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ORTHOGONAL LAURENT POLYNOMIALS OF TWO REAL VARIABLES

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Abstract. In this paper we consider an appropriate ordering of the Laurent monomials $x^i y^j$, $i, j \in \mathbb{Z}$ that allows us to study sequences of orthogonal Laurent polynomials of the real variables x and y with respect to a positive Borel measure μ defined on $\mathbb{R}^2$ such that $\{x = 0\} \cup \{y = 0\} \notin \operatorname{supp}(\mu)$. This ordering is suitable for considering the *multiplication plus inverse multiplication operator* on each varibale $\left(x + \frac{1}{x} \text{ and } y + \frac{1}{y}\right)$, and as a result we obtain five-term recurrence relations, Christoffel-Darboux and confluent formulas for the reproducing kernel and a related Favard's theorem. A connection with the one variable case is also presented.

1. Introduction

Orthogonal Laurent polynomials with respect to a positive Borel measure supported on the real line were introduced for the first time in [19], and also implicitly in [20] in relation to continued fractions and the solution of the strong Stieltjes moment problem (see also chronologically [25, 18, 26, 5, 17]). An extensive bibliography has been produced after these works, giving rise to a theory close to the well known theory of orthogonal polynomials on the real line (see e.g. [3, 10, 11, 9, 12, 7]), with applications in moment problems, recurrence relations, reproducing kernels, Favard's theorem, interpolation and quadrature formulas along with denseness and convergence, linear algebra and inverse eigenvalue problems, Krylov methods, model reduction, linear prediction, system identification,.... This theory has been also considered for positive Borel measures supported on the unit circle (for the first time [33]), giving rise in particular to the well known CMV theory ([4], see also e.g. [6, 8]) that has produced an important impulse in the theory of orthogonal polynomials on the unit circle (see [31]). In particular, in the context of quadrature formulas on the real line, the advantages of considering rules based on Laurent polynomials instead of ordinary polynomials have been shown deeply in the literature, theoretically and numerically.

In the one variable case, there are few weight functions on the real line that give rise to explicit expressions for the corresponding orthogonal Laurent polynomials. In practice these orthogonal Laurent polynomials are computed recursively from a three-term recurrence relation that holds for an arbitrarily ordered sequence of monomials $\{x^i\}_{i\in\mathbb{Z}}$ (induced from what is known in the literature as "generating sequence"). In the particular case of the "balanced" ordering

$$\mathscr{L} = \operatorname{span}\left\{1, x, \frac{1}{x}, x^2, \frac{1}{x^2}, \dots\right\}, \tag{1}$$

this recurrence is given by the following (see [11, 12])

Theorem 1.1. *Let ω be a positive Borel measure on $\mathbb{R}^+$ and let $\{\psi_k\}_{k\geq 0}$ be the sequence of orthonormal Laurent polynomials induced by the inner product $\langle f, g\rangle_\omega = \int_0^\infty f(x)g(x)d\omega(x)$ and the balanced ordering (1). Then, setting $\psi_{-1} \equiv 0$, there exist two sequences of positive real numbers $\{\Omega_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ such that for all $n \geq 0$,*

$$\begin{aligned} C_n\psi_{n+1}(x) &= (\Omega_n x - 1)\,\psi_n(x) - C_{n-1}\psi_{n-1}(x) && \text{if } n \text{ is even},\\ C_n\psi_{n+1}(x) &= \left(1 - \frac{\Omega_n}{x}\right)\psi_n(x) - C_{n-1}\psi_{n-1}(x) && \text{if } n \text{ is odd}. \end{aligned}$$

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Furthermore, $\psi_0 \equiv \frac{1}{\sqrt{m_0}}$, $\Omega_0 = \frac{m_0}{m_1}$ *and* $C_0 = \frac{\sqrt{m_2 m_0 - m_1^2}}{m_1}$, $m_k = \langle x^k, 1 \rangle$ *being corresponding moments for* ω, $k \in \mathbb{Z}$.

On the other hand, the general theory of multivariate orthogonal polynomials is still far from being considered an established field and has experienced delayed development, especially in fundamental aspects. In 1865, C. Hermite [15] explored a two-variable generalization of Legendre polynomials, marking the initial appearance of orthogonal polynomial families in multiple variables in the literature. However, it was not until 1926 that a study on families of orthogonal polynomials in two variables on the unit disk and the triangle appeared in the classic monograph by Appell and Kampé de Fériet [1]. Since that moment, various authors have contributed to the development of the general theory of polynomials in several variables; see, for example, [21, 24, 32].

Based on a vectorial representation, M. Kowalski ([23, 22]) proposed a novel approach in the study of polynomials in multiple variables. This perspective has allowed the development of a basic algebraic theory, which can be found in the monograph by C. F. Dunkl and Y. Xu ([14]). In particular, it has been possible to extend fundamental properties to multiple variables, such as the three-term relation, Favard's theorem or the Christoffel-Darboux formula. The monograph [14] comes highly recommended as reference for gaining insight into the current state of the art in multivariate orthogonal polynomials.

Orthogonal polynomials in several variables find diverse applications across fields like physics, quantum mechanics, and signal processing. One prominent application lies in optics and ophthalmology. Zernike polynomials are orthogonal polynomials on the unit disk [34] and were introduced by Fritz Zernike (Nobel prize for physics in 1953) in 1934 to address optical challenges related to telescopes and microscopes. In the year 2000, the Optical Society of America adopted them as the standard pattern in ophthalmic optics.

The purpose of this paper is to consider for the first time in the literature (as far as we know) the theory of sequences of orthogonal Laurent polynomials in several real variables. The advantages of considering orthogonal Laurent polynomials (or more generally, orthogonal rational functions) instead of ordinary orthogonal polynomials have been showed in a wide variety of contexts in the literature of the one variable case. The growing interest in the study of orthogonal polynomials in several variables undoubtedly motivates to consider generalizations to more general kind of functions than ordinary polynomials, mainly due to their possible applications in many problems like cubature rules, Fourier orthogonal series and summability of orthogonal expansions, moment problems,... In [2], multivariate orthogonal Laurent polynomials in the unit torus are studied.

For simplicity, we will restrict to the case of two real variables, but all the results can be extended to more variables by using a somewhat more involved notation. Here, the basic key is to start from an appropriate ordering for the Laurent monomials $x^i y^j$, $i, j \in \mathbb{Z}$ that is inspired on the "balanced case" (that is usually considered in the literature), but now for both real variables simultaneously. The vectorial representation of the Laurent polynomials is necessary for the proof of the main results.

The paper has been organized as follows. An appropriate ordering of the Laurent monomials $x^i y^j$, $i, j \in \mathbb{Z}$ for the construction of Laurent polynomials sequences of two real variables with respect to a linear functional is considered in Section 2. We concentrate in the positive-definite case, dealing with orthogonality with respect to a positive Borel measure μ defined on $\mathbb{R}^2$ such that $\{x = 0\} \cup \{y = 0\} \notin \operatorname{supp}(\mu)$. Five-term recurrence relations are obtained involving multiplication by $x + \frac{1}{x}$ and $y + \frac{1}{y}$. In Section 3 we deduce a Favard's theorem and Christoffel-Darboux and confluent formulas for the reproducing kernel, whereas in Section 4 we present a connection with the one variable case when μ is supported in a rectangle and it is of the form $d\mu(x, y) = d\mu_1(x) d\mu_2(y)$. Some conclusions are finally carried out.

