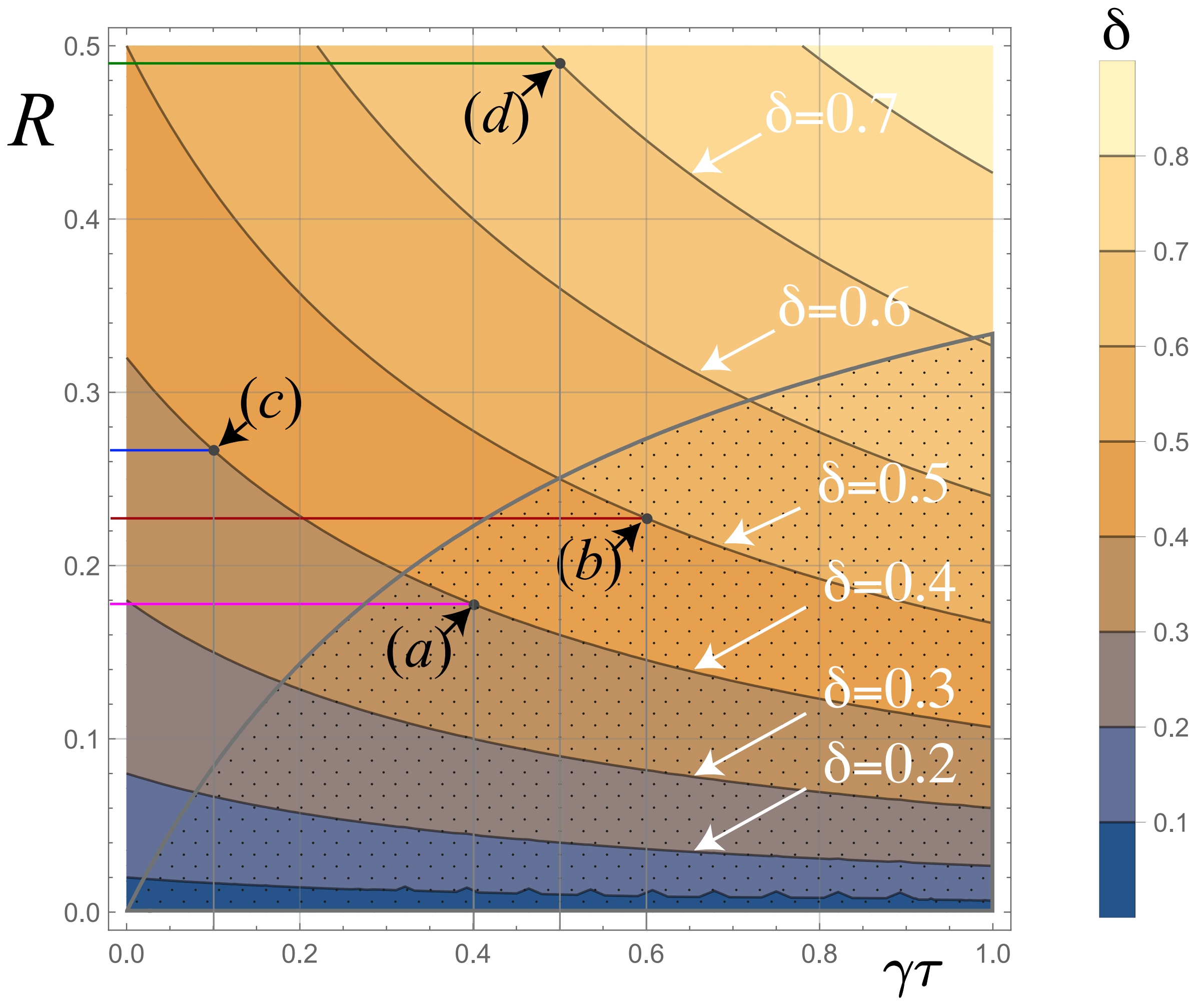

FIG. 2. Plot of R of (29) vs $\gamma\tau$, for various values of δ (distinct curves). We see that at fixed δ, as $\gamma\tau$ decreases, R increases. The same happens increasing δ, at $\gamma\tau$ fixed. The area with dotted background correspond to δ and $\gamma\tau$ values for which the variance of x is finite ($\gamma\tau - 2\delta^2 > 0$). The points in the graph labeled with the letters (a) and (b) ((c) and (d)) corresponds to the $\gamma\tau$ and δ values, used for the four plots of the PDF of figure 3 (figure 4).

Markovian systems, it's recognized as applicable to non-Markovian processes. Indeed, the FPE emerges by eliminating irrelevant or fast variables, weakly interacting with the part of interest, through perturbation techniques like Zwanzig and Mori's projective methods, or considerations on the order of magnitude of the generalized cumulants. Thus, it stands as the most important equation to derive the PDF time evolution in these approximations.

