The asymptotic stability on the line of ground states of the pure power NLS with $0 < |p-3| \ll 1$

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Abstract

For exponents p satisfying $0 < |p-3| \ll 1$ and only in the context of spatially even solutions we prove that the ground states of the nonlinear Schrödinger equation (NLS) with pure power nonlinearity of exponent p in the line are asymptotically stable. The proof is similar to a related result of Martel [46] for a cubic quintic NLS. Here we modify the second part of Martel's argument, replacing the second virial inequality for a transformed problem with a smoothing estimate on the initial problem, appropriately tamed by multiplying the initial variables and equations by a cutoff.

1 Introduction

We consider the pure power focusing Nonlinear Schrödinger Equation (NLS) on the line

$$\mathrm{i}\partial_t u + \partial_x^2 u = -f(u) \text{ where } f(u) = |u|^{p-1}u \text{ for } 0 < |p-3| \ll 1. \tag{1.1}$$

We consider only even solutions, eliminating translations and simplifying the problem. In particular, we will study Equation (1.1) in the space $H^1_{\mathrm{rad}}(\mathbb{R}) = H^1_{\mathrm{rad}}(\mathbb{R},\mathbb{C})$, of even functions in $H^1(\mathbb{R},\mathbb{C})$. It is well known that Equation (1.1) has standing waves, solutions with the form $u(t,x) = e^{\mathrm{i}\omega t}\phi_\omega(x)$. They are obtained from $\phi_\omega(x) = \omega^{\frac{1}{p-1}}\phi(\sqrt{\omega}x)$ with the explicit formula

$$\phi(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right), \tag{1.2}$$

see formula (3.1) of Chang et al. [7]. Energy $\mathbf{E}$ and Mass $\mathbf{Q}$ are invariants of (1.1), where

$$\mathbf{E}(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \int_{\mathbb{R}} F(u)\,dx \text{ where } F(u) = \frac{|u|^{p+1}}{p+1}, \tag{1.3}$$

$$\mathbf{Q}(u) = \frac{1}{2}\|u\|^2_{L^2(\mathbb{R})}. \tag{1.4}$$

It is well known that ϕ_ω minimizes $\mathbf{E}$ under the constraint $\mathbf{Q} = \mathbf{Q}(\phi_\omega) =: \mathbf{q}(\omega)$. Notice that $\mathbf{q}(\omega) = \omega^{\frac{2}{p-1}-\frac{1}{2}}\mathbf{q}(1)$. We have $\nabla\mathbf{E}(\phi_\omega) = \omega\nabla\mathbf{Q}(\phi_\omega)$ which reads also

$$-\phi''_\omega + \omega\phi_\omega - \phi^p_\omega = 0. \tag{1.5}$$

Set now for $\omega, \delta \in \mathbb{R}_+ := (0, \infty)$ the set

$$\mathcal{U}(\omega, \delta) := \bigcup_{\vartheta_0 \in \mathbb{R}} e^{\mathrm{i}\vartheta_0} D_{H^1_{\mathrm{rad}}(\mathbb{R})}(\phi_\omega, \delta),$$

where $D_X(u, r) := \{v \in X \mid \|u - v\|_X < r\}$. The following was shown by Cazenave and Lions [6], see also Shatah [54] and Weinstein [63].

Theorem 1.1 (Orbital Stability)**.** *Let $p \in (1, 5)$ and let $\omega_0 > 0$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in \mathcal{U}(\omega_0, \delta)$ then the corresponding solution satisfies $u \in C^0(\mathbb{R}, \mathcal{U}(\omega_0, \epsilon))$.*

□

In order to study the notion of asymptotic stability, like in finite dimension, it is useful to have information on the *linearization* of (1.1) at ϕ_ω, which we will see later has the following form

$$\partial_t \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{L}_\omega \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \text{ with } \mathcal{L}_\omega := \begin{pmatrix} 0 & L_{-\omega} \\ -L_{+\omega} & 0 \end{pmatrix}, \tag{1.6}$$

where

$$L_{+\omega} := -\partial_x^2 + \omega - p\phi_\omega^{p-1} \tag{1.7}$$

$$L_{-\omega} := -\partial_x^2 + \omega - \phi_\omega^{p-1}. \tag{1.8}$$

The linearization is better seen in the context of functions in $H^1_{\mathrm{rad}}(\mathbb{R}, \mathbb{R}^2)$ rather than in $H^1_{\mathrm{rad}}(\mathbb{R}, \mathbb{C})$, because it is $\mathbb{R}$–linear rather than $\mathbb{C}$–linear. For $p = 3$ the operator $\mathcal{L}_\omega$ is completely known very thoroughly, so for example all its plane waves are known explicitly, see section 10. Coles and Gustafson [8] proved that for $0 < |p - 3| \ll 1$ the linearization $\mathcal{L}_\omega$ has exactly one eigenvalue of the form $\mathrm{i}\lambda$ near $\mathrm{i}\omega$. We set $\lambda(p, \omega) := \lambda$. Furthermore $0 < \lambda(p, \omega) < \omega$ and $\dim \ker(\mathcal{L}_\omega - \mathrm{i}\lambda(p, \omega)) = 1$. Let $\xi_\omega \in H^1(\mathbb{R}, \mathbb{C}^2)$ be an appropriately normalized generator of $\ker(\mathcal{L}_\omega - \mathrm{i}\lambda(p, \omega))$, see §2.

In this paper we prove the following result.

Theorem 1.2. *There exists $p_1 < 3 < p_2$ s.t. for any $p \in (p_1, p_2) \backslash \{3\}$ and any $\omega_0 > 0$, any $a > 0$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in D_{H^1_{\mathrm{rad}}(\mathbb{R})}(\phi_{\omega_0}, \delta)$ there exist functions $\vartheta, \omega \in C^1(\mathbb{R}, \mathbb{R})$, $z \in C^1(\mathbb{R}, \mathbb{C})$ and $\omega_+ > 0$ s.t. the solution of (1.1) with initial datum u_0 can be written as*

$$u(t) = e^{\mathrm{i}\vartheta(t)} \left(\phi_{\omega(t)} + z(t)\xi_{\omega(t)} + \overline{z}(t)\overline{\xi}_{\omega(t)} + \eta(t) \right) \text{ with} \tag{1.9}$$

$$\int_{\mathbb{R}} \|e^{-a\langle x \rangle}\eta(t)\|^2_{H^1(\mathbb{R})} dt < \epsilon \text{ where } \langle x \rangle := \sqrt{1 + x^2} \tag{1.10}$$

$$\lim_{t \to \infty} \|e^{-a\langle x \rangle}\eta(t)\|_{L^2(\mathbb{R})} = 0 \tag{1.11}$$

$$\lim_{t \to \infty} z(t) = 0, \tag{1.12}$$

$$\lim_{t \to \infty} \omega(t) = \omega_+. \tag{1.13}$$

Remark 1.3. Standing wave solutions in the integrable case $p = 3$ are not asymptotically stable due the existence of *breathers*, see Borghese et al. [1, formula (1.21)], very close in $H^1(\mathbb{R})$ to a soliton (take for example $\eta_2 \longrightarrow 0$ in [1, formula (1.21)]). More broadly, when $p = 3$ it is possible to *add* solutions using Bäcklund transformations. In fact the situation resembles that of small energy solutions of NLS with a trapping linear potential with two or more eigenvalues when we treat the nonlinearity as a perturbation. Then the linear equation has quasiperiodic solutions, due to linear superposition, while generically a nonlinear equation does not, see [17] for a 1 D result and therein for references. However, some form of asymptotic stability holds also in the $p = 3$ case, using different norms and the theory of Integrable Systems, [1, 21, 53].

Remark 1.4. Notice that the fact that the $H^1(\mathbb{R})$ norm of η is uniformly bounded for all times, guaranteed by the orbital stability, and Theorem 1.2 imply that

$$\lim_{t\to+\infty} u(t)e^{-\mathrm{i}\vartheta(t)} = \phi_{\omega_+} \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}). \tag{1.14}$$

Remark 1.5. While we prove Theorem 1.2 in the case that $0 < |p_j - 3| \ll 1$ for $j = 1, 2$, it is possible to ease significantly these hypotheses. In fact, we emphasize that our theory is largely non perturbative. What we need in the proof are the following facts:

(i) $2\lambda(p, 1) > 1$;

(ii) we can take $\gamma(p, 1) \neq 0$, where this constant is related to the Fermi Golden Rule (FGR), see below, and is defined below in (5.3);

(iii) given the Jost functions $f_3(\cdot, 0)$ and $g_3(\cdot, 0)$ introduced later in Sect. 8, which depend analytically on p, their Wronskian is nonzero: $W[f_3(x, 0), g_3(x, 0)] \neq 0$.

According to numerical computations in Chang et al. [7], condition (**i**) holds for all $2 < p < 3$, where there are no other eigenvalues of the form $\mathrm{i}\lambda$ for $\lambda > 0$, no resonance is observed at the threshold of the continuous spectrum. Furthermore, both $\gamma(p, 1)$ and $W[f_3(x, 0), g_3(x, 0)]$ can be made to depend analytically on p. This would show that Theorem 1.2 holds for all the $p \in (2, 3)$ except for a discrete subset of $(2, 3)$. Similarly, when we consider $p > 3$, we have $\lambda(3, 1) = 1$ and $\lambda(5, 1) = 0$ and, according to the numerical computations in Chang et al. [7] (similar results were in part already known: the first author learned about them by personal communication by M.I.Weinstein in the year 2000), the function $(3, 5) \ni p \to \lambda(p, 1)$ is strictly decreasing, there are no other eigenvalues of the form $\mathrm{i}\lambda$ with $\lambda > 0$ and no resonance is observed at the threshold of the continuous spectrum. So there will be a $p_0 \in (3, 5)$ such that $2\lambda(p, 1) > 1$ for all $3 < p < p_0$ (notice that $p_0 \in (4, 5)$ in [7, fig. 1] with p_0 quite close to 5). Since also conditions (**ii**) and (**iii**) will be true for all the $p \in (3, p_0)$ outside a discrete subset of $(3, p_0)$ we conclude that outside a discrete subset of $(3, p_0)$ Theorem 1.2 continues to be true. That the FGR constants are nonzero and that condition (**iii**) holds, are expected to be generically true also for *large* perturbations of the cubic NLS. Furthermore, our framework could in principle be applied in higher dimensions.

Remark 1.6. Martel [46] conjectures that for generic small perturbations of the cubic NLS the asymptotic stability result in [45, 46] is true. Here we focus only on pure power NLS's but our method goes some way to prove this conjecture. For our method to work, we always need that the threshold of the continuous spectrum be not a resonance, which should be true for generic perturbations. For the smoothing estimate we need additionally condition (**iii**) in Remark 1.5, which is true for small perturbations. If there is no eigenvalue in $(0, \mathrm{i}\omega)$ like in Martel [45] and Rialland [52] and if the non resonance condition holds then our method proves Martel's conjecture. If there is one eigenvalue $\lambda \in (0, \mathrm{i}\omega)$ (it is easy to show, proceeding along the lines in [22] or, since this is 1 dimension, Coles and Gustafson [8], that there can be at most one such eigenvalue for small perturbations), condition (**i**) in Remark 1.5 will be true. If, as expected, generically the Nonlinear Fermi golden rule (FGR) condition (**ii**) in Remark 1.5 holds, then our method works. So, using our framework, to prove Martel's conjecture it remains only to prove that for generic small perturbations there is no threshold resonance and that if there is an eigenvalue the FGR is true. We think that also Martel's method in [46] yields a similar result.

Equation (1.1) is one of the most classical Hamiltonian systems in PDE's and the asymptotic stability of its ground states has been a longstanding open problem. Attempts at solving it date back at least to the 80's, see Soffer and Weinstein [57, 58]. The Vakhitov–Kolokolov stability criterion yields the orbital stability exactly for $p < 5$, while for $p \geqslant 5$ the ground states are orbitally unstable. On the other hand, proving asymptotic stability requires some form of spatial dispersion. It turns out that it is difficult to prove dispersion for $p < 5$, which is the opposite condition to the one utilized for example in Strauss [60] to prove a form of asymptotic stability of vacuum. In essence, dispersion is a linear phenomenon, but for $p < 5$ the nonlinearity is strong and makes it difficult to treat the problem as a perturbation of a linear equation. Whence the inability in the literature to deal with the asymptotic stability problem for equation (1.1), left unaddressed from Buslaev and Perelman [2, 3, 4] on. Another problem is the presence of nonzero eigenvalues of the linearization $\mathcal{L}_\omega$. These eigenvalues slow dispersion because the corresponding discrete modes tend to oscillate periodically and decay slowly, see [3, 4, 59, 28], and furthermore they are a drag to the dispersion of the continuous modes, on whose equation they exert a forcing. This is especially true in the case when there are eigenvalues close to 0, as happens for example for p close to 1 or to 5. A mechanism first discussed by Sigal [55], the Nonlinear Fermi Golden Rule (FGR), should allow to show that the discrete modes lose energy by nonlinear interaction with the continuous modes.

It is next to impossible to see the FGR, without utilizing the Hamiltonian structure of the NLS, see for example the complications in [27]. In [11] the FGR is seen using canonical coordinates and normal forms transformations. Recently papers like [14] have simplified significantly [11], eliminating the need of normal forms, thanks to the notion of Refined Profile, which is a generalization of the families of ground states, a sort of surrogate of a (here not existent) family of quasiperiodic solutions and encodes the discrete modes in the problem. Finding the Refined Profile is elementary, but requires Taylor expansions of the nonlinearity, with the order higher when there are eigenvalues of $\mathcal{L}_\omega$ closer to 0. Since $f(u)$ is not smooth in u, this is one of the main reasons why p needs here to stay close to 3, where the eigenvalue is not close to 0. Even more difficult appears the problem when the power p is such that $\mathcal{L}_\omega$ has resonances at the thresholds of the continuous spectrum, except in the integrable case $p = 3$. To see some of the difficulties, on a different and non integrable model involving a resonance, we refer to the partial results in [44, 50]. We stress that here the spectral configuration is as in Martel [46] and that we prove the FGR like in Martel [46].
Dispersion has played a crucial role in stabilization problems. The sequel [58] to [57] was only possible because a result on dispersion for Schrödinger by Journé et al. [34]. Strichartz estimates, in particular the 3 D endpoint Strichartz estimate of Keel and Tao [37], were introduced by Gustafson et. al. [33] and played an important role in the theory. Dimensions 1 and 2 were considered by Mizumachi [48, 47], whose use of smoothing estimates has provided us with crucial insights. But ultimately, in low dimensions Strichartz estimates have limited scope. A very important turning point in the theory in 1 D has been Kowalczyk et al. [39] which, along with the further developments and refinements in [40, 41, 38], has exploited very effectively virial inequalities. Recently Martel [45, 46] has applied and extended these ideas to the study of the asymptotic stability of two versions of the cubic–quintic NLS introduced by Pelinovsky et al. [51]. Rialland [52] has generalized [45]. One of the most striking features of the theory initiated by Kowalczyk et al. [40], is how easily the nonlinear term involving only the continuous mode of the solution is sorted out by what we might call the *high energy* virial inequality, see inequality is (6.5) below, by means of a clever but simple integration by parts. The same term is almost impossible to treat with perturbative methods involving the Duhamel formula. There exist also different approaches, some, but not exclusively, stemming from the theory of space–time resonances of Germain et al. [29]. For a partial sample we refer for example

to work of Delort [26], Germain et al. [30], Naumkin [49]. Recently Germain and Collot [9] have recovered and partially generalized Martel [45]. This theory requires a certain degree of smoothness of the nonlinearity $f(u)$, so it is not easily applicable to the specific model (1.1). We think that the framework in Kowalczyk et al., to which we return, is more robust and easier to apply in stability problems.
After the first *high energy* virial inequality, the papers [40, 41, 38, 45, 46] utilize what we might call a *low energy* virial inequality, which requires new coordinates where the linearization is nontrapping. This has some similarities with the subtraction of solitons to study dispersion by means of the Nonlinear Steepest Descent method of Deift and Zhou, as done for instance by Grunert and Teschl [43], although the details are very different. An interesting feature and a possible criticality of the low energy virial inequality, is that the virial inequality produces a different linear operator, which also needs to be non–trapping. While in [45, 46, 52], which deal with small perturbations of the cubic NLS, the two non–trapping conditions are shown to be equivalent, thanks to a result by Simon [56] on small perturbations of the Laplacian in dimensions 1 and 2, in general this might not be the case, so it is plausible that in some cases the second virial inequality method might require restrictions not intrinsic but rather due to the method of proof. To take a concrete example, in the first paper [17] of our own series inspired by the work of Kowalczyk et al.[40], the repulsivity Assumption 1.13 [17] is in fact unnecessary and is used only because of the method of proof. This is the main insight and motivation for this paper. In [17], besides the two virial inequalities, there is a smoothing estimate, inspired by Mizumachi [48, 47], which in [17] appears because the FGR rule is proved in an overly complicated way (a simplification appears in [19], motivated by [38]). The insight in the present paper, is that, while it is obviously a good idea to prove the FGR as simply as possible, it is possible to replace the the second virial inequality by smoothing estimates. We explain now some further reasons why this might be convenient. Kowalczyk et al.[39, 40, 41, 38] and Martel [45, 46] perform some Darboux transformations, which are almost isospectral transformations which allow to eliminate eigenvalues of the linearization in a controlled way. For scalar Schrödiger operators in the line the theory is fully developed in Deift and Trubowitz [25], with an important special case discussed in Chang et al. [7]. The analogue for the linearizations $\mathcal{L}_\omega$ is in Martel [45] and Rialland [52] in the case without internal modes and in Martel [46] with just one internal mode. It is not clear to us what are the Darboux transformations when the configuration of the internal modes of $\mathcal{L}_\omega$ is more complicated and if the space dimension is 2 or larger. So it is worth to develop some alternative method which does not use Darboux transformations. The Kato–smoothing estimates are a classical tool, originating in Kato [35], valid in any dimension. The smoothing estimates are perturbative, based on the Duhamel formula. But there is no issue here of too strong nonlinearity because we only need to bound the continuous mode multiplied by a spatial cutoff. This means that we can multiply the NLS by a cutoff, taming the nonlinearity. The cutoff appears also in the second virial inequalities in the theory of Kowalczyk et al. Obviously, in the equation we obtain an additional term, delicate for us, represented by the commutator of Laplacian and cutoff. We treat it via a specific smoothing estimate, see Lemma 2.3 below. In [17] we used some standard bounds on the Jost functions of Schrödinger operators in 1 D to prove an analogous lemma. Here, for $\mathcal{L}_\omega$ these bounds on the Jost functions are not as obvious and this is one of the points where we exploit that our problem is a small perturbation of the cubic NLS, the specific condition is (iii) in Remark 1.5 that appears generic and is in principle possible to check numerically in specific examples. Finally, for a rather long list of references on the subject up until 2020, we refer to our survey [16].

2 Linearization

We return to a discussion of the linearization (1.6). Weinstein [62] showed that for $1 < p < 5$ the generalized kernel $N_g(\mathcal{L}_\omega) := \cup_{j=1}^{\infty} \ker \mathcal{L}_\omega^j$ in $H^1_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$ is

$$N_g(\mathcal{L}_\omega) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ \phi_\omega \end{pmatrix}, \begin{pmatrix} \partial_\omega \phi_\omega \\ 0 \end{pmatrix} \right\}. \tag{2.1}$$

By symmetry reasons, it known that the spectrum $\sigma(\mathcal{L}_\omega) \subseteq \mathbb{C}$ is symmetric by reflection with respect of the coordinate axes. Furthermore, by Krieger and Schlag [42, p. 909] we know that $\sigma(\mathcal{L}_\omega) \subseteq \mathrm{i}\mathbb{R}$. By standard Analytic Fredholm theory the essential spectrum is $(-\infty\mathrm{i}, -\omega\mathrm{i}] \cup [\mathrm{i}\omega, +\infty\mathrm{i})$. As already mentioned $0 \in \sigma(\mathcal{L}_\omega)$. Numerical computations by Chang et al. [7] show that for $p \in (2,3) \cup (3,5)$, besides 0 there are two eigenvalues of $\mathcal{L}_\omega$, they are of the form $\pm\mathrm{i}\lambda$ with $\lambda > 0$ and if we set as above $\lambda(p,\omega) = \lambda$, we have $\lambda(3,\omega) = \omega$ and $\lambda(5,\omega) = 0$. As mentioned above Coles and Gustafson [8] corroborate rigorously the numerical computations of Chang et al. [7] for $0 < |p-3| \ll 1$. Furthermore, since at $p = 3$ the linearization $\mathcal{L}_\omega$ has only 0 as an eigenvalue, and $\pm\mathrm{i}\omega$ are resonances, Coles and Gustafson [8] imply that besides $-\mathrm{i}\lambda(p,\omega), 0, \mathrm{i}\lambda(p,\omega)$, for $0 < |p-3| \ll 1$ there are no other eigenvalues and that $\pm\mathrm{i}\omega$ are not resonances.
Let us consider the orthogonal decomposition

$$L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2) = N_g(\mathcal{L}_\omega) \bigoplus N_g^{\perp}(\mathcal{L}_\omega^*) \tag{2.2}$$

We have, for $\lambda = \lambda(p,\omega)$, a further decomposition

$$N_g^{\perp}(\mathcal{L}_\omega^*) = \ker(\mathcal{L}_\omega - \mathrm{i}\lambda) \bigoplus \ker(\mathcal{L}_\omega + \mathrm{i}\lambda) \bigoplus X_c(\omega) \text{ where} \tag{2.3}$$

$$X_c(\omega) := \left(N_g(\mathcal{L}_\omega^*) \bigoplus \ker(\mathcal{L}_\omega^* - \mathrm{i}\lambda) \bigoplus \ker(\mathcal{L}_\omega^* + \mathrm{i}\lambda) \right)^{\perp}. \tag{2.4}$$

We denote by P_c the projection of $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$ onto $X_c(\omega)$ associated with the above decompositions. The space $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$ and the action of $\mathcal{L}_\omega$ on it is obtained by first identifying $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}) = L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{R}^2)$ and then by extending this action to the completion of $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{R}^2) \bigotimes_{\mathbb{R}} \mathbb{C}$ which is identified with $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$. In $\mathbb{C}$ we consider the inner product

$$\langle z, w \rangle_{\mathbb{C}} = \mathrm{Re}\{z\overline{w}\} = z_1 w_1 + z_2 w_2 \text{ where } a_1 = \mathrm{Re}\, a, \quad a_2 = \mathrm{Im}\, a \text{ for } a = z, w.$$