We end this introduction with some remaining notation throughout the paper. We denote by $E[\cdot]$ the integer part function, by $\delta_{k,l}$ the Kronecker delta symbol, by $\mathcal{M}_{n,m}$ the space of (real) matrices of dimension $n \times m$, $\mathcal{M}_n$ being the space of square (real) matrices of dimension n, by $\mathcal{I}_n$ the identity matrix of dimension n, by $\mathcal{O}_{n,m}$ and $\mathcal{O}_n$ the zero matrices in $\mathcal{M}_{n,m}$ and $\mathcal{M}_n$, respectively, and by $\operatorname{diag}(a_1, \ldots, a_n) \in \mathcal{M}_n$ the diagonal matrix with ordered entries in the main diagonal $a_1, \ldots, a_n$.

2. Orthogonal Laurent polynomials of two real variables. Five-term relations

In the one-variable situation it is usual to consider a nested sequence of subspaces of Laurent polynomials $\{\mathscr{L}_n\}$ such that $\mathscr{L}_0 = \operatorname{span}\{1\}$, $\mathscr{L}_n \subset \mathscr{L}_{n+1}$, $\dim(\mathscr{L}_n) = n+1$ for all $n \geq 0$, and $\bigcup_{n\geq 0} \mathscr{L}_n = \mathscr{L}$. See e.g. [11, 13].

Having in mind the "balanced" ordering (1) in the one variable situation

$$\mathscr{L}_0 = \operatorname{span}\{1\}, \quad \mathscr{L}_{2k} = \operatorname{span}\left\{\frac{1}{x^k}, \dots, x^k\right\}, \quad \mathscr{L}_{2k-1} = \mathscr{L}_{2k-2} \oplus \operatorname{span}\{x^k\}, \quad \forall k \geq 1,$$

(see e.g. [5, 9, 12, 18] for the real line case, and [33, 31, 4, 8] for the unit circle case), we can proceed by defining the sequence

$$c_n = (-1)^{n+1} \cdot E\left[\frac{n+1}{2}\right], \quad \forall n \geq 0$$

and considering the Laurent monomials

$$p_{m,n}(x,y) = x^{c_m} y^{c_n}, \quad \forall m,n \geq 0$$

and the infinite matrix

$$\begin{array}{ccccc}
p_{0,0} = x^{c_0}y^{c_0} = 1 & p_{1,0} = x^{c_1}y^{c_0} = x & p_{2,0} = x^{c_2}y^{c_0} = \frac{1}{x} & p_{3,0} = x^{c_3}y^{c_0} = x^2 & \cdots \\
p_{0,1} = x^{c_0}y^{c_1} = y & p_{1,1} = x^{c_1}y^{c_1} = xy & p_{2,1} = x^{c_2}y^{c_1} = \frac{y}{x} & p_{3,1} = x^{c_3}y^{c_1} = x^2y & \cdots \\
p_{0,2} = x^{c_0}y^{c_2} = \frac{1}{y} & p_{1,2} = x^{c_1}y^{c_2} = \frac{x}{y} & p_{2,2} = x^{c_2}y^{c_2} = \frac{1}{xy} & p_{3,2} = x^{c_3}y^{c_2} = \frac{x^2}{y} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{2}$$

Setting $\mathcal{L} = \operatorname{span}\{x^iy^j : i,j \in \mathbb{Z}\}$, the space of Laurent polynomials of real variables x and y, we can order these elements $p_{n,m}$ by anti-diagonals in (2) as

$$\mathcal{L} = \operatorname{span}\{\underbrace{p_{0,0}}_{n+m=0}, \underbrace{p_{1,0}, p_{0,1}}_{n+m=1}, \underbrace{p_{2,0}, p_{1,1}, p_{0,2}}_{n+m=2}, \underbrace{p_{3,0}, p_{2,1}, p_{1,2}, p_{0,3}}_{n+m=3}, \underbrace{p_{4,0}, p_{3,1}, p_{2,2}, p_{1,3}, p_{0,4}}_{n+m=4}, \cdots\} \tag{3}$$

and define

$$\mathcal{L}_n = \operatorname{span}\{p_{i,j} : i+j \leq n\}, \quad \text{for all } n \geq 0, \qquad \dim(\mathcal{L}_n) = \frac{(n+1)(n+2)}{2}, \qquad \mathcal{L} = \bigcup_{n\geq 0} \mathcal{L}_n. \tag{4}$$

Consider

$$\phi_k(x,y) = \begin{pmatrix} p_{k,0}(x,y) \\ p_{k-1,1}(x,y) \\ \vdots \\ p_{0,k}(x,y) \end{pmatrix} \in \mathcal{M}_{k+1,1}, \quad \forall\ k \geq 0, \tag{5}$$

that is, the components of the vector ϕ_k are the $k+1$ linearly independent Laurent monomials of $\mathcal{L}_k \backslash \mathcal{L}_{k-1}$, ordered as they appear in the expansion (3). So, we can interpret $\mathcal{L}_n = \operatorname{span}\{\phi_0, \dots, \phi_n\}$, for all $n \geq 0$ so that if $\psi_k \in \mathcal{L}_k$ for some $k \geq 0$, then $\psi_k = \sum_{l=0}^k C_l \phi_l$ where $C_l \in \mathcal{M}_{1,l+1}$ are constant matrices (C_k being the leading coefficient matrix).

Observe that $\phi_0 \equiv 1$ and for all $l \geq 1$,

$$\phi_{2l}(x,y) = \begin{pmatrix} x^{-l}y^0 \\ x^ly \\ x^{-(l-1)}y^{-1} \\ \vdots \\ xy^l \\ x^0y^{-l} \end{pmatrix}, \qquad \phi_{2l-1}(x,y) = \begin{pmatrix} x^ly^0 \\ x^{-(l-1)}y \\ x^{l-1}y^{-1} \\ \vdots \\ xy^{-(l-1)} \\ x^0y^l \end{pmatrix}.$$

Thus,

$$x\phi_{2l}(x,y)=\begin{pmatrix} p_{2l-2,0}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{2l+1,1}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{2l-4,2}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ \vdots\\ p_{0,2l-2}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{3,2l-1}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{1,2l}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\end{pmatrix},\qquad \frac{1}{x}\phi_{2l}(x,y)=\begin{pmatrix} p_{2l+2,0}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{2l-3,1}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{2l,2}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ \vdots\\ p_{4,2l-2}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{0,2l-1}\in\mathcal{L}_{2l-1}\backslash\mathcal{L}_{2l-2}\\ p_{2,2l}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\end{pmatrix},\tag{6}$$

$$x\phi_{2l-1}(x,y)=\begin{pmatrix} p_{2l+1,0}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-4,1}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ p_{2l-1,2}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ \vdots\\ p_{0,2l-3}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ p_{3,2l-2}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{1,2l-1}\in\mathcal{L}_{2l}\backslash\mathcal{L}_{2l-1}\end{pmatrix},\qquad \frac{1}{x}\phi_{2l-1}(x,y)=\begin{pmatrix} p_{2l-3,0}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ p_{2l,1}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-5,2}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ \vdots\\ p_{4,2l-3}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{0,2l-2}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{2,2l-1}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\end{pmatrix},\tag{7}$$

$$y\phi_{2l}(x,y)=\begin{pmatrix} p_{2l,1}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-1,3}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{2l-2,0}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ \vdots\\ p_{2,2l-4}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{1,2l+1}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{0,2l-2}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\end{pmatrix},\qquad \frac{1}{y}\phi_{2l}(x,y)=\begin{pmatrix} p_{2l,2}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{2l-1,0}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-2,4}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ \vdots\\ p_{2,2l}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\\ p_{1,2l-3}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{0,2l+2}\in\mathcal{L}_{2l+2}\backslash\mathcal{L}_{2l+1}\end{pmatrix},\tag{8}$$

and

$$y\phi_{2l-1}(x,y)=\begin{pmatrix} p_{2l-1,1}\in\mathcal{L}_{2l}\backslash\mathcal{L}_{2l-1}\\ p_{2l-2,3}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-3,0}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ \vdots\\ p_{2,2l-1}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{1,2l-4}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ p_{0,2l+1}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\end{pmatrix},\qquad \frac{1}{y}\phi_{2l-1}(x,y)=\begin{pmatrix} p_{2l-1,2}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{2l-2,0}\in\mathcal{L}_{2l-2}\backslash\mathcal{L}_{2l-3}\\ p_{2l-3,4}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ \vdots\\ p_{2,2l-5}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\\ p_{1,2l}\in\mathcal{L}_{2l+1}\backslash\mathcal{L}_{2l}\\ p_{0,2l-3}\in\mathcal{L}_{2l-3}\backslash\mathcal{L}_{2l-4}\end{pmatrix}.\tag{9}$$

