A Blaschke–Petkantschin formula for linear and affine subspaces with application to intersection probabilities

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Abstract

Consider a uniformly distributed random linear subspace L and a stochastically independent random affine subspace E in $\mathbb{R}^n$, both of fixed dimension. For a natural class of distributions for E we show that the intersection $L \cap E$ admits a density with respect to the invariant measure. This density depends only on the distance $d(o, E \cap L)$ of $L \cap E$ to the origin and is derived explicitly. It can be written as the product of a power of $d(o, E \cap L)$ and a part involving an incomplete beta integral. Choosing E uniformly among all affine subspaces of fixed dimension hitting the unit ball, we derive an explicit density for the random variable $d(o, E \cap L)$ and study the behavior of the probability that $E \cap L$ hits the unit ball in high dimensions. Lastly, we show that our result can be extended to the setting where E is tangent to the unit sphere, in which case we again derive the density for $d(o, E \cap L)$. Our probabilistic results are derived by means of a new integral-geometric transformation formula of Blaschke–Petkantschin type.

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1 Introduction and motivation

Fix dimension parameters $n \geq 2$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$. Let L_1 be a q-dimensional random linear subspace and L_2 be an $(n-q+\gamma)$-dimensional random linear subspace of $\mathbb{R}^n$. We assume that both subspaces are stochastically independent and that L_1 and L_2 are selected according to the uniform distribution on the Grassmannian $G(n,k)$ of all k-dimensional linear subspaces of $\mathbb{R}^n$, with $k = q$ and $k = n-q+\gamma$, respectively. In other words, we use the normalized rotation invariant measures ν_k, $k = q$ and $k = n-q+\gamma$, on these spaces as our underlying probability measures; these and further concepts will formally be introduced in Section 2. The intersection $L_1 \cap L_2$ is almost surely a random subspace of $\mathbb{R}^n$ of dimension γ and its distribution is known to be the uniform distribution on the space $G(n,\gamma)$, see Figure 1.1.

Let us now change the set-up and let E_1 be a q-dimensional random affine subspace and E_2 be another $(n-q+\gamma)$-dimensional random affine subspace of $\mathbb{R}^n$. Since the motion invariant measure μ_k on $A(n,k)$, $k \in \{0, \dots, n-1\}$, is not finite, we restrict attention to the set

$$[B^n]_k = \{E \in A(n,k) : E \cap B^n \neq \emptyset\}$$

of k-dimensional affine subspaces hitting the unit ball B^n of $\mathbb{R}^n$. We thus take E_1 and E_2 as random affine subspaces distributed according to the normalized measures μ_q and $\mu_{n-q+\gamma}$ restricted to $[B^n]_q$ and $[B^n]_{n-q+\gamma}$, respectively. Assuming that E_1 and E_2 are stochastically independent, the intersection $E_1 \cap E_2$ is almost surely a random affine subspace of $\mathbb{R}^n$ with dimension γ.

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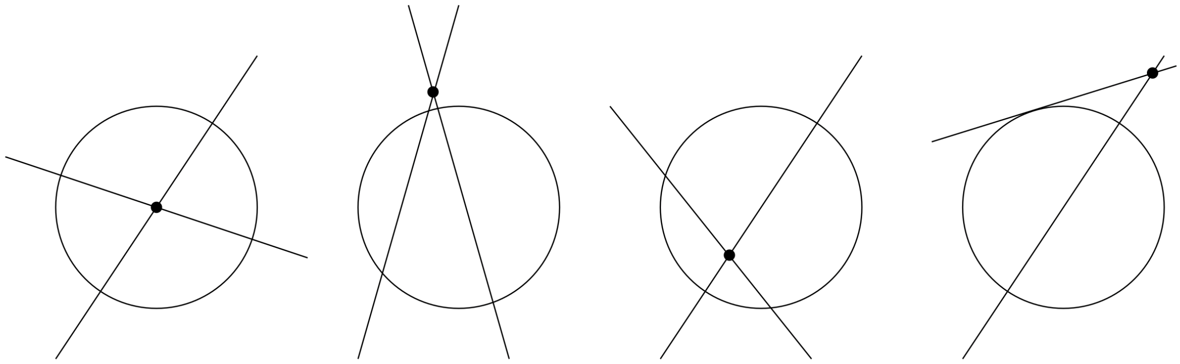

Figure 1.1: Illustration in the case $n = 2$, $q = 1$, $\gamma = 0$. From left to right: Intersection of two linear subspaces; intersection of two affine subspaces hitting the unit ball; intersection of a linear with an affine subspace hitting the unit ball; intersection of a linear subspace with an affine subspace tangent to the unit sphere.

However, the intersection of $E_1 \cap E_2$ with the unit ball B^n may or may not be empty with strictly positive probability, see Figure 1.1. Already this basic observation shows that – in contrast to the case of random linear subspace discussed above – the distribution of $E_1 \cap E_2$ cannot coincide with the normalized motion-invariant measure on $[B^n]_\gamma$. More precisely, since the distance of the intersection of E_1 and E_2 to the origin $o \in \mathbb{R}^n$ can be arbitrarily large, the distribution of $E_1 \cap E_2$ cannot even be supported on a compact subset of $A(n,\gamma)$. In fact, it is known from [22, Thm. 7.2.8] that the distribution of $E_1 \cap E_2$ has a non-trivial density with respect to the invariant measure on $A(n,\gamma)$, which is proportional to

$$E \mapsto \int_{A(E,q)\cap[B^n]_q} \int_{A(E,n-q+\gamma)\cap[B^n]_{n-q+\gamma}} [E_1, E_2]^{\gamma+1}\, \mu^E_{n-q+\gamma}(dE_2)\mu^E_q(dE_1),$$

where $A(E,q)$ is the set of q-dimensional affine subspaces containing E and μ^E_q is the invariant measure on that space (similarly for $A(E,n-q+\gamma)$ and $\mu^E_{n-q+\gamma}$). Moreover, $[E_1,E_2]$ stands for the so-called subspace determinant, describing the relative position of E_1 and E_2, see below for a detailed definition. We remark that in the special case $n = 2$, $q = 1$ and $\gamma = 0$ this is a classical result of M. Crofton discussed in [3, §7], whereas the case $n = 3$, $q = 2$ and $\gamma = 1$ goes back to W. Blaschke [3, §33].

On a more abstract level, both problems just mentioned naturally lead to the study of what is known in the literature as integral-geometric transformation formulas of Blaschke–Petkantschin type. Such formulas go back to the pioneering works of W. Blaschke [3] in dimensions $n = 2$, and $n = 3$ and have more systematically been investigated by his student B. Petkantschin [18]. They have further been developed in [21] using the language of differential forms and in [22] by means of a measure-theoretic approach. Blaschke–Petkantschin formulas are fundamental devices in integral and stochastic geometry and have found various applications in convex geometry and geometric analysis [4, 5, 6, 11, 16, 17, 20] as well as in stereology [7, 8, 9, 13, 14, 15]. For the intersection of two linear subspaces a formula of this type is given by

$$\begin{aligned}&\int_{G(n,q)} \int_{G(n,n-q+\gamma)} f(L_1,L_2)\, \nu_{n-q+\gamma}(dL_2)\nu_q(dL_1)\\ &= c_1(n,q,\gamma) \int_{G(n,\gamma)} \int_{G(L,q)} \int_{G(L,n-q+\gamma)} f(L_1,L_2)[L_1,L_2]^\gamma\, \nu^L_{n-q+\gamma}(dL_2)\nu^L_q(dL_1)\nu_\gamma(dL),\end{aligned} \tag{1.1}$$

where $f : G(n,q) \times G(n,n-q+\gamma) \to [0,\infty)$ is a measurable function and $G(L,q)$ is the relative Grassmannian of all q-dimensional linear subspaces of $\mathbb{R}^n$ containing L, whereas ν^L_q is the invariant probability measure on $G(L,q)$ which is invariant under all rotations of $\mathbb{R}^n$ that fix L (similarly for $G(L,n-q+\gamma)$ and $\nu^L_{n-q+\gamma}$). We refer to [22, Thm. 7.2.5] where also the value of the constant $c_1(n,q,\gamma)$ can be found, which only depends on the parameters in brackets. The corresponding formula for the intersection of two affine subspaces is a special case of [22, Thm. 7.2.8] and reads as follows:

$$\begin{aligned}&\int_{A(n,q)}\int_{A(n,n-q+\gamma)} f(E_1,E_2)\,\mu_{n-q+\gamma}(dE_2)\mu_q(dE_1)\\&=c_2(n,q,\gamma)\int_{A(n,\gamma)}\int_{A(E,q)}\int_{A(E,n-q+\gamma)} f(E_1,E_2)[E_1,E_2]^{\gamma+1}\,\mu^E_{n-q+\gamma}(dE_2)\mu^E_q(dE_1)\mu_\gamma(dE)\end{aligned} \tag{1.2}$$

for measurable functions $f: A(n,q)\times A(n,n-q+\gamma)\to[0,\infty)$. Here, $A(E,k)$ stands for the family of k-dimensional affine subspaces containing E and μ^E_k denotes the invariant measure on $A(E,k)$, $k=q$ and $k=n-q+\gamma$. The value of the constant $c_2(n,q,\gamma)$ only depends on the parameters in brackets and can be found in [22].

The present paper deals with a situation which in a sense is intermediate between (1.1) and (1.2), and combines the linear with the affine set-up. To the best of our knowledge, this has not found attention so far in the literature. More explicitly, let $L\in G(n,q)$ be a q-dimensional linear subspace of $\mathbb{R}^n$, $q\in\{1,\dots,n-1\}$, and for $\gamma\in\{0,\dots,q-1\}$ let $E\in G(n,n-q+\gamma)$ be an affine subspace of dimension $n-q+\gamma$. If E and L are in general position, their intersection $E\cap L$ is an affine subspace of dimension γ.