This obviously coincides with the inner product in $\mathbb{R}^2$ and expands as the standard sesquilinear $\langle X, Y \rangle_{\mathbb{C}^2} = X^{\intercal}\overline{Y}$ (row column product, vectors here are columns) form in $\mathbb{C}^2$. The operator of multiplication by i in $C = \mathbb{R}^2$ extends into the linear operator $J^{-1} = -J$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $u, v \in L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$ we set $\langle u, v \rangle := \int_{\mathbb{R}} \langle u(x), v(x) \rangle_{\mathbb{C}^2}\, dx$. We have a natural symplectic form given by $\Omega := \langle J^{-1}\cdot, \cdot \rangle$ in both $L^2(\mathbb{R}, \mathbb{C}^2)$ and $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{R}^2) = L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C})$, where equation (1.1) is the Hamiltonian system in $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C})$ with Hamiltonian the energy E in (1.3). As we mentioned we consider a generator $\xi_\omega \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda)$. Then for the complex conjugate $\overline{\xi}_\omega \in \ker(\mathcal{L}_\omega + \mathrm{i}\lambda)$. Notice the well known and elementary $J\mathcal{L}_\omega = -\mathcal{L}_\omega^* J$ implies that $\ker(\mathcal{L}_\omega^* + \mathrm{i}\lambda) = \operatorname{span}\{J\xi_\omega\}$ and $\ker(\mathcal{L}_\omega^* - \mathrm{i}\lambda) = \operatorname{span}\{J\overline{\xi}_\omega\}$. Notice that in Lemma 2.7 [13] it is shown that we can normalize ξ_ω so that

$$\Omega(\xi_\omega, \xi_\omega) = -\mathrm{i}, \tag{2.5}$$

consistently with the fact that the functional $\mathbf{E}(u) + \omega \mathbf{Q}(u)$ has a local minimum at $u = \phi_\omega$, see (3.13)–(3.14) later. Notice that (2.5) is the same as

$$\Omega(\operatorname{Re}\xi_\omega, \operatorname{Im}\xi_\omega) = \frac{1}{2} \text{ and } \Omega(\operatorname{Re}\xi_\omega, \operatorname{Re}\xi_\omega) = \Omega(\operatorname{Im}\xi_\omega, \operatorname{Im}\xi_\omega) = 0 \tag{2.6}$$

where the latter is immediate by the skewadjointness of J. Notice that

$$\xi_\omega = (\xi_1, \xi_2)^\intercal \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda) \Leftrightarrow \begin{cases} L_{-\omega}\xi_2 = \mathrm{i}\lambda\xi_1 \\ L_{+\omega}\xi_1 = -\mathrm{i}\lambda\xi_2 \end{cases} \tag{2.7}$$

$$\begin{cases} L_{-\omega}\xi_{2I} = \lambda\xi_{1R} \\ L_{+\omega}\xi_{1R} = -\lambda\xi_{2I} \end{cases} \text{ and } \begin{cases} L_{-\omega}\xi_{2R} = -\lambda\xi_{1I} \\ L_{+\omega}\xi_{1I} = \lambda\xi_{2R} \end{cases} \text{ with } \xi_{jR} = \operatorname{Re}\xi_j \text{ and } \xi_{jI} = \operatorname{Im}\xi_j.$$

This implies that we can normalize so that

$$\xi_\omega = (\xi_1, \xi_2)^\intercal \text{ with } \xi_1 = \operatorname{Re}\xi_1 \text{ and } \xi_2 = \mathrm{i}\operatorname{Im}\xi_2. \tag{2.8}$$

Hence condition (2.5) becomes

$$\int_{\mathbb{R}} \xi_1 \operatorname{Im}\xi_2 dx = 2^{-1}. \tag{2.9}$$

Notation 2.1. We will use the following miscellanea of notations and definitions.

1. We will set

$$\mathbf{e}(\omega) := \mathbf{E}(\phi_\omega),\ \mathbf{q}(\omega) := \mathbf{Q}(\phi_\omega) \text{ and } \mathbf{d}(\omega) := \mathbf{e}(\omega) + \omega\mathbf{q}(\omega). \tag{2.10}$$

2. We denote by $\operatorname{diag}(a, b)$ the diagonal matrix with first a and then b on the diagonal.

3. For $z \in \mathbb{C}$ we will use $z_1 = \operatorname{Re} z$ and $z_2 = \operatorname{Im} z$ and we will use the operators

$$\partial_z := \frac{1}{2}\left(\partial_{z_1} - \mathrm{i}\partial_{z_2}\right) \text{ and } \partial_{\overline{z}} := \frac{1}{2}\left(\partial_{z_1} + \mathrm{i}\partial_{z_2}\right).$$

4. Like in the theory of Kowalczyk et al. [40], we consider constants $A, B, \epsilon, \delta > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg 1. \tag{2.11}$$

Here we will take $A \sim B^3$, see Sect. 7 below, but in fact $A \sim B^n$ for any $n > 2$ would make no difference.

5. The notation $o_\varepsilon(1)$ means a constant with a parameter ε such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon \to 0^+} 0. \tag{2.12}$$

6. For $\kappa \in (0,1)$ fixed in terms of p and small enough, we consider

$$\|\eta\|_{L^{p,s}} := \|\langle x\rangle^s \eta\|_{L^p(\mathbb{R})} \text{ where } \langle x\rangle := \sqrt{1+x^2}, \tag{2.13}$$

$$\|\eta\|_{\Sigma_A} := \left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta'\right\|_{L^2(\mathbb{R})} + A^{-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta\right\|_{L^2(\mathbb{R})} \text{ and} \tag{2.14}$$

$$\|\eta\|_{\widetilde{\Sigma}} := \|\operatorname{sech}(\kappa\omega_0 x)\eta\|_{L^2(\mathbb{R})}. \tag{2.15}$$

7. We set
$$\mathbb{C}_{\pm} := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}. \tag{2.16}$$

8. We will consider the Pauli matrices
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

9. The point $\mathrm{i}\omega$ is a resonance for $\mathcal{L}_\omega$ if there exists a nonzero $v \in L^\infty(\mathbb{R}, \mathbb{C}^2)$ such that $\mathcal{L}_\omega v = \mathrm{i}\omega v$. Notice that for $p = 3$ the point $\mathrm{i}\omega$ is a resonance, see (10.7) for a v when $\omega = 1$. An elementary scaling yields the cases $\omega \neq 1$ from the $\omega = 1$ case.

10. Given two Banach spaces X and Y we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators from X to Y. We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

11. We have denoted by P_c the projection on the space (2.4) associated to the spectral decomposition (2.3) of the operator $\mathcal{L}_\omega$. Later in (7.2) we will introduce an operator H_ω which is an equivalent to $\mathcal{L}_\omega$ and obtained from $\mathcal{L}_\omega$ by a simple conjugation. By an abuse of notation we will continue to denote by P_c the analogous spectral projection to the continuous spectrum component, only of H_ω this time.

12. We have the following elementary formulas,
$$Df(u)X = \frac{d}{dt}\left(|u + tX|^{p-1}(u + tX)\right)\Big|_{t=0} = |u|^{p-1}X + (p-1)|u|^{p-3}u\langle u, X\rangle_{\mathbb{C}} \text{ and} \tag{2.17}$$
$$D^2 f(u)X^2 = \frac{d}{dt} Df(u + tX)X\Big|_{t=0}$$
$$= 2(p-1)|u|^{p-3}X\langle u, X\rangle_{\mathbb{C}} + (p-1)|u|^{p-3}u|X|^2 + (p-1)(p-3)|u|^{p-5}u\langle u, X\rangle_{\mathbb{C}}^2. \tag{2.18}$$

□

The group $e^{t\mathcal{L}_\omega}$ is well defined in $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$, leaves invariant $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{R}^2)$ and the terms of the direct sums in (2.2) and (2.3). The following result is an immediate consequence of a Proposition 8.1 in Krieger and Schlag [42], since $\mathcal{L}_\omega$ as an easy consequence of Coles and Gustafson [8] is for $0 < |p - 3| \ll 1$ admissible in the sense indicated in [42].

Proposition 2.2. *For any fixed $s > 3/2$ there is a constant C_ω such that*
$$\|P_c e^{t\mathcal{L}_\omega} : L^{2,s}(\mathbb{R}, \mathbb{C}^2) \to L^{2,-s}(\mathbb{R}, \mathbb{C}^2)\| \leqslant C_\omega \langle t\rangle^{-\frac{3}{2}} \text{ for all } t \in \mathbb{R}. \tag{2.19}$$

□

We will need a variation of the last result, which we will be rephrased later and proved as Lemma 8.9 and which is an analogue of Lemma 8.7 [17].

Proposition 2.3. *For $s > 3/2$ and $\tau > 1/2$ there exists a constant $C > 0$ such that*
$$\left\|\int_0^t e^{(t-t')\mathcal{L}_\omega} P_c(\omega) g(t') dt'\right\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leqslant C\|g\|_{L^2(\mathbb{R}, L^{2,\tau}_{\mathrm{rad}}(\mathbb{R}))} \text{ for all } g \in L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R})). \tag{2.20}$$

We will need the following result, whose proof is based on an argument in [24, Lemma 3.4].

Proposition 2.4 (Kato smoothing)**.** *For any ω and for any $s > 1$ there exists $c >$ such that*
$$\|e^{\mathrm{i}t\mathcal{L}_\omega} P_c u_0\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leqslant c\|u_0\|_{L^2(\mathbb{R})}. \tag{2.21}$$

3 Refined profile, modulation, continuation argument and proof of Theorem 1.2

It is well known, see Weinstein [62], that

$$\mathcal{S} = \left\{ e^{\mathrm{i}\vartheta}\phi_\omega : \vartheta \in \mathbb{R}, \omega > 0 \right\} \tag{3.1}$$

is a symplectic submanifold of $L^2_{\mathrm{rad}}(\mathbb{R}, \mathbb{C})$. We set

$$\phi[\omega, z] = \phi_\omega + \widetilde{\phi}[\omega, z] \text{ with } \widetilde{\phi}[\omega, z] := z\xi + \overline{z}\overline{\xi}. \tag{3.2}$$

For functions $\widetilde{z}_{\mathcal{R}}$, $\widetilde{\vartheta}_{\mathcal{R}}$ and $\widetilde{\omega}_{\mathcal{R}}$ to be determined below we introduce

$$\begin{aligned}
&\widetilde{z}[\omega, z] = \widetilde{z}_0[\omega, z] + \widetilde{z}_{\mathcal{R}}[\omega, z] \text{ with } \widetilde{z}_0[\omega, z] = \mathrm{i}\lambda z \\
&\widetilde{\vartheta}[\omega, z] = \omega + \widetilde{\vartheta}_{\mathcal{R}}[\omega, z] \text{ and } \widetilde{\omega}[\omega, z] = \widetilde{\omega}_{\mathcal{R}}[\omega, z].
\end{aligned}$$

Proposition 3.1. *There exist C^2 functions $\widetilde{z}_{\mathcal{R}}$, $\widetilde{\vartheta}_{\mathcal{R}}$ and $\widetilde{\omega}_{\mathcal{R}}$ defined in a neighborhood of $(\omega_0, 0) \in \mathbb{R}_+ \times \mathbb{C}$ with*

$$|\widetilde{\vartheta}_{\mathcal{R}}| + |\widetilde{\omega}_{\mathcal{R}}| + |\widetilde{z}_{\mathcal{R}}| \lesssim |z|^2 \tag{3.3}$$

such that, if we set

$$R[\omega, z] := \partial_x^2\phi[\omega, z] + f(\phi[\omega, z]) - \widetilde{\vartheta}\phi[\omega, z] + \mathrm{i}\widetilde{\omega}\partial_\omega\phi[\omega, z] + \mathrm{i}D_z\phi[\omega, z]\widetilde{z}, \tag{3.4}$$

we have

$$\|\cosh(\kappa\omega x)\, R[\omega, z]\|_{L^2(\mathbb{R})} \lesssim |z|^2, \tag{3.5}$$

with furthermore the following orthogonality conditions, for $z_1 = \operatorname{Re} z$ and $z_2 = \operatorname{Im} z$,

$$\langle \mathrm{i}R[\omega, z], \phi[\omega, z]\rangle = \langle \mathrm{i}R[\omega, z], \mathrm{i}\partial_\omega\phi[\omega, z]\rangle = \left\langle \mathrm{i}R[\omega, z], \mathrm{i}\partial_{z_j}\phi[\omega, z]\right\rangle = 0, \text{ for all } j = 1, 2. \tag{3.6}$$

Proof. From (1.5) and

$$D_z\widetilde{\phi}[\omega, z]\widetilde{z}_0 = \mathcal{L}_\omega\widetilde{\phi}[\omega, z] = -\mathrm{i}\left(-\partial_x^2\widetilde{\phi}[\omega, z] + \omega\widetilde{\phi}[\omega, z] - Df(\phi_\omega)\widetilde{\phi}[\omega, z]\right)$$

we obtain

$$\begin{aligned}
&\mathrm{i}D_z\phi[\omega, z]\widetilde{z}_0 = -\partial_x^2\widetilde{\phi}[\omega, z] + \omega\phi[\omega, z] - f(\phi[\omega, z]) + \widehat{R}[\omega, z] \text{ where} \\
&\widehat{R}[\omega, z] := f(\phi[\omega, z]) - f(\phi_\omega) - Df(\phi_\omega)\widetilde{\phi}[\omega, z].
\end{aligned}$$

Since

$$\widehat{R}[\omega, z] = \int_0^1\int_0^1 tD^2f(\phi_\omega + ts\widetilde{\phi}[\omega, z])dtds\widetilde{\phi}^2[\omega, z]$$

we conclude that

$$\|\cosh(\kappa\omega x)\, \widehat{R}[\omega, z]\|_{L^2(\mathbb{R})} \lesssim |z|^2. \tag{3.7}$$

Now we set

$$R[\omega,z]=\widehat{R}[\omega,z]-\mathrm{i}D_z\phi[\omega,z]\widetilde{z}_{\mathcal{R}}+\widetilde{\vartheta}_{\mathcal{R}}\phi[\omega,z]-\mathrm{i}\widetilde{\omega}_{\mathcal{R}}\partial_\omega\phi[\omega,z].$$

Setting $\phi=\phi[\omega,z]$ and $\widehat{R}=\widehat{R}[\omega,z]$, the orthogonality conditions (3.6) are equivalent to

$$\mathbf{A}\begin{pmatrix}\widetilde{\vartheta}_{\mathcal{R}}\\ \widetilde{\omega}_{\mathcal{R}}\\ \widetilde{z}_{\mathcal{R}}\end{pmatrix}=-\begin{pmatrix}\left\langle \widehat{R},\mathrm{i}\phi\right\rangle\\ \left\langle \widehat{R},\partial_\omega\phi\right\rangle\\ \left\langle \widehat{R},D_z\phi\right\rangle.\end{pmatrix} \tag{3.8}$$

where, by $D_{z_1}\widetilde{\phi}=2(\xi_1,0)^{\intercal}$ and $D_{z_2}\widetilde{\phi}=-2(0,\operatorname{Im}\xi_2)^{\intercal}$, with the right hand sides defined in (2.7),

$$\begin{aligned}\mathbf{A}&=\begin{pmatrix}\cancel{\langle\phi,\mathrm{i}\phi\rangle} & -\langle\partial_\omega\phi,\phi\rangle & -\cancel{\langle D_z\phi,\phi\rangle}\\ \langle\phi,\partial_\omega\phi\rangle & -\cancel{\langle\mathrm{i}\partial_\omega\phi,\partial_\omega\phi\rangle} & -\langle\mathrm{i}D_z\phi,\partial_\omega\phi\rangle\\ \cancel{\langle\phi,D_z\phi\rangle} & -\langle\mathrm{i}\partial_\omega\phi,D_z\phi\rangle & -\langle\mathrm{i}D_z\phi,D_z\phi\rangle\end{pmatrix} \qquad (3.9)\\ &=\begin{pmatrix}\mathbf{q}'(\omega)J^{-1} & O(z)\\ O(z) & \left\langle JD_{z_i}\widetilde{\phi},D_{z_j}\widetilde{\phi}\right\rangle\Big|_{i,j=1,2}\end{pmatrix}+o(z)=\begin{pmatrix}\mathbf{q}'(\omega)J^{-1} & 0\\ 0 & J\end{pmatrix}+O(z),\end{aligned}$$

where the cancelled terms are null. Since $\mathbf{A}|_{z=0}$ is invertible, we conclude that $\mathbf{A}$ is invertible also for small z. From (3.7) and (3.8) we obtain (3.3) and (3.5).

□

The proof of the following modulation is standard, see Stuart [61].

Lemma 3.2 (Modulation)**.** *Let $\omega_0>0$. There exists an $\delta_0>0$ and functions $\omega\in C^1(\mathcal{U}(\omega_0,\delta_0),\mathbb{R})$ and $\vartheta\in C^1(\mathcal{U}(\omega_0,\delta_0),\mathbb{R}/\mathbb{Z})$ and $z\in C^1(\mathcal{U}(\omega_0,\delta_0),\mathbb{C})$ such that for any $u\in\mathcal{U}(\omega_0,\delta_0)$*

$$\begin{aligned}&\eta(u):=e^{-\mathrm{i}\vartheta(u)}u-\phi[\omega(u),z(u)] \text{ satisfies} \qquad (3.10)\\ &\langle\eta(u),\mathrm{i}\phi[\omega(u),z(u)]\rangle=\langle\eta(u),\partial_\omega\phi[\omega(u),z(u)]\rangle=\langle\eta(u)],\partial_{z_j}\phi[\omega(u),z(u)]\rangle=0, \text{ for all } j=1,2.\end{aligned}$$

Furthermore we have the identities $\omega(\phi_\omega)=\omega$, $\vartheta(e^{\mathrm{i}\vartheta_0}u)=\vartheta(u)+\vartheta_0$ and $\omega(e^{\mathrm{i}\vartheta_0}u)=\omega(u)$ and $z(e^{\mathrm{i}\vartheta_0}u)=z(u)$.

□

We have now the ansatz

$$u=e^{\mathrm{i}\vartheta}\left(\phi[\omega,z]+\eta\right). \tag{3.11}$$

By orbital stability we can assume that there exists $\theta=\theta(t)$ such that

$$\|u-e^{\mathrm{i}\theta}\phi_{\omega_0}\|_{H^1}<\delta \text{ for all values of time.} \tag{3.12}$$

Then, using the notation in (2.10) and the fact that $\mathbf{d}'(\omega)=\mathbf{q}(\omega)$ it is standard to write

$$\begin{aligned}O(\delta)&=\mathbf{E}(u)+\omega\mathbf{Q}(u)-\mathbf{e}(\omega_0)-\omega\mathbf{q}(\omega_0)=\mathbf{d}(\omega)-\mathbf{d}(\omega_0)-\mathbf{d}'(\omega_0)(\omega-\omega_0)\\ &+2^{-1}\left\langle\left(\mathbf{d}^2\mathbf{E}(\phi_\omega)+\omega\mathbf{d}^2\mathbf{Q}(\phi_\omega)\right)r,r\right\rangle+o\left(\|r\|_{H^1}^2\right)+o\left(\|r\|_{H^1}(\omega-\omega_0)\right),\end{aligned} \tag{3.13}$$

for $r=z\xi+\overline{z}\overline{\xi}+\eta$. Now we have

$$\left\langle\left(d^2\mathbf{E}(\phi_\omega)+\omega d^2\mathbf{Q}(\phi_\omega)\right)r,r\right\rangle=\langle\mathcal{L}_\omega r,Jr\rangle=2\lambda|z|^2+\langle\mathcal{L}_\omega\eta,J\eta\rangle \tag{3.14}$$

Since $\langle \mathcal{L}_\omega \eta, J\eta\rangle \gtrsim \|\eta\|_{H^1}^2$, from the above and the strict convexity of $\mathbf{d}(\omega)$ we conclude that

$$|\omega - \omega_0| + |z| + \|\eta\|_{H^1} \lesssim \sqrt{\delta} \text{ for all values of time.} \tag{3.15}$$

We will set

$$\Theta := (\vartheta, \omega, z), \quad \widetilde{\Theta} := (\widetilde{\vartheta}, \widetilde{\omega}, \widetilde{z}) \text{ and } \widetilde{\Theta}_{\mathcal{R}} := (\widetilde{\vartheta}_{\mathcal{R}}, \widetilde{\omega}_{\mathcal{R}}, \widetilde{z}_{\mathcal{R}}). \tag{3.16}$$

The proof of Theorem 1.2 is mainly based on the following continuation argument.

Proposition 3.3. *There exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if*

$$\|\eta\|_{L^2(I,\Sigma_A)} + \|\eta\|_{L^2(I,\tilde{\Sigma})} + \|\dot{\Theta} - \widetilde{\Theta}\|_{L^2(I)} + \|z^2\|_{L^2(I)} \leqslant \epsilon \tag{3.17}$$

holds for $I = [0, T]$ for some $T > 0$ and for $\delta \in (0, \delta_0)$ then in fact for $I = [0, T]$ inequality (3.17) *holds for ϵ replaced by $o_\varepsilon(1)\epsilon$.*

Notice that this implies that in fact the result is true for $I = \mathbb{R}_+$. We will split the proof of Proposition 3.3 in a number of partial results obtained assuming the hypotheses of Proposition 3.3.