Moreover, the FPE has the advantage of being a second order classical parabolic PDE with

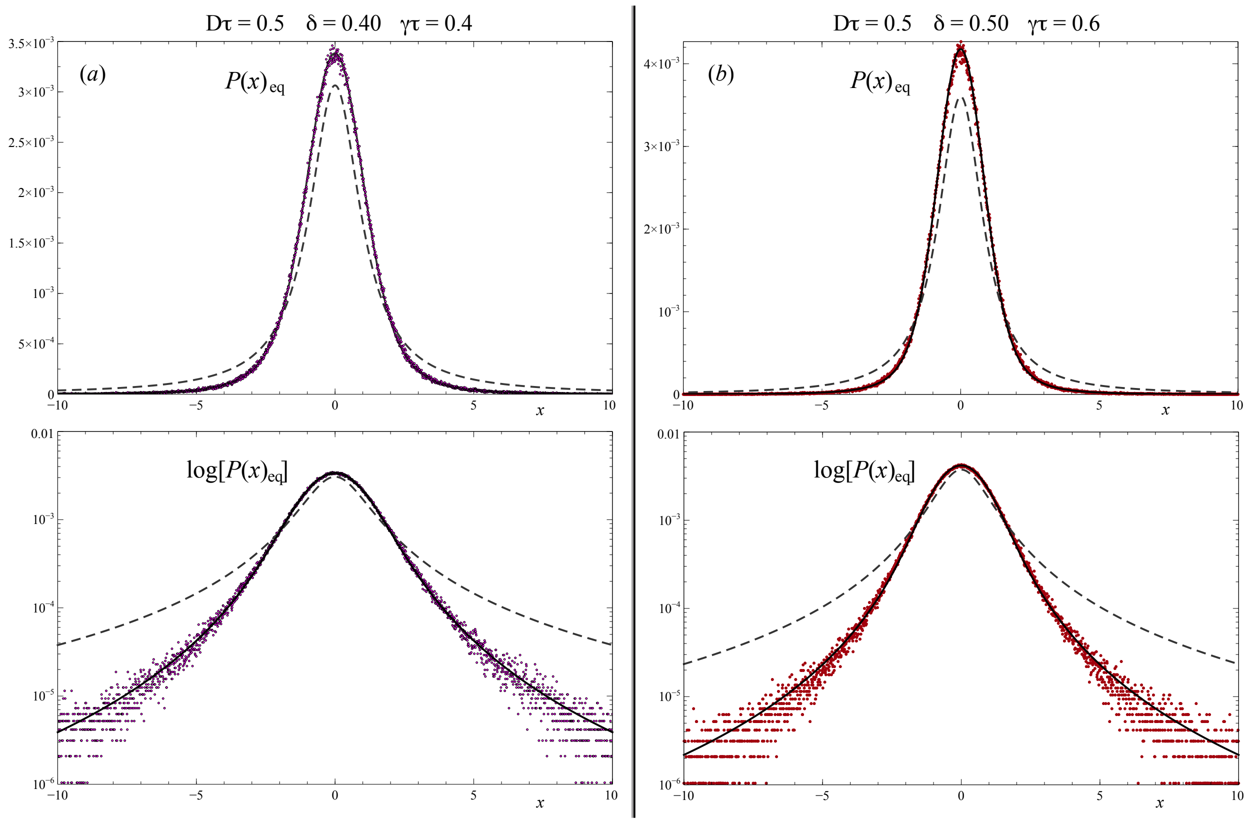

FIG. 3. Two vertical panels (a) and (b), respectively, displaying the Equilibrium PDF of the SDE (2) for the case in which the stochastic process $\xi(t)$ is the Ornstein Uhlenbeck process. Panel (a): $D_f\tau = 0.5$, $\delta = 0.4$ and $\gamma\tau = 0.4$. Panel (b): $D_f\tau = 0.5$, $\delta = 0.5$ and $\gamma\tau = 0.6$. In the bottom part, the same data as the upper part are presented in semi-logarithmic scale. Circles are the results of the numerical simulation. Solid lines represent the theoretical result (30), i.e., the equilibrium solution of the PDE (19), with r given in (28). Dashed lines depict $P_{eq,FPE}(x) \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$, the solution for the vanishing "Fick" current of (7) (or the equilibrium PDF of the corresponding FPE). In these two cases, corresponding to the two points (a) and (b) in the diagram of figure 2, we have $\gamma\tau > 2\delta^2$, thus the variance of x is finite (see text for details).

well-studied properties on solution existence and positivity. Its importance and widespread use are undeniable.