Our principal goal is the following. Find, for a given rotation invariant measure $\tilde\mu_{n-q+\gamma}$ on $A(n,n-q+\gamma)$ that is absolutely continuous with respect to $\mu_{n-q+\gamma}$, a measurable function $J: A(n,\gamma)\to[0,\infty)$ such that

$$\int_{G(n,q)}\int_{A(n,n-q+\gamma)} f(E\cap L)\,\tilde\mu_{n-q+\gamma}(dE)\nu_q(dL)=\int_{A(n,\gamma)} f(E)\,J(E)\,\mu_\gamma(dE) \tag{1.3}$$

holds for all measurable functions $f: A(n,\gamma)\to[0,\infty)$. The main result of this paper provides an explicit description of $J(E)$ and its dependence on $\tilde\mu_{n-q+\gamma}$. In the important particular case $\tilde\mu_{n-q+\gamma}=\mu_{n-q+\gamma}$, it turns out that $J(E)=cd(o,E)^{-(n-q)}$, where c is a known constant and $d(o,E)$ stands for the distance of E to the origin o. Another interesting case arises when $\tilde\mu_{n-q+\gamma}$ is the restriction of $\mu_{n-q+\gamma}$ to $[hB^n]_{n-q+\gamma}$ for some fixed $h>0$. Then, the left-hand side of (1.3) is

$$\int_{G(n,q)}\int_{A(n,n-q+\gamma)} f(E\cap L)\,\mathbf{1}_{\{d(o,E)\le h\}}\,\mu_{n-q+\gamma}(dE)\nu_q(dL), \tag{1.4}$$

and $J(E)$ involves, besides of $d(o,E)^{-(n-q)}$, an additional factor that can be expressed in terms of an incomplete beta function. In probabilistic terms, our new integral-geometric transformation formula (1.3) will allow us to determine the density with respect to the invariant measure μ_γ on $A(n,\gamma)$ of the intersection of a random linear subspace L of dimension q and a stochastically independent random subspace E of dimension $n-q+\gamma$ hitting the unit ball in $\mathbb{R}^n$, see Figure 1.1. Moreover, we will also be able to determine the corresponding density with respect to μ_γ if the random affine subspace E is only tangent to the unit sphere, see again Figure 1.1.

The remaining parts of this paper are structured as follows. In Section 2 we set up the notation and gather some background material. Some preliminary considerations are contained in Section 3 and in Section 4 we formulate our main theorems, especially the new integral-geometric transformation formula of Blaschke–Petkantschin-type. In Sections 5 and 6 we present the two applications to intersection probabilities mentioned above. Finally, Section 7 contains the proofs of our main results.

2 Notation and background material

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space for some fixed dimension $n\ge1$. The *Euclidean norm* will always be denoted by $\|\cdot\|$ and by λ_n we indicate the *Lebesgue measure* on $\mathbb{R}^n$.

The *Euclidean unit ball* and *sphere* are denoted by B^n and S^{n-1} and their volume and surface content are given by

$$\kappa_n = \lambda_n(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \qquad \text{and} \qquad \omega_n = \mathcal{H}^{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{2.1}$$

respectively. Here, $\mathcal{H}^{n-1}$ is the $(n-1)$-*dimensional Hausdorff measure* in $\mathbb{R}^n$. Further, for a set $E \subset \mathbb{R}^n$ we define the distance $d(o,E) = \inf_{x\in E} \|x\|$ of E to the origin $o \in \mathbb{R}^n$. We will make use of the *incomplete beta function*

$$B(x;\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \qquad 0 \le x \le 1,$$

with real parameters $\alpha,\beta > 0$. Notice that the *complete beta integral* satisfies

$$B\big(\tfrac{m}{2},\tfrac{k}{2}\big) = B\big(1;\tfrac{m}{2},\tfrac{k}{2}\big) = 2\frac{\omega_{m+k}}{\omega_m\omega_k}, \qquad m,k \in \mathbb{N}. \tag{2.2}$$

For $n \ge 1$ and $q \in \{0,\dots,n\}$ we let $G(n,q)$ be the *Grassmannian* of all q-dimensional linear subspaces of $\mathbb{R}^n$ and write $A(n,q)$ for the family of q-dimensional affine subspaces of $\mathbb{R}^n$. We endow these spaces with the usual Borel σ-algebras and describe now shortly invariant measures on these families. Details can be found e.g. in [22], where also measurablility issues are discussed.

The group SO_n of rotations in $\mathbb{R}^n$ carries a unique invariant (or *Haar*) probability measure ν. This group acts naturally on the space $G(n,q)$ and we denote by ν_q the unique SO_n-invariant probability measure on $G(n,q)$. Both, SO_n and the group of translations act naturally on the *affine Grassmannian* $A(n,q)$. There exists a motion invariant measure on $A(n,q)$, and this measure is unique up to a multiplicative constant. We will use the motion invariant measure μ_q on the affine Grassmannian $A(n,q)$ given by

$$\mu_q(\,\cdot\,) = \int_{G(n,q)} \int_{L^\perp} \mathbb{1}_{\{L+x\in\,\cdot\,\}}\, \lambda_{L^\perp}(dx)\nu_q(dL), \tag{2.3}$$

where $\lambda_{L^\perp}$ denotes the Lebesgue measure on the orthogonal complement $L^\perp$ of $L \in G(n,q)$. Finally, for $M \in A(n,q)$ and $p \in \{0,\dots,q\}$ we denote by $G(M,p)$ and $A(M,p)$ the *relative Grassmannian* of all p-dimensional linear and affine subspaces contained in M, respectively. If, on the other hand, $p \in \{q,\dots,n\}$ then $G(M,p)$ and $A(M,p)$ are the sets of linear and affine subspaces of dimension p that contain M. These spaces carry natural invariant measures ν_p^M and μ_p^M as described in [22, Sec. 7.1]. In particular, these measures satisfy

$$\int_{G(n,q)} \int_{G(M,p)} f(L)\, \nu_p^M(dL)\, \nu_q(dM) = \int_{G(n,p)} f(L)\, \nu_p(dL)$$

for all measurable functions $f : G(n,p) \to [0,\infty)$, and similarly

$$\int_{A(n,q)} \int_{A(F,p)} f(E)\, \mu_p^F(dE)\, \mu_q(dF) = \int_{A(n,p)} f(E)\, \mu_p(dE)$$

for all measurable functions $f : A(n,p) \to [0,\infty)$, see [22, Thm. 7.1.1 and Thm. 7.1.2].

Let $0 \le p,q \le n-1$ and fix $L \in G(n,p)$ and $M \in G(n,q)$. If $p+q \le n$ the *subspace determinant* $[L,M]$ is defined as the $(p+q)$-volume of a parallelepiped spanned by the union of an orthonormal basis in L and an orthonormal basis in M. If $p+q \ge n$ we define $[L,M] = [L^\perp, M^\perp]$. If $p+q = n$, both definitions coincide and $[L,M]$ is the factor by which the p-volume is multiplied under the orthogonal projection from L onto $M^\perp$ thus,

$$\int_{L^\perp} f(x)\, \lambda_{L^\perp}(dx) = [M,L] \int_M f(x|L^\perp)\, \lambda_M(dx), \tag{2.4}$$

for measurable $f : L^\perp \to [0,\infty)$. Here, $x|L^\perp$ denotes the orthogonal projection of a point $x \in \mathbb{R}^n$ onto $L^\perp$. For further background on subspace determinants, we refer to [22, Sec. 14.1].

3 Preliminary considerations

Let $n \geq 1$ and $q \in \{1, \dots, n-1\}$. The rotation group SO_n acts on the space of real-valued functions f on $A(n,q)$ by

$$(\eta f)(E) = f(\eta^{-1}E), \qquad E \in A(n,q),$$

for $\eta \in SO_n$. The function f is called *rotation invariant* if $\eta f = f$ for any rotation $\eta \in SO_n$. The *rotational mean* of f, given by

$$f_{\mathrm{rot}}(E) = \int_{SO_n} (\eta f)(E)\, \nu(d\eta), \qquad E \in A(n,q),$$

is rotation invariant.

Assume that f is rotation invariant and $n \geq 2$. Then, we have $f(E) = f(E')$ for any two affine subspaces $E, E' \in A(n,q)$ with $d(o,E) = d(o,E')$, since there is a rotation $\eta \in SO_n$ with $\eta E = E'$. This implies that there is a function $f_I : [0,\infty) \to \mathbb{R}$, such that

$$f(E) = f_I(d(o,E)), \qquad E \in A(n,q). \tag{3.1}$$

The following lemma shows that for our purposes, it is essentially enough to consider the class of rotation invariant functions.

Lemma 3.1. *Let $n \geq 2$, $q \in \{1, \dots, n-1\}$, $\gamma \in \{0, \dots, q-1\}$, a rotation invariant measure $\tilde{\mu}_{n-q+\gamma}$ on $A(n, n-q+\gamma)$ and a rotation invariant function $J : A(n,\gamma) \to [0,\infty)$ be given. Then*

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L)\, \tilde{\mu}_{n-q+\gamma}(dE)\, \nu(dL) = \int_{A(n,\gamma)} f(E)\, J(E)\, \mu_\gamma(dE) \tag{3.2}$$

holds for all measurable functions $f : A(n,\gamma) \to [0,\infty)$, if it holds for those f that are in addition rotation invariant.