It follows from (6)-(7) and (8)-(9) that if $\psi_k\in\mathcal{L}_k\backslash\mathcal{L}_{k-1}$, then $x\psi_k\in\mathcal{L}_{k+2}$, $\frac{1}{x}\psi_k\in\mathcal{L}_{k+2}$ and $y\psi_k\in\mathcal{L}_{k+2}$, $\frac{1}{y}\psi_k\in\mathcal{L}_{k+2}$, respectively. However, the key fact in what follows is that $\left(x+\frac{1}{x}\right)\psi_k\in\mathcal{L}_{k+2}\backslash\mathcal{L}_{k+1}$ and $\left(y+\frac{1}{y}\right)\psi_k\in\mathcal{L}_{k+2}\backslash\mathcal{L}_{k+1}$.

A *Laurent system* in two variables $\{\varphi_n\}_{n\geq 0}$ is a sequence of vectors of increasing size

$$\varphi_n\in\mathcal{M}_{n+1,1},\quad \varphi_n\in\mathcal{L}_n\backslash\mathcal{L}_{n-1},\quad \forall\ n\geq 0$$

such that the components in the vector φ_n are linearly independent. It is clear that in this case

$$\varphi_n=\sum_{i=0}^{n}A_i\phi_i,\qquad \text{with } A_n \text{ invertible.}$$

Let us consider a linear functional L defined in $\mathcal{L}$ by $L(x^iy^j)=\mu_{i,j}$ for $i,j\in\mathbb{Z}$ and extended by linearity. It can be defined over product of vectors in the following way

$$L(f\,g^T)=\left(L(f_i\,g_j)\right)_{i=1,\dots,k;\,j=1,\dots,m}\in\mathcal{M}_{k,m},\ \text{ where } f=[f_1,\dots,f_k]^T \text{ and } g=[g_1,\dots,g_m]^T.\tag{10}$$

Definition 2.1. *A Laurent system* $\{\varphi_n\}_{n\geq 0}$ *is a* system of orthogonal Laurent polynomials *with respect to the linear functional* L *if for all* $n\geq 0$

$$\begin{aligned} &L(\varphi_n\varphi_k^T)=0,\quad k=0,\dots,n-1\\ &L(\varphi_n\varphi_n^T)=\mathcal{H}_n\quad \textit{with } \mathcal{H}_n \textit{ an invertible matrix.}\end{aligned}\tag{11}$$

In the case when $\mathcal{H}_n=\mathcal{I}_{n+1}$ *for all* $n\geq 0$, $\{\varphi_n\}_{n\geq 0}$ *is called a* system of orthonormal Laurent polynomials.

Observe that the orthogonality conditions are equivalent to

$$\begin{aligned} &L(\phi_k \varphi_n^T) = 0, \quad k = 0, \dots, n-1 \\ &L(\phi_n \varphi_n^T) = \mathcal{S}_n \quad \text{with } \mathcal{S}_n \text{ an invertible matrix.} \end{aligned} \tag{12}$$

For $n \geq 0$, $k, l \geq 0$, we define the matrices

$$\mathrm{M}_{k,l} = L(\phi_k\, \phi_l^T)$$

and the matrix

$$\mathrm{M}_n = (\mathrm{M}_{k,l})_{k,l=0}^n \quad \text{with} \quad \Delta_n = \det M_n.$$

We call M_n a *moment matrix*. Observe that

$$\Delta_0 = |\mu_{0,0}|, \quad \Delta_1 = \left|\begin{array}{c|cc} \mu_{0,0} & \mu_{1,0} & \mu_{0,1} \\ \hline \mu_{1,0} & \mu_{2,0} & \mu_{1,1} \\ \mu_{0,1} & \mu_{1,1} & \mu_{0,2} \end{array}\right|, \quad \Delta_2 = \left|\begin{array}{c|cc|ccc} \mu_{0,0} & \mu_{1,0} & \mu_{0,1} & \mu_{-1,0} & \mu_{1,1} & \mu_{0,-1} \\ \hline \mu_{1,0} & \mu_{2,0} & \mu_{1,1} & \mu_{0,0} & \mu_{2,1} & \mu_{1,-1} \\ \mu_{0,1} & \mu_{1,1} & \mu_{0,2} & \mu_{-1,1} & \mu_{1,2} & \mu_{0,0} \\ \hline \mu_{-1,0} & \mu_{0,0} & \mu_{-1,1} & \mu_{-2,0} & \mu_{0,1} & \mu_{-1,-1} \\ \mu_{1,1} & \mu_{2,1} & \mu_{1,2} & \mu_{0,1} & \mu_{2,2} & \mu_{1,0} \\ \mu_{0,-1} & \mu_{1,-1} & \mu_{0,0} & \mu_{-1,-1} & \mu_{1,0} & \mu_{0,-2} \end{array}\right|, \dots$$

Proposition 2.2. *A system of orthogonal Laurent polynomials with respect to the linear functional L exists, if and only if, $\Delta_n \neq 0$ for all $n \geq 0$.*

Proof. Using that $\varphi_n = \sum_{i=0}^n A_i \phi_i$ we have

$$L(\phi_k \varphi_n^T) = \sum_{k=0}^n L(\phi_k \phi_i^T) A_i^T = \sum_{k=0}^n M_{k,i} A_i^T$$

The orthogonality conditions (12) are equivalent to the following linear system of equations:

$$M_n \begin{pmatrix} A_0^T \\ \vdots \\ A_{n-1}^T \\ A_n^T \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathcal{S}_n \end{pmatrix}$$

The system has a unique solution if the matrix M_n is invertible, that is, if $\Delta_n \neq 0$. □

Definition 2.3. *A linear functional L defined in $\mathcal{L}$ given by (3) is called* quasi-definite *if there exists a system of orthogonal Laurent polynomials with respect to L. L is* positive definite *if it is quasi-definite and $L(\psi^2) > 0$, $\forall \psi \in \mathcal{L}$, $\psi \neq 0$.*

Proposition 2.4. *If L is a positive definite moment functional then there exists a system of orthonormal Laurent polynomials with respect to L.*

Proof. Suppose that L is positive definite. Let $\mathbf{a} = (\mathbf{a_0}, \dots, \mathbf{a_n})$, ($\mathbf{a_j}$ with $j+1$ components) be an eigenvector of the matrix M_n corresponding to eigenvalue λ. Then, on the one hand, $\mathbf{a^T}\, \mathbf{M_n}\, \mathbf{a} = \lambda \|\mathbf{a}\|^\mathbf{2}$. On the other hand, $\mathbf{a}^T\, M_n\, \mathbf{a} = L(\psi^2) > 0$, where $\psi = \sum_{j=0}^n \mathbf{a_j}^\mathbf{T} \phi_\mathbf{j}$. It follows that $\lambda > 0$. Since all the eigenvalues are positive, $\Delta_n = \det(M_n) > 0$.