The connection between FPE and Fick's law is not coincidental. The FPE, when expressed as a continuity equation, reveals that the current linked to the stochastic process involves a diffusion term that is proportional to the gradient of the PDF, constituting Fick's law. Conversely, assuming Fick's law holds, the continuity equation emerges as a second-

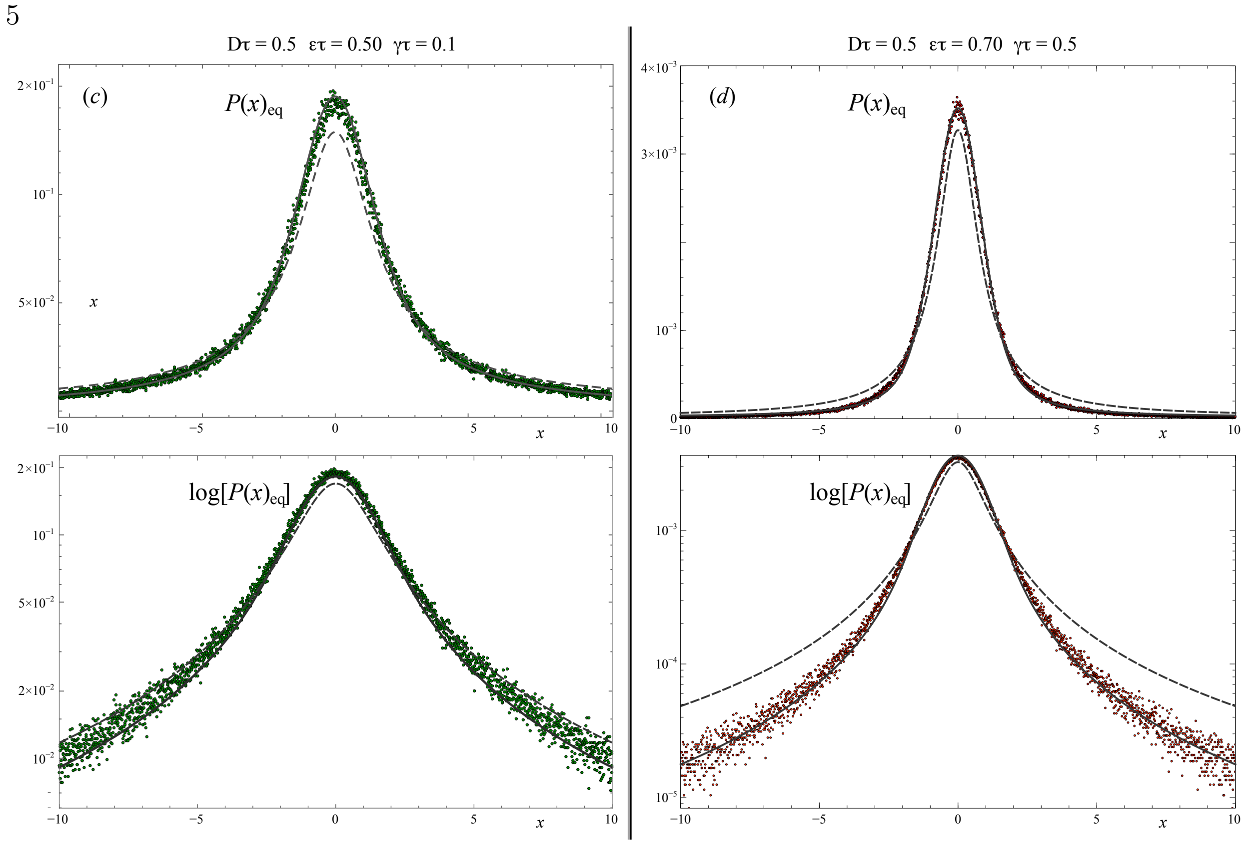

FIG. 4. The same as figure 3, but for different values of $\gamma\tau$ and δ as indicated in the header of the panels. In these two cases, corresponding to the two points (c) and (d) in the diagram of figure 2, we have $\gamma\tau < 2\delta^2$, thus, at equilibrium all the moments of x diverge.

order PDE, exhibiting the structure of an FPE. Hence, whether Fick's law holds or not, and the ME with FPE structure structure are intricately connected.

The extensive use of the FPE has led to the development of numerous methods for extracting crucial statistical information. Standard spectral analysis procedures, similar to those applied to the Schrödinger equation in quantum mechanics, can be employed. Additionally, the diffusion and drift coefficients of the FPE allow for the derivation of an analytical expression for the mean first-passage time. This important quantity represents the average time for a trajectory, starting from an initial position x_0, to reach a specified target point x_T for the first time.