Proof. Suppose that (3.2) holds true for all non-negative, measurable and rotation invariant functions. Fix an arbitrary measurable function $f : A(n,q) \to [0,\infty)$. Using the rotation invariance of the measures $\tilde{\mu}_{n-q+\gamma}$ and ν_q and Tonelli's theorem, the left-hand side of (3.2) is

$$\begin{aligned} &\int_{SO_n} \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(\eta^{-1}(E \cap L))\, \tilde{\mu}_{n-q+\gamma}(dE)\, \nu_q(dL) \nu(d\eta) \\ &= \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f_{\mathrm{rot}}(E)\, \tilde{\mu}_{n-q+\gamma}(dE)\, \nu_q(dL). \end{aligned}$$

Using the rotation invariance of μ_γ and J, a similar argument shows that the right-hand side of (3.2) is

$$\int_{A(n,\gamma)} f_{\mathrm{rot}}(E) J(E)\, \mu_\gamma(dE).$$

By assumption, (3.2) holds for the rotation invariant function f_{rot}, so the last two displayed expressions coincide and the assertion is shown. □

In the proof of our main result, the following lemma will turn out to be of crucial importance. It can be seen a generalization of [12, Lem. 4.4]. In that result the authors prove that

$$A(n,k,r,\alpha) = \int_{G(n,k)} [F,L]^\alpha\, \nu_k(dL) = \prod_{i=0}^{n-r-1} \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{k-i+\alpha}{2})}{\Gamma(\frac{n-i+\alpha}{2})\Gamma(\frac{k-i}{2})}, \tag{3.3}$$

for $\alpha \geq 0$, $r,k \in \{1, \dots, n\}$ with $r + k \geq n$ and $F \in G(n,r)$, where the right-hand side is interpreted as 1 if $r = n$. The next lemma is a counterpart for the Grassmannian associated to a hyperplane. (We note that the definition of the subspace determinant in [12] is equivalent to our definition as $r + k \geq n$.)

Lemma 3.2. *Let $n \geq 2$, $\alpha \geq 0$ and $p, q \in \{1, \dots, n-1\}$ with $p + q \leq n$ be given. Then, for any $u \in S^{n-1}$ and any fixed $M \in G(n, p)$ we have that*

$$\int_{G(\operatorname{span} u, q)} [L, M]^{\alpha} \, \nu_q^{\operatorname{span} u}(dL) = a(n, p, q, \alpha)[u, M]^{\alpha} \tag{3.4}$$

with

$$a(n, p, q, \alpha) = \prod_{i=1}^{p} \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{n-q-i+\alpha+1}{2})}{\Gamma(\frac{n-i+\alpha}{2})\Gamma(\frac{n-q-i+1}{2})}.$$

Proof. If $u \in M$, then (3.4) holds trivially as both sides vanish. Hence, we may assume $u \notin M$. Suppose first that we also have $u \notin M^{\perp}$ and fix $L \in G(\operatorname{span} u, q)$. Since $\dim M^{\perp} + \dim L^{\perp} = 2n - (p + q) \geq n$, [19, Lem. 4.1] implies that

$$\begin{aligned}[L^{\perp}, M^{\perp}] &= [L^{\perp}, M^{\perp} \cap u^{\perp}] \| u | M^{\perp} \| \\ &= [L^{\perp}, M^{\perp} \cap u^{\perp}][u, M],\end{aligned}$$

where we recall that $u|M^{\perp}$ is the projection of u onto $M^{\perp}$. Here, we used that fact that the subspace determinant $[L', M']$, as defined in [19], coincides with our definition whenever $\dim M' + \dim L' \geq n$, but differs otherwise, causing [19] to consider subspace determinants relative to $u^{\perp}$, which is not necessary using our definition. We thus obtain

$$[L, M] = [L^{\perp}, M^{\perp}] = [L^{\perp}, M^{\perp} \cap u^{\perp}][u, M],$$

a relation that is also true for $u \in M^{\perp}$, implying that the left-hand side of (3.4) coincides with

$$\int_{G(\operatorname{span} u, q)} [L^{\perp}, M^{\perp} \cap u^{\perp}]^{a} \, \nu_q^{\operatorname{span} u}(dL) \, [u, M]^{\alpha} = \int_{G(u^{\perp}, n-q)} [L, M^{\perp} \cap u^{\perp}]^{a} \, \nu_{q-1}^{u^{\perp}}(dL) \, [u, M]^{\alpha}.$$

The last integral is now of the form (3.3), but with $u^{\perp}$ instead of $\mathbb{R}^n$ as the ambient space. It is thus equal to $A(n-1, n-q, n-p-1, \alpha)$. This constant coincides with $a(n, p, q, \alpha)$, and the assertion is proven. □

4 Presentation of the main results

Having established the basic notions and concepts in Section 3, we will now state our main result: a general reduction of integrals of the form (1.3). We remark that we imposed the dimensional constraints $0 \leq \gamma < q < n$ at the beginning of the introduction, since $\gamma = q$ or $q = n$, would imply that the left-hand side of (1.3) becomes trivial. These constraints imply the assumption $n \geq 2$, which we now adopt for the rest of the paper.
Recall that (1.3) involves a rotation invariant measure $\tilde{\mu}_{n-q+\gamma}$ on $A(n, n-q+\gamma)$ that is dominated by $\mu_{n-q+\gamma}$. Hence, the Radon–Nikodym theorem guarantees the existence of a $\mu_{n-q+\gamma}$-density $\tilde{H} \geq 0$ for $\tilde{\mu}_{n-q+\gamma}$. The assumed rotation invariance implies that $H = \tilde{H}_{\mathrm{rot}}$ is also a $\mu_{n-q+\gamma}$-density for $\tilde{\mu}_{n-q+\gamma}$. We will state our results using this density, as the function $J(r) = J_H(r)$ can explicitly be expressed in terms of H. For specific choices of the density H, the function $J_H(r)$ in the statement of this theorem can be simplified and be made more explicit, as illustrated in Corollaries 4.2 and 4.4 below. The proof of Theorem 4.1 is postponed to Section 7 at the end of this paper.

Theorem 4.1. *Fix $n \geq 2$, $q \in \{1, \dots, n-1\}$, $\gamma \in \{0, \dots, q-1\}$, and let $H : A(n, n-q+\gamma) \to [0, \infty)$ be a measurable and rotation invariant function. Then*

$$\begin{aligned}&\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL) \\ &= D(n, q, \gamma) \int_{A(n,\gamma)} f(E) d(o, E)^{-(n-q)} J_H(d(o, E)) \, \mu_{\gamma}(dE)\end{aligned}$$

for all measurable $f : A(n,\gamma) \to [0,\infty)$. *Here,*

$$J_H(r) = \int_0^1 H_I(rz) z^q (1-z^2)^{\frac{n-q}{2}-1}\, dz, \tag{4.1}$$

and the constant is given by

$$D(n,q,\gamma) = \frac{\omega_{\gamma+1}\omega_{q-\gamma}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}\omega_{n-\gamma}}. \tag{4.2}$$

We shall now discuss two special cases in which the function $J_H(r)$ and thus the integral relation in Theorem 4.1 can be simplified. We start by considering the special case, where H is the function $H_h(E) = \mathbb{1}_{\{d(o,E)\le h\}}$ for some $h > 0$, corresponding to $\tilde{\mu}_{n-q+\gamma}$ in (1.3) being the restriction of $\mu_{n-q+\gamma}$ to $[hB^n]_{n-q+\gamma}$. Definition (4.1) and a substitution yield

$$J_{H_h}(r) = \frac{1}{2}\int_0^{(\min\{\frac{h}{r},1\})^2} z^{\frac{q+1}{2}-1}(1-z)^{\frac{n-q}{2}-1}\, dz = \begin{cases} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}} & : r \le h, \\ \frac{1}{2}B\big((\frac{h}{r})^2; \frac{q+1}{2}, \frac{n-q}{2}\big) & : r > h, \end{cases} \tag{4.3}$$

where (2.2) was used at the second equality sign. This implies the following result.

Corollary 4.2. *Fix* $n \ge 2$, $q \in \{1,\dots,n-1\}$, $\gamma \in \{0,\dots,q-1\}$, *and* $h > 0$. *Then*

$$\begin{aligned}&\int_{G(n,q)}\int_{[hB^n]_{n-q+\gamma}} f(E\cap L)\,\mu_{n-q+\gamma}(dE)\,\nu_q(dL)\\ &= D(n,q,\gamma)\int_{A(n,\gamma)} f(E) d(o,E)^{-(n-q)} J(d(o,E))\,\mu_\gamma(dE),\end{aligned}$$

where $J = J_{H_h}$ *is given by* (4.3) *and where* $D(n,q,\gamma)$ *is the constant given in Theorem* 4.1.

The incomplete beta integral, and hence the function J in Corollary 4.2, can be expressed in terms of a hypergeometric function. Also, weight functions J_H associated to more general densities H can be expressed in terms of – possibly several – hypergeometric functions. For instance, if H_I is an even polynomial restricted to $[0,\infty)$, Euler's integral relation implies such a representation, see e.g. [1] for details.
Using the monotone convergence theorem when $h \to \infty$ in Corollary 4.2, gives an explicit integral relation of the form (1.3) with $\tilde{\mu}_{n-q+\gamma} = \mu_{n-q+\gamma}$.

Corollary 4.3. *Fix* $n \ge 2$, $q \in \{1,\dots,n-1\}$ *and* $\gamma \in \{0,\dots,q-1\}$. *Then*

$$\begin{aligned}&\int_{G(n,q)}\int_{A(n,n-q+\gamma)} f(E\cap L)\,\mu_{n-q+\gamma}(dE)\,\nu_q(dL)\\ &= \tilde{D}(n,q,\gamma)\int_{A(n,\gamma)} f(E) d(o,E)^{-(n-q)}\,\mu_\gamma(dE)\end{aligned}$$

for all measurable functions $f : A(n,\gamma) \to [0,\infty)$. *Here,*

$$\tilde{D}(n,q,\gamma) = \frac{\omega_{n+1}\omega_{\gamma+1}\omega_{q-\gamma}}{\omega_{n-(q-\gamma)+1}\omega_{n-\gamma}\omega_{q+1}}.$$

Let us also report that Theorem 4.1 allows an extension to multiple intersections in the following way. The proof is postponed to Section 7.