Proposition 3.4. *We have*

$$\|\dot{\vartheta} - \widetilde{\vartheta}\|_{L^1(I)} + \|\dot{\omega} - \widetilde{\omega}\|_{L^1(I)} \lesssim \epsilon^2, \tag{3.18}$$

$$\|\dot{z} - \widetilde{z}\|_{L^2(I)} \lesssim \sqrt{\delta}\epsilon, \tag{3.19}$$

$$\|\dot{z}\|_{L^\infty(I)} \lesssim \sqrt{\delta}. \tag{3.20}$$

Proposition 3.5 (Fermi Golden Rule (FGR) estimate)**.** *We have*

$$\|z^2\|_{L^2(I)} \lesssim A^{-1/2}\epsilon. \tag{3.21}$$

Proposition 3.6 (Virial Inequality)**.** *We have*

$$\|\eta\|_{L^2(I,\Sigma_A)} \lesssim A\delta + \|z^2\|_{L^2(I)} + \|\eta\|_{L^2(I,\tilde{\Sigma})} + \epsilon^2. \tag{3.22}$$

Proposition 3.7 (Smoothing Inequality)**.** *We have*

$$\|\eta\|_{L^2(I,\tilde{\Sigma})} \lesssim o_{B^{-1}}(1)\epsilon. \tag{3.23}$$

Proof of Theorem 1.2. It is straightforward that Propositions 3.4–3.7 imply Proposition 3.3 and thus the fact that we can take $I = \mathbb{R}_+$ in all the above inequalities. This in particular implies (1.12). By $z \in L^4(\mathbb{R}_+)$ and $\dot{z} \in L^\infty(\mathbb{R}_+)$ we have (1.13).
We next focus on the limit (1.11). We first rewrite our equation, entering the ansatz (3.11) in (1.1), to obtain, for $\phi = \phi[\omega, z]$,

$$\mathrm{i}\dot{\eta} - \dot{\vartheta}\eta - \dot{\vartheta}\phi + \mathrm{i}\dot{\omega}\partial_\omega\phi + \mathrm{i}D_z\phi\dot{z} = -\partial_x^2\eta - \partial_x^2\phi - f(\phi + \eta).$$

Then, adding and subtracting and using (3.4), for $R = R[\omega, z]$ we obtain

$$\begin{aligned}
&\mathrm{i}\dot{\eta} - (\dot{\vartheta} - \widetilde{\vartheta})\eta - (\dot{\vartheta} - \widetilde{\vartheta})\phi + \mathrm{i}(\dot{\omega} - \widetilde{\omega})\partial_\omega\phi + \mathrm{i}D_z\phi(\dot{z} - \widetilde{z}) = -\partial_x^2\eta + \widetilde{\vartheta}\eta \\
&- (f(\phi + \eta) - f(\phi)) - R \\
&-\partial_x^2\phi - f(\phi) + \widetilde{\vartheta}\phi - \mathrm{i}\widetilde{\omega}\partial_\omega\phi - \mathrm{i}D_z\phi\widetilde{z} + R,
\end{aligned} \tag{3.24}$$

where the last line equals 0, because of the definition of R in (3.4). Equation (3.24) rewrites

$$\begin{aligned}&\dot\eta + \mathrm{i}(\dot\vartheta - \widetilde{\vartheta})\eta + e^{-\mathrm{i}\vartheta}\mathrm{i}D_\Theta\phi[\Theta]\left(\dot\Theta - \widetilde{\Theta}\right) \\ &= \mathrm{i}\left(\partial_x^2 + Df(\phi[\omega,z])\right)\eta - \mathrm{i}\widetilde{\vartheta}\eta + \mathrm{i}\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z]) - Df(\phi[\omega,z])\eta\right) + \mathrm{i}R[\omega,z].\end{aligned} \tag{3.25}$$

Let

$$\mathbf{a}(t) := 2^{-1}\|e^{-a\langle x\rangle}\eta(t)\|^2_{L^2(\mathbb{R})}.$$

Then by the Orbital Stability

$$\begin{aligned}\dot{\mathbf{a}} &= -\frac{1}{2}\left\langle\left[e^{-2a\langle x\rangle}, \mathrm{i}\partial_x^2\right]\eta,\eta\right\rangle \\ &-\left\langle e^{-a\langle x\rangle}\left(\mathrm{i}\dot\vartheta\eta + e^{-\mathrm{i}\vartheta}\mathrm{i}D_\Theta\phi[\Theta]\left(\dot\Theta-\widetilde{\Theta}\right)\right), e^{-a\langle x\rangle}\eta\right\rangle \\ &+\left\langle e^{-a\langle x\rangle}\left(\mathrm{i}\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z])\right) + \mathrm{i}R[\omega,z]\right), e^{-a\langle x\rangle}\eta\right\rangle = O(\epsilon^2) \text{ for all times.}\end{aligned} \tag{3.26}$$

Since we already know from (1.12) that $\mathbf{a} \in L^1(\mathbb{R})$, we conclude that $\mathbf{a}(t) \xrightarrow{t\to+\infty} 0$. Notice that the integration by parts in (3.26) can be made rigorous considering that if $u_0 \in H^2(\mathbb{R})$ by the well known regularity result by Kato, see [5], we have $\eta \in C^0\left(\mathbb{R}, H^2(\mathbb{R})\right)$ and the above argument is correct and by a standard density argument the result can be extended to $u_0 \in H^1(\mathbb{R})$.

Finally, we prove (1.13). Since $\mathbf{Q}(\phi_\omega) = \mathbf{q}(\omega)$ is monotonic, it suffices to show $\mathbf{Q}(\phi_\omega)$ converge as $t \to \infty$. Next, from the conservation of $\mathbf{Q}$, the exponential decay of $\phi[\omega,z]$, (1.11) and (1.12), we have

$$\lim_{t\to\infty}\left(\mathbf{Q}(u_0) - \mathbf{Q}(\phi_{\omega(t)}) - \mathbf{Q}(\eta(t))\right) = 0.$$

Here, notice that we can take

$$e^{-a\langle x\rangle} \gtrsim \max\{|\phi[\omega,z]|, |\phi[\omega,z]|^{p-1}, |\phi[\omega,z]|^{p-2}\}.$$

Thus, our task is now to prove $\frac{d}{dt}\mathbf{Q}(\eta) \in L^1$, which is sufficient to show the convergence of $\mathbf{Q}(\eta)$. Now, from (3.25), we have

$$\begin{aligned}\frac{d}{dt}\mathbf{Q}(\eta) = \langle\eta,\dot\eta\rangle &= -\left\langle \eta, e^{-\mathrm{i}\vartheta}\mathrm{i}D_\Theta\phi[\Theta]\left(\dot\Theta-\widetilde{\Theta}\right)\right\rangle + \langle\eta, \mathrm{i}Df(\phi[\omega,z])\eta\rangle \\ &+ \langle \eta, \mathrm{i}\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z]) - Df(\phi[\omega,z])\eta\right)\rangle + \langle\eta, \mathrm{i}R[\omega,z]\rangle \\ &= I + II + III + IV.\end{aligned}$$

By the bound of the 1st and the 3rd term of (3.17), we have $I \in L^1(\mathbb{R}_+)$. Next, by (2.17), we have $|Df(\phi[\omega,z])\eta| \lesssim e^{-a\langle x\rangle}|\eta|$ for some small a. Therefore, $II \in L^1(\mathbb{R}_+)$. $IV \in L^1(\mathbb{R}_+)$ follows from (3.5) and $|z|^2 \in L^2(\mathbb{R}_+)$. For III, since f is at least C^2 (we only consider $p > 2$) we have In analogy to similar partitions in [20] which allow to offset the lack of differentiability of $f(u)$, we partition the line where x lives as

$$\begin{aligned}&\Omega_{1,t,s} = \{x \in \mathbb{R} \mid |s\eta(t,x)| \leqslant 2|\phi[\omega(t),z(t)]|\} \text{ and} \\ &\Omega_{2,t,s} = \mathbb{R}\backslash\Omega_{1,t,s} = \{x \in \mathbb{R} \mid |s\eta(t,x)| > 2|\phi[\omega(t),z(t)]|\},\end{aligned}$$

Then, we have

$$II(t) = \sum_{j=1,2}\mathrm{Re}\int_0^1 ds\int_{\Omega_{j,t,s}} \mathrm{i}\overline{\eta}(t)D^2 f(\phi[\omega(t),z(t)] + s\eta)(\eta(t),\eta(t))\,dx =: II_1(t) + II_2(t).$$

For II_1, by $\|\eta\|_{L^\infty} \lesssim \epsilon \leqslant 1$ and by (2.18), we have

$$|II_1(t)| \lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |\eta(t)|(|\phi[\omega(t),z(t)]| + |s\eta|)^{p-2}|\eta|^2\, dx \lesssim \int_{\mathbb{R}} |\phi[\omega(t),z(t)]|^{p-2}|\eta(t)|^2\, dx.$$

Thus, we see $II_1 \in L^1(\mathbb{R}_+)$. For II_2, since $|s\eta + \phi[\omega,z]| > \frac{1}{2}|s\eta| > 0$ we can expand the integrand as

$$\begin{aligned} D^2 f(\phi[\omega(t),z(t)] + s\eta(t))(\eta(t),\eta(t)) &= D^2 f(s\eta(t))(\eta(t),\eta(t)) \\ &\quad + \int_0^1 D^3 f(\tau\phi[\omega(t),z(t)] + s\eta(t))(\phi[\omega,z],\eta(t),\eta(t))d\tau. \end{aligned}$$

Now, since $D^2 f(s\eta)(\eta,\eta) = s^{p-2}p(p-1)|\eta|^{p-1}\eta$, we have

$$\mathrm{Re}\int_0^1 ds \int_{\Omega_{2,t,s}} \mathrm{i}\overline{\eta} D^2 f(s\eta(t))(\eta(t),\eta(t))\, dx = 0,$$

because the integrand becomes purely imaginary. Therefore, from the bound

$$|D^3 f(\psi)(w_1,w_2,w_3)| \lesssim |\psi|^{p-3}|w_1||w_2||w_3|$$

and from

$$|\tau\phi[\omega(t),z(t)] + s\eta(t)| \sim |s\eta(t)| \text{ for } \tau \in [0,1] \text{ in } \Omega_{2,t,s}$$

we have

$$|II_2(t)| \lesssim \int_0^1 ds \int_{\Omega_{2,t,s}} |\eta(t)||s\eta(t)|^{p-3}|\phi[\omega(t),z(t)]||\eta(t)|^2\, dx \lesssim \int_0^1 s^{p-3}\, ds \int_{\mathbb{R}} |\phi[\omega(t),z(t)]||\eta(t)|^2\, dx.$$

Since $p > 2$ we have $\int_0^1 s^{p-3}\, ds < \infty$ and we see $II_2 \in L^1(\mathbb{R}_+)$. Therefore, we have the conclusion. $\square$

4 Proof of Proposition 3.4

Lemma 4.1. *We have the estimates*

$$|\dot{\vartheta} - \widetilde{\vartheta}| + |\dot{\omega} - \widetilde{\omega}| \lesssim \left(|z|^2 + \|\eta\|_{\widetilde{\Sigma}}\right)\|\eta\|_{\widetilde{\Sigma}} \tag{4.1}$$

$$|\dot{z} - \widetilde{z}| \lesssim \left(|z| + \|\eta\|_{\widetilde{\Sigma}}\right)\|\eta\|_{\widetilde{\Sigma}}. \tag{4.2}$$

Proof. Applying $\left\langle \cdot, \mathrm{i}e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle$ with $\mathbf{\Theta} \in \mathbb{R}^4$ to (3.25) and by the cancelations (3.6), we get

$$\begin{aligned} &\left\langle D_\Theta\phi[\Theta](\dot{\Theta} - \widetilde{\Theta}), \mathrm{i}D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle - \left\langle \eta, e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle(\dot{\vartheta} - \widetilde{\vartheta}) - \left\langle \eta, \mathrm{i}e^{-\mathrm{i}\vartheta} D^2_\Theta\phi[\Theta](\mathbf{\Theta}, \dot{\Theta} - \widetilde{\Theta})\right\rangle \\ &\cancel{-\left\langle \eta, e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle}\widetilde{\vartheta} - \left\langle \eta, \mathrm{i}e^{-\mathrm{i}\vartheta} D^2_\Theta\phi[\Theta](\mathbf{\Theta}, \widetilde{\Theta})\right\rangle = \cancel{-\widetilde{\vartheta}\left\langle \eta, e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle} \\ &+ \left\langle \eta, \left(\partial_x^2 + Df(\phi[\omega,z])\right) e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle - \left\langle f(\phi+\eta) - f(\phi) - Df(\phi)\eta, e^{-\mathrm{i}\vartheta} D_\Theta\phi[\Theta]\mathbf{\Theta}\right\rangle. \end{aligned} \tag{4.3}$$

Setting also $R[\Theta] = e^{\mathrm{i}\vartheta} R[\omega,z]$, notice that equation (3.4) can be written as

$$\partial_x^2\phi[\Theta] + f(\phi[\Theta]) + \mathrm{i}D_\Theta\phi[\Theta]\widetilde{\Theta} = R[\Theta].$$

Differentiating in Θ, we obtain

$$\left(\partial_x^2 + Df(\phi[\Theta])\right) D_\Theta \phi[\Theta]\boldsymbol{\Theta} + \mathrm{i} D_\Theta^2 \phi[\Theta](\widetilde{\Theta}, \boldsymbol{\Theta}) + \mathrm{i} D_\Theta \phi[\Theta] D_\Theta \widetilde{\Theta} \boldsymbol{\Theta} = D_\Theta R[\Theta]\boldsymbol{\Theta}.$$

Since from $f(e^{\mathrm{i}\vartheta}\phi) = f(\phi)$ we have $Df(e^{\mathrm{i}\vartheta}\phi)X = Df(\phi)e^{-\mathrm{i}\vartheta}X$, we get

$$\begin{aligned}
&\left\langle \eta, \left(\partial_x^2 + Df(\phi)\right) e^{-\mathrm{i}\vartheta} D_\Theta \phi[\Theta]\boldsymbol{\Theta} \right\rangle = \left\langle e^{\mathrm{i}\vartheta}\eta, \left(\partial_x^2 + Df(\phi[\Theta])\right) D_\Theta \phi[\Theta]\boldsymbol{\Theta} \right\rangle \\
&= -\left\langle e^{\mathrm{i}\vartheta}\eta, \mathrm{i} D_\Theta^2 \phi[\Theta](\widetilde{\Theta}, \boldsymbol{\Theta}) + \cancel{\mathrm{i} D_\Theta \phi[\Theta] D_\Theta \widetilde{\Theta} \boldsymbol{\Theta}} - D_\Theta R[\Theta]\boldsymbol{\Theta} \right\rangle
\end{aligned}$$

where the cancellation follows by the modulation orthogonality (3.10). Entering this inside (4.3) we get

$$\begin{aligned}
&\left\langle D_\Theta \phi[\Theta](\dot{\Theta} - \widetilde{\Theta}), \mathrm{i} D_\Theta \phi[\Theta]\boldsymbol{\Theta} \right\rangle - \left\langle \eta, e^{-\mathrm{i}\vartheta} D_\Theta \phi[\Theta]\boldsymbol{\Theta} \right\rangle (\dot{\vartheta} - \widetilde{\vartheta}) - \left\langle \eta, \mathrm{i} e^{-\mathrm{i}\vartheta} D_\Theta^2 \phi[\Theta](\boldsymbol{\Theta}, \dot{\Theta} - \widetilde{\Theta}) \right\rangle \\
&- \cancel{\left\langle \eta, \mathrm{i} e^{-\mathrm{i}\vartheta} D_\Theta^2 \phi[\Theta](\boldsymbol{\Theta}, \widetilde{\Theta}) \right\rangle} = - \cancel{\left\langle e^{\mathrm{i}\vartheta}\eta, \mathrm{i} D_\Theta^2 \phi[\Theta](\widetilde{\Theta}, \boldsymbol{\Theta}) \right\rangle} + \left\langle e^{\mathrm{i}\vartheta}\eta, D_\Theta R[\Theta]\boldsymbol{\Theta} \right\rangle \\
&- \left\langle f(\phi + \eta) - f(\phi) - Df(\phi)\eta, e^{-\mathrm{i}\vartheta} D_\Theta \phi[\Theta]\boldsymbol{\Theta} \right\rangle,
\end{aligned}$$

where the cancellation is obvious, since we have equal terms. So, from this we get

$$\begin{aligned}
&-(\dot{\omega} - \widetilde{\omega}) \langle \partial_\omega \phi, \phi \rangle + \langle D_z \phi(\dot{z} - \widetilde{z}), \phi \rangle + O\left(\|\eta\|_{\widetilde{\Sigma}} |\dot{\Theta} - \widetilde{\Theta}| \right) \\
&= \langle \eta, \mathrm{i}R \rangle - \langle f(\phi + \eta) - f(\phi) - Df(\phi)\eta, \mathrm{i}\phi \rangle
\end{aligned}$$

which implies

$$\begin{aligned}
&-(\dot{\omega} - \widetilde{\omega}) \langle \partial_\omega \phi, \phi \rangle + O\left(|z||\dot{z} - \widetilde{z}|\right) + O\left(\|\eta\|_{\widetilde{\Sigma}} |\dot{\Theta} - \widetilde{\Theta}| \right) \\
&= O\left(\|\eta\|_{\widetilde{\Sigma}} |z|^2 \right) + O\left(\|\eta\|_{\widetilde{\Sigma}}^2 \right).
\end{aligned} \tag{4.4}$$

Similarly, using $\|\cosh(\kappa\omega x)\, \partial_\omega \widehat{R}[\omega, z]\|_{L^2(\mathbb{R})} \lesssim |z|^2$,

$$\begin{aligned}
&(\dot{\vartheta} - \widetilde{\vartheta}) \langle \phi, \partial_\omega \phi \rangle + O\left(|z||\dot{z} - \widetilde{z}|\right) + O\left(\|\eta\|_{\widetilde{\Sigma}} |\dot{\Theta} - \widetilde{\Theta}| \right) \\
&= O\left(\|\eta\|_{\widetilde{\Sigma}} |z|^2 \right) + O\left(\|\eta\|_{\widetilde{\Sigma}}^2 \right).
\end{aligned} \tag{4.5}$$

Finally we get the following which along the other formulas yields the lemma

$$\left\langle D_z \phi(\dot{z} - \widetilde{z}), \mathrm{i}\partial_{z_j} \phi \right\rangle + O\left(|z||\dot{\Theta} - \widetilde{\Theta}| \right) = \left\langle \eta, \partial_{z_j} R \right\rangle + O\left(\|\eta\|_{\widetilde{\Sigma}}^2 \right). \tag{4.6}$$

$\square$

Proof of Proposition 3.4. Lemma 4.1 and (3.17) imply immediately (3.18)–(3.19). Entering this, (3.3) and (3.15) in (4.6) we obtain (3.20).

$\square$

5 The Fermi Golden Rule: proof of Proposition 3.5

The nonlinear Fermi Golden Rule (FGR) was an idea initiated by Sigal [55] and further developed by Buslaev and Perelman [3] and by Soffer and Weinstein [59]. More complicated configurations were discussed in [11], where the deep connection of the FGR with the Hamiltonian nature of the NLS was clarified. This comes about because the FGR has to do with the fact that the integral of certain coefficients on appropriate spheres of the phase space associated to $\mathcal{L}_\omega$ are strictly positive. The positivity is due to the fact that the coefficients are essentially squares, that is the product of pairs of factors which are complex conjugates to each other. It turns out that the factors are like this thanks to the Hamiltonian structure of the NLS, which gives relations between coefficients of the system, since they are partial derivatives of a fixed given function, the Hamiltonian $\mathbf{E}$, see [11, pp. 287–288] for a heuristical explanation. The rigorous argument in [11] needed various changes of variables, to get into canonical coordinates and normal forms. However the notion of Refined Profile and the related modulation ansatz provide a framework to prove the FGR in a direct way, without any need of a search of canonical coordinates and of normal forms, see for example [14]. The proof involves differentiating a Lyapunov functional which lately in the literature, especially for space dimension 1, is simpler than what would be analogous here to the energy $\mathbf{E}(\phi[\Theta])$ used up until [14, 17]. A good reference for the simpler Lyapunov functional is Kowalczyk and Martel [38] where the spectrum is rather simple while for a version with a more complicated spectral configuration we refer to [19]. In our current paper the spectrum is like in Kowalczyk and Martel [38] and Martel [46] and involves the functional

$$\mathcal{J}_{\mathrm{FGR}} := \left\langle J\eta, \chi_A \left(z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)} \right) \right\rangle, \tag{5.1}$$

with a nonzero $g^{(\omega)} \in L^\infty(\mathbb{R}, \mathbb{C}^2)$ satisfying

$$\mathcal{L}_\omega g^{(\omega)} = 2\mathrm{i}\lambda(p,\omega) g^{(\omega)}. \tag{5.2}$$

That $g^{(\omega)}$ exists is known since Krieger and Schlag [42], see Lemma 6.3, or earlier Buslaev and Perelman [2]. Notice that if g solves (5.2) for $\omega = 1$ then $g^{(\omega)}(x) := g(\sqrt{\omega}x)$ solves (5.2), where $\lambda(p,\omega) = \omega\lambda(p,1)$. We define the FGR constant $\gamma(\omega,p)$ by

$$\gamma(\omega,p) := \left\langle \phi_\omega^{p-2} \left(p\xi_1^2 + \xi_2^2 \right), g_1^{(\omega)} \right\rangle + 2 \left\langle \phi_\omega^{p-2} \xi_1 \xi_2, g_2^{(\omega)} \right\rangle. \tag{5.3}$$

The non-degeneracy of this constant, which is usually assumed, but proved in this paper, is important.