As a consequence, there exists a system $\{\varphi_n\}_n$ of orthogonal Laurent polynomials with respect to L with $\mathcal{H}_n = L(\varphi_n\, \varphi_n^T)$. For any nonzero vector $\mathbf{v}$, $\psi = \mathbf{v}\varphi_n$ is a nonzero element of $\mathcal{L}_n$. Then, $\mathbf{v}\, \mathcal{H}_n\, \mathbf{v}^T = L(\psi^2) > 0$, so $\mathcal{H}_n$ is a positive definite matrix. If we define $\tilde{\varphi}_n = (\mathcal{H}_n^{1/2})^{-1} \varphi_n$, then $L(\tilde{\varphi}_n\, \tilde{\varphi}_n^T) = (\mathcal{H}_n^{1/2})^{-1} L(\varphi_n\, \varphi_n^T) (\mathcal{H}_n^{1/2})^{-1} = \mathcal{I}$. This proves that $\{\tilde{\varphi}_n\}_n$ is a system of orthonormal Laurent polynomials with respect to L. □

From now on, we will deal with a positive Borel measure $\mu(x, y)$ on $\mathbb{R}^2$ such that $\{x = 0\} \cup \{y = 0\} \notin \operatorname{supp}(\mu) =: D$. We consider the induced inner product

$$\langle f, g \rangle_\mu = \iint_D f(x,y) g(x,y) d\mu(x,y), \quad f, g \in L_2^\mu = \left\{ h : \mathbb{R}^2 \to \mathbb{R} \ : \ \iint_D h^2(x,y) d\mu(x,y) < \infty \right\}, \tag{13}$$

and we suppose the existence of the moments

$$\mu_{i,j} = \langle x^i, y^j \rangle_\mu, \qquad \forall\ i, j \in \mathbb{Z}. \tag{14}$$

Consider the inner product

$$\langle f, g^T\rangle = (\langle f_i, g_j\rangle_\mu)_{i=1,\ldots,k;\, j=1,\ldots,m} \in \mathcal{M}_{k,m}, \quad \text{where} \quad f = [f_1,\ldots,f_k]^T \text{ and } g = [g_1,\ldots,g_m]^T. \tag{15}$$

From orthogonalization to $\mathcal{L}_n = \operatorname{span}\{\phi_0,\ldots,\phi_n\}$ with respect to the inner product (15), for all $n \geq 0$, we can obtain an equivalent system $\mathcal{L}_n = \operatorname{span}\{\varphi_0,\ldots,\varphi_n\}$ verifying $\varphi_0 \in \mathcal{L}_0$, $\varphi_l \in \mathcal{L}_l \backslash \mathcal{L}_{l-1}$, $\varphi_l \perp \mathcal{L}_{l-1}$, for all $l = 1,\ldots,n$, and $\langle \varphi_k, \varphi_k^T\rangle = \mathcal{I}_{k+1}$, for all $k = 0,\ldots,n$. If this procedure is repeated for all $n \geq 0$, we get $\{\varphi_k\}_{k\geq 0}$, a family of orthonormal Laurent polynomials of two real variables with respect to the measure μ.

Remark 2.5. *Observe that φ_n is uniquely determined up to left multiplication by orthogonal matrices. Indeed, if $Q_{n+1} \in \mathcal{M}_{n+1}$ is an orthogonal matrix and $\tilde{\varphi}_n = Q_{n+1}\varphi_n$ then $\tilde{\varphi}_n \perp \mathcal{L}_{n-1}$ and*

$$\langle \tilde{\varphi}_n, \tilde{\varphi}_n^T\rangle = \langle Q_{n+1}\varphi_n, \varphi_n^T Q_{n+1}^T\rangle = Q_{n+1}\langle \varphi_n, \varphi_n^T\rangle Q_{n+1}^T = \mathcal{I}_n.$$

We can write

$$\varphi_n = \sum_{i=0}^{n} A_i^{(n)}\phi_i, \tag{16}$$

with $A_i^{(n)} \in \mathcal{M}_{n+1,i+1}$ constant matrices, $A_n^{(n)}$ being regular.

From (6)-(7) we get

$$\left(x + \frac{1}{x}\right)\phi_n = B_{n+2,1}^{(n)}\phi_{n+2} + B_{n+1,1}^{(n)}\phi_{n+1} + B_{n-1,1}^{(n)}\phi_{n-1} + B_{n-2,1}^{(n)}\phi_{n-2},$$

where by introducing $z_s = (0 \ \cdots \ 0 \ 1 \ 0) \in \mathcal{M}_{1,s}$ for all $s \geq 3$, it follows for all $n \geq 2$ that

$$\begin{aligned} B_{n-2,1}^{(n)} &= \left[\begin{array}{c} \mathcal{I}_{n-1} \\ \hline \mathcal{O}_{2,n-1} \end{array}\right] \in \mathcal{M}_{n+1,n-1}, \qquad & B_{n-1,1}^{(n)} &= \left[\mathcal{O}_{n+1,n-1} | z_{n+1}^T\right] \in \mathcal{M}_{n+1,n}, \\ B_{n+1,1}^{(n)} &= \left[\begin{array}{c} \mathcal{O}_{n,n+2} \\ \hline z_{n+2} \end{array}\right] \in \mathcal{M}_{n+1,n+2}, \qquad & B_{n+2,1}^{(n)} &= \left[\mathcal{I}_{n+1} | \mathcal{O}_{n+1,2}\right] \in \mathcal{M}_{n+1,n+3}. \end{aligned} \tag{17}$$

These formulas are also valid to define $B_{i,1}^{(0)}$ and $B_{j,1}^{(1)}$ for $i = 1,2$ and $j = 0,2,3$ if we interpret $\mathcal{O}_{0,2} = \mathcal{O}_{2,0} = \varnothing$ and $z_2 = (1 \ 0)$. Here, the second subindex in the $B_{s,1}^{(n)} \in \mathcal{M}_{n+1,s+1}$ matrices with $s \in \{n-2, n-1, n+1, n+2\}$ is used to separate the case of multiplication by $\left(y + \frac{1}{y}\right)$, see further. So, it is clear that

$$\left(x + \frac{1}{x}\right)\varphi_n(x,y) = A_n^{(n)}\left[B_{n+2,1}^{(n)}\phi_{n+2} + B_{n+1,1}^{(n)}\phi_{n+1} + B_{n-1,1}^{(n)}\phi_{n-1} + B_{n-2,1}^{(n)}\phi_{n-2}\right] + \text{l. t.}, \tag{18}$$

where by "l. t." we understand linear combinations of $\{\phi_0,\ldots,\phi_{n+1}\}$.

The main reason why multiplication by $x + \frac{1}{x}$ should be considered is the fact that $B_{n+2,1}^{(n)}$ is full rank. So, for certain constant matrices $C_k^{(n+2)} \in \mathcal{M}_{n+3,k+1}$, it holds that $\phi_{n+2}(x,y) = \sum_{k=0}^{n+2} C_k^{(n+2)}\varphi_k(x,y)$ with $C_{n+2}^{(n+2)} = \left(A_{n+2}^{(n+2)}\right)^{-1}$ and then

$$\begin{aligned} \left(x + \tfrac{1}{x}\right)\varphi_n(x,y) &= A_n^{(n)}B_{n+2,1}^{(n)}\phi_{n+2}(x,y) + \text{lower terms} \\ &= A_n^{(n)}B_{n+2,1}^{(n)}\left(\textstyle\sum_{k=0}^{n+2} C_k^{(n+2)}\varphi_k(x,y)\right) + \text{lower terms} \\ &= A_n^{(n)}B_{n+2,1}^{(n)}C_{n+2}^{(n+2)}\varphi_{n+2}(x,y) + \text{lower terms.} \end{aligned}$$

If we define

$$D_{n+2,1}^{(n)} := A_n^{(n)}B_{n+2,1}^{(n)}C_{n+2}^{(n+2)} = A_n^{(n)}B_{n+2,1}^{(n)}\left(A_{n+2}^{(n+2)}\right)^{-1} \in \mathcal{M}_{n+1,n+3},$$

and it has (full) rank $n+1$.