Corollary 4.4. *Fix* $\ell, m, n \ge 1$, $q_1,\dots,q_\ell \in \{0,\dots,n\}$, *put* $q = q_1+\dots+q_\ell-(\ell-1)n$. *Suppose that* $q \in \{1,\dots,n-1\}$ *and fix* $\gamma \in \{0,\dots,q-1\}$ *and* $p_1,\dots,p_m \in \{0,\dots,n\}$ *such that* $p_1+\dots+p_m-(m-1)n = n-q+\gamma$. *Further, let* $f : A(n,\gamma) \to [0,\infty)$ *be a measurable function, and let* $H : A(n,n-q+\gamma) \to [0,\infty)$ *be a rotation invariant measurable function. Define*

$$\begin{aligned}I_{\ell,m} = \int_{G(n,q_1)}\cdots\int_{G(n,q_\ell)}\int_{A(n,p_1)}\cdots\int_{A(n,p_m)} & f(E_1\cap\ldots\cap E_m\cap L_1\cap\ldots\cap L_\ell)\\ &\times H(E_1\cap\ldots\cap E_m)\,\mu_{p_m}(dE_m)\ldots\mu_{p_1}(dE_1)\nu_{q_\ell}(dL_\ell)\ldots\nu_{q_1}(dL_1).\end{aligned}$$

Then

$$I_{\ell,m} = D(n,q,\gamma)\frac{\omega_{n-q+\gamma+1}\omega_{n+1}^m}{\omega_{n+1}\omega_{p_1}\cdots\omega_{p_m}}\int_{A(n,\gamma)} f(E)d(o,E)^{-(n-q)}J_H(d(o,E))\,\mu_\gamma(dE),$$

where J_H is given by (4.1) and the leading constant by (4.2).

Finally, we state an alternative version of Theorem 4.1, which appears to be more general, as the intersecting linear subspace now is fixed. Closer investigation shows, however, that the two statements are equivalent, if a suitable Blaschke–Petkantschin relation is applied. The proof of this equivalence will be given in Section 7.

Theorem 4.5. *Fix $n \geq 2$, $q \in \{1,\dots,n-1\}$, $\gamma \in \{0,\dots,q-1\}$ and $L_0 \in G(n,q)$. Let $H : A(n,n-q+\gamma) \to [0,\infty)$ be a rotation invariant, measurable function. Then*

$$\int_{A(n,n-q+\gamma)} f(E\cap L_0)H(E)\,\mu_{n-q+\gamma}(dE) = \frac{\omega_{\gamma+1}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}}\int_{A(L_0,\gamma)} f(E)J_H(d(o,E))\,\mu_\gamma^{L_0}(dE) \tag{4.4}$$

for all measurable $f : A(L_0,\gamma) \to [0,\infty)$. Here, J_H is given in Theorem 4.1.

Similar versions, that is, versions where the linear space is hold fixed, are possible also for Corollaries 4.2-4.4.

5 Intersection probabilities for linear and affine subspaces hitting the unit ball

In this section we return to the problem from stochastic geometry already mentioned in the introduction. Namely, we consider a random linear subspace L of dimension $q \in \{1,\dots,n-1\}$ in $\mathbb{R}^n$ with distribution ν_q and a random affine subspace E of dimension $n-q+\gamma$, where $\gamma \in \{0,\dots,q-1\}$ is a fixed number. Recalling that

$$[B^n]_{n-q+\gamma} = \{E \in A(n,n-q+\gamma) : E\cap B^n \neq \emptyset\},$$

we use the restriction of $\kappa_{q-\gamma}^{-1}\mu_{n-q+\gamma}$ to $[B^n]_{n-q+\gamma}$ as distribution for E and suppose that L and E are stochastically independent. In fact, it follows from (2.3) that $\kappa_{q-\gamma}^{-1}\mu_{n-q+\gamma}$ is indeed a probability measure on $[B^n]_{n-q+\gamma}$. We are interested in the distribution of the random affine subspace $E\cap L$, which is almost surely of dimension γ. Since its distribution is invariant under rotations of $\mathbb{R}^n$, all relevant information is contained in the distribution of the random variable $d(o,E\cap L)$ describing the distance of $E\cap L$ to the origin. This distribution turns out to have a density $f_{n,q,\gamma}$ with respect to Lebesgue measure on $[0,\infty)$. The next result shows that it is heavy tailed and asymptotically of Pareto type with shape parameter $\gamma+2$.

Theorem 5.1. *In the set-up just introduced, the probability density $f_{n,q,\gamma}(\delta)$ of the random variable $d(o,E\cap L)$ is given by*

$$f_{n,q,\gamma}(\delta) = (q-\gamma)\frac{\omega_{\gamma+1}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}}\begin{cases}\frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}}\delta^{q-\gamma-1} & : 0\leq\delta\leq 1\\ \frac{1}{2}\delta^{q-\gamma-1}B(\frac{1}{\delta^2};\frac{q+1}{2},\frac{n-q}{2}) & : \delta > 1.\end{cases}$$

Proof. Fix $\delta > 0$ and consider the probability of the event $d(o,E\cap L)\leq\delta$. Using Corollary 4.2 with $h=1$, we obtain

$$\begin{aligned}\mathbb{P}[d(o,E\cap L)\leq\delta] &= \frac{1}{\kappa_{q-\gamma}}\int_{G(n,q)}\int_{A(n,n-q+\gamma)} \mathbf{1}_{\{d(o,E)\leq 1\}}\,\mathbf{1}_{\{d(o,E\cap L)\leq\delta\}}\,\mu_{n-q+\gamma}(dE)\nu_q(dL)\\ &= \frac{D(n,q,\gamma)}{\kappa_{q-\gamma}}\int_{A(n,\gamma)} \mathbf{1}_{\{d(o,E)\leq\delta\}}d(o,E)^{-(n-q)}J_{H_1}(d(o,E))\,\mu_\gamma(dE),\end{aligned}$$

with

$$J_{H_1}(r) = \begin{cases} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}} & : r \le 1 \\ \frac{1}{2}B(\frac{1}{r^2}; \frac{q+1}{2}, \frac{n-q}{2}) & : r > 1. \end{cases}$$

We now apply the decomposition (2.3), use $d(o, L + x) = \|x\|$ whenever $x \in L^\perp$, and then introduce spherical coordinates in $L^\perp$ to see that

$$\begin{aligned} &\mathbb{P}[d(o, E \cap L) \le \delta] \\ &= \frac{D(n,q,\gamma)}{\kappa_{q-\gamma}} \int_{G(n,\gamma)} \int_{L^\perp} \mathbf{1}_{\{\|x\| \le \delta\}} \|x\|^{-(n-q)} J_{H_1}(\|x\|)\, \lambda_{L^\perp}(dx) \nu_\gamma(dL) \\ &= \frac{D(n,q,\gamma)\omega_{n-\gamma}}{\kappa_{q-\gamma}} \int_0^\delta r^{q-\gamma-1} J_{H_1}(r)\, dr. \end{aligned}$$

Taking the derivative with respect to δ in the last expression, inserting the value of the constant $D(n,q,\gamma)$ of Theorem 4.1 and then using (2.1) yields the result. □

Although the density $f_{n,q,\gamma}(\delta)$ is defined piecewise, we remark that it is continuous at the point $\delta = 1$. In fact, this follows from the continuity of the incomplete beta function in combination with (2.2). Let us also mention at this point that since we are working with the unit ball B^n as a reference set, the results of this section and also the next one continue to hold if the random linear subspace L is replaced by a deterministic linear subspace of the same dimension.
Since the density of $d(o, E \cap L)$ is asymptotically of Pareto type with shape parameter $\gamma + 2$, the moment properties of the random variable $d(o, E \cap L)$ depend on the intersection dimension γ. The next result delivers a precise description.

Corollary 5.2. *Let $\alpha \in \mathbb{R}$. In the set-up just introduced, one has that $\mathbb{E}d(o, E \cap L)^\alpha < \infty$ if and only if $\alpha \in (\gamma - q, \gamma + 1)$. In particular, if $\gamma = 0$ the random variable $d(o, E \cap L)$ has infinite expectation.*

Proof. We have to check under which conditions on α the product of δ^α with the probability density $f_{n,q,\gamma}(\delta)$ of Theorem 5.1 is integrable at $\delta = 0$ and $\delta = \infty$. The function $\delta \mapsto \delta^{q-\gamma-1+\alpha}$ is integrable at $\delta = 0$, if and only if $\alpha > \gamma - q$. Moreover, since $B(\delta^{-2}; a, b) = \frac{\delta^{-2a}}{a} + O(\delta^{-2(a+1)})$ as $\delta \to \infty$, see [1, §6.6.8 and §15.7], the required integrability is satisfied, whenever the function $\delta^{\alpha+q-\gamma-1-(q+1)} = \delta^{\alpha-\gamma-2}$ is integrable at $\delta = \infty$. The latter holds if and only if $\alpha - \gamma - 2 < -1$, or equivalently, $\alpha < \gamma + 1$. This completes the proof. □

As anticipated above, the intersection of E and L may or may not hit the unit ball. By Theorem 5.1 the probability for the first of these events is given by

$$\mathbb{P}[E \cap L \cap B^n \neq \emptyset] = (q - \gamma) \frac{\omega_{\gamma+1}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}} \int_0^1 \delta^{q-\gamma-1}\, d\delta.$$

Simplification leads to the following result.

Corollary 5.3. *In the set-up just introduced, it holds that*

$$p_{n,q,\gamma} = \mathbb{P}[E \cap L \cap B^n \neq \emptyset] = \frac{\omega_{\gamma+1}\omega_{n+1}}{\omega_{q+1}\omega_{n-(q-\gamma)+1}}.$$

This result allows quantifying the asymptotic behavior of the intersection probability $p_{n,q,\gamma}$ for fixed q and γ in high dimensions, that is, as $n \to \infty$. Using (2.1), we can rewrite $p_{n,q,\gamma}$ in terms of gamma functions as

$$p_{n,q,\gamma} = \frac{\Gamma(\frac{q+1}{2})\Gamma(\frac{n-(q-\gamma)+1}{2})}{\Gamma(\frac{\gamma+1}{2})\Gamma(\frac{n+1}{2})}.$$

Since q and γ are assumed to be fixed, we can apply Stirling's formula [1, §6.1.37] for the gamma function to see that

$$p_{n,q,\gamma} = \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{\gamma+1}{2})}\Big(\frac{2}{n}\Big)^{\frac{q-\gamma}{2}}(1+o_n(1)) = \frac{\omega_{\gamma+1}}{\omega_{q+1}}\Big(\frac{2}{\pi\, n}\Big)^{\frac{q-\gamma}{2}}(1+o_n(1)),$$

as $n \to \infty$, where we write $o_n(1)$ for a sequence that converges to zero with n. Hence, in high dimensions, the intersection point of the random affine subspace E and the random linear subspace L will asymptotically almost surely be outside the unit ball B^n.