Lemma 5.1. *For $|p-3| \ll 1$, we can choose $g^{(\omega)}$ so that $\gamma(\omega,p) \neq 0$.*

The proof of Lemma 5.1 is given in section 11. Notice that once we have have $\gamma(\omega,p) \neq 0$, we can multiply g by a constant to get

$$2(p-1)\gamma(\omega,p) = 1. \tag{5.4}$$

In the next lemma we will need the following reformulation of equation (3.25), where we identify $J = -\mathrm{i}$,

$$\begin{aligned}\dot\eta = \mathcal{L}_\omega \eta - J(\widetilde{\vartheta}_{\mathcal{R}} + \widetilde{\vartheta} - \dot\vartheta)\eta - e^{J\vartheta} D_\Theta \phi[\Theta](\dot\Theta - \widetilde{\Theta}) + J\left(Df(\phi[\omega,z]) - Df(\phi_\omega)\right)\eta \\ - J\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z]) - Df(\phi[\omega,z])\eta\right) - JR[\omega,z].\end{aligned} \tag{5.5}$$

We have the following

Lemma 5.2. *There is a C^1 in time function $\mathcal{I}_{\mathrm{FGR}}$, which satisfies $|\mathcal{I}_{\mathrm{FGR}}| \lesssim \sqrt{\delta}$ such that*

$$\left| \dot{\mathcal{J}}_{\mathrm{FGR}} + \dot{\mathcal{I}}_{\mathrm{FGR}} - |z|^4 \right| \lesssim A^{-1} \left(|z|^4 + \|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\tilde\Sigma}^2 \right). \tag{5.6}$$

Proof. Differentiating $\mathcal{J}_{\mathrm{FGR}}$, we have

$$\begin{aligned}\dot{\mathcal{J}}_{\mathrm{FGR}} =& \left\langle J\dot\eta, \chi_A \left(z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)} \right) \right\rangle + \left\langle J\eta, \chi_A \left(2z\widetilde{z} g^{(\omega)} + 2\overline{z}\overline{\widetilde{z}}\overline{g}^{(\omega)} \right) \right\rangle \\ &+ \left\langle J\eta, \chi_A \left(2z(\dot z - \widetilde{z}) g^{(\omega)} + 2\overline{z}(\dot{\overline{z}} - \overline{\widetilde{z}})\overline{g}^{(\omega)} \right) \right\rangle \\ &+ \left\langle J\eta, \chi_A \left(z^2 \partial_\omega g^{(\omega)} + \overline{z}^2 \partial_\omega \overline{g}^{(\omega)} \right) \right\rangle (\dot\omega - \widetilde{\omega}) + \left\langle J\eta, \chi_A \left(z^2 \partial_\omega g^{(\omega)} + \overline{z}^2 \partial_\omega \overline{g}^{(\omega)} \right) \right\rangle \widetilde{\omega} \\ &=: A_1 + A_2 + A_3 + A_4 + A_5.\end{aligned}$$

We consider first the last three terms, the simplest ones. By (4.2) we have

$$|A_3| \lesssim \|\eta\chi_A\|_{L^1} |z| \; |\dot z - \widetilde{z}| \lesssim A^{\frac{3}{2}} A^{-1} \left\| \operatorname{sech}\left(\frac{2}{A} x \right) \eta \right\|_{L^2} |\dot z - \widetilde{z}| \lesssim \sqrt{\delta} A^{\frac{3}{2}} \left(\|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\widetilde{\Sigma}}^2 \right).$$

Since $\|\partial_\omega g^{(\omega)}\|_{L^\infty} \lesssim \langle x \rangle$, using also (4.1) we have

$$\begin{aligned}|A_4| &\lesssim A \|\eta\chi_A\|_{L^1} |z|^4 |\dot\omega - \widetilde{\omega}| \lesssim A^{\frac{5}{2}} |z|^4 \|\eta\|_{\Sigma_A} |\dot\omega - \widetilde{\omega}| \\ &\lesssim \delta^5 A^{\frac{5}{2}} \left(\|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\widetilde{\Sigma}}^2 \right).\end{aligned}$$

Finally, using (3.3) we have

$$\begin{aligned}|A_5| &\lesssim A \|\eta\chi_A\|_{L^1} |z|^2 |\widetilde{\omega}| \lesssim A^{\frac{5}{2}} |z|^4 \|\eta\|_{\Sigma_A} \\ &\lesssim \delta^2 A^{\frac{5}{2}} \left(\|\eta\|_{\Sigma_A}^2 + |z|^4 \right).\end{aligned}$$

Turning to the main terms, we have

$$A_2 = \left\langle J\eta, 2\mathrm{i}\lambda\chi_A \left(z^2 g^{(\omega)} - \overline{z}^2 \overline{g}^{(\omega)} \right) \right\rangle + \left\langle J\eta, \chi_A \left(2z\widetilde{z}_{\mathcal{R}} g^{(\omega)} + 2\overline{z}\, \overline{\widetilde{z}}_{\mathcal{R}}\, \overline{g}^{(\omega)} \right) \right\rangle = A_{21} + A_{22}.$$

By (3.3) proceeding like for A_3,

$$|A_{22}| \lesssim \|\eta\chi_A\|_{L^1} |z|^3 \lesssim \sqrt{\delta} A^{\frac{3}{2}} \|\eta\|_{\Sigma_A} |z|^2 \lesssim \sqrt{\delta} A^{\frac{3}{2}} \left(\|\eta\|_{\Sigma_A}^2 + |z|^4 \right).$$

By (5.5) and by (5.2) for the cancellation, we have

$$\begin{aligned}A_1 + A_{21} =& -\cancel{\left\langle J\eta, \chi_A \mathcal{L}_\omega (z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)}) \right\rangle} + \cancel{A_{21}} + (\widetilde{\vartheta}_{\mathcal{R}} + \widetilde{\vartheta} - \dot\vartheta) \left\langle \eta, \chi_A (z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)}) \right\rangle \\ &- \left\langle J e^{-\mathrm{i}\vartheta} D_\Theta \phi[\Theta](\dot\Theta - \widetilde{\Theta}), \chi_A (z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)}) \right\rangle \\ &+ \left\langle (2\chi_A' \partial_x + \chi_A'')\eta, z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)} \right\rangle - \left\langle (Df(\phi[\omega, z]) - Df(\phi_\omega))\, \eta, \chi_A (z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)}) \right\rangle \\ &+ \left\langle f(\phi[\omega, z] + \eta) - f(\phi[\omega, z]) - Df(\phi[\omega, z])\eta, \chi_A (z^2 g^{(\omega)} + \overline{z}^2 \overline{g}) \right\rangle + \left\langle R, \chi_A (z^2 g + \overline{z}^2 \overline{g}^{(\omega)})) \right\rangle \\ &= A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16}.\end{aligned}$$

It is easy to see, and a rather routine computation repeated often in the literature, also using Lemma 4.1, that

$$\sum_{j=1}^{5} |A_{1j}| \lesssim \sqrt{\delta} A^{\frac{3}{2}} \left(|z|^4 + \|\eta\|_{\Sigma_A}^2 + \|\eta\|_{\widetilde{\Sigma}}^2 \right).$$

The key term for the FGR is A_{16}. We claim we have

$$\begin{aligned}A_{16} &= \left\langle f(\phi[\omega,z]) - f(\phi_\omega) - Df(\phi_\omega)\left(z\xi + \overline{z}\overline{\xi}\right), \chi_A(z^2 g^{(\omega)} + \overline{z}^2 \overline{g}^{(\omega)}))\right\rangle \\ &- \left\langle \mathrm{i} D_z \phi[\omega,z]\widetilde{z}_{\mathcal{R}} - \widetilde{\vartheta}_{\mathcal{R}}\phi[\omega,z] + \mathrm{i}\widetilde{\omega}_{\mathcal{R}}\partial_\omega \phi[\omega,z], \chi_A(z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)}))\right\rangle =: A_{161} + A_{162}.\end{aligned}$$

We have

$$|A_{162}| \lesssim A^{-1}|z|^4. \tag{5.7}$$

Indeed, for example we have

$$\begin{aligned}&\widetilde{\omega}_{\mathcal{R}}\left\langle \mathrm{i}\partial_\omega \phi[\omega,z], \chi_A(z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)})\right\rangle = \widetilde{\omega}_{\mathcal{R}}\left\langle \mathrm{i}\left(\cancel{\partial_\omega \phi_\omega} + z\partial_\omega \xi + +\overline{z}\partial_\omega \xi\right), z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)}\right\rangle \\ &+ \widetilde{\omega}_{\mathcal{R}}\left\langle \mathrm{i}\partial_\omega \phi[\omega,z], (1-\chi_A)\left(z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)}\right)\right\rangle = O\left(z^5\right) + O\left(e^{-\kappa\omega_0 A} z^4\right)\end{aligned}$$

where we used the orthogonality (2.2)–(2.3), the bound (3.3) and the exponential decay of ξ. The other terms forming A_{162} can be bounded similarly, yielding (5.7). We have

$$\begin{aligned}A_{161} &= 2^{-1}\left\langle D^2 f(\phi_\omega)\left(z\xi + \overline{z}\overline{\xi}\right)^2, \chi_A(z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)}))\right\rangle \\ &+ \int_{[0,1]^2} t\left\langle \left(D^2 f(\phi_\omega + t\ s\left(z\xi + \overline{z}\overline{\xi}\right)) - D^2 f(\phi_\omega)\right)\left(z\xi + \overline{z}\overline{\xi}\right)^2, \chi_A(z^2 g^{(\omega)} + \overline{z}^2\overline{g}^{(\omega)}))\right\rangle =: A_{1611} + A_{1612}.\end{aligned}$$

We have, taking $\delta > 0$ small enough,

$$|A_{1612}| \leqslant o_\delta(1)|z|^4 \leqslant A^{-1}|z|^4.$$

Next, by (2.18) for $\xi = (\xi_1, \xi_2)^\intercal$, $X = (z\xi_1 + \overline{z}\overline{\xi}_1) + \mathrm{i}(z\xi_2 + \overline{z}\overline{\xi}_2)$, $u = \phi_\omega$ and identifying $\mathbb{C} = \mathbb{R}^2$, we have

$$D^2 f(\phi_\omega)\left(z\xi + \overline{z}\overline{\xi}\right)^2 = (p-1)\phi_\omega^{p-2}\begin{pmatrix} p(z\xi_1 + \overline{z}\overline{\xi}_1)^2 + (z\xi_2 + \overline{z}\overline{\xi}_2)^2 \\ 2(z\xi_1 + \overline{z}\overline{\xi}_1)(z\xi_2 + \overline{z}\overline{\xi}_2)\end{pmatrix}. \tag{5.8}$$

Then, by (2.18) we have

$$\begin{aligned}A_{1611} &= 2(p-1)|z|^4\gamma(\omega,p) \\ &+ 4(p-1)|z|^2\left(\left\langle \phi_\omega^{p-2}\left(p|\xi_1|^2 + |\xi_2|^2\right), z^2 g_1^{(\omega)}\right\rangle + 2\left\langle \phi_\omega^{p-2}(\xi_1\overline{\xi}_2 + \overline{\xi}_1\xi_2), z^2 g_2^{(\omega)}\right\rangle\right) \\ &+ 2(p-1)\left(\left\langle \phi_\omega^{p-2}\left(p\overline{\xi}_1^2 + \overline{\xi}_2^2\right), z^4 g_1^{(\omega)}\right\rangle + 2\left\langle \phi_\omega^{p-2}\overline{\xi}_1\overline{\xi}_2, z^4 g_2^{(\omega)}\right\rangle\right) \\ &=: A_{16111} + A_{16112} + A_{16113}.\end{aligned}$$

We claim that

$$\sum_{j=2}^{3}|A_{1511j}| \lesssim |z|^5 + |z|^3|\dot{\Theta} - \widetilde{\Theta}|.$$

To $n = 2, 4$ write

$$\frac{d}{dt}z^n = n\widetilde{z}z^{n-1} + n\left(\dot{z} - \widetilde{z}\right)z^{n-1} = n\mathrm{i}\lambda z^n + n\widetilde{z}_{\mathcal{R}}z^{n-1} + n\left(\dot{z} - \widetilde{z}\right)z^{n-1}. \tag{5.9}$$

Then for example

$$\left\langle \phi_\omega^{p-2}\overline{\xi}_1\overline{\xi}_2, z^4 g_2^{(\omega)}\right\rangle = \left\langle \phi_\omega^{p-2}\overline{\xi}_1\overline{\xi}_2, \frac{1}{4\mathrm{i}\lambda}\left(\frac{d}{dt}z^4 - 4\widetilde{z}_{\mathcal{R}}z^3 - 4\left(\dot{z}-\widetilde{z}\right)z^3\right) g_2^{(\omega)}\right\rangle$$

and applying the Leibnitz rule for the time derivative, it is easy to obtain the claim, from which we conclude, since the other terms can be treated similarly,

$$\sum_{j=2}^{3}|A_{1511j}| \lesssim A^{-1}\left(|z|^4 + \|\eta\|^2_{\Sigma_A} + \|\eta\|^2_{\widetilde{\Sigma}}\right).$$

Using also the normalization in (5.4) we obtain (5.6).

□

Proof of Proposition 3.5. Integrating (5.6) we obtain $\|z^2\|^2_{L^2(I)} \lesssim \sqrt{\delta} + A^{-1}\epsilon^2$ yielding (3.21).

□

6 High energies: proof of Proposition 3.6

The power of the method used by Kowalczyk et al. [40] is seen at high energies, thanks to a striking computation that deals with great ease with the $|\eta|^{p-1}\eta$ term in the equation (3.25), see in particular formula (3.12) in Kowalczyk et al. [40]. Notice that methods of proof of dispersion based on the Duhamel formula, run into great trouble when dealing with the $|\eta|^{p-1}\eta$ at low p's.
Following the framework in Kowalczyk et al. [40] we fix an even function $\chi \in C_0^\infty(\mathbb{R},[0,1])$ satisfying

$$1_{[-1,1]} \leqslant \chi \leqslant 1_{[-2,2]} \text{ and } x\chi'(x) \leqslant 0 \text{ and set } \chi_C := \chi(\cdot/C) \text{ for a } C>0. \tag{6.1}$$

We consider the function

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1-\chi(x))\right) \text{ and } \varphi_A(x) := \int_0^x \zeta_A^2(y)\,dy \tag{6.2}$$

and the vector field

$$S_A := \varphi_A' + 2\varphi_A\partial_x. \tag{6.3}$$

Next, we set

$$\mathcal{I} := 2^{-1}\langle \mathrm{i}\eta, S_A\eta\rangle.$$

Lemma 6.1. *There exists a fixed constant $C>0$ s.t. for an arbitrary small number*

$$\|\eta\|^2_{\Sigma_A} \leqslant C\left[-\dot{\mathcal{I}}_{1\mathrm{st},1} + \|\eta\|^2_{\widetilde{\Sigma}} + |\dot{\Theta}-\widetilde{\Theta}|^2 + |z|^2\right]. \tag{6.4}$$

Proof. From (3.25)

$$\begin{aligned}\dot{\mathcal{I}} = -\langle\dot{\eta}, \mathrm{i}S_A\eta\rangle &= \left\langle\partial_x^2\eta, S_A\eta\right\rangle + \langle R, S_A\eta\rangle + \langle f(\phi+\eta)-f(\phi), S_A\eta\rangle\\ &+ O\left(|\dot{\Theta}-\widetilde{\Theta}|\|\eta\|_{\widetilde{\Sigma}}\right).\end{aligned}$$

From [45] we have

$$\left\langle\partial_x^2\eta, S_A\eta\right\rangle \leqslant -2\|(\zeta_A\eta)'\|^2_{L^2} + \frac{C}{A}\|\eta\|^2_{\widetilde{\Sigma}}.$$

Like in [45]

$$\left\langle f(\phi+\eta)-f(\phi),S_A\eta\right\rangle = -2\left\langle F(\phi+\eta)-F(\phi)-f(\phi)\eta,\zeta_A^2\right\rangle$$
$$-2\left\langle f(\phi+\eta)-f(\phi)-f'(\phi)\eta,\phi'\varphi_A\right\rangle+\left\langle f(\phi+\eta)-f(\phi),\zeta_A^2\eta\right\rangle=\sum_{j=1}^{3}B_j.$$

We have

$$B_1=-2\int_{[0,1]^3}t_1\left\langle D^2f(t_3\phi+t_1t_2\eta)(\eta,\phi),\eta\zeta_A^2\right\rangle dt_1dt_2dt_3+2\left\langle F(\eta),\zeta_A^2\right\rangle$$
$$B_3=\int_{[0,1]^2}\left\langle D^2f(t_2\phi+t_1\eta)(\eta,\phi),\eta\zeta_A^2\right\rangle dt_1dt_2-\left\langle f(\eta),\eta\zeta_A^2\right\rangle$$

This yields

$$|B_1|+|B_3|\lesssim\|\eta\|_{\widetilde{\Sigma}}^2+\delta^{p-1}A^2\|\eta\|_{\Sigma_A}^2,$$

where the crucial bound is

$$\int_{\mathbb{R}}|\eta|^{p+1}\zeta_A^2dx\lesssim A^2\|\eta\|_{L^\infty(\mathbb{R})}^{p-1}\|(\zeta_A\eta)'\|_{L^2(\mathbb{R})}^2, \tag{6.5}$$

see Kowalczyk et al. [40], see also [15]. We have

$$|B_2|\lesssim\|\eta\|_{\widetilde{\Sigma}}^2.$$

Notice, see Lemma 6.2 [19], that the following holds, completing the proof,

$$\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta'\|_{L^2}^2+A^{-2}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta\|_{L^2}^2\lesssim\|(\zeta_A\eta)'\|_{L^2(\mathbb{R})}^2+A^{-1}\|\eta\|_{\widetilde{\Sigma}}^2.$$

□

Proof of Proposition 3.6. Integrating in I inequality (6.1) we obtain (3.22).

□

7 Low energies: proof of Proposition 3.7

While very effective and efficient at proving dispersion at high energies thanks to inequality (6.5), the virial inequality of Kowalczyk et al. [40] is somewhat inefficient at low energies, because it places some restrictions on the system that seem due to the method of proof. In fact, as we show below, we can replace the virial inequality with smoothing estimates. This because we only need to bound $\|\eta\|_{\widetilde{\Sigma}}=\|\mathrm{sech}\,(\kappa\omega_0x)\,\eta\|_{L^2(\mathbb{R})}$, which has the rapidly decaying weight $\mathrm{sech}\,(\kappa\omega_0x)$. It is enough to bound $\|\mathrm{sech}\,(\kappa\omega_0x)\,\chi_B\eta\|_{L^2(\mathbb{R})}$ because the difference is, choosing $B\sim\sqrt[3]{A}\ll\sqrt{A}$, a small fraction of $\|\eta\|_{\Sigma_A}$. To get a bound for $\|\mathrm{sech}\,(\kappa\omega_0x)\,\chi_B\eta\|_{L^2(\mathbb{R})}$, we can multiply equation (5.5) by χ_B. The cutoff χ_B tames the term $\chi_B|\eta|^{p-1}\eta$, which is very small and easy to bound. There is a new term due to the commutator of χ_B and $\mathcal{L}_\omega$ which requires a new smoothing estimate, see Proposition 2.3, not already contained in Krieger and Schlag [42]. Notice that in the case of scalar Schrödinger operators in $\mathbb{R}$ a version of Proposition 2.3 is implied by Deift and Trubowitz [25] and is easy to prove, see Sect. 8 [17]. For convenience, in the study of dispersive and smoothing estimates of (5.5) it is customary to use a different coordinate system. We consider the matrix U defined by

$$U = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}, \quad U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}. \tag{7.1}$$

We have

$$U^{-1}JU = \mathrm{i}\sigma_3 \text{ where } \sigma_3 := \mathrm{diag}(1,-1).$$

By elementary computations

$$\begin{aligned} &U^{-1}\mathcal{L}_\omega U = \mathrm{i}H_\omega \text{ where } H_\omega = \sigma_3\left(-\partial_x^2 + \omega\right) + V_\omega, \\ &V_\omega := -\frac{p+1}{2}\phi_\omega^{p-1}\sigma_3 - \mathrm{i}\frac{p-1}{2}\phi_\omega^{p-1}\sigma_2 \text{ where } \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \end{aligned} \tag{7.2}$$

Notice that we have the symmetries

$$\sigma_1 H_\omega = -H_\omega \sigma_1 \text{ and } \sigma_3 H_\omega = H_\omega^* \sigma_3. \tag{7.3}$$

Applying U^{-1} to equation (5.5) we get

$$\begin{aligned} \partial_t(U^{-1}\eta) &= \mathrm{i}H_\omega U^{-1}\eta - \mathrm{i}\sigma_3(\widetilde{\vartheta}_{\mathcal{R}} + \widetilde{\vartheta} - \dot{\vartheta})U^{-1}\eta \\ &\quad - e^{\mathrm{i}\sigma_3\vartheta}U^{-1}D_\Theta\phi[\Theta](\dot{\Theta} - \widetilde{\Theta}) + \mathrm{i}\sigma_3 U^{-1}\left(Df(\phi[\omega,z]) - Df(\phi_\omega)\right)\eta \\ &\quad - \mathrm{i}\sigma_3 U^{-1}\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z]) - Df(\phi[\omega,z])\eta\right) - \mathrm{i}\sigma_3 U^{-1}R[\omega,z]. \end{aligned} \tag{7.4}$$

Set $v := \chi_B U^{-1}\eta$. We denote $P_d(\omega)$ the discrete spectrum projection and $P_c(\omega)$ the continuous spectrum projection associated to H_ω, which are closely related to the corresponding projections for $\mathcal{L}_\omega$. Then we write

$$v = P_c(\omega_0)v + [P_d(\omega_0), \chi_B]U^{-1}\eta \text{ where it is easy to check that} \tag{7.5}$$

$$\|[P_d(\omega_0), \chi_B]U^{-1}\eta\|_{L^{2,s}(\mathbb{R})} \leqslant o_{B^{-1}}(1)\|\eta\|_{\widetilde{\Sigma}} \text{ for any } s \in \mathbb{R}. \tag{7.6}$$

Setting $w = P_c v(\omega_0)$, we have

$$\begin{aligned} \partial_t w &= \mathrm{i}H_{\omega_0}w - \mathrm{i}\varpi P_c(\omega_0)\sigma_3 w + \mathrm{i}P_c(\omega_0)\sigma_3\left(2\chi_B'\partial_x + \chi_B''\right)U^{-1}\eta \\ &\quad + \mathrm{i}P_c(V_{\omega_0} - V_\omega)w + \mathrm{i}P_c(V_{\omega_0} - V_\omega)[P_d(\omega_0), \chi_B]U^{-1}\eta \\ &\quad - P_c(\omega_0)\chi_B e^{\mathrm{i}\sigma_3\vartheta}U^{-1}D_\Theta\phi[\Theta](\dot{\Theta} - \widetilde{\Theta}) + \mathrm{i}P_c(\omega_0)\sigma_3\chi_B U^{-1}\left(Df(\phi[\omega,z]) - Df(\phi_\omega)\right)\eta \\ &\quad - \mathrm{i}P_c(\omega_0)\chi_B\sigma_3 U^{-1}\left(f(\phi[\omega,z]+\eta) - f(\phi[\omega,z]) - Df(\phi[\omega,z])\eta\right) - \mathrm{i}P_c(\omega_0)\sigma_3\chi_B U^{-1}R[\omega,z] \end{aligned} \tag{7.7}$$

where

$$\varpi := \widetilde{\vartheta}_{\mathcal{R}} + \widetilde{\vartheta} - \dot{\vartheta} + \omega - \omega_0. \tag{7.8}$$

There is a splitting $P_c(\omega) = P_+(\omega) + P_-(\omega)$ with $P_\pm(\omega)$ the spectral projections in $\mathbb{R}_\pm \cap \sigma_e(H_\omega)$. Specifically we have the following for which we refer to [24].

Lemma 7.1. *The following are bounded operators $P_\pm(\omega)$ in $L^2(\mathbb{R})$*

$$\begin{aligned} P_+(\omega)u &= \lim_{M\to+\infty}\lim_{\epsilon\to0^+}\frac{1}{2\pi\mathrm{i}}\int_{[\omega,M]}\left[R_{H_\omega}(E+\mathrm{i}\epsilon) - R_{H_\omega}(E-\mathrm{i}\epsilon)\right]u\,dE \\ P_-(\omega)u &= \lim_{M\to+\infty}\lim_{\epsilon\to0^+}\frac{1}{2\pi\mathrm{i}}\int_{[-M,-\omega]}\left[R_{H_\omega}(E+\mathrm{i}\epsilon) - R_{H_\omega}(E-i\epsilon)\right]u\,dE \end{aligned} \tag{7.9}$$

and for any $M>0$ and $N>0$ and for $C=C(N,M,\omega)$ we have

$$\|(P_+(\omega)-P_-(\omega)-P_c(H_\omega)\sigma_3)f\|_{L^{2,M}(\mathbb{R})}\leqslant C\|f\|_{L^{2,-N}(\mathbb{R})}. \tag{7.10}$$

□

A version of Lemma 7.1 was introduced by Buslaev and Perelman [3], see also [4].