In short, we have proved that

$$\left(x+\frac{1}{x}\right)\varphi_n(x,y)=\sum_{k=0}^{n+2}D_{k,1}^{(n)}\varphi_k(x,y)\quad\text{where}\quad D_{k,1}^{(n)}\in\mathcal{M}_{n+1,k+1}$$

and $D_{n+2,1}^{(n)}$ being of (full) rank $n+1$. Now, using the orthogonality conditions and the property $\left(x+\frac{1}{x}\right)\varphi_k(x,y)\in\mathcal{L}_{k+2}$, it follows that

$$D_{k,1}^{(n)}=\langle\left(x+\frac{1}{x}\right)\varphi_n(x,y),\varphi_k^T(x,y)\rangle=\langle\varphi_n(x,y),\left(x+\frac{1}{x}\right)\varphi_k^T(x,y)\rangle=0\quad\text{if}\quad k<n-2.$$

This implies the following five-term recurrence relation that holds for $n\geq 2$:

$$\begin{aligned}\left(x+\tfrac{1}{x}\right)\varphi_n(x,y)=&D_{n+2,1}^{(n)}\varphi_{n+2}(x,y)+D_{n+1,1}^{(n)}\varphi_{n+1}(x,y)+D_{n,1}^{(n)}\varphi_n(x,y)\\&+D_{n-1,1}^{(n)}\varphi_{n-1}(x,y)+D_{n-2,1}^{(n)}\varphi_{n-2}(x,y),\end{aligned}\tag{19}$$

with

$$D_{s,1}^{(n)}=\langle\left(x+\frac{1}{x}\right)\varphi_n(x,y),\varphi_s^T(x,y)\rangle\in\mathcal{M}_{n+1,s+1},\qquad s\in\{n-2,\dots,n+2\}\tag{20}$$

and leading coefficient matrix $D_{n+2,1}^{(n)}$ of (full) rank $n+1$. Moreover, for $s\in\{n-2,\dots n+2\}$

$$\begin{aligned}D_{s,1}^{(n)}=&\langle\left(x+\frac{1}{x}\right)\varphi_n(x,y),\varphi_s^T(x,y)\rangle=\langle\varphi_n(x,y),\left(x+\frac{1}{x}\right)\varphi_s^T(x,y)\rangle\\=&\langle\varphi_n,\left(D_{s+2,1}^{(s)}\varphi_{s+2}+D_{s+1,1}^{(s)}\varphi_{s+1}+D_{s,1}^{(s)}\varphi_s+D_{s-1,1}^{(s)}\varphi_{s-1}+D_{s-2,1}^{(s)}\varphi_{s-2}\right)^T\rangle=\left(D_{n,1}^{(s)}\right)^T\end{aligned}$$

so we have proved that

$$D_{s,1}^{(n)}=\left(D_{n,1}^{(s)}\right)^T,\quad s\in\{n-2,\dots n+2\}.\tag{21}$$

This implies, in particular, that $D_{n,1}^{(n)}$ is symmetric and that the tailed coefficient matrix $D_{n-2,1}^{(n)}\in\mathcal{M}_{n+1,n-1}$ in (19) is also of full rank, equal to $n-1$.

Concerning the initial conditions, we may observe that the recurrence (19) is also valid for $n=0,1$ by setting $\varphi_{-1}\equiv\varphi_{-2}\equiv 0$. Indeed, recall first that

$$\phi_0(x,y)\equiv 1,\quad\phi_1(x,y)=\begin{pmatrix}x\\y\end{pmatrix},\quad\phi_2(x,y)=\begin{pmatrix}1/x\\xy\\1/y\end{pmatrix},\quad\phi_3(x,y)=\begin{pmatrix}x^2\\y/x\\x/y\\y^2\end{pmatrix}.$$

For $n=0$, we can find matrices $D_{i,1}^{(0)}\in\mathcal{M}_{1,i+1}$, $i=0,1,2$ such that (19) holds. From (16) we can write

$$\left(x+\frac{1}{x}\right)A_0^{(0)}=\left[D_{2,1}^{(0)}A_2^{(2)}\right]\phi_2+\left[D_{1,1}^{(0)}A_1^{(1)}+D_{2,1}^{(0)}A_1^{(2)}\right]\phi_1+\left[D_{0,1}^{(0)}A_0^{(0)}+D_{1,1}^{(0)}A_0^{(1)}+D_{2,1}^{(0)}A_0^{(2)}\right]\phi_0$$

where $A_i^{(i)}$ are regular, for $i=0,1,2$. Hence, since

$$\left(x+\frac{1}{x}\right)A_0^{(0)}=B_{2,1}^{(0)}A_0^{(0)}\phi_2+B_{1,1}^{(0)}A_0^{(0)}\phi_1,$$

it follows that

$$\begin{aligned}D_{2,1}^{(0)}&=B_{2,1}^{(0)}A_0^{(0)}\left(A_2^{(2)}\right)^{-1}\\D_{1,1}^{(0)}&=\left(B_{1,1}^{(0)}A_0^{(0)}-D_{2,1}^{(0)}A_1^{(2)}\right)\left(A_1^{(1)}\right)^{-1}\\D_{0,1}^{(0)}&=-\left(D_{1,1}^{(0)}A_0^{(1)}+D_{2,1}^{(0)}A_0^{(2)}\right)\left(A_0^{(0)}\right)^{-1}.\end{aligned}\tag{22}$$

Similarly when $n=1$, we can find matrices $D_{i,1}^{(1)} \in \mathcal{M}_{2,i+1}$, $i=0,1,2,3$ such that (19) holds. By one hand, we can write from (16)

$$\begin{aligned}\left(x+\frac{1}{x}\right)\left[A_0^{(1)}\phi_0 + A_1^{(1)}\phi_1\right] = &\left[D_{3,1}^{(1)}A_3^{(3)}\right]\phi_3 + \left[D_{3,1}^{(1)}A_2^{(3)} + D_{2,1}^{(1)}A_2^{(2)}\right]\phi_2\\ &+\left[D_{3,1}^{(1)}A_1^{(3)} + D_{2,1}^{(1)}A_1^{(2)} + D_{1,1}^{(1)}A_1^{(1)}\right]\phi_1\\ &+\left[D_{3,1}^{(1)}A_0^{(3)} + D_{2,1}^{(1)}A_0^{(2)} + D_{1,1}^{(1)}A_0^{(1)} + D_{0,1}^{(1)}A_0^{(0)}\right]\phi_0\end{aligned}$$

with $A_i^{(i)}$ regular matrices for $i=0,\dots,3$. By other hand,

$$\left(x+\frac{1}{x}\right)\left[A_0^{(1)}\phi_0 + A_1^{(1)}\phi_1\right] = A_1^{(1)}B_{3,1}^{(1)}\phi_3 + \left[A_0^{(1)}B_{2,1}^{(0)} + A_1^{(1)}B_{2,1}^{(1)}\right]\phi_2 + A_0^{(1)}B_{1,1}^{(0)}\phi_1 + A_1^{(1)}B_{0,1}^{(1)}\phi_0.$$