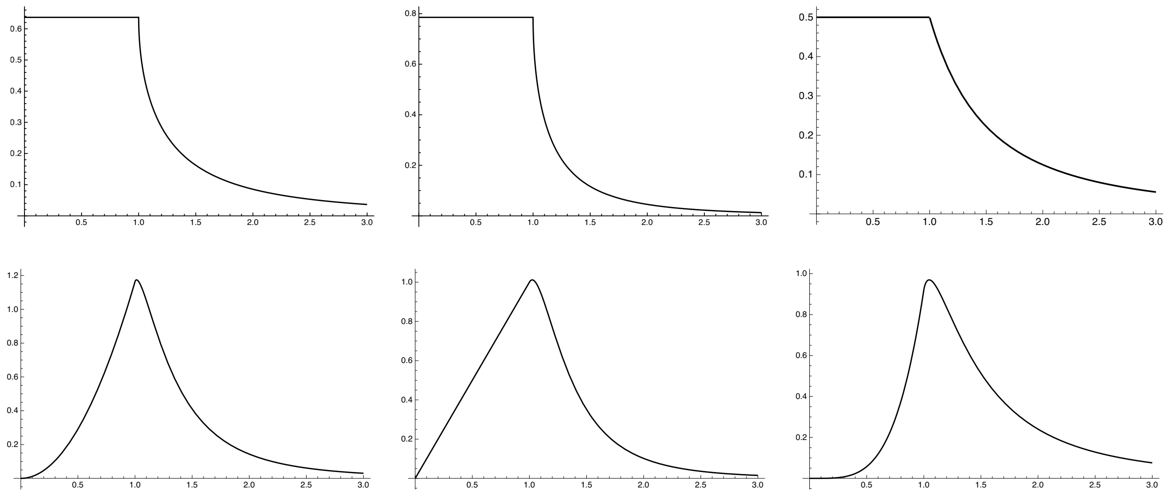

Figure 5.1: First line: The probability densities $f_{2,1,0}$, $f_{3,2,1}$ and $f_{3,1,0}$ from left to right. Second line: The probability densities $f_{8,5,2}$, $f_{9,5,3}$ and $f_{9,6,1}$ from left to right.

In the planar case $n = 2$ we can take $q = 1$ and $\gamma = 2$ and obtain

$$f_{2,1,0}(\delta) = \frac{2}{\pi}\begin{cases}1 & : 0 \le \delta \le 1\\ 1-\sqrt{1-\delta^{-2}} & : \delta > 1,\end{cases} \qquad p_{2,1,0} = \frac{2}{\pi}.$$

Moreover, if $n = 3$ we can consider the case $q = 2$ and $\gamma = 1$, which yields

$$f_{3,2,1}(\delta) = \begin{cases}\frac{\pi}{4} & : 0 \le \delta \le 1\\ \frac{1}{2}(\operatorname{arccsc}(\delta) - \sqrt{1-\delta^{-2}}) & : \delta > 1,\end{cases} \qquad p_{3,2,1} = \frac{\pi}{4},$$

and the case $q = 1$ and $\gamma = 0$, where the probability density reduces to

$$f_{3,1,0}(\delta) = \frac{1}{2}\begin{cases}1 & : 0 \le \delta \le 1\\ \delta^{-2} & : \delta > 1,\end{cases} \qquad p_{3,1,0} = \frac{1}{2},$$

see also Figure 5.1. In particular, if $n = 3$, $q = 2$ and $\gamma = 1$, we obtain

$$\mathbb{E}d(o, E\cap L) = \int_0^\infty \delta f_{3,2,1}(\delta)\, d\delta = \frac{\pi}{4},$$

whereas in the other two cases $\mathbb{E}d(o, E\cap L) = \infty$. To illustrate the potential complexity of the density functions $f_{n,q,\gamma}(\delta)$ we also record the following values:

n,q,γ	$f_{n,q,\gamma}$	$p_{n,q,\gamma}$	$\mathbb{E}d(o,E\cap L)$
$8,5,2$	$\begin{cases} \frac{128}{35\pi}\delta^2 & : 0\le\delta\le 1 \\ \frac{48}{105\pi}\frac{8\delta^7-(8\delta^6+4\delta^4+3\delta^2-15)\sqrt{\delta^2-1}}{\delta^5} & : \delta>1, \end{cases}$	$\frac{128}{105\pi}$	$\frac{4}{\pi}$
$9,5,3$	$\begin{cases} \delta & : 0\le\delta\le 1 \\ \frac{4\delta^2-3}{\delta^7} & : \delta>1, \end{cases}$	$\frac{1}{2}$	$\frac{16}{15}$
$9,6,1$	$\begin{cases} \frac{75\pi}{265}\delta^4 & : 0\le\delta\le 1 \\ \frac{75\delta^8\operatorname{arccsc}(\delta)-5(16\delta^6+10\delta^4+8\delta^2-48)\sqrt{\delta^2-1}}{128\delta^4} & : \delta>1. \end{cases}$	$\frac{15\pi}{256}$	$\frac{5\pi}{8}$

We see that the complexity of the probability density $f_{n,q,\gamma}$ varies depending on n,q,γ. We notice in particular that $p_{9,5,3}=\frac{1}{2}$, which is the same as $p_{3,1,0}$ from before. In fact, for all odd n, there are (often multiple) ways we can choose q and γ such that $p_{n,q,\gamma}=\frac{1}{2}$. For instance, if we take $\gamma=\frac{n-1}{2}-1$ and $q=\frac{n-1}{2}+1$ or $q=n-2$, then $p_{n,q,\gamma}=\frac{1}{2}$.

6 Intersection probabilities for linear and affine subspaces tangent to the unit sphere

In this section, we consider another application of Theorem 4.1 to intersection probabilities in stochastic geometry. As in the previous section, we consider a random linear subspace L of dimension $q\in\{1,\dots,n-1\}$ in $\mathbb{R}^n$ with distribution ν_q. In contrast, E is now a stochastically independent random affine subspace E in $\mathbb{R}^n$ of dimension $n-q+\gamma$ such that $d(o,E)=1$. Thus, E is tangent to the unit sphere S^{n-1} in $\mathbb{R}^n$. The rotation invariant probability measure on that space is given by

$$\sigma_{n-q+\gamma}(A)=\frac{1}{\omega_{q-\gamma}}\int_{G(n,n-q+\gamma)}\int_{S^{n-1}\cap M^\perp}\mathbb{1}_A(M+u)\,\mathcal{H}^{q-\gamma-1}(du)\,\nu_{n-q+\gamma}(dM) \tag{6.1}$$

for a Borel sets $A\subseteq\{E'\in A(n,n-q+\gamma)\ :\ d(o,E')=1\}$. As in the previous section, we are interested in the distribution of $L\cap E$. Again, all relevant information is contained in the distribution of $d(o,E\cap L)$ due to rotation invariance. The next result is a stepping stone in the derivation of the probability density of the random variable $d(o,E\cap L)$.

Lemma 6.1. *Fix $n\ge 2$, $q\in\{1,\dots,n-1\}$, $\gamma\in\{0,\dots,q-1\}$ and let $f:A(n,\gamma)\to[0,\infty)$ be a measurable, rotation invariant and bounded function. Then*

$$\begin{aligned}\int_{G(n,q)}\int_{G(n,n-q+\gamma)}\int_{S^{n-1}\cap M^\perp}& f((M+hu)\cap L)\,\mathcal{H}^{q-\gamma-1}(du)\,\nu_{n-q+\gamma}(dM)\,\nu_q(dL)\\ &=D(n,q,\gamma)\omega_{n-\gamma}h^{\gamma+1}\int_h^\infty f_I(r)r^{-(\gamma+2)}\left(1-\frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1}dr\end{aligned}$$

for almost every $h>0$.