In our study, we demonstrated that when the system of interest is inherently noisy-featuring sources like Nyquist noise in electric circuits, various thermal fluctuations, rapid internal dynamics, intrinsic measurement errors, etc., the mentioned standard procedures for eliminating fast or weakly interacting variables (often, but not necessarily, modeled as stochastic processes) lead to a third-order PDE, instead of an FPE. In fact, an additive third-order partial differential operator emerges from the interplay between the standard diffusion process due to internal noise and the diffusion process due to the external colored stochastic process (or to the irrelevant degrees of freedom we project out).

Given the inevitability of such internal noise (of varying intensity), we conclude that the third-order PDE should be considered more fundamental than the FPE in statistical physics. This fact also implies the breakdown of the Fick's law.

While this approach can be extended to accommodate more general drift fields, our current focus in this work is on the simpler linear drift case, which is widely employed across various disciplines. The analytical expressions of the moments of the PDF reveals that the unexpected third derivative term significantly tightens the equilibrium PDF, in comparison to what we would obtain if we dropped this term, maintaining just the standard structure of the FPE. Figures 3 and 4 support this observation, showing perfect agreement between numerical simulations of the SDE and the third-order PDE. In particular, the figures clearly illustrate that the actual PDFs, effectively captured by the third-order PDE, exhibit a significantly more narrow equilibrium PDFs compared to those derived from the FPE. This observation agrees with the general finding, emphasized in section II, that the third-order differential contribution to ME leads to a reduction in the moments of x. Consequently, the tails of the actual equilibrium PDF (and those of the equilibrium PDF of the third-order PDE) decay more rapidly than those of the FPE, indicating that crucial statistical quantities, such as the average first-passage time, computed using standard FPE techniques, would yield inaccurate results.

Thus, the fact that for the statistical behavior of a specific part of a complex system, the third-order PDE should be considered more fundamental than the FPE raises the question of how to extend to this PDE the general methods and results, such like those which allow the derivation of relevant statistical information from the FPE. For example, in the 1-D case, it would be interesting to obtain an analytical expression for the equilibrium PDF or a closed expression for the mean first-passage time. All these are matter of future works.

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Appendix A: The multidimensional case

In this Appendix we generalize the result (19) to the multi-dimensional case. For the reader convenience, we reapt here the general N-D extension of the SDE (2), already introduced in (1):

$$\dot{\boldsymbol{x}} = -\mathbb{E} \cdot \boldsymbol{x} + \boldsymbol{f}(t) - \Xi(t) \cdot \boldsymbol{x} \tag{A1}$$

where $\boldsymbol{x} := (x_1, \dots, x_N)$, $\mathbb{E}$ and $\Xi(t)$ are $N \times N$ matrices with constant and stochastic components, respectively. Moreover, $\boldsymbol{f}(t)$ is a multidimensional white noise with correlation, or diffusion matrix given by $\mathbb{D}$.

As for the 1-D case, to any realization $\Xi(\cdot)$ of the matrix stochastic process $\Xi(u)$, $0 \leq u \leq t$, from (A1) we can write the following Liouville equation for the PDF of $\boldsymbol{x}$, that we indicate with $P_{\Xi(\cdot)}(\boldsymbol{x}, t)$:

$$\begin{aligned} &\partial_t P_{\Xi(\cdot)}(\boldsymbol{x}, t) \\ &= \{\mathcal{L}_a + \mathcal{L}_{\Xi(t)}\} P_{\Xi(\cdot)}(\boldsymbol{x}, t), \end{aligned} \tag{A2}$$

in which the unperturbed Liouvillian is ($\boldsymbol{\partial}$ is the N-D gradient operator and the superscript "T" means "transpose"):

$$\mathcal{L}_a := \boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} + +\boldsymbol{\partial}^T \cdot \mathbb{D} \cdot \boldsymbol{\partial} \tag{A3}$$

and the Liouville perturbation operator is

$$\mathcal{L}_{\Xi(t)} := \boldsymbol{\partial}^T \cdot \Xi(t) \cdot \boldsymbol{x}. \tag{A4}$$

We rewrite the Liouville equation (A2) in interaction representation:

$$\partial_t \tilde{P}_{\Xi(\cdot)}(\boldsymbol{x}, t) = \tilde{\mathcal{L}}_{\Xi(t)}(t) \tilde{P}_{\Xi(\cdot)}(\boldsymbol{x}, t), \tag{A5}$$

where

$$\tilde{P}_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x},t) := e^{-\mathcal{L}_a t} P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x},t) \tag{A6}$$

and

$$\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t) := e^{-\mathcal{L}_a t} \mathcal{L}_{\boldsymbol{\Xi}(t)} e^{\mathcal{L}_a t} = e^{-\mathcal{L}_a^{\times} t}[\mathcal{L}_{\boldsymbol{\Xi}(t)}]. \tag{A7}$$