We rewrite (7.7) as

$$\begin{aligned}\partial_t w &= \mathrm{i}H_{\omega_0}w-\mathrm{i}\varpi(P_+(\omega_0)-P_-(\omega_0))w+\mathrm{i}\sigma_3\left(2\chi_B'\partial_x+\chi_B''\right)w\\ &-\mathrm{i}\varpi P_c(\omega_0)\sigma_3[P_d(\omega_0),\chi_B]U^{-1}\eta-\mathrm{i}\varpi(P_c(\omega_0)\sigma_3-P_+(\omega_0)+P_-(\omega_0))w && (7.11)\\ &+\mathrm{i}P_c(V_{\omega_0}-V_\omega)w+\mathrm{i}P_c(V_{\omega_0}-V_\omega)[P_d(\omega_0),\chi_B]U^{-1}\eta && (7.12)\\ &-P_c(\omega_0)\chi_B e^{\mathrm{i}\sigma_3\vartheta}U^{-1}D_\Theta\phi[\Theta](\dot\Theta-\widetilde\Theta)+\mathrm{i}P_c(\omega_0)\sigma_3\chi_B U^{-1}\left(Df(\phi[\omega,z])-Df(\phi_\omega)\right)\eta && (7.13)\\ &-\mathrm{i}P_c(\omega_0)\chi_B\sigma_3U^{-1}\left(f(\phi[\omega,z]+\eta)-f(\phi[\omega,z])-Df(\phi[\omega,z])\eta\right)-\mathrm{i}P_c(\omega_0)\sigma_3\chi_BU^{-1}R[\omega,z]. && (7.14)\end{aligned}$$

We have

$$\begin{aligned}w&=\mathcal{U}(t,0)w(0)+\mathrm{i}\int_0^t\mathcal{U}(t,t')P_c(\omega_0)\sigma_3\left(2\chi_B'\partial_x+\chi_B''\right)U^{-1}\eta dt' && (7.15)\\ &+\int_0^t\mathcal{U}(t,t')\,(\text{lines (7.11)– (7.14)})\,dt', && (7.16)\end{aligned}$$

where $\mathcal{U}$ is the generator associated with the linear evolution $\partial_t w=\mathrm{i}H_{\omega_0}w-\mathrm{i}\varpi(P_+(\omega_0)-P_-(\omega_0))w$. For $\alpha_{tt'}=\int_{t'}^t\varpi(\upsilon)d\upsilon$, $P_\pm=P_\pm(\omega_0)$ and $P_c=P_c(\omega_0)$, expanding the exponential we get

$$\mathcal{U}(t,t'):=e^{\mathrm{i}(t-t')H_{\omega_0}}P_ce^{\mathrm{i}\alpha_{tt'}(P_+-P_-)}=P_ce^{\mathrm{i}(t-t')H_{\omega_0}}\left(\cos\left(\alpha_{tt'}\right)P_c+\mathrm{i}\sin\left(\alpha_{tt'}\right)(P_+-P_-)\right). \tag{7.17}$$

Lemma 7.2. *For $S>3/2$ we have*

$$\|w\|_{L^2(I,L^{2,-S}(\mathbb{R}))}\leqslant o_{B^{-1}}(1)\epsilon. \tag{7.18}$$

Proof. Let us take $S>3/2$. Taking the expansion in (7.17), we have

$$\begin{aligned}\|\mathcal{U}(t,0)w(0)\|_{L^2(I,L^{2,-S}(\mathbb{R}))}&\leqslant\|e^{\mathrm{i}tH_{\omega_0}}w(0)\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))}\\ &+\|e^{\mathrm{i}tH_{\omega_0}}(P_+-P_-)w(0)\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))}.\end{aligned}$$

By the analogue for H_ω of (2.21), see (9.1), which is what we actually prove in Sect. 9 below, we have

$$\|e^{\mathrm{i}tH_{\omega_0}}w(0)\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))}\lesssim\|w(0)\|_{L^2(\mathbb{R})}.$$

Similarly

$$\|e^{\mathrm{i}tH_{\omega_0}}(P_+-P_-)w(0)\|_{L^2(\mathbb{R},L^{2,-S}(\mathbb{R}))}\lesssim\|(P_+-P_-)w(0)\|_{L^2(\mathbb{R})}\lesssim\|w(0)\|_{L^2(\mathbb{R})}.$$

The second term in (7.15) is more delicate for more than one reason. First of all, by (7.17) and $\alpha_{tt'}=\alpha_{t0}-\alpha_{t'0}$, we write the integrand as the sum

$$\begin{aligned}&e^{\mathrm{i}(t-t')H_{\omega_0}}P_c\left[\cos\left(\alpha_{t0}\right)\cos\left(\alpha_{t'0}\right)+\sin\left(\alpha_{t0}\right)\sin\left(\alpha_{t'0}\right)\right.\\ &\left.+\mathrm{i}\left(\cos\left(\alpha_{t0}\right)\sin\left(\alpha_{t'0}\right)-\sin\left(\alpha_{t0}\right)\cos\left(\alpha_{t'0}\right)\right)(P_+-P_-)\right]\sigma_3\left(2\chi_B'\partial_x+\chi_B''\right)U^{-1}\eta\end{aligned}$$

This yields various terms, that can be bounded all in the same way, so that we bound only the last of them. We proceed like in [17]. We have

$$
\begin{aligned}
&\| \sin(\alpha_{t0}) \int_0^t e^{\mathrm{i}(t-t')H_{\omega_0}} P_c (P_+ - P_-)\sigma_3 \left(2\chi_B' \partial_x + \chi_B''\right) \cos(\alpha_{t'0}) U^{-1} \eta dt' \|_{L^2(I, L^{2,-S}(\mathbb{R}))} \\
&\lesssim \| \int_0^t e^{\mathrm{i}(t-t')H_{\omega_0}} (P_+ - P_- - P_c\sigma_3)\sigma_3 \left(2\chi_B' \partial_x + \chi_B''\right) \cos(\alpha_{t'0}) U^{-1} \eta dt' \|_{L^2(I, L^{2,-S}(\mathbb{R}))} \\
&+ \| \int_0^t e^{\mathrm{i}(t-t')H_{\omega_0}} P_c \left(2\chi_B' \partial_x + \chi_B''\right) \cos(\alpha_{t'0}) U^{-1} \eta dt' \|_{L^2(I, L^{2,-S}(\mathbb{R}))} =: I_1 + I_2.
\end{aligned}
$$

For I_1 can use the estimate (2.19) derived by Krieger and Schlag [42] and write

$$
\begin{aligned}
I_1 &\lesssim \| \int_0^t \| e^{\mathrm{i}(t-t')H_{\omega_0}} P_c \|_{L^{2,S} \to L^{2,-S}} \| P_+ - P_- - P_c\sigma_3 \|_{L^2 \to L^{2,S}} \| \left(2\chi_B' \partial_x + \chi_B''\right) U^{-1} \eta \|_{L^2(\mathbb{R})} dt' \|_{L^2(I)} \\
&\lesssim \| \int_0^t \langle t - t' \rangle^{-3/2} \| \left(2\chi_B' \partial_x + \chi_B''\right) U^{-1} \eta \|_{L^2(\mathbb{R})} dt' \|_{L^2(I)} \lesssim \| \left(2\chi_B' \partial_x + \chi_B''\right) \eta \|_{L^2(I, L^2(\mathbb{R}))}, \qquad (7.19)
\end{aligned}
$$

where we postpone completion of the analysis. The term I_2 is more delicate and is bounded by Lemma 8.9, expressed for H_ω instead of $\mathcal{L}_\omega$, which is the same. So for any $s > 1/2$

$$
I_2 \lesssim \| \left(2\chi_B' \partial_x + \chi_B''\right) \eta \|_{L^2(I, L^{2,s}(\mathbb{R}))}. \qquad (7.20)
$$

Now we have

$$
\begin{aligned}
&\| \left(2\chi_B' \partial_x + \chi_B''\right) \eta \|_{L^2(I, L^{2,s}(\mathbb{R}))} \lesssim B^{s-1} \| \mathrm{sech}\left(\frac{2}{A} x\right) \eta' \|_{L^2(I, L^2(\mathbb{R}))} \\
&+ B^{s-2} \| 1_{B \leqslant |x| \leqslant 2B} \mathrm{sech}\left(\frac{2}{A} x\right) \eta \|_{L^2(I, L^2(\mathbb{R}))} \lesssim B^{s-1} \| \eta \|_{L^2(I, \Sigma_A)} \\
&+ B^{s-1} \left(\left\| \left(\mathrm{sech}\left(\frac{2}{A} x\right) \eta \right)' \right\|_{L^2(I, L^2(\mathbb{R}))} + \| \eta \|_{L^2(I, \widetilde{\Sigma})} \right) = o_{B^{-1}}(1) \epsilon,
\end{aligned}
$$

where we used $s \in (1/2, 1)$ and, see [17],

$$
\| 1_{B \leqslant |x| \leqslant 2B} u \|_{L^2(\mathbb{R})} \lesssim \sqrt{\| 1_{B \leqslant |x| \leqslant 2B} |x| \|_{L^1(\mathbb{R})}} \left(\left\| u' \right\|_{L^2(\mathbb{R})} + \| u \|_{\widetilde{\Sigma}} \right).
$$

This implies the following, yielding good bounds for the terms in the right hand side of line (7.15),

$$
I_1 + I_2 = o_{B^{-1}}(1) \epsilon.
$$

The terms in line (7.16) can be similarly bounded using in particular the analogue for H_ω of Proposition 2.2. The estimates are elementary and similar to [17, Sect. 8].

$\square$

Proof of Proposition 3.7. From (7.5), (7.6) and (7.18) we have $\| v \|_{L^2(I, \widetilde{\Sigma})} \lesssim o_{B^{-1}}(1) \epsilon$. Next, from $v = \chi_B U^{-1} \eta$, and thanks to the relation $A \sim B^3$ set in (2.11) we have

$$
\begin{aligned}
\| \eta \|_{\widetilde{\Sigma}} &\lesssim \| v \|_{\widetilde{\Sigma}} + \| (1 - \chi_B) \eta \|_{\widetilde{\Sigma}} \lesssim \| v \|_{\widetilde{\Sigma}} + A^{-2} \| \mathrm{sech}\left(\frac{2}{A} x\right) \eta \|_{L^2} \\
&\lesssim \| v \|_{\widetilde{\Sigma}} + A^{-1} \| \eta \|_{\Sigma_A} \qquad (7.21)
\end{aligned}
$$

So by (3.22) and (3.21) we get the following, which implies (3.23),

$$\begin{aligned}\|\eta\|_{L^2(I,\widetilde{\Sigma})} &\lesssim \|v\|_{L^2(I,\widetilde{\Sigma})} + A^{-1}\|\eta\|_{L^2(I,\Sigma_A)} \\ &\lesssim o_{B^{-1}}(1)\epsilon + A^{-1}\left(\|z^2\|_{L^2(I)} + \|\eta\|_{L^2(I,\widetilde{\Sigma})}\right) \lesssim o_{B^{-1}}(1)\epsilon + A^{-1}\|\eta\|_{L^2(I,\widetilde{\Sigma})}.\end{aligned}$$

□

8 The resolvent of the linearized operator

We will focus on the operator H_ω in (7.2). For the discussion it is enough to consider $\omega = 1$, since the operators for other values of ω are obtained by a scaling transformation. We will set $H = H_1$ with vector potential $V = V_1$. We will set

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Given two (column) functions $f, g : \mathbb{R} \to \mathbb{C}^2$, using the row column product, we consider the Wronskian

$$W[f,g](x) := f'(x)^\intercal g(x) - f(x)^\intercal g'(x).$$

It is well known that H has Jost functions, discussed in [2, 42], which we subsume here.

Proposition 8.1. *For any $k \in \mathbb{R}$ there exists solutions $f_j(x,k)$ for $j = 1,2,3,4$ of*

$$Hu = (1+k^2)u \tag{8.1}$$

with for a fixed $C > 0$ and for $x \geqslant 0$

$$f_j(x,k) = e^{\mathrm{i}(-1)^{j+1}xk} m_j(x,k) \text{ with } |m_j(x,k) - e_1| \leqslant C\langle k\rangle^{-1} e^{-(p-1)x} \text{ for } j = 1,2, \tag{8.2}$$

$$f_3(x,k) = e^{-\sqrt{2+k^2}x} m_3(x,k) \text{ with } |m_3(x,k) - e_2| \leqslant C\langle k\rangle^{-1} e^{-(p-1)x}. \tag{8.3}$$

There is a solution $\widetilde{f}_4(x,k)$ of (8.1) with

$$\widetilde{f}_4(x,k) = e^{\sqrt{2+k^2}x} \widetilde{m}_4(x,k) \text{ with } |\widetilde{m}_4(x,k) - e_2| \leqslant C\langle k\rangle^{-1} e^{-(p-1)x}. \tag{8.4}$$

We have

$$W[f_1,f_2] = 2\mathrm{i}k \ , \ W[f_2,\widetilde{f}_4] = 2\sqrt{2+k^2} \ , \ W[f_j,f_3] = 0 \text{ for } j = 1,2. \tag{8.5}$$

There is a unique choice of $c_1, c_2 \in \mathbb{C}$ such that for

$$f_4(x,k) := -c_1 f_1(x,k) - c_2 f_2(x,k) + \widetilde{f}_4 \Longrightarrow W[f_j,f_4] = 0 \text{ for } j = 1,2. \tag{8.6}$$

□

For the proof see [42]. Since the potential $V(x)$ is even, writing

$$g_j(x,k) := f_j(-x,k) \tag{8.7}$$

yields analogous Jost functions with prescribed behavior as $x \to -\infty$. Notice that since the potential $V(x)$ is exponentially decreasing, all the above Jost functions extend in the region $|\operatorname{Im} k| \leqslant \delta_p$ for a small $\delta_p > 0$.

Remark 8.2. For $p = 3$ it is possible to write explicit formulas for the above Jost functions, for $f_j(x,k)$ for $j = 1, 2$ see [36]. We write the formulas in Sect. 10.

We consider the matrices

$$\begin{aligned} F_1(x,k) &= (f_1(x,k), f_3(x,k)) \quad , \quad F_2(x,k) = (f_2(x,k), f_4(x,k)), \\ G_1(x,k) &= (g_2(x,k), g_4(x,k)) \quad , \quad G_2(x,k) = (g_1(x,k), g_3(x,k)). \end{aligned}$$

For matrix valued functions $F = (\phi_1, \phi_2)$ and $G = (\psi_1, \psi_2)$ we set

$$W[F,G] := F'(x)^\intercal G(x) - F(x)^\intercal G'(x).$$

By direct computation, see [42],

$$W[F,G] = \begin{pmatrix} W[\phi_1,\psi_1] & W[\phi_1,\psi_2] \\ W[\phi_2,\psi_1] & W[\phi_2,\psi_2] \end{pmatrix}.$$

Still quoting from [42], we have the following.

Lemma 8.3. *For any $k \in \mathbb{R}\backslash\{0\}$ there matrices $A(k)$ and $B(k)$, smooth in k and s.t.*

$$F_1(x,k) = G_1(x,k)A(k) + G_2(x,k)B(k), \tag{8.8}$$

with $A(-k) = \overline{A}(k)$, $B(-k) = \overline{B}(k)$ and

$$G_2(x,k) = F_2(x,k)A(k) + F_1(x,k)B(k), \tag{8.9}$$

$$W[F_1(x,k), G_2(x,k)] = A(k)^\intercal \mathrm{diag}(2\mathrm{i}k, -2\sqrt{2+k^2}) \tag{8.10}$$

$$W[F_1(x,k), G_1(x,k)] = -B(k)^\intercal \mathrm{diag}(2\mathrm{i}k, -2\sqrt{2+k^2}). \tag{8.11}$$

Furthermore

$$\begin{aligned} G_1(x,k) &= F_2(-x,k) \quad , \quad G_2(x,k) = F_1(-x,k) \\ \overline{F_1(x,k)} &= F_1(x,-k) \quad , \quad \overline{F_2(x,k)} = F_2(x,-k). \end{aligned} \tag{8.12}$$

□

We set

$$D(k) := W[F_1(x,k), G_2(x,k)] = \begin{pmatrix} W[f_1(x,k), g_1(x,k)] & W[f_1(x,k), g_3(x,k)] \\ W[f_3(x,k), g_1(x,k)] & W[f_3(x,k), g_3(x,k)] \end{pmatrix}. \tag{8.13}$$

The following holds, see [42].

Lemma 8.4. *For $k \neq 0$ the following are equivalent:*

- $\det A(k) = 0$*;*
- *$E = 1 + k^2$ is an eigenvalue of H;*
- $\det D(k) = 0$.

Furthermore $E = 1$ is neither nor an eigenvalue of H if and only if $\det D(0) \neq 0$ and we have $D(-k) = \overline{D(k)}$ and $D(k)^\intercal = D(k)$.

The following holds, see [42].

Lemma 8.5. *For $k \geqslant 0$ the following extensions of the resolvent $R_H(E)$ from above and from below the real line hold, for $E = 1 + k^2$:*

$$R_H^+(x,y,E) = \begin{cases} -F_1(x,k)D^{-1}(k)G_2(y,k)^\intercal\sigma_3 & \text{if } x \geqslant y \\ -G_2(x,k)D^{-1}(k)F_1(y,k)^\intercal\sigma_3 & \text{if } x \leqslant y \end{cases} \tag{8.14}$$

$$R_H^-(x,y,E) = \begin{cases} -F_1(x,-k)D^{-1}(-k)G_2(y,-k)^\intercal\sigma_3 & \text{if } x \geqslant y \\ -G_2(x,-k)D^{-1}(-k)F_1(y,-k)^\intercal\sigma_3 & \text{if } x \leqslant y. \end{cases} \tag{8.15}$$

We set $x^\pm = \max\{\pm x, 0\}$. The main result of this section is the following.

Proposition 8.6. *There exists a small constant $\delta_3 > 0$ such that for any p with $0 < |p-3| < \delta_3$ there exists a constant C such that for any $E \in (-\infty,-1] \cup [1,+\infty)$ we have*

$$\left|R_H^\pm(x,y,E)\right| \leqslant C \begin{cases} (1 + x^- + y^+) & \text{if } x \geqslant y \\ (1 + x^+ + y^-) & \text{if } x \leqslant y. \end{cases} \tag{8.16}$$

Assuming Proposition 8.6, we have the following.

Lemma 8.7. *For $S > 3/2$ and $\tau > 1/2$ we have*

$$\sup_{E\in\mathbb{R}} \|R_H^\pm(E)P_c\|_{L^{2,\tau}(\mathbb{R})\to L^{2,-S}(\mathbb{R})} < \infty. \tag{8.17}$$

Proof. First of all, from the proof of Proposition 8.6 it will be clear that (8.16) holds for any $E \in (-\infty,-a] \cup [a,+\infty)$ for an $a \in (0,1)$ sufficiently close to 1. Then in such a set we proceed like in [17], we can ignore P_c and consider the square of the Hilbert–Schmidt norm

$$\begin{aligned} &\int_{\mathbb{R}} dx\, \langle x\rangle^{-2S} \int_{\mathbb{R}} |R_{H_{N+1}}^+(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dy = \int_{\mathbb{R}} dx\, \langle x\rangle^{-2S} \int_{-\infty}^{x} |R_{H_{N+1}}^+(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dy \\ &+ \int_{\mathbb{R}} dx\, \langle x\rangle^{-2S} \int_{x}^{+\infty} |R_{H_{N+1}}^+(x,y,z)|^2 \langle y\rangle^{-2\tau}\, dy. \end{aligned} \tag{8.18}$$

The second term in the right hand side is bounded by

$$\begin{aligned} &\int_{x<y} \langle x\rangle^{-2S} \langle y\rangle^{-2\tau} \left(1 + x^+ + y^-\right)^2 dxdy \leqslant \int_{x<y<0} \langle x\rangle^{-2S} \langle y\rangle^{-2\tau+2} dxdy \\ &+ \int_{0<x<y} \langle x\rangle^{-2S+2} \langle y\rangle^{-2\tau} dxdy + \int_{x<0<y} \langle x\rangle^{-2S} \langle y\rangle^{-2\tau} dxdy =: \sum_{j=1}^{3} I_j. \end{aligned}$$

Then

$$I_1 \leqslant \int_{\mathbb{R}} \langle x\rangle^{-2S+2} dx \int_{\mathbb{R}} \langle y\rangle^{-2\tau} dy =: I_4 < \infty \text{ for } S > 3/2 \text{ and } \tau > 1/2.$$

Similarly $I_j < I_4$ for $j = 2,3$. Similar estimates hold for the term in the first line in the right hand side of (8.18). So now we need to consider the inequality in (8.17) only for $E \in [-a,a]$, in which case we can drop the superscript $\pm$. Then the result is trivial, because

$$\begin{aligned} &\sup_{E\in[-a,a]} \|R_H(E)P_c\|_{L^{2,\tau}(\mathbb{R})\to L^{2,-S}(\mathbb{R})} \leqslant \sup_{E\in[-a,a]} \|R_H(E)P_c\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \\ &= \sup_{E\in[-a,a]} \|R_{HP_c}(E)\|_{R(P_c)\to R(P_c)} < \infty \end{aligned}$$

by the invariance of $R(P_c)$, the Range of P_c, and by $\sigma(HP_c) \cap (-1,1) = \varnothing$.

∎

The following formula is inspired by Mizumachi [48, Lemma 4.5].

Lemma 8.8. *Let for $g \in \mathcal{S}(\mathbb{R} \times \mathbb{R}, \mathbb{C}^2)$ with $P_c g(t) = g(t)$*

$$U(t, \cdot) := \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{i}Et} \left(R_H^-(E) + R_H^+(E) \right) g^\vee(E, \cdot) dE$$

where $g^\vee$ is the inverse Fourier transform in t of g. Then

$$2 \int_0^t e^{-\mathrm{i}(t-t')H} g(t') dt' = U(t,x) - \int_{\mathbb{R}_-} e^{-\mathrm{i}(t-t')H} g(t') dt' + \int_{\mathbb{R}_+} e^{-\mathrm{i}(t-t')H} g(t') dt'. \tag{8.19}$$

We postpone the proof of Lemma 8.8 until the end of this section. From Lemmas 8.7 and 8.8 we conclude the following, inspired by Mizumachi [48].