Proof. Let $f:A(n,\gamma)\to[0,\infty)$ satisfy the above assumptions and define the function $F:(0,\infty)\to[0,\infty)$ by

$$\begin{aligned}F(h)&=D(n,q,\gamma)\int_{A(n,\gamma)}f(E)d(o,E)^{-(n-q)}J_{H_h}(d(o,E))\,\mu_\gamma(dE)\\ &=D(n,q,\gamma)\omega_{n-\gamma}\int_0^\infty f_I(r)r^{q-\gamma-1}J_{H_h}(r)\,dr,\end{aligned}$$

with $D(n,q,\gamma)$ as in Theorem 4.1 and J_{H_h} given by (4.3). Since

$$\frac{\partial}{\partial h}J_{H_h}(r)=\begin{cases}0 & : 0\le r<h\\ r^{-q-1}h^q\left(1-\frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} & : r>h,\end{cases}$$

we conclude that $F(h)$ is differentiable for almost all $h > 0$ with derivative

$$F'(h) = D(n,q,\gamma)\omega_{n-\gamma} \int_h^\infty f_I(r) r^{-\gamma-2} h^q \left(1 - \frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} dr.$$

Here, differentiation under the integral sign can be justified as follows. For fixed $r > 0$, the integrand $g(r,h) = f_I(r) r^{q-\gamma-1} J_{H_h}(r)$ is absolutely continuous as function of h. Since $(r,h) \mapsto \partial/(\partial h) g(r,h)$ is integrable on $(0,\infty) \times (a,b)$ for any $0 < a < b < \infty$, Fubini's theorem and Lebesgue differentiation theorem imply

$$F'(h) = \frac{d}{dh} \int_1^h \int_0^\infty \frac{\partial}{\partial h} g(r,t) dr dt = \int_0^\infty \frac{\partial}{\partial h} g(r,h) dr$$

for a.e. $h \in (a,b)$, yielding the intermediate assertion.
On the other hand, Theorem 4.1, relation (2.3) and spherical integration yield

$$F(h) = \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1}\cap M^\perp} \int_0^h f((M+ru)\cap L) r^{q-\gamma-1}\, dr\, \mathcal{H}^{q-\gamma-1}(du)\, \nu_{n-q+\gamma}(dM) \nu_q(dL),$$

and hence

$$F'(h) = \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1}\cap M^\perp} f((M+hu)\cap L) h^{q-\gamma-1}\, \mathcal{H}^{q-\gamma-1}(dy)\, \nu_{n-q+\gamma}(dM) \nu_q(dL)$$

for almost every $h > 0$ due to the Lebesgue differentiation theorem. This completes the proof of the lemma. $\square$

Theorem 6.2. *Fix $n \geq 2$, $q \in \{1,\dots,n-1\}$ and $\gamma \in \{0,\dots,q-1\}$. Let L be a random linear subspace with distribution ν_q and let E be a stochastically independent random affine subspace, tangent to the unit sphere, with distribution $\sigma_{n-q+\gamma}$ given by (6.1). Then the random variable $d(o, E\cap L)^{-2}$ has a beta distribution with shape parameters*

$$a = \frac{\gamma+1}{2} \qquad \text{and} \qquad b = \frac{n-q}{2}.$$

More explicitly,

$$g_{n,q,\gamma}(r) = \frac{\omega_{\gamma+1}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}} r^{-(\gamma+2)} \left(1 - \frac{1}{r^2}\right)^{\frac{n-q}{2}-1} \mathbb{1}_{\{r>1\}}$$

is a probability density for $d(o, E\cap L)$.

Figure 6.1 illustrates the variety of density functions $g_{n,q,\gamma}$.

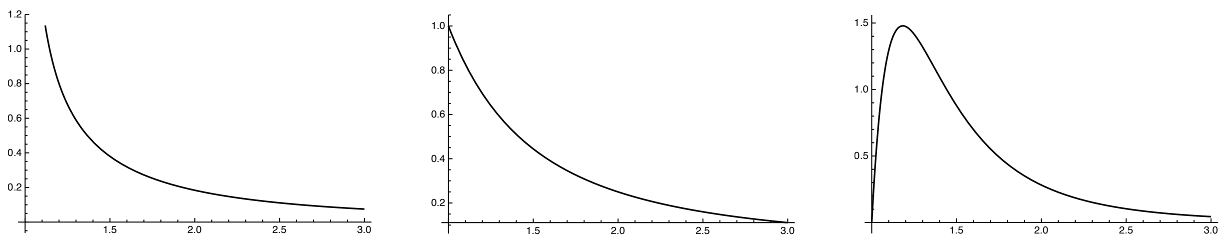

Figure 6.1: From left to right: The probability densities $g_{2,1,0}$, $g_{3,1,0}$ and $g_{9,5,3}$.

Proof. Let X be a beta-distributed random variable on $(0,1)$ with shape parameters $a, b > 0$. Then $X^{-1/2}$ has tail probabilities

$$\begin{aligned} 1 - F_{X^{-1/2}}(t) &= \mathbb{P}\Big[X \le \frac{1}{t^2}\Big] \\ &= \frac{1}{B(a,b)} \int_0^{1/t^2} x^{a-1}(1-x)^{b-1}\, dx \\ &= \frac{2}{B(a,b)} \int_t^\infty r^{-2a-1}\Big(1 - \frac{1}{r^2}\Big)^{b-1} dr, \qquad t > 1. \end{aligned} \tag{6.2}$$

On the other hand, applying Lemma 6.1 with $f(E) = 1_{\{d(o,E)\ge t\}}$ yields

$$\begin{aligned} \frac{1}{\omega_{q-\gamma}} \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1}\cap M^\perp} &1_{\{d(o,(M+hu)\cap L)\ge t\}}\, \mathcal{H}^{q-\gamma-1}(du)\, \nu_{n-q+\gamma}(dM)\nu_q(dL) \\ &= D(n,q,\gamma)\frac{\omega_{n-\gamma}}{\omega_{q-\gamma}} h^{\gamma+1} \int_h^\infty 1_{\{r\ge t\}} r^{-(\gamma+2)}\Big(1 - \frac{h^2}{r^2}\Big)^{\frac{n-q}{2}-1} dr \end{aligned} \tag{6.3}$$

for almost all $h > 0$. Using monotone convergence, one can show that both sides of (6.3) are continuous in h, so this relation extends to all $h > 0$. Letting $h = 1$, we see that the tail distributions of $d(o, E \cap L)$ are proportional to

$$\int_t^\infty r^{-(\gamma+2)}\Big(1 - \frac{1}{r^2}\Big)^{\frac{n-q}{2}-1} dr, \qquad t > 1.$$

A comparison with (6.2) shows the first claim. The explicit density $g_{n,q,\gamma}$ is obtained by differentiation of (6.3) when $h = 1$. □

Let us remark in this context that random affine subspaces with the property that their squared distance to the origin follows a beta distribution arise naturally in the theory of random beta polytopes. For example, let $X_0, \dots, X_r$ with $r \in \{1, \dots, n\}$ be stochastically independent random points in B^n with probability density proportional to $(1 - \|x\|^2)^{\frac{\nu-2}{2}}$ for some $\nu > 0$, and consider the random variable $d(o, M)^2$, where $M \in A(n,r)$ is the affine hull of $X_0, \dots, X_r$. Then it follows from [10, Thm. 2.7] that this random variable has the beta distribution $\mathrm{Beta}(\frac{n-r}{2}, \frac{\nu(r+1)+r(n-1)}{2})$. From the explicit density of $d(o, E \cap L)$ in Theorem 6.2, the existence of moments can directly be read off.

Corollary 6.3. *Let $\alpha \in \mathbb{R}$. In the set-up just introduced, one has that $\mathbb{E}d(o, E\cap L)^\alpha < \infty$ if and only if $\alpha < \gamma + 1$. In particular, if $\gamma = 0$ then the random variable $d(o, E \cap L)$ has infinite expectation. If $\gamma \ge 1$ we have*

$$\mathbb{E}d(o, E \cap L) = \frac{\omega_{\gamma+1}\omega_{n-q+\gamma}}{\omega_\gamma \omega_{n-(q-\gamma)+1}} = \frac{\omega_{n-q+\gamma}}{\omega_\gamma}(2\pi)^{-(n-q)}.$$

Applying Stirling's formula [1, §6.1.37] we conclude that for fixed $\gamma \ge 1$,

$$\frac{\mathbb{E}d(o, E\cap L)}{\sqrt{n}} \longrightarrow \frac{1}{\sqrt{2\pi}} \frac{\omega_{\gamma+1}}{\omega_\gamma},$$

as $n \to \infty$, independently of q.

7 Proofs of Theorem 4.1, Corollary 4.4 and Theorem 4.5

Proof of Theorem 4.1. Given $E \in A(n,q)$, $q \in \{0, \dots, n\}$, we will write $\mathrm{lin}(E)$ for the linear subspace in $G(n,q)$ parallel to E. In other words, $\mathrm{lin}(E) = E - x$ for all $x \in E$.

For the proof of Theorem 4.1, fix $n \geq 1$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$, and let $f : A(n, \gamma) \to [0, \infty)$ be a measurable function. Due to Lemma 3.1, we may assume without loss of generality that f is rotation invariant. The integral of interest is

$$I = \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E)\, \mu_{n-q+\gamma}(dE)\, \nu_q(dL).$$

First note that I is indeed well-defined as for ν_q-almost all $L \in G(n, q)$ and $\mu_{n-q+\gamma}$-almost all $E \in A(n, n-q+\gamma)$ we have $\dim(E \cap L) = \gamma$. This follows from [22, Lem. 13.2.1], applied to L and $L' = \mathrm{lin}(E)$.
Define the function $g : A(n, q) \times A(n, n-q+\gamma) \to \mathbb{R}$ by

$$g(E_1, E_2) = f\big(\mathrm{lin}(E_1) \cap E_2\big) 1_{\{d(o,E_1) \leq 1\}} H(E_2),$$

where we let $f(\mathrm{lin}(E_1) \cap E_2) = 0$ if $\dim(\mathrm{lin}(E_1) \cap E_2) \neq \gamma$. Definition (2.3) implies

$$I = \frac{1}{\kappa_{n-q}} \int_{A(n,q)} \int_{A(n,n-q+\gamma)} g(E_1, E_2)\, \mu_{n-q+\gamma}(dE_2) \mu_q(dE_1).$$

Applying [22, Thm. 7.2.8] with $s_1 = q$ and $s_2 = n-q+\gamma$ yields

$$\begin{aligned} I = \frac{\bar{b}}{\kappa_{n-q}} \int_{A(n,\gamma)} \int_{A(E,q)} \int_{A(E,n-q+\gamma)} & g(E_1, E_2) [\mathrm{lin}(E_1), \mathrm{lin}(E_2)]^{\gamma+1} \\ & \times\, \mu^E_{n-q+\gamma}(dE_2)\, \mu^E_q(dE_1)\, \mu_\gamma(dE), \end{aligned} \tag{7.1}$$

with the constant $\bar{b}$ given by

$$\bar{b} = b_{n,n-\gamma} \frac{b_{n-\gamma,n-q} b_{n-\gamma,q-\gamma}}{b_{n,n-q} b_{n,q-\gamma}}, \qquad b_{i,j} = \frac{\omega_{i-j+1} \cdots \omega_i}{\omega_1 \cdots \omega_j},\ i \in \mathbb{N}, j \in \{1, \dots, i\}. \tag{7.2}$$