Integrating (A5) and averaging over the realization of $\boldsymbol{\Xi}(t)$, we get

$$\tilde{P}(\boldsymbol{x};t) = \langle \overleftarrow{\exp}\left[\int_0^t du\, \tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(u)\right]\rangle_{\boldsymbol{\Xi}}\, P(\boldsymbol{x};0) \tag{A8}$$

in which $\overleftarrow{\exp}[...]$ is the standard chronological ordered exponential (from right to left) and $\tilde{P}(\boldsymbol{x};t) := e^{-\mathcal{L}_a t} P(\boldsymbol{x};t)$ with $P(\boldsymbol{x};t) := \langle P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x},t)\rangle_{\boldsymbol{\Xi}}$. By using the generalized cumulant approach and retaining only the second cumulant we get the following ME for the PDF of $\boldsymbol{x}$:

$$\partial_t P(\boldsymbol{x};t) = \mathcal{L}_a P(x;t) + \int_0^{\infty} \mathrm{d}u\, \langle \mathcal{L}_{\boldsymbol{\Xi}(t)} \tilde{\mathcal{L}}_{\boldsymbol{\Xi}(-u)}(-u)\rangle_{\boldsymbol{\Xi}}\, P(\boldsymbol{x};t), \tag{A9}$$

corresponding to the N-D version of Eq. (12). To obtain the explicit expression, as PDE, of the ME (A9), we must solve the Lie evolution of $\mathcal{L}_{\boldsymbol{\Xi}(t)}$ along the Liouvillian $\mathcal{L}_a$, i.e. we have to explicitly evaluate $\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(u)}(u)$ of (A7), in which $\mathcal{L}_a$ and $\mathcal{L}_{\boldsymbol{\Xi}(t)}$ are given in (A3) and (A4), respectively. For that, let us start considering the operator identity $e^{(\mathcal{L}_A+\mathcal{L}_B)t} = e^{\mathcal{L}_A t}\cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u\, \bar{\mathcal{L}}_B(u)\right)$ in which $\mathcal{L}_A$ and $\mathcal{L}_B$ are operators that in general do not commute with each other, and where $\bar{\mathcal{L}}_B(u) := e^{-\mathcal{L}_A u}\mathcal{L}_B e^{\mathcal{L}_A u}$. From this identity, by making the associations $\mathcal{L}_A = \boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}$ and $\mathcal{L}_B = \boldsymbol{\partial}^T\cdot\mathbb{D}\cdot\boldsymbol{\partial}$ (thus, $\mathcal{L}_A+\mathcal{L}_B = \mathcal{L}_a$), with a few algebra we easily obtain:

$$e^{\mathcal{L}_a t} = e^{\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\, t}\cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u\, \boldsymbol{\partial}^T\cdot e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^T u}\cdot\boldsymbol{\partial}\right). \tag{A10}$$

By using (A10) and (A3) in (A7) we get

$$\begin{aligned}
\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t) :=& e^{-\mathcal{L}_a t}\mathcal{L}_{\boldsymbol{\Xi}(t)} e^{\mathcal{L}_a t}\\
=& e^{-\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\, t}\cdot \overleftarrow{\exp}\left(-\int_{-t}^0 \mathrm{d}u\, \boldsymbol{\partial}^T\cdot e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^T u}\cdot\boldsymbol{\partial}\right)\\
&\cdot \mathcal{L}_{\boldsymbol{\Xi}(t)}\cdot e^{\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\, t}\cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u\, \boldsymbol{\partial}^T\cdot e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^T u}\cdot\boldsymbol{\partial}\right)\\
=& \overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u\, \boldsymbol{\partial}^T\cdot e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^T u}\cdot\boldsymbol{\partial}\right)^{\times}\left[e^{-\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\, t^{\times}}\left[\mathcal{L}_{\boldsymbol{\Xi}(t)}\right]\right].
\end{aligned} \tag{A11}$$

In the last side of the following equation we have exploited the following identity, easily demonstrated:

$$e^{-\mathcal{L}_A\theta}\cdot\overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u\,\bar{\mathcal{L}}_B(u)\right)=e^{-\mathcal{L}_A\theta\times}\left[\overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u\,\bar{\mathcal{L}}_B(u)\right)\right]e^{-\mathcal{L}_A\theta}$$
$$=\overleftarrow{\exp}\left(-\int_\theta^{t+\theta}\mathrm{d}u\,\bar{\mathcal{L}}_B(u)\right)e^{-\mathcal{L}_A\theta}. \tag{A12}$$