Hence,

$$\begin{aligned}D_{3,1}^{(1)} &= A_1^{(1)}B_{3,1}^{(1)}\cdot\left(A_3^{(3)}\right)^{-1}\\ D_{2,1}^{(1)} &= \left(A_0^{(1)}B_{2,1}^{(0)} + A_1^{(1)}B_{2,1}^{(1)} - D_{3,1}^{(1)}A_2^{(3)}\right)\cdot\left(A_2^{(2)}\right)^{-1}\\ D_{1,1}^{(1)} &= \left(A_0^{(1)}B_{1,1}^{(0)} - D_{3,1}^{(1)}A_1^{(3)} - D_{2,1}^{(1)}A_1^{(2)}\right)\cdot\left(A_1^{(1)}\right)^{-1}\\ D_{0,1}^{(1)} &= \left(A_1^{(1)}B_{0,1}^{(1)} - D_{3,1}^{(1)}A_0^{(3)} - D_{2,1}^{(1)}A_0^{(2)} - D_{1,1}^{(1)}A_0^{(1)}\right)\cdot\left(A_0^{(0)}\right)^{-1}.\end{aligned}\tag{23}$$

As a consequence, we can give from (21) a matrix representation with respect to $\{\varphi_k\}_{k=0}^{\infty}$ of the *multiplication plus inverse multiplication* operator:

$$\left(x+\frac{1}{x}\right)\cdot\left(\varphi_0\ \varphi_1\ \varphi_2\ \varphi_3\ \varphi_4\ \varphi_5\ \cdots\right)^T = \mathcal{F}_1\cdot\left(\varphi_0\ \varphi_1\ \varphi_2\ \varphi_3\ \varphi_4\ \varphi_5\ \cdots\right)^T\tag{24}$$

where $\mathcal{F}_1$ is the block symmetric matrix given by

$$\mathcal{F}_1 = \begin{pmatrix} D_{0,1}^{(0)} & D_{1,1}^{(0)} & D_{2,1}^{(0)} & \mathcal{O}_{1,4} & \mathcal{O}_{1,5} & \mathcal{O}_{1,6} & \mathcal{O}_{1,7} & \mathcal{O}_{1,8} & \cdots\\ \left(D_{1,1}^{(0)}\right)^T & D_{1,1}^{(1)} & D_{2,1}^{(1)} & D_{3,1}^{(1)} & \mathcal{O}_{2,5} & \mathcal{O}_{2,6} & \mathcal{O}_{2,7} & \mathcal{O}_{2,8} & \cdots\\ \left(D_{2,1}^{(0)}\right)^T & \left(D_{2,1}^{(1)}\right)^T & D_{2,1}^{(2)} & D_{3,1}^{(2)} & D_{4,1}^{(2)} & \mathcal{O}_{3,6} & \mathcal{O}_{3,7} & \mathcal{O}_{3,8} & \cdots\\ \mathcal{O}_{4,1} & \left(D_{3,1}^{(1)}\right)^T & \left(D_{3,1}^{(2)}\right)^T & D_{3,1}^{(3)} & D_{4,1}^{(3)} & D_{5,1}^{(3)} & \mathcal{O}_{4,7} & \mathcal{O}_{4,8} & \cdots\\ \mathcal{O}_{5,1} & \mathcal{O}_{5,2} & \left(D_{4,1}^{(2)}\right)^T & \left(D_{4,1}^{(3)}\right)^T & D_{4,1}^{(4)} & D_{5,1}^{(4)} & D_{6,1}^{(4)} & \mathcal{O}_{5,8} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.\tag{25}$$

A very similar analysis can be done from (8)-(9) when considering multiplication by $y+\frac{1}{y}$, we omit most of the details. It is important to point out that the corresponding $B_{s,2}^{(n)}$ matrices in the relation

$$\left(y+\frac{1}{y}\right)\phi_n = B_{n+2,2}^{(n)}\phi_{n+2} + B_{n+1,2}^{(n)}\phi_{n+1} + B_{n-1,2}^{(n)}\phi_{n-1} + B_{n-2,2}^{(n)}\phi_{n-2}$$

are given in this case (compare with (17)) for all $n\geq 2$ by

$$B_{n-2,2}^{(n)} = \left[\begin{array}{c}\mathcal{O}_{2,n-1}\\ \hline \mathcal{I}_{n-1}\end{array}\right]\in\mathcal{M}_{n+1,n-1},\qquad B_{n-1,2}^{(n)} = \left[\tilde{z}_{n+1}^T|\mathcal{O}_{n+1,n-1}\right]\in\mathcal{M}_{n+1,n},$$

$$B_{n+1,2}^{(n)} = \left[\begin{array}{c}\tilde{z}_{n+2}\\ \hline \mathcal{O}_{n,n+2}\end{array}\right]\in\mathcal{M}_{n+1,n+2},\qquad B_{n+2,2}^{(n)} = \left[\mathcal{O}_{n+1,2}|\mathcal{I}_{n+1}\right]\in\mathcal{M}_{n+1,n+3},$$

where $\tilde{z}_s = (0\ 1\ 0\ \cdots\ 0)\in\mathcal{M}_{1,s}$ for all $s\geq 3$ and $\tilde{z}_2 = (0\ 1)$. These formulas are again valid for $n=0$, $s\in\{1,2\}$ and $n=1$, $s\in\{0,2,3\}$. Thus,

$$D_{n+2,2}^{(n)} := \left[\mathcal{O}_{n+1,2}|A_n^{(n)}\right]\cdot C_{n+2}^{(n+2)} = A_n^{(n)}\cdot B_{n+2,2}^{(n)}\cdot\left(A_{n+2}^{(n+2)}\right)^{-1}\in\mathcal{M}_{n+1,n+3},$$

also has (full) rank $n+1$. The analog of (18) is

$$\left(y+\frac{1}{y}\right)\varphi_n(x,y) = A_n^{(n)}\left[B_{n+2,2}^{(n)}\phi_{n+2} + B_{n+1,2}^{(n)}\phi_{n+1} + B_{n-1,2}^{(n)}\phi_{n-1} + B_{n-2,2}^{(n)}\phi_{n-2}\right] + \text{lower terms.} \tag{26}$$

The corresponding five-term recurrence is now, for $n \geq 2$:

$$\begin{aligned}\left(y+\tfrac{1}{y}\right)\varphi_n(x,y) &= D_{n+2,2}^{(n)}\varphi_{n+2}(x,y) + D_{n+1,2}^{(n)}\varphi_{n+1}(x,y) + D_{n,2}^{(n)}\varphi_n(x,y)\\ &\quad + D_{n-1,2}^{(n)}\varphi_{n-1}(x,y) + D_{n-2,2}^{(n)}\varphi_{n-2}(x,y),\end{aligned} \tag{27}$$

with

$$D_{s,2}^{(n)} = \langle\left(y+\frac{1}{y}\right)\varphi_n(x,y), \varphi_s^T(x,y)\rangle \in \mathcal{M}_{n+1,s+1}, \qquad s \in \{n-2,\dots,n+2\}. \tag{28}$$

Equation (27) is again valid for $n = 0, 1$. The analog of (22) is

$$\begin{aligned} D_{2,2}^{(0)} &= B_{2,2}^{(0)} A_0^{(0)} \left(A_2^{(2)}\right)^{-1}\\ D_{1,2}^{(0)} &= \left(B_{1,2}^{(0)} A_0^{(0)} - D_{2,2}^{(0)} A_1^{(2)}\right)\left(A_1^{(1)}\right)^{-1}\\ D_{0,2}^{(0)} &= -\left(D_{1,2}^{(0)} A_0^{(1)} + D_{2,2}^{(0)} A_0^{(2)}\right)\left(A_0^{(0)}\right)^{-1}. \end{aligned} \tag{29}$$

As in (21) it holds

$$D_{s,2}^{(n)} = \left(D_{n,2}^{(s)}\right)^T \quad \text{for} \quad s \in \{n-2,\dots,n+2\} \tag{30}$$

and hence, $D_{n,2}^{(n)}$ is symmetric and the tailed coefficient matrix $D_{n-2,2}^{(n)} \in \mathcal{M}_{n+1,n-1}$ in (27) is also of full rank, equal to $n-1$.