Expanding (7.1) by decomposing the measure μ_γ according to (2.3), applying [22, Eq. (13.14)] and Tonelli's theorem yields

$$I = \frac{\bar{b}}{\kappa_{n-q}} \int_{G(n,\gamma)} I_1(L_0, f)\, \nu_\gamma(dL_0) \tag{7.3}$$

with

$$\begin{aligned} I_1(L_0, f) = \int_{G(L_0,q)} \int_{G(L_0,n-q+\gamma)} \int_{L_0^\perp} & f((M+t) \cap L) 1_{\{d(o,L+t) \leq 1\}} H(M+t) \\ & \times\, \lambda_{L_0^\perp}(dt) [M, L]^{\gamma+1}\, \nu^{L_0}_{n-q+\gamma}(dM)\, \nu^{L_0}_q(dL). \end{aligned}$$

Since

$$\int_{G(L_0,p)} h(L)\, \nu^{L_0}_p(dL) = \int_{G(L_0^\perp,p-\gamma)} h(L_0 + L)\, \nu^{L_0^\perp}_{p-\gamma}(dL)$$

for all $L_0 \in G(n, \gamma)$ and for any measurable function $h : G(L_0, p) \to [0, \infty)$ and integers $\gamma < p < n$, we conclude

$$\begin{aligned} I_1(L_0, f) = \int_{G(L_0^\perp,q-\gamma)} \int_{G(L_0^\perp,n-q)} \int_{L_0^\perp} & f\Big(\big((M+t) \cap L\big) + L_0\Big) 1_{\{d(o,L+t) \leq 1\}} \\ & \times H(M + L_0 + t)\, \lambda_{L_0^\perp}(dt) [M, L]^{\gamma+1}\, \nu^{L_0^\perp}_{n-q}(dM)\, \nu^{L_0^\perp}_{q-\gamma}(dL), \end{aligned} \tag{7.4}$$

where we also have used $(M + L_0 + t) \cap (L + L_0) = \big((M + t) \cap L\big) + L_0$, $d(o, L + L_0 + t) = d(o, L + t)$ and $[M + L_0, L + L_0] = [M, L]$.

Now consider (7.4) for fixed $L_0 \in G(n,\gamma)$, $M \in G(L_0^\perp, n-q)$ and $L \in G(L_0^\perp, q-\gamma)$. As f and H are both rotation invariant, we may write

$$f(((M+t)\cap L)+L_0) = f_I\big(d(o,(M+t)\cap L)\big),$$
$$H(M+L_0+t) = H_I(\|t|M^\perp\|),$$

with f_I and H_I satisfying (3.1). Thus, we may identify $L_0^\perp$ with $\mathbb{R}^{n-\gamma}$ and conclude that

$$I_1(L_0,f) = \int_{G(n-\gamma,q-\gamma)}\int_{G(n-\gamma,n-q)} I_2(L,M)[M,L]^{\gamma+1}\nu_{n-q}(dM)\,\nu_{q-\gamma}(dL), \tag{7.5}$$

with

$$\begin{aligned} I_2(M,L) &= \int_{\mathbb{R}^{n-\gamma}} f_I(d(o,(M+t)\cap L))\mathbf{1}_{\{d(o,L+t)\le 1\}}H_I(\|t|M^\perp\|)\,\lambda_{n-\gamma}(dt)\\ &= \int_M\int_{M^\perp} f_I(d(o,(M+x)\cap L))H_I(\|x\|)\mathbf{1}_{\{d(o,L+x+y)\le 1\}}\,\lambda_{M^\perp}(dx)\,\lambda_M(dy) \end{aligned}$$

for $M \in G(n-\gamma,n-q)$ and $L \in G(n-\gamma,q-\gamma)$. Applying (2.4) to the Lebesgue integral over $M^\perp$ we get

$$\begin{aligned} I_2(M,L) &= [M,L]\int_M\int_L f_I(d(o,(M+t)\cap L))H_I(\|t|M^\perp\|)\mathbf{1}_{\{d(o,L+t|M^\perp+y)\le 1\}}\,\lambda_L(dt)\,\lambda_M(dy)\\ &= \int_L f_I(d(o,(M+t)\cap L))H_I(\|t|M^\perp\|)I_3(M,L,t)\,\lambda_L(dt), \end{aligned}$$

where we first used the fact that $M+t|M^\perp = M+t$ holds for any $t \in L$, and then Tonelli's theorem. Here,

$$I_3(M,L,t) = [M,L]\int_M \mathbf{1}_{\{\|(t|M^\perp)|L^\perp+y|L^\perp\|\le 1\}}\,\lambda_M(dy).$$

Another application of (2.4) reveals that

$$I_3(M,L,t) = \int_{L^\perp}\mathbf{1}_{\{\|(t|M^\perp)|L^\perp+z\|\le 1\}}\,\lambda_{L^\perp}(dz) = \kappa_{n-q}$$

is the volume of a unit ball in $L^\perp$, centered at $(t|M^\perp)|L^\perp$. Inserting this into I_2 gives

$$I_2(M,L) = \kappa_{n-q}\int_L f_I(d(o,(M+t)\cap L))H_I(\|t|M^\perp\|)\,\lambda_L(dt).$$

Since $t \in L$ and $M\cap L = \{o\}$ for almost all L and M, we have

$$d(o,(M+t)\cap L) = d(o,\{t\}) = \|t\|.$$

This, and the use of spherical coordinates in L give

$$I_2(M,L) = \kappa_{n-q}\int_0^\infty f_I(r)r^{q-\gamma-1}\int_{S^{n-\gamma-1}\cap L} H_I(r[u,M])\,\mathcal{H}^{q-\gamma-1}(du)\,dr.$$

Inserting this into (7.5) and the result into (7.3), we get after an application of Tonelli's theorem that

$$I = \bar{b}\int_0^\infty f_I(r)r^{q-\gamma-1}\int_{G(n-\gamma,n-q)} J_H(M,r)\,\nu_{n-q}(dM)\,dr, \tag{7.6}$$

with

$$J_H(M,r) = \int_{G(n-\gamma,q-\gamma)}\int_{S^{n-\gamma-1}\cap L} H_I(r[u,M])\mathcal{H}^{q-\gamma-1}(du)[M,L]^{\gamma+1}\,\nu_{q-\gamma}(dL).$$

An invariance argument and [22, Thm. 7.1.1] imply

$$J_H(M,r) = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \int_{S^{n-\gamma-1}} H_I(r[u,M]) \int_{G(\operatorname{span} u, q-\gamma)} [M,L]^{\gamma+1}\, \nu_{q-\gamma}^{\operatorname{span} u}(dL)\, \mathcal{H}^{n-\gamma-1}(du).$$

Applying Lemma 3.2 in $\mathbb{R}^{n-\gamma}$ to the innermost integral in $J_H(M,r)$ yields

$$J_H(M,r) = c_1 \int_{S^{n-\gamma-1}} H_I(r[u,M])[u,M]^{\gamma+1}\, \mathcal{H}^{n-\gamma-1}(du)$$

with the constant $c_1 = a(n-\gamma, n-q, q-\gamma, \gamma+1)\frac{\omega_{q-\gamma}}{\omega_{n-\gamma}}$. To simplify $J_H(M,r)$ further, we use the fact that $[u,M] = \|u|M^\perp\|$ and apply [2, Lem. 1] with $B_p = M^\perp$, $p = q-\gamma$ and $d = n-\gamma$. This yields

$$\begin{aligned} J_H(M,r) &= \frac{c_1}{2}\omega_{n-q}\omega_{q-\gamma} \int_0^1 H_I(rt^{\frac{1}{2}})t^{\frac{q-1}{2}}(1-t)^{\frac{n-q}{2}-1}\, dt \\ &= c_1\omega_{n-q}\omega_{q-\gamma} \int_0^1 H_I(rz)z^q(1-z^2)^{\frac{n-q}{2}-1}\, dz, \end{aligned}$$

using the substitution $z = \sqrt{t}$ in the last step. A comparison with (4.1) gives $J_H(M,r) = c_1\omega_{n-q}\omega_{q-\gamma}J_H(r)$, so abbreviating

$$c_2 = \bar{b}c_1\frac{\omega_{n-q}\omega_{q-\gamma}}{\omega_{n-\gamma}},$$

(7.6) becomes

$$\begin{aligned} I &= c_2\omega_{n-\gamma}\int_0^\infty f_I(r)r^{q-\gamma-1}J_H(r)\, dr \\ &= c_2\int_{A(n,\gamma)} f(E)d(o,E)^{-(n-q)}J_H(d(o,E))\, \mu_\gamma(dE), \end{aligned}$$

where we used (2.3) and spherical coordinates in $\operatorname{lin}(E)$. Hence, the theorem is shown once we have confirmed that

$$c_2 = D(n,q,\gamma). \tag{7.7}$$

We have

$$c_1 = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}}\prod_{i=1}^{n-q}\frac{\Gamma(\frac{n-\gamma-i}{2})\Gamma(\frac{n-q-i+\gamma}{2}+1)}{\Gamma(\frac{n-i+1}{2})\Gamma(\frac{n-q-i+1}{2})} = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}}\prod_{i=1}^{n-q}\frac{\omega_{n-i+1}\omega_{n-q-i+1}}{\omega_{n-\gamma-i}\omega_{n-q+\gamma+2-i}},$$

so direct insertion gives

$$\begin{aligned} \bar{b}c_1 &= \left(\frac{\omega_{\gamma+1}\cdots\omega_n}{\omega_1\cdots\omega_{n-\gamma}}\cdot\frac{\omega_{q-\gamma+1}\cdots\omega_{n-\gamma}}{\omega_{q+1}\cdots\omega_n}\cdot\frac{\omega_{n-q+1}\cdots\omega_{n-\gamma}}{\omega_{n-(q-\gamma)+1}\cdots\omega_n}\right) \\ &\qquad\qquad \times\left(\frac{\omega_{q+1}\cdots\omega_n}{\omega_{q-\gamma}\cdots\omega_{n-\gamma-1}}\cdot\frac{\omega_1\cdots\omega_{n-q}}{\omega_{\gamma+2}\cdots\omega_{n-(q-\gamma)+1}}\right)\frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \\ &= \left(\frac{\omega_{q+1}\cdots\omega_n}{\omega_{q+1}\cdots\omega_n}\cdot\frac{\omega_{\gamma+1}\cdots\omega_n}{\omega_{\gamma+2}\cdots\omega_n}\cdot\frac{1}{\omega_{n-(q-\gamma)+1}}\right) \\ &\qquad\qquad \times\left(\frac{\omega_{q-\gamma}\cdots\omega_{n-\gamma}}{\omega_{q-\gamma}\cdots\omega_{n-\gamma}}\cdot\frac{\omega_1\cdots\omega_{n-\gamma}}{\omega_1\cdots\omega_{n-\gamma}}\right) \\ &= \frac{\omega_{\gamma+1}}{\omega_{n-(q-\gamma)+1}}, \end{aligned}$$

where the products were suitably sorted at the second equality sign, and simplified at the third equality sign. We thus get

$$c_2 = \frac{\omega_{\gamma+1}\omega_{q-\gamma}\omega_{n-q}}{\omega_{n-(q-\gamma)-1}\omega_{n-\gamma}} = D(n,q,\gamma).$$