By using (A4) and the results of[30], in particular those in Section VIA, we have

$$e^{-\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\,t\times}\left[\mathcal{L}_{\boldsymbol{\Xi}(t)}\right]=e^{-\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}\,t\times}\left[\boldsymbol{\partial}^T\cdot\boldsymbol{\Xi}(t)\cdot\boldsymbol{x}\right]=\boldsymbol{\partial}^T\cdot e^{\mathbb{E}t\times}\left[\boldsymbol{\Xi}(t)\right]\cdot\boldsymbol{x}. \tag{A13}$$

Inserting this result in (A11) we obtain

$$\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t)=$$
$$\overleftarrow{\exp}\left(-\int_0^t\mathrm{d}u\,\boldsymbol{\partial}^T\cdot e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\cdot\boldsymbol{\partial}\right)^{\times}\left[\boldsymbol{\partial}^T\cdot e^{\mathbb{E}t\times}\left[\boldsymbol{\Xi}(t)\right]\cdot\boldsymbol{x}\right]. \tag{A14}$$

By expanding the above series of nexted commutators we see that all the terms are zero, apart the zeroth and the first ones. Therefore, we get

$$\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t)=\boldsymbol{\partial}^T\cdot e^{\mathbb{E}t\times}\left[\boldsymbol{\Xi}(t)\right]\cdot\boldsymbol{x}$$
$$-\boldsymbol{\partial}^T\cdot e^{\mathbb{E}t\times}\left[\boldsymbol{\Xi}(t)\right]\cdot\int_0^t\mathrm{d}u\,\left\{\left(e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\right)^T+e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\right\}\cdot\boldsymbol{\partial}, \tag{A15}$$

that, given the symmetry property of the diffusion coefficient matrix, yields the final explicit differential form for the interaction representation of the Liouvillian $\mathcal{L}_{\boldsymbol{\Xi}(t)}$:

$$\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t)=\boldsymbol{\partial}^T\cdot e^{\mathbb{E}t\times}\left[\boldsymbol{\Xi}(t)\right]\cdot\left\{\boldsymbol{x}-2\int_0^t\mathrm{d}u\,e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\cdot\boldsymbol{\partial}\right\}. \tag{A16}$$

Thus, by using this expression in the ME (A9), together with Eqs. (A3) and (A4), we arrive to the final general PDE of third order for the PDF of $\boldsymbol{x}$ for the multi-dimensional case:

$$\partial_t P(\boldsymbol{x};t)=\left\{\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}++\boldsymbol{\partial}^T\cdot\mathbb{D}\cdot\boldsymbol{\partial}\right\}P(\boldsymbol{x};t)+\int_0^\infty\mathrm{d}u$$
$$\times\,\langle\boldsymbol{\partial}^T\cdot\boldsymbol{\Xi}(t)\cdot\boldsymbol{x}\left(\boldsymbol{\partial}^T\cdot e^{-\mathbb{E}u\times}\left[\boldsymbol{\Xi}(-u)\right]\cdot\left\{\boldsymbol{x}+2\int_0^u\mathrm{d}u\,e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\cdot\boldsymbol{\partial}\right\}\right)\rangle_{\boldsymbol{\Xi}}\,P(\boldsymbol{x};t), \tag{A17}$$

In the simplified case in which $\boldsymbol{\Xi}(t)=\epsilon\mathbb{G}\xi(t)$, with $\langle\xi(t)\xi\rangle_{\boldsymbol{\Xi}}=\varphi(t)$ then we have

$$\partial_t P(\boldsymbol{x};t)=\left\{\boldsymbol{\partial}^T\cdot\mathbb{E}\cdot\boldsymbol{x}++\boldsymbol{\partial}^T\cdot\mathbb{D}\cdot\boldsymbol{\partial}\right\}P(x;t)+\frac{\delta^2}{\tau^2}\int_0^\infty\mathrm{d}u\,\varphi(u)$$
$$\times\,\boldsymbol{\partial}^T\cdot\mathbb{G}\cdot\boldsymbol{x}\left(\boldsymbol{\partial}^T\cdot e^{-\mathbb{E}u\times}\left[\mathbb{G}\right]\cdot\left\{\boldsymbol{x}+2\int_0^u\mathrm{d}u\,e^{\mathbb{E}u}\cdot\mathbb{D}\cdot e^{\mathbb{E}^Tu}\cdot\boldsymbol{\partial}\right\}\right)\,P(\boldsymbol{x};t), \tag{A18}$$

where we have also used the definition of the adimensional parameter $\delta:=\epsilon\tau$, that is the relevant small quantity in the cumulant expansion.