Lemma 8.9. *For $S > 3/2$ and $\tau > 1/2$ there exists a constant $C(S, \tau)$ such that we have*

$$\left\| \int_0^t e^{-\mathrm{i}(t-t')H} P_c g(t') dt' \right\|_{L^2(\mathbb{R}, L^{2,-S}(\mathbb{R}))} \leqslant C(S, \tau) \|g\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))}. \tag{8.20}$$

Proof. The proof is verbatim in [17]. We can use formula (8.19) and bound U, with the bound on the last two terms in the right hand side of (8.19) similar. Taking Fourier transform in t,

$$\begin{aligned} \|U\|_{L^2_t L^{2,-S}} &\leqslant 2 \sup_{\pm} \|R^{\pm}_{H_{N+1}}(\lambda) \widehat{g}(\lambda, \cdot)\|_{L^2_\lambda L^{2,-S}} \leqslant \\ &\leqslant 2 \sup_{\pm} \sup_{\lambda \in \mathbb{R}} \|R^{\pm}_{H_{N+1}}(\lambda)\|_{L^{2,\tau} \to L^{2,-S}} \|\widehat{g}(\lambda, x)\|_{L^{2,\tau} L^2_\lambda} \lesssim \quad \|g\|_{L^2_t L^{2,\tau}}. \end{aligned}$$

$\square$

Proof of Proposition 8.6. From $R_H^-(x,y,E) = \overline{R_H^+(x,y,E)}$ for the bounds it is enough consider the case of R_H^+. From

$$R_H^+(x,y,E) = \sigma_3 R_H^+(-y,-x,E)^\intercal \sigma_3$$

it is enough to consider case $x \geqslant y$. Finally

$$R_H^+(x,y,E) = -\sigma_1 R_H^-(x,y,-E)\sigma_1,$$

it is enough to focus on $E = 1 + k^2$.
After the above reductions, we remark that it is enough to consider the case when k is close to 0. This is because for k away from 0 a better estimate, without the term $1 + x^+ + y^-$, is already contained in [42]. Notice that the estimates in [42] there are some subtleties because, while for $x \geqslant 0 \geqslant y$ the desired estimate follows directly form the bounds here stated in Proposition 8.1, for say $0 > x \geqslant y$ the bounds rely on (8.8)–(8.9) and on formulas (8.10)–(8.11) which yield formulas like $A(k)D^{-1}(k) = \mathrm{diag}\left(\frac{1}{2\mathrm{i}k}, -\frac{1}{\sqrt{2+k^2}} \right)$ which are responsible for some crucial cancellations.
In the sequel we consider only the case R_H^+ for $E = 1 + k^2$ and $x \geqslant y$ for k close to 0. We will follow the argument in [12], which unfortunately has some mistakes, but contains some useful insights that we will review, avoiding errors. We will prove the following where to simplify notation we set $f(x,k) = f_1(x,k)$. We will write $D_k(x) = \frac{1-e^{-2\mathrm{i}kx}}{2k\mathrm{i}}$ with $D_0(x) = x$.

Lemma 8.10. *For $0 < |p-3| \ll 1$ and $k \in \overline{\mathbb{C}}_+$ close to 0, it is possible to write $f(x,k) = e^{\mathrm{i}kx} m(x,k)$ where*

$$\begin{aligned} m(x,k) =& e_1 - \int_x^\infty D_k(x-y) \mathrm{diag}(1,0) V(y) m(y,k) dt \\ & - \int_{\mathbb{R}} \frac{e^{-\sqrt{k^2+2}|x-y| - \mathrm{i}k(x-y)}}{2\sqrt{k^2+2}} \mathrm{diag}(0,1) V(y) m(y,k) dt. \end{aligned} \tag{8.21}$$

In particular there exists a constant C such that

$$|m(x,k) - e_1| \leqslant C(1 + x^-) \text{ for all } k \text{ near } 0 \text{ and } x \in \mathbb{R}. \tag{8.22}$$

Proof. First of all it is easy to see that if $m(x,k)$ satisfies (8.21), then $f(x,k) = e^{\mathrm{i}kx} m(x,k)$ satisfies (8.1) and can be taken as the $f_1(x,k)$ in Proposition 8.1. Now let us write $m(x,k) = (m_1(x,k), m_2(x,k))^\intercal$, where here m_1 and m_2 are the two components of m and should not be confused with the m_1 and m_2 in Proposition 8.1. For the first component of m we have

$$m_1(x,k) + \int_x^\infty D_k(x-y) V_{11}(y) m_1(y,k) dy = 1 - \int_x^\infty D_k(x-y) V_{12}(y) m_2(y,k) dy.$$

It is elementary that the operators

$$A_{ij}(k) u := - \int_x^\infty D_k(x-y) V_{ij}(y) u(y) dy$$

are bounded within the space $\langle x^- \rangle L^\infty(\mathbb{R})$ endowed with norm $\|v\|_{\langle x^- \rangle L^\infty(\mathbb{R})} := \| \langle x^- \rangle^{-1} v \|_{L^\infty(\mathbb{R})}$. The following is standard and follows from [25, Lemma 1 p.130].

Claim 8.11. The operators $(1 - A_{ij}(k)) : \langle x^- \rangle L^\infty(\mathbb{R}) \to \langle x^- \rangle L^\infty(\mathbb{R})$ can be inverted with norm of the inverse uniformly bounded in $k \in \overline{\mathbb{C}}_+$.

Proof. Consider $(1 - A_{ij}(k))u = v$ and write formally the series

$$\sum_{n=0}^\infty u_n \text{ with } u_n = A_{ij}(k) u_{n-1} \text{ and } u_0 = v.$$

Then, like in [25, p.132] and by $|D_k(x-y)| \leqslant |x-y|$, for $x_0 = x$

$$\begin{aligned} |u_n(x)| &\leqslant \int_{x \leqslant x_1 \leqslant \ldots \leqslant x_n} dx_1 \ldots dx_n \prod_{\ell=1}^n (x_\ell - x_{\ell-1}) |V_{ij}(x_\ell)| \langle x_n \rangle \| \langle \cdot \rangle^{-1} v \|_{L^\infty(\mathbb{R})} \\ &\leqslant \frac{1}{n!} \left(\int_x^\infty (y-x) \langle y \rangle |V_{ij}(y)| dy \right)^n \|v\|_{\langle x^- \rangle L^\infty(\mathbb{R})}. \end{aligned}$$

This means that the series is uniformly convergent in half–lines and that for any x we have

$$u(x) = v(x) - \int_x^\infty D_k(x-y) V_{ij}(y) u(y) dy.$$

Then

$$\begin{aligned} |u(x)| &\leqslant |v(x)| + \int_x^\infty y |V_{ij}(y)| |u(y)| dy - x \int_x^\infty |V_{ij}(y)| |u(y)| dy \\ &\leqslant |v(x)| + \int_0^\infty y |V_{ij}(y)| |u(y)| dy + x^- \int_x^\infty |V_{ij}(y)| |u(y)| dy. \end{aligned}$$

This implies

$$\left\langle x^- \right\rangle^{-1} |u(x)| \leqslant \|v\|_{\langle x^- \rangle L^\infty(\mathbb{R})} + \int_x^\infty 2 \langle y \rangle^2 |V_{ij}(y)| \left\langle y^- \right\rangle^{-1} |u(y)| dy$$

which in turn implies the following, by an application of Gronwall's inequality,

$$\left\langle x^- \right\rangle^{-1} |u(x)| \leqslant \|v\|_{\langle x^- \rangle L^\infty(\mathbb{R})} \exp\left(\int_x^\infty 2 \langle y \rangle^2 |V_{ij}(y)| dy \right).$$

□

Thanks to Claim 8.11 we can write

$$m_1(\cdot, k) = (1 - A_{11}(k))^{-1} 1 + (1 - A_{11}(k))^{-1} A_{12}(k) m_2(\cdot, k) \tag{8.23}$$

where if

$$|m_2(y, k)| \leqslant C \text{ for all } y \in \mathbb{R} \text{ and for } k \text{ close to } 0, \tag{8.24}$$

then

$$|m_1(x, k)| \leqslant C \left\langle x^- \right\rangle \text{ for all } x \in \mathbb{R} \text{ and for } k \text{ close to } 0. \tag{8.25}$$

For the second component of m we have

$$\begin{aligned} m_2(x, k) = &- \int_{\mathbb{R}} \frac{e^{-\sqrt{k^2+2}|x-y| - \mathrm{i}k(x-y)}}{2\sqrt{k^2+2}} V_{21}(y) m_1(y, k) dy \\ &- \int_{\mathbb{R}} \frac{e^{-\sqrt{k^2+2}|x-y| - \mathrm{i}k(x-y)}}{2\sqrt{k^2+2}} V_{22}(y) m_2(y, k) dy. \end{aligned}$$

Using formula (8.23), we can eliminate in the last equation $m_1(\cdot, k)$, obtaining an equation of the form

$$\begin{aligned} (1 + \mathbf{A}(k)) m_2(\cdot, k) &= - \int_{\mathbb{R}} \frac{e^{-\sqrt{k^2+2}|x-y| - \mathrm{i}k(x-y)}}{2\sqrt{k^2+2}} V_{21}(y) (1 - A_{11}(k))^{-1} 1 dy \text{ where} \\ \mathbf{A}(k) u_2 &:= \int_{\mathbb{R}} \frac{e^{-\sqrt{k^2+2}|x-y| - \mathrm{i}k(x-y)}}{2\sqrt{k^2+2}} \left(V_{22}(y) u_2(y) + V_{21}(y) (1 - A_{11}(k))^{-1} A_{12}(k) u_2 \right) dy. \end{aligned} \tag{8.26}$$

We want to solve this equation in $L^\infty(\mathbb{R})$. The operator $\mathbf{A}(k)$ is compact from $L^\infty(\mathbb{R})$ into itself. If $\ker(1 + \mathbf{A}(k)) = 0$ for $k = 0$ then the same is true for k close to 0 and by Fredholm theory, equation (8.26) is solvable and Lemma 8.10 is proved. So suppose now that there exists a nonzero $u_2 \in L^\infty(\mathbb{R})$ such that $(1 + \mathbf{A}(0)) u_2 = 0$. Then setting

$$u_1 := (1 - A_{11}(0))^{-1} A_{12}(0) u_2,$$

with $u_1 \in \langle x^- \rangle L^\infty(\mathbb{R})$ by Claim 8.11, the pair $u := (u_1, u_2)^\intercal$ solves (recall that $D_0(x) = x$)

$$\begin{aligned} u(x) = &- \int_x^\infty (x - y) \mathrm{diag}(1, 0) V(y) u(y) dt \\ &- \int_{\mathbb{R}} \frac{e^{-\sqrt{2}|x-y|}}{2\sqrt{2}} \mathrm{diag}(0, 1) V(y) u(y) dt \end{aligned} \tag{8.27}$$

and $u(x)$ is a solution of (8.1) for $k=0$. Since $u(x) \xrightarrow{x\to+\infty} 0$, it follows that $u(x)=cf_3(x,0)$ for a non zero constant $c\in\mathbb{C}$. This, $u_1 \in \langle x^-\rangle L^\infty(\mathbb{R})$ and $u_2\in L^\infty(\mathbb{R})$ yield

$$|f_3(x,0)| \lesssim 1+x^-. \tag{8.28}$$

The latter is equivalent to

$$W[f_3(x,0),g_3(x,0)]=0. \tag{8.29}$$

If this is not true for $p=3$, by continuity of the dependence on the parameter p of the solutions of system (8.1), (8.28) is not true for any p close to 3.

Claim 8.12. For $p=3$ we have $|f_3(x,0)| \sim e^{\sqrt{2}\,|x|}$ as $x\to-\infty$.

Proof. Follows immediately from formula (10.9) below which yields a function proportional to $g_3(x,0)$ and from $f_3(x,0)=g_3(-x,0)$. □

From Claim 8.12 we conclude that $W[f_3(x,0),g_3(x,0)]\neq 0$ for $p=3$ and also for p close to 3. Hence $\ker(1+\mathbf{A}(0))=0$. This completes the proof of Lemma 8.10. □

Proof of Proposition 8.6: continuation and end. We have already discussed the fact that the desired bound for the kernel $R_H^+(x,y,E)$ for $E=1+k^2$ and $x\geqslant y$ and k outside a neighborhood of 0 are true by Krieger and Schlag [42]. So now we consider the case when k is small. Then by the bound (8.22) for the $f_1(x,k)$ in (8.2) and by the exponential decay to 0 for $x\to-\infty$ of $g_3(x,k)=f_3(-x,k)$, it follows that

$$W[f_1(x,k),g_3(x,k)]=W[f_3(x,k),g_1(x,k)]=0.$$

So the matrix $D(k)$ in (8.13) is diagonal. This then implies, similarly to the proof in Kriger and Schlag [42], that for $x\geqslant y$ and for k small,

$$R_H^+(x,y,1+k^2)=(f_1(x,k),0)D^{-1}(k)(g_1(y,k),0)^\intercal+(0,f_3(x,k))D^{-1}(k)(0,g_3(y,k))^\intercal. \tag{8.30}$$

We bound this for $0>x>y$. The first term can be bounded by a constant times $\langle x^-\rangle$ because $|g_1(y,k)|\lesssim 1$ for $y<0$ and, by (8.22), $|f_1(x,k)|\lesssim\langle x^-\rangle$ for $x\leqslant 0$. The second term is uniformly bounded, because $|f_3(x,k)|\lesssim e^{\sqrt{2+k^2}|x|}$ for $x\leqslant 0$ and $|g_3(y,k)|\lesssim e^{-\sqrt{2+k^2}|y|}$ for $y\leqslant 0$ when k is sufficiently small. So this yields

$$|R_H^+(x,y,1+k^2)|\lesssim\left\langle x^-\right\rangle \text{ for } 0\geqslant x\geqslant y \text{ and for } k \text{ close to } 0.$$

By exploiting the symmetries due to $g_j(x,k)=f_j(-x,k)$ and by similar estimates, we obtain also the estimate

$$|R_H^+(x,y,1+k^2)|\lesssim\left\langle y^+\right\rangle \text{ for } x\geqslant y\geqslant 0 \text{ and for } k \text{ close to } 0.$$

So we obtained the estimate (8.16) for all $x\geqslant y$. This completes the proof of Proposition 8.6. □

Proof of Lemma 8.8. The group $e^{\mathrm{i}tH}$ is continuous and, see [42, Lemma 6.11] equibounded, with infinitesimal generator $\mathrm{i}H$. Then for $a>0$ and $u_0,v_0\in L^2(\mathbb{R},\mathbb{C}^2)$ by the Hille Yoshida theorem, Goldstein [31, p. 17], we have

$$\langle \mathrm{i}R_H(\lambda-\mathrm{i}a)u_0,v_0\rangle=\int_0^{+\infty}\left\langle e^{\mathrm{i}t(H-\lambda+\mathrm{i}a)}u_0dt,v_0\right\rangle=\int_0^{+\infty}e^{-\mathrm{i}t\lambda}\left\langle e^{\mathrm{i}t(H+\mathrm{i}a)}u_0,v_0\right\rangle dt \text{ and}$$
$$\langle -\mathrm{i}R_H(\lambda+\mathrm{i}a)u_0,v_0\rangle=\left\langle\int_{-\infty}^0 e^{\mathrm{i}t(H-\lambda-\mathrm{i}a)}u_0dt,v_0\right\rangle=\int_{-\infty}^0 e^{-\mathrm{i}t\lambda}\left\langle e^{\mathrm{i}t(H-\mathrm{i}a)}u_0,v_0\right\rangle dt.$$

Then

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\left\langle \mathrm{i}R_H(\lambda-\mathrm{i}a)u_0,v_0\right\rangle d\lambda = \chi_{\mathbb{R}_+}(t)\left\langle e^{\mathrm{i}t(H+\mathrm{i}a)}u_0,v_0\right\rangle \text{ and}$$
$$-\frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\left\langle \mathrm{i}R_H(\lambda+\mathrm{i}a)u_0,v_0\right\rangle d\lambda = \chi_{\mathbb{R}_-}(t)\left\langle e^{\mathrm{i}t(H-\mathrm{i}a)}u_0,v_0\right\rangle .$$

So for g, which for convenience we take in $\mathcal{S}(\mathbb{R}\times\mathbb{R},\mathbb{C}^2)\cap C_c(\mathbb{R}_t,L^2(\mathbb{R}_x,\mathbb{C}^2))$, we have

$$\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\mathrm{i}R_H(\lambda-\mathrm{i}a)g^{\vee}(\lambda,\cdot)d\lambda = \int_{-\infty}^{t} e^{\mathrm{i}(t-t')(H+\mathrm{i}a)}g(t')dt' \text{ and}$$
$$-\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\mathrm{i}R_H(\lambda+\mathrm{i}a)g^{\vee}(\lambda,\cdot)d\lambda = \int_{t}^{+\infty} e^{\mathrm{i}(t-t')(H-\mathrm{i}a)}g(t')dt'.$$

Summing up and after an elementary manipulation, for $t>0$ we have

$$\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\left(R_H(\lambda-\mathrm{i}a)+R_H(\lambda+\mathrm{i}a)\right)g^{\vee}(\lambda,\cdot)d\lambda \tag{8.31}$$
$$= \int_{-\infty}^{0} e^{-(t-t')a}e^{\mathrm{i}(t-t')H}g(t')dt' - \int_{0}^{+\infty} e^{(t-t')a}e^{\mathrm{i}(t-t')H}g(t')dt'$$
$$+\int_{0}^{t} e^{-(t-t')a}e^{\mathrm{i}(t-t')H}g(t')dt' + \int_{0}^{t} e^{(t-t')a}e^{\mathrm{i}(t-t')H}g(t')dt' \xrightarrow{a\to 0^+}$$
$$\int_{-\infty}^{0} e^{\mathrm{i}(t-t')H}g(t')dt' - \int_{0}^{+\infty} e^{\mathrm{i}(t-t')H}g(t')dt' + 2\int_{0}^{t} e^{\mathrm{i}(t-t')H}g(t')dt' \tag{8.32}$$

where the limit of the right hand side holds in $L^2(\mathbb{R})$ by $e^{\pm at'}g(t') \xrightarrow{a\to 0^+} g$ in $L^1(\mathbb{R},L^2(\mathbb{R}_x,\mathbb{C}^2))$ and by the Strichartz estimates, see Keel and Tao [37, Theorem 1.2]. We now focus at the limit of line (8.31) as $a\to 0^+$ when $P_cg(t)=g(t)$ for all times. We claim that

$$\lim_{a\to 0^+} \text{line (8.31)} = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}t\lambda}\left(R_H^-(\lambda)+R_H^+(\lambda)\right)g^{\vee}(\lambda,\cdot)d\lambda \text{ in } L^{2,-s}(\mathbb{R}) \tag{8.33}$$

for $s>3/2$. We distinguish between three cases. For $\lambda\in[-1+\alpha,1-\alpha]$ for any fixed $\alpha\in(0,1)$ we have uniform convergence of the resolvents in the operator norm, where $P_cg^{\vee}(\lambda)=g^{\vee}(\lambda)$ avoids the singularities of the resolvent. For such that $\operatorname{Re} z\in(-\infty,-1-\alpha]\cup[1+\alpha,\infty)$ and $\operatorname{Im} z\geqslant 0$ (resp. $\operatorname{Im} z\leqslant 0$) it is possible to apply the 3 dimensional theory in [10] to conclude that $R_H(z)$ is continuous as a function with values in the space of of bounded operators operator from $L^{2,s}(\mathbb{R})$ to $L^{2,-s}(\mathbb{R})$ for $s>1/2$. We finally consider the case when λ is close to $\{1,-1\}$. For symmetry reasons it is not restrictive to consider the limit of $R_H(\lambda+\mathrm{i}a)$ for λ close to 1 focusing on the corresponding integral kernel in the region $x\geqslant y$. Notice that Lemma 8.10 continues to be true for $k\in\mathbb{C}\backslash[0,+\infty)$ with k near 0. The resolvent of $R_H(\lambda+\mathrm{i}a)$ is given by (8.14) with $\lambda+ia=1+k^2$ and we continue to have the diagonalization of $D(k)$ yielding to (8.30). Then $R_H(x,y,\lambda+\mathrm{i}a)$ satisfies the estimate (8.16) and by dominated convergence we obtain the desired limit.