The matrix representation with respect to $\{\varphi_k\}_{k=0}^{\infty}$ of the *multiplication plus inverse multiplication* operator in the variable y is

$$\left(y+\frac{1}{y}\right)\cdot\begin{pmatrix}\varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \cdots\end{pmatrix}^T = \mathcal{F}_2 \cdot \begin{pmatrix}\varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \cdots\end{pmatrix}^T \tag{31}$$

where $\mathcal{F}_2$ is a block symmetric matrix like $\mathcal{F}_1$ in (25) but replacing the second subindexes 1 in the $D_{s,1}^{(n)}$ matrices by 2.

The results of this section can be summarized in the following

Theorem 2.6. *Let $\{\varphi_k\}_{k\geq 0}$ be a family of orthonormal Laurent polynomials with respect to the measure μ. Then, for all $n \geq 0$ there exist constant matrices $D_{s,i}^{(n)} \in \mathcal{M}_{n+1,s+1}$, $s \in \{n-2,\dots,n+2\}$, $i \in \{1,2\}$ given by (20) and (28) when $n \geq 2$, $s \in \{0,1,2\}$ if $n = 0$ and $s \in \{0,1,2,3\}$ if $n = 1$, such that the following five-term relations hold:*

$$\begin{aligned}\left(x+\frac{1}{x}\right)\varphi_n(x,y) &= D_{n+2,1}^{(n)}\varphi_{n+2}(x,y) + D_{n+1,1}^{(n)}\varphi_{n+1}(x,y)\\ &\quad + D_{n,1}^{(n)}\varphi_n(x,y) + D_{n-1,1}^{(n)}\varphi_{n-1}(x,y) + D_{n-2,1}^{(n)}\varphi_{n-2}(x,y),\\ \left(y+\frac{1}{y}\right)\varphi_n(x,y) &= D_{n+2,2}^{(n)}\varphi_{n+2}(x,y) + D_{n+1,2}^{(n)}\varphi_{n+1}(x,y)\\ &\quad + D_{n,2}^{(n)}\varphi_n(x,y) + D_{n-1,2}^{(n)}\varphi_{n-1}(x,y) + D_{n-2,2}^{(n)}\varphi_{n-2}(x,y),\end{aligned} \tag{32}$$

with $\varphi_{-1} \equiv \varphi_{-2} \equiv 0$. Moreover, for $i \in \{1,2\}$, $D_{s,i}^{(n)} = \left(D_{n,i}^{(s)}\right)^T$, for all $n \geq 2$, $s \in \{n-2,n-1,n\}$, $D_{0,i}^{(1)} = \left(D_{1,i}^{(0)}\right)^T$ and the matrices $D_{n+2,i}^{(n)}$ and $D_{n-2,i}^{(n)}$ are full rank.

We conclude this section with the following elementary result that will be used in the next section.

Lemma 2.7. *For all $n \geq 1$, the matrices*

$$D_{n+2}^{(n)} := \begin{bmatrix} D_{n+2,1}^{(n)} \\ \hline D_{n+2,2}^{(n)} \end{bmatrix} \in \mathcal{M}_{2(n+1),n+3} \qquad \textit{and} \qquad B_{n+2}^{(n)} := \begin{bmatrix} B_{n+2,1}^{(n)} \\ \hline B_{n+2,2}^{(n)} \end{bmatrix} \in \mathcal{M}_{2(n+1),n+3}$$

have full rank, equal to $n+3$. This full rank properties are also valid for $n=0$, but in this case they are equal to 2, instead of 3.

Proof. The result trivially follows for the matrix $B_{n+2}^{(n)}$ since we have already seen that

$$B_{n+2,1}^{(n)} = [\mathcal{I}_{n+1}|\mathcal{O}_{n+1,2}] \in \mathcal{M}_{n+1,n+3}, \qquad B_{n+2,2}^{(n)} = [\mathcal{O}_{n+1,2}|\mathcal{I}_{n+1}] \in \mathcal{M}_{n+1,n+3}.$$

We can write

$$D_{n+2}^{(n)} = \left(\begin{array}{c|c} A_n^{(n)} & \mathcal{O}_{n+1,2} \\ \mathcal{O}_{n+1,2} & A_n^{(n)} \end{array} \right) \cdot \left(A_{n+2}^{(n+2)}\right)^{-1},$$

so the result follows directly by using Sylvester inequality (see e.g. [16, p. 13]). Finally, the result for $n=0$ holds since

$$D_2^{(0)} = B_2^{(0)} \cdot \left(A_2^{(2)}\right)^{-1}.$$

□

3. Favard's theorem and Christoffel-Darboux formula

From the results of Section 2 we have all the necessary technical modifications to adapt the proofs of Favard's theorem and Christoffel-Darboux formula presented in [14, Section 3.3] for the ordinary polynomials in several variables to the Laurent case.

Our first observation is that since for all $n \geq 1$, the matrix $D_{n+2}^{(n)} \in \mathcal{M}_{2(n+1),n+3}$ defined in Lemma 2.7 is of full rank $n+3$, it has a generalized inverse

$$\left(\overline{D}_{n+2}^{(n)}\right)^T = \left(\left(\overline{D}_{n+2,1}^{(n)}\right)^T | \left(\overline{D}_{n+2,2}^{(n)}\right)^T\right) \in \mathcal{M}_{n+3,2(n+1)}, \qquad \left(\overline{D}_{n+2,i}^{(n)}\right)^T \in \mathcal{M}_{n+3,n+1}, \quad i=1,2,$$

that is not unique. This means

$$\left(\overline{D}_{n+2}^{(n)}\right)^T \cdot D_{n+2}^{(n)} = \left(\overline{D}_{n+2,1}^{(n)}\right)^T \cdot D_{n+2,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T \cdot D_{n+2,2}^{(n)} = \mathcal{I}_{n+3}.$$

We need also the following auxiliary result.

Proposition 3.1. *Let $\left(\overline{D}_{n+2}^{(n)}\right)^T$ be a generalized inverse of $\overline{D}_{n+2}^{(n)}$. Then, there exists constant matrices $E_n^i \in \mathcal{M}_{n+3,n+3-i}$, $i=1,2,3,4$ such that*

$$\varphi_{n+2} = \left[\left(x+\frac{1}{x}\right)\left(\overline{D}_{n+2,1}^{(n)}\right)^T + \left(y+\frac{1}{y}\right)\left(\overline{D}_{n+2,2}^{(n)}\right)^T\right]\varphi_n + E_n^1\varphi_{n+1} + E_n^2\varphi_n + E_n^3\varphi_{n-1} + E_n^4\varphi_{n-2}.$$

Proof. If we add (19) multiplied by the left by $\left(\overline{D}_{n+2,1}^{(n)}\right)^T$ and (27) multiplied by the left by $\left(\overline{D}_{n+2,2}^{(n)}\right)^T$, we get

$$\begin{aligned}
\left[\left(x+\tfrac{1}{x}\right)\left(\overline{D}_{n+2,1}^{(n)}\right)^T + \left(y+\tfrac{1}{y}\right)\left(\overline{D}_{n+2,2}^{(n)}\right)^T\right]\varphi_n &= \left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n+2,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n+2,2}^{(n)}\right]\varphi_{n+2} \\
&+ \left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n+1,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n+1,2}^{(n)}\right]\varphi_{n+1} + \left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n,2}^{(n)}\right]\varphi_n \\
&+ \left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n-1,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n-1,2}^{(n)}\right]\varphi_{n-1} + \left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n-2,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n-2,2}^{(n)}\right]\varphi_{n-2}.
\end{aligned}$$

So, the result follows by considering

$$E_n^i = -\left[\left(\overline{D}_{n+2,1}^{(n)}\right)^T D_{n+2-i,1}^{(n)} + \left(\overline{D}_{n+2,2}^{(n)}\right)^T D_{n+2-i,2}^{(n)}\right] \in \mathcal{M}_{n+3,n+3-i}, \qquad i=1,2,3,4.$$

□

Now we are in position to prove a Favard-type theorem by following the ideas presented in [14, Section 3.3]. We concentrate in the positive-definite case.