This shows (7.7) and completes the proof. $\square$

Proof of Corollary 4.4. For a Borel set $B \subseteq G(n,q)$, define

$$\tilde{\nu}_q(B) = \int_{G(n,q_1)} \cdots \int_{G(n,q_\ell)} \mathbb{1}_B(L_1 \cap \ldots \cap L_\ell)\, \nu_{q_\ell}(dL_\ell) \ldots \nu_{q_1}(dL_1).$$

This gives rise to an invariant probability measure $\tilde{\nu}_q$ on $G(n,q)$. However, by [22, Thm. 13.2.11] there is only one such measure, which implies that $\tilde{\nu}_q = \nu_q$. This allows us in a first step to reduce the outer integral over $G(n,q_1),\ldots,G(n,q_m)$ in $I_{\ell,m}$ to a single integral over $G(n,q)$:

$$\begin{aligned} I_{\ell,m} = \int_{G(n,q)} \int_{A(n,p_1)} \cdots \int_{A(n,p_m)} & f(E_1 \cap \ldots \cap E_m \cap L) \\ & \times H(E_1 \cap \ldots \cap E_m)\, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1) \nu_q(dL). \end{aligned} \tag{7.8}$$

To also convert the inner integrals over $A(n,p_1),\ldots,A(n,p_m)$ to a single integral, let B be a Borel set in $A(n,n-q+\gamma)$ and define

$$\widehat{\mu}_{n-q+\gamma}(B) = \int_{A(n,p_1)} \cdots \int_{A(n,p_m)} \mathbb{1}_B(E_1 \cap \ldots \cap E_m)\, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1).$$

The measure $\widehat{\mu}_{n-q+\gamma}$ is motion invariant on $A(n,n-q+\gamma)$. However, by [22, Thm. 13.1.3 and Thm. 13.2.12] all such measures are constant multiples of the invariant measure $\mu_{n-q+\gamma}$, so there exists some constant $c > 0$ such that $\widehat{\mu}_{n-q+\gamma} = c\mu_{n-q+\gamma}$. To determine the value of the constant c, we employ a special case of the Crofton formula [22, Thm. 5.1.1]. It says that

$$\int_{A(n,k)} \mathcal{H}^i(E \cap W)\, \mu_k(dE) = \frac{\omega_{n+1}\omega_{i+1}}{\omega_{k+1}\omega_{n-k+i+1}} \mathcal{H}^{n-k+i}(W)$$

for $0 \le i \le k \le n-1$ and where $W \subset \mathbb{R}^n$ is a convex set of dimension $n-k+i$, that is, the affine hull of W has dimension $n-k+i$. Applying Crofton's formula with $i = k = n-q+\gamma$ and $W = B^n$ we obtain

$$\begin{aligned} \int_{A(n,n-q+\gamma)} \mathcal{H}^{n-q+\gamma}(E \cap B^n)\, \widehat{\mu}_{n-q+\gamma}(dE) &= c \int_{A(n,n-q+\gamma)} \mathcal{H}^{n-q+\gamma}(E \cap B^n)\, \mu_{n-q+\gamma}(dE) \\ &= c\mathcal{H}^n(B^n) = c\kappa_n. \end{aligned}$$

On the other hand, by applying Crofton's formula repeatedly to each of the integrals in its definition, the integral on the left is

$$\begin{aligned} &\int_{A(n,p_1)} \cdots \int_{A(n,p_m)} \mathcal{H}^{n-q+\gamma}(E_1 \cap \ldots \cap E_m \cap B^n)\, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1) \\ &= \frac{\omega_{n+1}\omega_{n-q+\gamma+1}}{\omega_{p_m+1}\omega_{2n-p_m-q+\gamma+1}} \int_{A(n,p_1)} \cdots \int_{A(n,p_{m-1})} \mathcal{H}^{2n-p_m-q+\gamma}(E_1 \cap \ldots \cap E_{m-1} \cap B^n) \\ &\qquad\qquad \mu_{p_{m-1}}(dE_{m-1}) \ldots \mu_{p_1}(dE_1) \\ &\ \vdots \\ &= \frac{\omega_{n+1}\omega_{n-q+\gamma+1}}{\omega_{p_m+1}\omega_{2n-p_m-q+\gamma+1}} \frac{\omega_{n+1}\omega_{2n-p_m-q+\gamma}}{\omega_{p_{m-1}}\omega_{3n-p_m-p_{m-1}-q+\gamma+1}} \cdots \\ &\qquad\qquad \cdots \frac{\omega_{n+1}\omega_{mn-p_m-\ldots-p_2-q+\gamma}}{\omega_{p_1+1}\omega_{n-p_1+mn-p_m-\ldots-p_2-q+\gamma+1}} \mathcal{H}^n(B^n) \\ &= \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m}\omega_{n+1}} \kappa_n. \end{aligned}$$

Here, we simplified the telescopic product of the ω-terms and used our assumption $p_1 + \ldots + p_m - (m-1)n = n - 1 + \gamma$ in the last step. A comparison of these two expressions implies

$$\widehat{\mu}_{n-q+\gamma} = \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m} \omega_{n+1}} \mu_{n-q+\gamma}.$$

As a consequence, we can reduce the inner integrals in (7.8) to a single integral over $A(n, n-q+\gamma)$:

$$I_{\ell,m} = \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m} \omega_{n+1}} \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E)\, \mu_{n-q+\gamma}(dE) \nu_q(dL).$$

The result of Corollary 4.4 can now be concluded from Theorem 4.1. $\square$

The proof that the results of Theorems 4.1 and 4.5 are equivalent, is based on the following Blaschke–Petkantschin formula:

$$\int_{A(n,k)} g(E) \mu_k(dE) = \frac{\omega_{n-k}}{\omega_{r-k}} \int_{G(n,r)} \int_{A(L,k)} g(E) d(o,E)^{n-r} \mu_k^L(dE) \nu_r(dL), \tag{7.9}$$

with integers $0 \le k < r < n$ and measurable $g : A(n,k) \to [0, \infty)$, see e.g. [14, p. 33].

Proof of the equivalence of Theorems 4.1 and 4.5. That Theorem 4.5 implies Theorem 4.1 follows rather directly by invariant integration of (4.4) with respect to L_0 and an application of the Blaschke–Petkantschin formula (7.9) with $r = q$ and $k = \gamma$.
Assume now that Theorem 4.1 holds, and let $f : A(L_0, \gamma) \to [0, \infty)$ be measurable. Assume first that f is rotation invariant under all rotations fixing L_0. The function $\tilde{f}(E) = f_I(d(o,E))$, $E \in A(n,\gamma)$, is the rotation invariant extension of f to $A(n,\gamma)$. Hence, the invariance of $\mu_{n-q+\gamma}$ gives

$$\begin{aligned}
&\int_{G(n,q)} \int_{A(n,n-q+\gamma)} \tilde{f}(E \cap L) H(E)\, \mu_{n-q+\gamma}(dE)\, \nu_q(dL) \\
&= \int_{SO(n)} \int_{A(n,n-q+\gamma)} \tilde{f}(E \cap \vartheta L_0) H(E)\, \mu_{n-q+\gamma}(dE)\, \nu(d\vartheta) \\
&= \int_{SO(n)} \int_{A(n,n-q+\gamma)} \tilde{f}\big(\vartheta(E \cap L_0)\big) H(\vartheta^{-1} E)\, \mu_{n-q+\gamma}(dE)\, \nu(d\vartheta) \\
&= \int_{A(n,n-q+\gamma)} f(E \cap L_0) H(E)\, \mu_{n-q+\gamma}(dE).
\end{aligned}$$

On the other hand, the Blaschke–Petkantschin relation (7.9) and a similar reasoning shows

$$\int_{A(n,\gamma)} \tilde{f}(E) d(o,E)^{-(n-q)} J_H(d(o,E))\, \mu_\gamma(dE) = \frac{\omega_{n-\gamma}}{\omega_{q-\gamma}} \int_{A(L_0,\gamma)} f(E) J_H(d(o,E))\, \mu_\gamma^{L_0}(dE).$$

Theorem 4.1 states that these two displayed expressions coincide up to multiplication with the constant $D(n,q,\gamma)$, and thus (4.4) holds for the function f chosen.
With arguments as in the proof of Lemma 3.1, one can show that (4.4) holds for all measurable $f : A(L_0, \gamma) \to [0, \infty)$ if it holds for all such functions which are in addition invariant under all rotations fixing L_0. This proves that Theorem 4.5 holds and concludes the proof of the equivalence. $\square$

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