□

9 Proof of Proposition 2.4

We will prove the following version for H, that is in the case $\omega=1$, of (2.21), where P_c is the spectral projection associated to the continuous spectrum of H, for any $u_0\in L^2(\mathbb{R},\mathbb{C}^2)$ and for a fixed constant $c > 0$,

$$\|e^{\mathrm{i}t\mathcal{L}} P_c u_0\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leqslant c\|u_0\|_{L^2(\mathbb{R})}. \tag{9.1}$$

Let $g(t,x) \in \mathcal{S}(\mathbb{R}^2)$ with $g(t) = P_c(H)g(t)$. Then

$$\begin{aligned}\int_{\mathbb{R}} \left\langle e^{-\mathrm{i}tH} u_0, \sigma_3 g\right\rangle dt &= \frac{1}{\sqrt{2\pi}\mathrm{i}} \int_{\mathbb{R}} \left\langle (R_H^+(E) - R_H^-(E))u_0, \sigma_3 \overline{\widehat{g}}(E)\right\rangle_x dE \\ &= \frac{1}{\sqrt{2\pi}\mathrm{i}} \int_{\sigma_c(H)} \left\langle (R_H^+(E) - R_H^-(E))u_0, \sigma_3 \overline{\widehat{g}}(E)\right\rangle_x dE.\end{aligned}$$

Then from Fubini we have

$$\left|\int_{\mathbb{R}} \left\langle e^{-\mathrm{i}tH} u_0, \sigma_3 g\right\rangle dt\right| \leqslant \|(R_H^+(E) - R_H^-(E))u_0\|_{L_x^{2,-s} L_E^2(\sigma_c(H))} \|g\|_{L_x^{2,s} L_t^2}.$$

So now we need to show that

$$\|(R_H^+(E) - R_H^-(E))u_0\|_{L_x^{2,-s} L_E^2(\sigma_c(H))} \lesssim \|u_0\|_{L^2(\mathbb{R})},$$

where the subscripts x and E indicate the variables of integration. We can split between E away from the thresholds of $\sigma_c(H)$, where the corresponding bound is obtained thanks to the corresponding bound for the flat operator $\sigma_3(-\partial_x^2 + 1)$ like in the 3 dimensional case, proved in [23], and the case when E is close to the thresholds ± 1. More generally, we will show that for $s > 1$ there is a constant C_s

$$\|R_H^{\pm}(E)u_0\|_{L^2(|E-1| \ll 1, L^{2,-s}(\mathbb{R}))} \leqslant C_s \|u_0\|_{L^2(\mathbb{R})}, \tag{9.2}$$

with an analogous estimate valid near -1. Let us consider the scalar Schrödinger operator $h = -\partial_x^2 + \operatorname{sech}^2\left(\frac{p-1}{2}x\right)$. Then we claim that $\langle x\rangle^{-s}$ is h–smoothing in the sense of Kato [35], which implies that for $s > 1$ there is a constant C_s such that

$$\|R^{\pm}_{\sigma_3(h+1)}(E)u_0\|_{L_E^2(\mathbb{R}, L_x^{2,-s}(\mathbb{R}))} \leqslant C_s \|u_0\|_{L^2(\mathbb{R})}. \tag{9.3}$$

Since $\sigma_3(h+1)$ is selfadjoint, by (5.3) in Theorem 5.1 [35], (9.3) will follow if for a fixed $C > 0$

$$\|\langle x\rangle^{-s} R_{\sigma_3(h+1)}(z) \langle x\rangle^{-s}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R}))} < C \text{ for all } z \text{ with } 0 < |\operatorname{Im} z|. \tag{9.4}$$

From $R_{\sigma_3(h+1)}(z) = \operatorname{diag}(R_h(z-1), -R_h(z+1))$, (9.4) follows from

$$\|\langle x\rangle^{-s} R_h(z) \langle x\rangle^{-s}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R}))} < C \text{ for all } z \text{ with } 0 < |\operatorname{Im} z|. \tag{9.5}$$

The kernel of $R_h(z)$ for $x < y$, with an analogous formula for $x > y$, for $\arg\sqrt{z} \in [0, \pi]$ is

$$R_h(z)(x,y) = \frac{T(\sqrt{z})}{2\mathrm{i}\sqrt{z}} f_-(x,\sqrt{z}) f_+(y,\sqrt{z}) = \frac{T(\sqrt{z})}{2\mathrm{i}\sqrt{z}} e^{\mathrm{i}\sqrt{z}(x-y)} m_-(x,\sqrt{z}) m_+(y,\sqrt{z}),$$

where the Jost functions $f_{\pm}(x,\sqrt{z}) = e^{\pm \mathrm{i}\sqrt{z}x} m_{\pm}(x,\sqrt{z})$ solve $hu = zu$ with

$$\lim_{x\to+\infty} m_+(x,\sqrt{z}) = 1 = \lim_{x\to-\infty} m_-(x,\sqrt{z}).$$

These functions satisfy, see Lemma 1 p. 130 [25],

$$|m_\pm(x,\sqrt{z})-1| \leqslant C_1\langle \max\{0,\mp x\}\rangle \langle \sqrt{z}\rangle^{-1}\left|\int_x^{\pm\infty}\langle y\rangle \mathrm{sech}^2\left(\frac{p-1}{2}y\right)dy\right|$$
$$|m_\pm(x,k)-1| \leqslant \left\langle \sqrt{z}\right\rangle^{-1}\left|\int_x^{\pm\infty}\mathrm{sech}^2\left(\frac{p-1}{2}y\right)dy\right|\exp\left(\left\langle \sqrt{z}\right\rangle^{-1}\left|\int_x^{\pm\infty}\mathrm{sech}^2\left(\frac{p-1}{2}y\right)dy\right|\right),$$

while, since h has no 0 resonance, $T(k)=\alpha k(1+o(1))$ near $k=0$ for some $\alpha\neq 0$ and $T(k)=1+O(1/k)$ for $k\to\infty$ and $T\in C^0(\mathbb{R})$, see Theorem 1 [25]. Then (here for $z\in\mathbb{R}$ we are taking $R_h^+(z)$)

$$\begin{aligned}\|\langle x\rangle^{-s}R_h(z)\langle x\rangle^{-s}\|^2_{L^2(\mathbb{R})\to L^2(\mathbb{R}))} \leqslant &\int_{x<y}\langle x\rangle^{-2s}\langle y\rangle^{-2s}|R_h(z)(x,y)|^2dxdy\\ &+\int_{x>y}\langle x\rangle^{-2s}\langle y\rangle^{-2s}|R_h(z)(x,y)|^2dxdy =: \mathcal{A}+\mathcal{B}.\end{aligned}$$

The two terms on the right can be estimated similarly, so we bound only the first. It is easy to see that, like in Proposition 8.6,

$$|R_h(z)(x,y)| \leqslant C\begin{cases}(1+x^-+y^+) \text{ if } x\geqslant y\\ (1+x^++y^-) \text{ if } x\leqslant y,\end{cases} \tag{9.6}$$

where in fact these estimates are what inspired (8.16). Then

$$\begin{aligned}\mathcal{A} &\lesssim \int_{0<x<y}\langle x\rangle^{2-2s}\langle y\rangle^{-2s}dxdy+\int_{x<y<0}\langle x\rangle^{-2s}\langle y\rangle^{2-2s}dxdy+\int_{x<0<y}\langle x\rangle^{-2s}\langle y\rangle^{-2s}dxdy\\ &=: A_1+A_2+A_3.\end{aligned}$$

Then

$$A_1=\int_{\mathbb{R}_+}dx\langle x\rangle^{2-2s}\int_x^{+\infty}\langle y\rangle^{-2s}dy \lesssim \int_{\mathbb{R}_+}\langle x\rangle^{3-4s}dx<+\infty \text{ for } 3-4s<-1 \Longleftrightarrow s>1.$$

Similarly A_2, obviously also A_3, and $\mathcal{B}$ are bounded for $s>1$. So this yields (9.5) and (9.4). In particular this implies (9.3). Now we can express

$$H=\sigma_3(h+1)+\widetilde{V}$$

where $\widetilde{V}=M_0\ \mathrm{sech}^2\left(\frac{p-1}{2}x\right)$, for M_0 a constant matrix. We can factor

$$\widetilde{V}=B^*A \text{ with } B^*=\langle x\rangle^s\widetilde{V} \text{ and } A=\langle x\rangle^{-s}.$$

Now, for $\operatorname{Im} z>0$, for $Q_0(z)=AR_{\sigma_3(h+1)}(z)B^*$, we have

$$AR_H(z)=(1+Q_0(z))^{-1}AR_{\sigma_3(h+1)}(z).$$

The function $Q_0(z)$ extends as an element of $C^0(\overline{\mathbb{C}}_+\backslash\sigma_p(H),\mathcal{L}(L^2))$ with values in the space of compact operators of $L^2=L^2(\mathbb{R},\mathbb{C}^2)$ in itself. Furthermore, $(1+Q_0(z))^{-1}$ extends into a bounded operator except for those $z\in\mathbb{R}$ for which $\ker(1+Q_0(z))\neq 0$. For z near but not equal to 1, by standard arguments that can be seen in [23], this implies that z is an eigenvalue of H, but this is not possible since here our z's are taken much closer to 1 than $\lambda(p,1)$, which is the only positive eigenvalue of H. The other possibility is that $\ker(1+Q_0(1)) \neq 0$. We exclude this proceeding by contradiction. If

$$(1+Q_0(1))\, w = 0 \text{ with } w \neq 0$$

then $\psi = R^{+}_{\sigma_3(h+1)}(1)B^* w$ satisfies

$$(\sigma_3(h+1)-1)\,\psi = B^* w = -B^* A R^{+}_{\sigma_3(h+1)}(1) B^* w = -\widetilde{V}\psi$$

and so $\psi \neq 0$ is a nontrivial distributional solution of $(H-1)u = 0$. We claim that $\psi \in L^\infty(\mathbb{R})$. In fact, for $g = B^* w$,

$$|\psi(x)| \leqslant \int_{x<y} |R^{+}_{\sigma_3(h+1)}(x,y,1)|\ |g(y)|dy + \int_{x>y} |R^{+}_{\sigma_3(h+1)}(x,y,1)|\ |g(y)|dy =: B_1(x) + B_2(x).$$

Then by (9.6) we have

$$B_1(x) \leqslant \int_{x<y} (1+x^+ + y^-)\ |g(y)|dy$$

If now $x<0$ then by the rapid decay of g we get

$$B_1(x) \leqslant \int_{\mathbb{R}} (1+|y|)\ |g(y)|dy < \infty.$$

If $x>0$ we write

$$B_1(x) \leqslant \int_{x<y} (1+|x|)\ |g(y)|dy \leqslant \int_{\mathbb{R}} (1+|y|)\ |g(y)|dy < \infty.$$

So $B_1 \in L^\infty(\mathbb{R})$. By a similar argument we obtain $B_2 \in L^\infty(\mathbb{R})$ and hence also $\psi \in L^\infty(\mathbb{R})$. But then 1 is a resonance for H, which is not true. So we conclude that $\ker(1+Q_0(1)) = 0$. Then

$$\begin{aligned}
&\|R^{\pm}_{H}(E)u_0\|_{L^2(|E-1|\ll 1, L^{2,-s}(\mathbb{R}))} \\
&= \|\,(1+Q_0(E))^{-1}\langle x\rangle^{-s} R^{\pm}_{\sigma_3(h+1)}(E)u_0\|_{L^2(|E-1|\ll 1, L^2(\mathbb{R}))} \\
&\lesssim \|\,\langle x\rangle^{-s} R^{\pm}_{\sigma_3(h+1)}(E)u_0\|_{L^2(|E-1|\ll 1, L^2(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})}
\end{aligned}$$

and (9.2) for the $+$ and for $-$. This completes the proof of (9.1) and so also of Proposition 2.4.

10 Explicit Jost functions of the linearization for $p=3$.

When $p=3$ the Jost functions discussed in §8 have been explicitly known since Kaup [36]. In fact it was shown that these Jost functions can be expressed in terms of the solutions of the Lax pair system. In turn, the latter ones can be expressed in terms of the solutions of the Lax pair system for the null solution of the NLS using Bäcklund transformations. However here we will use some transformations in Martel [45, 46] to write these explicit formulas. It is not restrictive to take $\omega = 1$. For $\omega = 1$ we have $L_+ = L_0$ and $L_- = L_1$ where, Chang et al. [7],

$$L_j := -\partial_x^2 + 1 - k_{j-1}(p)k_j(p)\frac{2}{p+1}\phi^{p-1} \text{ for } j = 0,1,2,\ldots \text{ and } k_j(p) := \frac{p+1}{2} - \frac{j(p-1)}{2}. \quad (10.1)$$

Notice that

$$k_1(p) = 1\,, \quad k_2(p) = \frac{3-p}{2} \text{ and } k_3(p) = 2 - p.$$

When $p = 3$, $L_2 = L_3 = -\partial_x^2 + 1$. Let

$$S_1 = S(k_1(p)) := \partial_x + k_1(p) \tanh\left(\frac{p-1}{2}x\right) = \partial_x + \tanh\left(\frac{p-1}{2}x\right).$$

Martel [45, 46] exploits the following formula, which we derive for reader's sake,

$$S_1^2 L_0 L_1 = S_1^2 L_0 S_1^* S_1 = S_1 S_1^* L_3 S_1^2 = L_2 L_3 S_1^2,$$

where the first and last equalities follow from (3.14) and the second from (3.24) in Chang et al. [7]. Taking the adjoint we obtain

$$L_1 L_0 (S_1^*)^2 = (S_1^*)^2 L_3 L_2. \tag{10.2}$$

This formula is exploited in Martel [45, 46] to show that starting from

$$\begin{cases} L_2 w_1 = \lambda w_2 \\ L_3 w_2 = -\lambda w_1 \end{cases} \tag{10.3}$$

we get

$$\begin{cases} \xi_1 := (S_1^*)^2 w_1 \\ \xi_2 := -\frac{1}{\lambda} L_0 v_1 \end{cases} \Longrightarrow \begin{cases} L_1 \xi_2 = \lambda \xi_1 \\ L_0 \xi_1 = -\lambda \xi_2, \end{cases} \tag{10.4}$$

where $L_0 \xi_1 = -\lambda \xi_2$ is true by definition and

$$L_1 \xi_2 = -\frac{1}{\lambda} L_1 L_0 (S_1^*)^2 w_1 = -\frac{1}{\lambda} (S_1^*)^2 L_3 L_2 w_1 = -(S_1^*)^2 L_3 w_2 = \lambda (S_1^*)^2 w_1 = \lambda \xi_1.$$

For $p = 3$, by $L_2 = L_3 = -\partial_x^2 + 1$ for $\lambda = \mathrm{i}(1 + k^2)$ we consider solutions to (10.3) of the form

$$(w_1, w_2)^\intercal = \begin{cases} e^{\mathrm{i}kx}(1, -\mathrm{i})^\intercal \\ e^{\mu x}(1, \mathrm{i})^\intercal \text{ where } \mu := \sqrt{2 + k^2} \end{cases} \tag{10.5}$$

and by (10.4), after elementary computations, we obtain Jost functions for $\mathcal{L}_\omega\big|_{\omega=1}$ for $p = 3$,

$$\begin{cases} e^{\mathrm{i}kx}\left(1 - k^2 - 2\mathrm{i}k \tanh(x) - 2\mathrm{sech}^2(x), \mathrm{i}\left(1 - k^2 - 2\mathrm{i}k \tanh(x)\right)\right)^\intercal \\ e^{\mu x}\left(\mu^2 + 1 - 2\mu \tanh(x) - 2\mathrm{sech}^2(x), \mathrm{i}\left(\mu^2 + 1 - 2\mu \tanh(x)\right)\right)^\intercal. \end{cases} \tag{10.6}$$

Notice that

$$e^{\mathrm{i}kx}\left(1 - k^2 - 2\mathrm{i}k \tanh(x) - 2\mathrm{sech}^2(x), \mathrm{i}\left(1 - k^2 - 2\mathrm{i}k \tanh(x)\right)\right)^\intercal\Big|_{k=0} = \left(1 - 2\mathrm{sech}^2(x), \mathrm{i}\right)^\intercal \tag{10.7}$$

yields the resonance at the threshold i, see formula (3.54) Chang at al. [7]. Eigenfunctions for the operator H are obtained applying to the vectors in (10.6) the matrix U^{-1} yielding

$$(u_1, u_2)^\intercal := 2^{-1}(\xi_1 - \mathrm{i}\xi_2, \xi_1 + \mathrm{i}\xi_2)^\intercal.$$

Entering in this formula the functions in (10.6) we obtain

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{\mathrm{i}kx} \begin{pmatrix} 1 - k^2 - 2\mathrm{i}k \tanh(x) - \mathrm{sech}^2(x) \\ -\mathrm{sech}^2(x) \end{pmatrix} \text{ and} \tag{10.8}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{\mu x} \begin{pmatrix} -\mathrm{sech}^2(x) \\ \mu^2 + 1 - 2\mu \tanh(x) - \mathrm{sech}^2(x) \end{pmatrix}, \tag{10.9}$$

which are the Jost functions of H for $p = 3$.

11 The linear approximation of $\gamma(p,1)$ at $p=3$.

In this section we prove Lemma 5.1 by following Martel [46]. We will focus only on $\gamma(p) := \gamma(p,1)$, since the general $\omega > 0$ case follows from the $\omega = 1$ by scaling. We write ϕ_p to denote ϕ given in (1.2). Similarly, when it is necessary to stress the dependence on p, we write $g^{(1)} = g_p = (g_{p,1}, g_{p,2})^\intercal$, $\xi_{\omega=1} = \xi_p = (\xi_{p,1}, \xi_{p,2})^\intercal$, $\mathcal{L}_\omega|_{\omega=1} = \mathcal{L}_p$ and $L_{\pm\omega}|_{\omega=1} = L_{p\pm}$. For g_p, like in (2.8), we take $g_{p,1} = \operatorname{Re} g_{p,1}$ and $g_{p,2} = \mathrm{i}\operatorname{Im} g_{p,2}$. In the following, we choose

$$\xi_3 = \left(1-\phi_3^2, \mathrm{i}\right)^\intercal \text{ and } g_3 = \left(\frac{1}{2}\phi_3^2\cos(x) + \frac{\phi_3'}{\phi_3}\sin(x), \mathrm{i}\frac{\phi_3'}{\phi_3}\sin(x)\right)^\intercal,$$

where ξ_3 is just a resonance and not an eigenfunction.

Remark 11.1. Notice that here for ξ_p we are not using the normalization in (2.9) and instead we are defining it as a solution of (11.3) which will be defined in Lemma 11.5. On the other hand, g_3 is given multiplying by $-\frac{1}{2}$ the vector given in (10.6) with $k=1$ and then taking the real part for the first component and imaginary part for the second component respectively.

Also, for g_p we have the following lemma which is a variant of Lemma 19 of Martel [46].

Lemma 11.2. *We can choose g_p so that*

$$\|g_p - \left(\frac{1}{2}\phi_3^2\cos(\tau x) + \frac{\phi_3'}{\phi_3}\sin(\tau x), \mathrm{i}\frac{\phi_3'}{\phi_3}\sin(\tau x)\right)\|_{L^\infty} \lesssim |p-3|,$$

where $\tau = \sqrt{1-\lambda(p,1)^2}$.

Proof. The proof is parallel to Lemma 19 of Martel [46]. □

We will start by writing an expansion in p of the eigenfunction ξ_p. Along the way we give a new proof, based on Martel [46], of the result by Coles and Gustafson [8] about the existence of an eigenvalue.

Lemma 11.3. *There exists a small $\delta_1 > 0$ and a function $\alpha \in C^\infty(D_{\mathbb{R}}(3,\delta_1), \mathbb{R})$ such that*

$$\alpha(p) = (p-3)^2\left(2^{-2} + 2^{-5}2^{-\frac{1}{2}}\left\langle\phi_3^2, \mathbf{T}\right\rangle\right) + O\left((p-3)^3\right), \tag{11.1}$$

where

$$\mathbf{T} := \frac{e^{-\sqrt{2}|\cdot|}}{2} * \phi_3^2, \tag{11.2}$$

and such that $\mathrm{i}(1-\alpha^2(p))$ *is an eigenvalue of* $\mathcal{L}_p$ *for* $0 < |p-3| < \delta_1$. *That is,* $\lambda(p,1) = 1-\alpha(p)^2$.

Remark 11.4. Notice that $(-\partial_x^2 + 2)\mathbf{T} = \sqrt{2}\phi_3^2$.

Proof. We are looking to solutions to

$$\begin{cases} L_-\xi_{p,2} = \mathrm{i}(1-\alpha^2)\xi_{p,1} \\ L_+\xi_{p,1} = -\mathrm{i}(1-\alpha^2)\xi_{p,2} \end{cases}. \tag{11.3}$$

By (10.3), this is equivalent to the existence of $w_p = (w_{p,1}, w_{p,2})^\intercal$ such that

$$\begin{aligned} &\begin{pmatrix} 0 & -L_3 \\ L_2 & 0 \end{pmatrix}\begin{pmatrix} w_{p,1} \\ w_{p,2} \end{pmatrix} \\ &= \left(-J(-\partial_x^2+1) + k_2(p)\frac{2}{p+1}\phi_p^{p-1}\begin{pmatrix} 0 & k_3(p) \\ -1 & 0 \end{pmatrix}\right)w_p = \mathrm{i}(1-\alpha^2)w_p. \end{aligned} \tag{11.4}$$

Applying U^{-1}, recall U is given in (7.1), to this equation and introducing $Z_p =: U^{-1}w_{p,}$, after elementary computations we get the equivalent problem

$$\left(-\sigma_3(-\partial_x^2+1)+k_2(p)\frac{1}{p+1}\phi_p^{p-1}\begin{pmatrix}k_3(p)+1 & 1-k_3(p)\\ k_3(p)-1 & -(k_3(p)+1)\end{pmatrix}\right)Z_p=(1-\alpha^2)Z_p.$$

Substituting the values of $k_2(p)$ and $k_3(p)$ and multiplying by $-\sigma_3$, this can be written

$$\left((-\partial_x^2+1)+(p-3)\mathbf{P}_p(x)\right)Z_p=-\sigma_3(1-\alpha^2)Z_p$$

with

$$\mathbf{P}_p(x)=\begin{pmatrix}3-p & p-1\\ p-1 & 3-p\end{pmatrix}\frac{1}{2(p+1)}\phi_p^{p-1}(x).$$

Notice in particular that

$$\mathbf{P}_3(x)=\sigma_1\frac{1}{4}\phi_3^2(x)=\sigma_1\frac{1}{2}\mathrm{sech}^2(x). \tag{11.5}$$

For

$$H_\alpha=\begin{pmatrix}-\partial_x^2+\kappa^2 & 0\\ 0 & -\partial_x^2+\alpha^2\end{pmatrix}\text{ with }\kappa=\sqrt{2-\alpha^2},$$

we can write

$$\left(H_\alpha+(p-3)\mathbf{P}_p\right)Z_p=0,$$

which is equivalent to

$$\left(1+(p-3)H_\alpha^{-1}\mathbf{P}_p\right)Z_p=0.$$

If we set

$$|\mathbf{P}_p(x)|^{\frac12}:=\begin{pmatrix}1+\sqrt{p-2} & 1-\sqrt{p-2}\\ 1-\sqrt{p-2} & 1+\sqrt{p-2}\end{pmatrix}\frac{1}{2\sqrt{p+1}}\phi_p^{\frac{p-1}{2}}(x)\text{ and}$$
$$\mathbf{P}_p^{\frac12}(x):=\sigma_1|\mathbf{P}_p(x)|^{\frac12}=\begin{pmatrix}1-\sqrt{p-2} & 1+\sqrt{p-2}\\ 1+\sqrt{p-2} & 1-\sqrt{p-2}\end{pmatrix}\frac{1}{2\sqrt{p+1}}\phi_p^{\frac{p-1}{2}}(x),$$

from the elementary computation

$$\begin{aligned}\begin{pmatrix}1-c & 1+c\\ 1+c & 1-c\end{pmatrix}\begin{pmatrix}1+c & 1-c\\ 1-c & 1+c\end{pmatrix}&=\begin{pmatrix}2(1-c^2) & (1-c)^2+(1+c)^2\\ (1-c)^2+(1+c)^2 & 2(1-c^2)\end{pmatrix}\\ &=2\begin{pmatrix}1-c^2 & 1+c^2\\ 1+c^2 & 1-c^2\end{pmatrix},\end{aligned}$$

it follows that $\mathbf{P}_p(x)=\mathbf{P}_p^{\frac12}(x)|\mathbf{P}_p(x)|^{\frac12}$ and furthermore these matrices commute.
Setting

$$\Psi_p:=\mathbf{P}_p^{\frac12}Z_p, \tag{11.6}$$

the equation for Z_p writes

$$\left(1+(p-3)K_{\alpha p}\right)\Psi_p=0\text{ where }K_{\alpha p}=\mathbf{P}_p^{\frac12}H_\alpha^{-1}|\mathbf{P}_p|^{\frac12}.$$