Appendix B: The cumulant approach as a systematic way to obtain a ME for the reduced PDF of x

In this Appendix we outline a few minima key steps to obtain the FPE (3) and the ME (12), starting from the generalized cumulant (or M-cumulant) approach formally presented in[32]. We begin with the generic Liouville equation (9) (the stochastic process is one-dimensional, but the extension to multi-dimensional cases is straightforward), expressed in interaction representation:

$$\partial_t \tilde{P}_{\xi(\cdot)}(x,t) = \epsilon\xi(t), \tilde{\mathcal{L}}_I(t)\tilde{P}_{\xi(\cdot)}(x,t). \tag{B1}$$

Here,

$$\tilde{P}_{\xi(\cdot)}(x,t) := e^{-\mathcal{L}_a t} P_{\xi(\cdot)}(x,t) \tag{B2}$$

and

$$\tilde{\mathcal{L}}_I(t) := e^{-\mathcal{L}_a t}\tilde{\mathcal{L}}_I e^{\mathcal{L}_a t} = e^{-\mathcal{L}_a^{\times} t}[\mathcal{L}_I]. \tag{B3}$$

In[30], $\tilde{\mathcal{L}}_I(t)$ of (B3) is also referred to as the Lie evolution of the operator $\mathcal{L}_I$ along the Liouvillian $\mathcal{L}_a$, for a time $-t$.

Integrating (B1) and averaging over the realization of $\xi(t)$, we get

$$\tilde{P}(x;t) = \langle \overleftarrow{\exp}\left[\epsilon\int_0^t du\;\xi(u)\tilde{\mathcal{L}}_I(u)\right]\rangle_\xi P(x;0) \tag{B4}$$

in which $\overleftarrow{\exp}[...]$ is the standard chronological ordered exponential (from right to left) and $\tilde{P}(x;t) := e^{-\mathcal{L}_a t}P(x;t)$ with $P(x;t) := \langle P_{\xi(\cdot)}(x,t)\rangle_\xi$. Moreover, we have exploited the assumption that at the initial time $t=0$ the total PDF factorizes as $P_{\xi(\cdot)}(x,0) = P(x;0)p(\xi)$. This is equivalent to stating that at the initial time the PDF of x does not depend on the possible values of the process ξ, or alternatively, we wait long enough so that the initial conditions became irrelevant. Apart that, Eq. (B4) is exact; no approximations have been introduced at this level.

We can look at the r.h.s. of (B4) as a sort of characteristic function, or moment generating function, with wave number $k := \mathrm{i}\epsilon$, for the stochastic operator

$$\Omega(u) := \xi(u)\tilde{\mathcal{L}}_I(u). \tag{B5}$$

Formally, we can then introduce a generalized cumulant generating function[32]:

$$\langle \overleftarrow{\exp}\left[\epsilon\int_0^t du\;\xi(u)\tilde{\mathcal{L}}_I(u)\right]\rangle_\xi := \overleftarrow{\exp}\left[\mathcal{K}(\epsilon,t)\right] \tag{B6}$$

with

$$\mathcal{K}(\epsilon,t)=\sum_{i=1}^{\infty}\epsilon^i\mathcal{K}_i(t). \tag{B7}$$

As for standard stochastic processes, we define the n-times joint M-cumulant of $\Omega(u)$, that we indicate as $\langle\langle\Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle$, by setting

$$\mathcal{K}_i(t):=\int_0^t \mathrm{d}u_1\int_0^{u_1}\mathrm{d}u_2...\int_0^{u_{n-1}}\mathrm{d}u_n\langle\langle\Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle. \tag{B8}$$

Using (B8) in the r.h.s. of (B6) and expanding both exponential functions, we get the standard relationship among cumulants and moments. For example, the joint two and four times M-cumulants are given in terms of moments as (to improve readability, until the end of this paragraph we will avoid putting the subscript "ξ' to the angle brackets):

$$\langle\langle\Omega(u_1)\,\Omega(u_2)\rangle\rangle=\langle\Omega(u_1)\,\Omega(u_2)\rangle=\tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_2)\langle\xi(u_1)\,\xi(u_2)\rangle \tag{B9}$$

and

$$\begin{aligned}
&\langle\langle\Omega(u_1)\,\Omega(u_2)\Omega(u_3)\,\Omega(u_4)\rangle\rangle=\\
&\langle\Omega(u_1)\,\Omega(u_2)\Omega(u_3)\,\Omega(u_4)\rangle-\langle\Omega(u_1)\,\Omega(u_2)\rangle\langle\Omega(u_3)\,\Omega(u_4)\rangle\\
&-\langle\Omega(u_1)\,\Omega(u_3)\rangle\langle\Omega(u_2)\,\Omega(u_4)\rangle-\langle\Omega(u_1)\,\Omega(u_4)\rangle\langle\Omega(u_2)\,\Omega(u_3)\rangle=\\
&\tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_3)\tilde{\mathcal{L}}_I(u_4)\left[\langle\xi(u_1)\,\xi(u_2)\,\xi(u_3)\,\xi(u_4)\rangle-\langle\xi(u_1)\xi(u_2)\langle\xi(u_3)\xi(u_4)\rangle\right]\\
&-\tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_3)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_4)\langle\xi(u_1)\xi(u_3)\rangle\langle\xi(u_2)\xi(u_4)\rangle\\
&-\tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_4)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_3)\langle\xi(u_1)\xi(u_4)\rangle\langle\xi(u_2)\xi(u_3)\rangle,
\end{aligned} \tag{B10}$$

respectively. From (B10) it is clear that the Gaussian nature of $\xi(t)$ does not implies the same for $\Omega(t)$ of Eq. (B5), as the time-dependent Liouvillian $\tilde{\mathcal{L}}_I(u)$ generally does not commute with itself evaluated at different times. However, when the unperturbed Liouvillian $\mathcal{L}_a$ and perturbation Liouvillian $\mathcal{L}_I$ commute with each other, as in the case of Eq. (10) with $D_f=0$ and $\mathcal{L}_I$ of Eq. (11), we have $\tilde{\mathcal{L}}_I(u)=\mathcal{L}_I$, that does not depend on time. Hence, in this case the Gaussian nature of $\xi(t)$ is transferred to the stochastic operator $\Omega(t)$. Therefore, in this specific scenario, the M-cumulant series appearing in the exponential function of (B6) reduces to only the second term containing the second M-cumulant, simplifying to (without loss of generality, we consider the average value of $\xi(t)$ to be zero):

$$\tilde{P}(x;t)=\exp\left[\epsilon^2\mathcal{L}_I\mathcal{L}_I\int_0^t\mathrm{d}u_1\int_0^{u_1}\mathrm{d}u_2\langle\xi(u_1)\,\xi(u_2)\rangle\right]P(x;0). \tag{B11}$$

Time-deriving this result we obtain

$$
\begin{aligned}
\partial_t \tilde{P}(x;t) =& \epsilon^2 \mathcal{L}_I \mathcal{L}_I \int_0^t \mathrm{d}u \langle \xi(t)\, \xi(u)\rangle \ \tilde{P}(x;t) \\
=& \epsilon^2 \mathcal{L}_I \mathcal{L}_I \tau \tilde{P}(x;t).
\end{aligned} \tag{B12}
$$

Getting rid of the interaction representation and by using (10) with $D_f = 0$ and (11), Eq. (B12) becomes exactly the FPE (3).

In the more general case, the Liouvillians $\mathcal{L}_a$ and $\mathcal{L}_I$ do not commute with each other, so $\tilde{\mathcal{L}}_I(u)$ of (B3) depends on time. The advantage of utilizing the M-cumulants lies in the fact that, similar to standard cumulants, they are exactly zero when referring to independent random variables[32]. Thus, the time lag between two events increases until they become independent of each other, any joint M-cumulant containing these two events must tend to zero. To model this situation more realistically, we assume that independence does not occur abruptly at a fixed time lag $\bar{\tau}$ but instead follows a smoother pattern, characterized by an exponential trend. Formally, for a series of events $\xi(t_1), \xi(t_2), ..., \xi(t_n)$ with $t_1 \geq t_2 \geq ... \geq t_n$, we assume that the corresponding joint n-cumulant decays at least exponentially with the time lag $u_1 - u_n$::

$$
|\langle\langle \Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle| \lesssim \exp(-(u_1-u_n)/\bar{\tau}). \tag{B13}
$$

In this scenario, along with the definitions (B7) and (B8), it is evident that the argument of the exponential function in the right-hand side of (B6) now yields a power series of $\bar{\delta} := \epsilon\bar{\tau}$. For a sufficiently small $\bar{\delta}$, we can truncate this series to the first non-zero term, which is the second one. Thus, Eq. (B6), combined with Eq. (B6) and (B9), gives

$$
\tilde{P}(x;t) = \overleftarrow{\exp}\left[\epsilon^2 \int_0^t \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 \tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_2)\langle \xi(u_1)\, \xi(u_2)\rangle + O(\bar{\delta}^4)\right] P(x;0). \tag{B14}
$$

Time-deriving this result we obtain

$$
\partial_t \tilde{P}(x;t) = \epsilon^2 \int_0^t \mathrm{d}u \tilde{\mathcal{L}}_I(t)\tilde{\mathcal{L}}_I(u)\langle \xi(t)\, \xi(u)\rangle \ \tilde{P}(x;t) + O(\bar{\delta}^4 t/\bar{\tau}) \tag{B15}
$$

Getting rid of the interaction representation and by using again (10) (but now letting $D_f \neq 0$) and (11), Eq. (B15) becomes the approximate ME (12).

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