We expand

$$K_{\alpha p} = L_{\alpha p} + M_{\alpha p} \text{ with } M_{\alpha p} := \mathbf{P}_p^{\frac{1}{2}} N_\alpha |\mathbf{P}_p|^{\frac{1}{2}} \text{ with integral kernels}$$
$$N_\alpha(x,y) = \frac{1}{2\alpha}\begin{pmatrix} \frac{\alpha}{\kappa} e^{-\kappa|x-y|} & 0 \\ 0 & e^{-\alpha|x-y|}-1 \end{pmatrix} \text{ and}$$
$$L_{\alpha p}(x,y) = \frac{1}{2\alpha}\mathbf{P}_p^{\frac{1}{2}}(x) \operatorname{diag}(0,1) \, |\mathbf{P}_p(y)|^{\frac{1}{2}}.$$

Here $(p,\alpha) \to M_{\alpha p}$ is in $C^\infty(D_{\mathbb{R}^2}((3,0),\delta_1), L^2(\mathbb{R},\mathbb{C}^2))$ for a small enough $\delta_1 > 0$. The equation for Ψ_p becomes

$$\frac{1}{2\alpha}(1+(p-3)M_{\alpha p})^{-1}\mathbf{P}_p^{\frac{1}{2}}(x) e_2 \left\langle e_2, |\mathbf{P}_p|^{\frac{1}{2}}\Psi_p \right\rangle = -\frac{1}{p-3}\Psi_p. \tag{11.7}$$

To have a solution in (11.7) it is not restrictive to posit

$$\Psi_p = (1+(p-3)M_{\alpha p})^{-1}\mathbf{P}_p^{\frac{1}{2}} e_2 \in C^\infty(D_{\mathbb{R}^2}((3,0),\delta_1), L^2(\mathbb{R},\mathbb{C}^2)) \tag{11.8}$$
$$\Psi_p = \mathbf{P}_p^{\frac{1}{2}}(1+(p-3)N_\alpha \mathbf{P}_p)^{-1} e_2, \tag{11.9}$$

where the two formulas for Ψ_p are equivalent. With them, (11.7) is equivalent to

$$\alpha = -\frac{p-3}{2} s(p,\alpha) \text{ with } s(p,\alpha) := \left\langle e_2, |\mathbf{P}_p|^{\frac{1}{2}}(1+(p-3)M_{\alpha p})^{-1}\mathbf{P}_p^{\frac{1}{2}} e_2 \right\rangle. \tag{11.10}$$

Notice that $s(\cdot,\cdot) \in C^\infty(D_{\mathbb{R}^2}((3,0),\delta_1),\mathbb{R})$ with

$$\begin{aligned} s(p,\alpha) &= \langle e_2, \mathbf{P}_p e_2\rangle - (p-3)\left\langle e_2, |\mathbf{P}_p|^{\frac{1}{2}} M_{\alpha p}\mathbf{P}_p^{\frac{1}{2}} e_2 \right\rangle + O\left((p-3)^2\right) \\ &= \langle e_2, \mathbf{P}_p e_2\rangle - (p-3)\langle e_2, \mathbf{P}_p N_\alpha \mathbf{P}_p e_2\rangle + O\left((p-3)^2\right). \end{aligned}$$

Since, we have

$$\begin{aligned} \langle e_2, \mathbf{P}_p e_2\rangle &= \frac{p-3}{2(p+1)}\left\langle e_2, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \phi_p^{p-1} e_2 \right\rangle + \frac{1}{p+1}\cancel{\left\langle e_2, \phi_p^{p-1}\sigma_1 e_2\right\rangle} \\ &= -\frac{p-3}{2(p+1)}\int_{\mathbb{R}} \phi_p^{p-1} dx = -\frac{p-3}{2(p-1)}\int_{\mathbb{R}} \operatorname{sech}^2(x)\, dx = -\frac{p-3}{p-1}, \end{aligned}$$

with the canceled term null, and

$$\begin{aligned} \langle e_2, \mathbf{P}_p N_\alpha \mathbf{P}_p e_2\rangle &= \langle e_2, \mathbf{P}_3 N_0 \mathbf{P}_3 e_2\rangle + O((p-3)) + O(\alpha) \\ &= 4^{-2}\left\langle e_2, \phi_3^2\sigma_1 N_0 \phi_3^2 \sigma_1 e_2\right\rangle + O((p-3)) + O(\alpha) \\ &= 2^{-4}2^{-\frac{1}{2}}\left\langle \phi_3^2, \mathbf{T}\right\rangle + O((p-3)) + O(\alpha), \end{aligned}$$

we obtain

$$s(p,\alpha) = -(p-3)\left(\frac{1}{p-1} + 2^{-5}2^{-\frac{1}{2}}\left\langle \phi_3^2, \mathbf{T}\right\rangle\right) + O\left((p-3)^3\right) + O\left((p-3)\alpha\right).$$

Applying the implicit function theorem to (11.10) we get (11.1).

$\square$

In analogy to Martel [46] we give an expansion of a ξ_p.

Lemma 11.5. *There exists an open interval $\mathcal{I}$ containing 3 and for each $p \in \mathcal{I}$ there exists a solution $\xi_p = (\xi_{p,1}, \xi_{p,2})^\intercal$ of (11.3) of the form*

$$\xi_{p,1} = 1 - \phi_3^2 + (p-3)R_1 + (p-3)^2\tilde{\xi}_{p,1}, \tag{11.11}$$

$$\xi_{p,2} = \mathrm{i}\left(1 + (p-3)R_2 + (p-3)^2\tilde{\xi}_{p,2}\right), \tag{11.12}$$

where

$$R_1 = -x\phi_3\phi_3' - \frac{1}{4\sqrt{2}}(3-\phi_3^2)\mathbf{T} - \frac{\phi_3'}{2\sqrt{2}\phi_3}\mathbf{T}' \text{ and}$$
$$R_2 = \frac{1}{2}\phi_3^2 + \frac{3}{4\sqrt{2}}\mathbf{T} + \frac{\phi_3'}{2\sqrt{2}\phi_3}\mathbf{T}'$$

and where furthermore, for any $k \geqslant 0$ there exists a constant C_k such that

$$|\widetilde{\xi}_{p,j}^{(k)}(x)| \leqslant C_k \langle x\rangle^3 \text{ for all } x \in \mathbb{R} \text{ and all } p \in \mathcal{I}. \tag{11.13}$$

Proof. From (11.6) and (11.9), and in particular expanding the latter, we have

$$\begin{aligned} Z_p &= (1 + (p-3)N_\alpha \mathbf{P}_p)^{-1}e_2 = e_2 - (p-3)N_\alpha\mathbf{P}_p e_2 + (p-3)^2\widetilde{Z}_2 \\ &= e_2 - (p-3)2^{-2}2^{-\frac{1}{2}}\mathbf{T}e_1 + (p-3)^2\widetilde{Z} \text{ with } \widetilde{Z} = \widetilde{Z}_1 + \widetilde{Z}_2 \\ \widetilde{Z}_1 &:= -\int_0^1 \partial_{p'}\left(N_{\alpha(p')}\mathbf{P}_{p'}\right)\Big|_{p'=3+t(p-3)} e_2 dt \\ \widetilde{Z}_2 &:= N_\alpha\mathbf{P}_p N_\alpha |\mathbf{P}_p|^{\frac{1}{2}}(1 + (p-3)M_{\alpha p})^{-1}\mathbf{P}_p^{\frac{1}{2}} e_2 \end{aligned}$$

where we used $N_0\mathbf{P}_3 = 2^{-2}\mathbf{T}e_1$. By standard computations for any $k \geqslant 0$ there exists a constant C_k such that

$$|\partial_x^k \widetilde{Z}_2| \leqslant C_k \langle x\rangle \text{ for all } x \in \mathbb{R} \text{ and all } p \text{ near } 3.$$

It is also elementary to see that for any $k \geqslant 0$ there exists a constant C_k such that

$$|\partial_p\left(N_{\alpha(p)}(x,y)\partial_y^k\mathbf{P}_p(y)\right)e_2| \lesssim \langle x-y\rangle^3 \operatorname{sech}\left(\frac{p-1}{2}y\right) \text{ for all } x, y \in \mathbb{R},$$

which implies for any $k \geqslant 0$ there exists a constant C_k such that

$$|\partial_x^k \widetilde{Z}| \leqslant C_k \langle x\rangle^3 \text{ for all } x \in \mathbb{R} \text{ and all } p \text{ near } 3. \tag{11.14}$$

Going back to $w_p = (w_{p,1}, w_{p,2})^\intercal$ and for $\widetilde{w} = U\widetilde{Z}$ we have

$$w_p = UZ_p = \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} - (p-3)2^{-2}2^{-\frac{1}{2}}\mathbf{T}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + (p-3)^2\widetilde{w}.$$

Notice that the first term in the expansion is exactly what we get entering $k = 0$ in (10.5).

Going back to $\xi_p = (\xi_{p,1}, \xi_{p,2})^\intercal$ by means of (10.4) we have

$$\begin{aligned} \xi_{p,1} = (S_1^*)^2 w_{p,1} &= \left(\partial_x + \frac{\phi_p'}{\phi_p}\right)^2 w_{p,1} = \left(\partial_x^2 + 2\frac{\phi_p'}{\phi_p}\partial_x + \left(\frac{\phi_p'}{\phi_p}\right)' + \frac{\phi_3'^2}{\phi_p^2}\right) w_{p,1} \\ &= \left(\partial_x^2 + 2\frac{\phi_p'}{\phi_p}\partial_x + \frac{\phi_p''}{\phi_p}\right) w_{p,1}. \end{aligned}$$

We will use also the expansion in [8]

$$\phi_p^{p-1} = \phi_3^2 + (p-3)q_1 + (p-3)^2 q_R(x) \text{ with } q_1(x) = \operatorname{sech}^2(x)\left(2^{-1} - 2x\tanh(x)\right) \tag{11.15}$$

where

$$q_R(x) = \int_0^1 \partial_{p'}^2 \phi_{p'}^{p'-1}(x)\Big|_{p'=(1-t)3+tp} tdt = \int_0^1 \partial_{p'}^2 \left(\frac{p'+1}{2}\operatorname{sech}^2\left(\frac{p'-1}{2}x\right)\right)\Big|_{p'=(1-t)3+tp} tdt.$$

Notice that for any $k \geqslant 0$ there exists a constant C_k such that

$$|q_R^{(k)}(x)| \leqslant C_k \langle x\rangle^2 \operatorname{sech}^2\left(\min\left\{\frac{p-1}{2}, 1\right\} x\right) \text{ for all } x \in \mathbb{R} \text{ and all } p \text{ near } 3. \tag{11.16}$$

Recalling the identities

$$\begin{aligned} &-\phi_p'' + \phi_p - \phi_p^p = 0 \text{ and} \\ &-\phi_p'^2 + \phi_p^2 - \frac{2}{p+1}\phi_p^{p+1} = 0 \end{aligned}$$

we get

$$\frac{\phi_p''}{\phi_p} = 1 - \phi_p^{p-1} = 1 - \phi_3^2 - (p-3)q_1 - (p-3)^2 q_R,$$

so that using also the expansion in (11.15) we have

$$\xi_{p,1} = 1 - \phi_3^2 + (p-3)R_1 + (p-3)^2\widetilde{\xi}_1$$

where, using the equation in (11.2),

$$\begin{aligned} R_1 &:= -q_1 - 2^{-2}2^{-\frac{1}{2}}\left(\partial_x^2 + 2\frac{\phi_3'}{\phi_3}\partial_x + 1 - \phi_3^2\right)\mathbf{T} \\ &= -q_1 + 2^{-2}\phi_3^2 - 2^{-2}2^{-\frac{1}{2}}(3-\phi_3^2)\mathbf{T} + 2^{-1}2^{-\frac{1}{2}}\tanh(x)\mathbf{T}' \end{aligned}$$

which by (11.15) yields the desired expression of R_1 and

$$\widetilde{\xi}_1 := -\frac{2^{-\frac{1}{2}}}{4(p-3)}\left(\frac{\phi_p'}{\phi_p} - \frac{\phi_3'}{\phi_3}\right)\mathbf{T}' + 2^{-2}2^{-\frac{1}{2}}q_R\mathbf{T} + (S_1^*)^2\widetilde{w}_1.$$

By (11.14) and (11.16), we have (11.13) for $j=1$. Next, by (10.4), we have $-\mathrm{i}(1-\alpha^2)\xi_{p,2} = L_{+p}\xi_{p,1}$. Substituting the expansions (11.11) and

$$L_{+p} = L_{+3} - (p-3)(\phi_3^2 + 3q_1) - (p-3)^2\left(3q_R + q_1 + (p-3)q_1q_R\right),$$

which follows from (11.15), we have

$$\begin{aligned} -\mathrm{i}\xi_{p,2} &= L_{+p}\xi_{p1} + (p-3)^2\left(\frac{\alpha^2}{(p-3)^2}\frac{1}{1-\alpha^2}L_{+p}\xi_{p,1}\right) \\ &= L_{+3}(1-\phi_3^2) + (p-3)\left(L_{+3}R_1 - (\phi_3^2+3q_1)(1-\phi_3^2)\right) \\ &\quad + (p-3)^2\left(L_{+3}\tilde{\xi}_1 - (\phi_3^2+3q_1)\left(R_1 + (p-3)\tilde{\xi}_1\right) - \left(3q_R + q_1 + (p-3)q_1q_R\right)\xi_{p,1}\right) \\ &\quad + (p-3)^2\left(\frac{\alpha^2}{(p-3)^2}\frac{1}{1-\alpha^2}L_{+p}\xi_{p,1}\right) \end{aligned}$$

By looking the coefficients of $(p-3)^0$ and $(p-3)^1$ we have (11.12). Further, the estimate of $\tilde{\xi}_2$ follows from the estimate of $\tilde{\xi}_1$ given by (11.13), (11.16) and the explicit form of R_1 and q_1.

$\square$

Now, Lemma 5.1 is a direct consequence of the following lemma.

Lemma 11.6. *For $|p-3| \ll 1$, we have*

$$\gamma(p,1) = \frac{\pi}{\sqrt{2}\cosh(\pi/2)}(p-3) + o(p-3).$$

Proof. We set

$$\begin{aligned} E &:= \partial_p|_{p=3}\, \phi_p = \frac{1}{2}\phi_3\left(\frac{1}{4} - \log\phi_3\right) + \frac{1}{2}x\phi_3', \\ F &:= \partial_p|_{p=3}\, \phi_p^{p-2} = E + \phi_3 \log \phi_3. \end{aligned}$$

Further, we set

$$\tilde{F} = \phi_p^{p-2} - \phi_3 - (p-3)F. \tag{11.17}$$

Recall,

$$\gamma(p,1) = \left\langle \phi_p^{p-2}(p\xi_{p,1}^2 + \xi_{p,2}^2), g_{p,1}\right\rangle + 2\left\langle \phi_p^{p-2}\xi_{p,1}\xi_{p,2}, g_{p,2}\right\rangle$$

and setting, following notation and argument in Martel [46],

$$\begin{aligned} G_{p,1} &:= \phi_p^{p-2}(p\xi_{p,1}^2 + \xi_{p,2}^2), \\ G_{p,2} &:= -2\mathrm{i}\phi_p^{p-2}\xi_{p,1}\xi_{p,2}, \end{aligned}$$

we have

$$\begin{aligned} G_{p,1} &= \phi_3\left(3(1-\phi_3^2)^2 - 1\right) + (p-3)\Delta_1 + \tilde{\Delta}_1, \\ \frac{1}{2}G_{p,2} &= \phi_3(1-\phi_3^2) + (p-3)\Delta_2 + \tilde{\Delta}_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= F\left(3(1-\phi_3^2)^2 - 1\right) + \phi_3(1-\phi_3^2)^2 + 6\phi_3(1-\phi_3^2)R_1 - 2\phi_3 R_2, \\ \Delta_2 &= F(1-\phi_3^2) + \phi_3 R_1 + \phi_3(1-\phi_3^2)R_2 \end{aligned}$$

and $\tilde{\Delta}_1, \tilde{\Delta}_2$ are remainder terms of $(p-3)^2$ order. Since it is easy to verify that the $\tilde{F}$ in (11.17) is decaying exponentially, we see from Lemma 11.5 that the contribution of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ to $\gamma(p,1)$ are $(p-3)^2$ order. Thus, we can ignore these terms.
We have $\langle \phi_p, g_{p1}\rangle = 0$ like in Martel [46]. Now, by $g_{p,2} = \frac{\mathrm{i}}{2\lambda(p,1)}L_{p+}g_{p,1}$, we have

$$\begin{aligned} \gamma(p,1) &= \langle G_{p,1}, g_{p,1}\rangle + \left\langle G_{p,2}, \frac{1}{2\lambda(p,1)}L_{p+}g_{p,1}\right\rangle \\ &= \left\langle G_{p,1} + \frac{1}{2\lambda(p,1)}L_{p+}G_{p,2}, g_{p,1}\right\rangle \end{aligned}$$

By direct computation, see the proof of Lemma 20 of [46], we have

$$G_{3,1} + \frac{1}{2}(-\partial_x^2 + 1 - 3\phi_3^2)G_{3,2} = \phi_3 \left(3(1-\phi_3^2)^2 - 1\right) + \frac{1}{2}(-\partial_x^2 + 1 - 3\phi_3^2)\left(\phi_3(1-\phi_3)^2\right) = 2\phi_3.$$

Thus, we see $\gamma(3,1) = 0$. Further, expanding $G_{p,1}, G_{p,2}$ and L_{p+}, we have

$$\begin{aligned}
\gamma(p,1) &= \left\langle 2\phi_3 + (p-3)\left(\Delta_1 + L_{3+}\Delta_2 + \frac{1}{2}\left(\partial_p|_{p=3} L_{p+}\right) G_{3,2}\right), g_{p,1}\right\rangle + o(p-3) \\
&= (p-3)\left(-2\langle E, g_{3,1}\rangle + \langle \Delta_1, g_{3,1}\rangle + 2\langle \Delta_2, -\mathrm{i}g_{3,2}\rangle - \frac{1}{2}\left\langle \phi_3\left(\frac{7}{4}\phi_3 + 3x\phi_3'\right) G_{3,2}, g_{3,1}\right\rangle\right) \\
&\quad + o(p-3).
\end{aligned}$$

Thus, it suffices to compute the coefficient of $p-3$, which we denote γ_1 (i.e. $\gamma(p,1) = (p-3)\gamma_1 + o(p-3)$). Following Martel [46] we will consider the following constants,

$$\begin{aligned}
p_k &= \int \operatorname{sech}^k \cos, \\
q_k &= \int \operatorname{sech}^k \log \circ \operatorname{sech} \cos \\
r_k &= \int \operatorname{sech}^k \mathbf{T} \cos \\
s_k &= \int \operatorname{sech}^k \mathbf{T} \tanh \sin \\
a_k &= \int x \operatorname{sech}^k \tanh \cos \\
b_k &= \int \operatorname{sech}^k \tanh \sin \\
c_k &= \int \operatorname{sech}^k \log \circ \operatorname{sech} \tanh \sin \\
d_k &= \int x \operatorname{sech}^k \sin \\
e_k &= \int \operatorname{sech}^k \tanh \mathbf{T}' \cos \\
f_k &= \int \operatorname{sech}^k \mathbf{T}' \sin .
\end{aligned}$$

Then, after a quite long but elementary computation, we arrive to

$$
\begin{aligned}
\gamma_1 &= \sqrt{2}\left(-\frac{3}{2}(2\log 2+1)p_5+\frac{3}{2}(2\log 2+1)p_7+\frac{3}{2}q_3-6q_5+6q_7-\frac{3}{2}a_3+6a_5-6a_7\right)\\
&+\sqrt{2}\left(\frac{3}{2}(2\log 2+1)b_3-\frac{3}{2}(2\log 2+1)b_5-\frac{3}{2}c_1+6c_3-6c_5\right)\\
&+\sqrt{2}\left(\frac{3}{2}d_1-\frac{3}{2}d_3-6d_3+6d_5+6d_5-6d_7)\right)\\
&+\sqrt{2}\left(-\frac{1}{2}q_3+\frac{1}{2}a_3+\frac{1}{2}c_1-\frac{1}{2}d_1+\frac{1}{2}d_3\right)\\
&+\sqrt{2}\left(-4p_5+4p_7+4b_3-4b_5+12a_5-24a_7-12d_3+36d_5-24d_7-2p_5+2b_3\right)\\
&-\frac{9}{2}r_3+12r_5-6r_7+3e_3-6e_5+\frac{9}{2}s_1-12s_3+6s_5-3f_1+9f_3-6f_5-\frac{3}{2}r_3+e_3+\frac{3}{2}s_1-f_1+f_3.
\end{aligned}
$$

We can further simplify this quantity. First, we can eliminate b_k, c_k, d_k, e_k and f_k by the identities obtained by integration by parts,

$$
\begin{aligned}
b_k &= (k+1)p_{k+2}-kp_k,\\
c_k &= (k+1)q_{k+2}-kq_k+p_{k+2}-p_k,\\
d_k &= -ka_k+p_k,\\
e_k &= s_k+kr_k-(k+1)r_{k+2},\\
f_k &= -r_k+ks_k.
\end{aligned}
$$

The expression given by p_k, q_k, r_k, s_k and a_k $(k=1,3,5,7)$ can be reduced to p_1, q_1, r_k, s_1 and a_1 by the identities, again obtained by integration by parts,

$$
\begin{aligned}
p_{k+2} &= \frac{1+k^2}{k(k+1)}p_k,\\
q_{k+2} &= \frac{1}{k(k+1)}\left((1+k^2)q_k-(2k+1)p_{k+2}+(k+1)p_k\right),\\
r_{k+2} &= \frac{1}{k(k+1)}\left((k^2-3)r_k+2ks_k+2\sqrt{2}p_{k+2}\right),\\
s_{k+2} &= \frac{1}{(k+1)(k+2)}\left((k^2-3)s_k+2(k+1)r_{k+2}-2kr_k+2\sqrt{2}(k+3)p_{k+4}-2\sqrt{2}(k+2)p_{k+2}\right),\\
a_{k+2} &= \frac{1}{(k+1)(k+2)}\left((k^2+1)a_k-2kp_k+2(k+1)p_{k+2}\right).
\end{aligned}
$$

Now, as in Martel [46], a quite surprising simplification occurs. That is, after reducing the expression to a linear combination of p_1, q_1, r_1, s_1 and a_1 by means of lengthy but nonetheless very elementary computations, the coefficients of q_1, r_1, s_1 and a_1 vanish and we are left with the very simple formula

$$
\gamma_1 = \frac{1}{\sqrt{2}}p_1,
$$

like in Martel [46]. We have $p_1 = \pi/\cosh(\pi/2)$, see Martel [46]. We have the relation $p_1 = p_{1,\mathrm{Martel}}/\sqrt{2}$. This completes the proof of Lemma 11.6. We will provide all the elementary computations which we have skipped here in the forthcoming note [18]. □

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