Theorem 3.2 (Favard)**.** *Let $\mathcal{L}$ and $\mathcal{L}_n$ be given by (2)-(4), $\{\varphi_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $\mathcal{L}$ written in the form (16) with ϕ_k defined in (5) for all $k \geq 0$, $\varphi_n \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$, for all $n \geq 1$ where $\varphi_0 \equiv 1$ and set $\varphi_{-2} \equiv \varphi_{-1} \equiv 0$.*

Suppose that for all $n \geq 0$, there exist matrices $D_{k,i}^{(n)} \in \mathcal{M}_{n+1,k+1}$, $i = 1,2$, $k \in \{n-2,\dots,n+2\}$ when $n \geq 2$, $k \in \{0,1,2\}$ when $n = 0$ and $k \in \{0,1,2,3\}$ when $n = 1$, such that

1. *the L-polynomials φ_n satisfy the recurrences* (32) *with $D_{s,i}^{(n)} = \left(D_{n,i}^{(s)}\right)^T$, for all $n \geq 2$, $s \in \{n-2,n-1,n\}$, $D_{0,i}^{(1)} = \left(D_{1,i}^{(0)}\right)^T$ and $i \in \{1,2\}$.*
2. *the matrices in the relation satisfy the rank conditions:*
$$\begin{aligned}
&\operatorname{rank} D_{n+2,i}^{(n)} = \operatorname{rank} D_{n,i}^{(n+2)} = n+1, \qquad n \geq 0, \quad i = 1,2,\\
&\operatorname{rank} D_{n+2}^{(n)} = n+3, \qquad n \geq 1\\
&\operatorname{rank} D_1^{(0)} = 2,
\end{aligned}$$
with $D_{n+2}^{(n)} \in \mathcal{M}_{2(n+1),n+3}$ introduced in Lemma 2.7 and $D_1^{(0)} \in \mathcal{M}_2$ given by
$$D_1^{(0)} := \left[\begin{array}{c} D_{1,1}^{(0)} \\ \hline D_{1,2}^{(0)} \end{array}\right] \in \mathcal{M}_2\,. \tag{33}$$

Then, there exist a linear functional L which defines a positive-definite functional on $\mathcal{L}$ and which makes $\{\varphi_n\}_{n=0}^{\infty}$ an orthonormal basis in $\mathcal{L}$.

Proof. We first prove that $\{\varphi_n\}_{n=0}^{\infty}$ forms a basis of $\mathcal{L}$. Using the expression (16), it suffices to prove that the leading coefficient $A_n^{(n)}$ is regular, for all $n \geq 0$. For $n \geq 2$ we see that comparing the coefficient matrices of ϕ_{n+2} in (18), (26) and (19), (27) we get
$$\operatorname{diag}\left(A_n^{(n)}\ ,\ A_n^{(n)}\right) \cdot B_{n+2}^{(n)} = D_{n+2}^{(n)} \cdot A_{n+2}^{(n+2)},$$
where $B_{n+2}^{(n)}, D_{n+2}^{(n)}$ are the matrices of rank $n+3$ that have been introduced in Lemma 2.7. To prove that $\operatorname{rank} A_n^{(n)} = n+1$ we proceed by induction by showing that from the two initial conditions $n = 0,1$ (that holds by hypothesis) we get that if $\operatorname{rank} A_n^{(n)} = n+1$, then $\operatorname{rank} A_{n+2}^{(n+2)} = n+3$. Indeed, if $A_n^{(n)}$ is invertible, then $\operatorname{diag}\left(A_n^{(n)}\ ,\ A_n^{(n)}\right)$ is also invertible,
$$\operatorname{rank}\left(\operatorname{diag}\left(A_n^{(n)}\ ,\ A_n^{(n)}\right) \cdot B_{n+2}^{(n)}\right) = \operatorname{rank} B_{n+2}^{(n)} = n+3,$$
and hence, $\operatorname{rank}\left(D_{n+2}^{(n)} \cdot A_{n+2}^{(n+2)}\right) = n+3$. By using Sylvester inequality we get
$$\operatorname{rank} A_{n+2}^{(n+2)} \geq \operatorname{rank}\left(D_{n+2}^{(n)} \cdot A_{n+2}^{(n+2)}\right) \geq \operatorname{rank} D_{n+2}^{(n)} + \operatorname{rank} A_{n+2}^{(n+2)} - (n+3) = \operatorname{rank} A_{n+2}^{(n+2)}.$$
So, we conclude $\operatorname{rank} A_{n+2}^{(n+2)} = \operatorname{rank}\left(D_{n+2}^{(n)} \cdot A_{n+2}^{(n+2)}\right) = n+3$ and the induction is complete.

Since $\{\varphi_n\}_{n=0}^{\infty}$ is a basis of $\mathcal{L}$, the linear functional L defined on $\mathcal{L}$ by $L(1) = 1$ and $L(\varphi_n) = 0$, for all $n \geq 1$ is well defined. We now use induction to prove that
$$L(\varphi_k \varphi_j^T) = 0, \qquad \text{for all } \ k \neq j. \tag{34}$$
For $n \geq 0$ assume that (34) hold $\forall k,j$ such that $0 \leq k \leq n$ and $j > k$. The induction process is directly obtained from Proposition 3.1 since we have for all $l > n+1$ that
$$L(\varphi_{n+1}\varphi_l^T) = L\left[\left(\overline{D}_{n+1,1}^{(n-1)}\right)^T \varphi_{n-1}\left(x + \frac{1}{x}\right)\varphi_l^T\right] + L\left[\left(\overline{D}_{n+1,2}^{(n-1)}\right)^T \varphi_{n-1}\left(y + \frac{1}{y}\right)\varphi_l^T\right] = 0.$$
Let us see finally that $\mathcal{H}_n := L(\varphi_n\varphi_n^T) = \mathcal{I}_{n+1}$. Notice from (19) that
$$\begin{aligned}
D_{n+2,1}^{(n)}\mathcal{H}_{n+2} &= L\left[\left(x + \tfrac{1}{x}\right)\varphi_n\varphi_{n+2}^T\right] = L\left[\varphi_n\left(\left(x + \tfrac{1}{x}\right)\varphi_{n+2}\right)^T\right] = \mathcal{H}_n\left(D_{n,1}^{(n+2)}\right)^T\\
&= \mathcal{H}_n D_{n+2,1}^{(n)}, \quad \forall n \geq 0.
\end{aligned}$$

We get a similar result from (27) when multiplying by $\left(y+\frac{1}{y}\right)$, and both relations can be written together as

$$D_{n+2}^{(n)}\cdot\mathcal{H}_{n+2}=\operatorname{diag}\left(\mathcal{H}_n\ ,\ \mathcal{H}_n\right)\cdot D_{n+2}^{(n)}. \tag{35}$$

We proceed again by induction over n. It is clear from construction that it holds for $n=0$ and the proof is concluded if we prove it for $n=1$. Indeed, in such case if we suppose that the property holds for all $0\le k\le n+1$, it follows from (35) that $D_{n+2}^{(n)}\cdot\mathcal{H}_{n+2}=D_{n+2}^{(n)}$. Since $D_{n+2}^{(n)}$ is of full rank it has a generalized inverse, so $\mathcal{H}_{n+2}=\mathcal{I}_{n+2}$.

Taking $n=0,1$ in (19) we see that

$$L\left[\left(x+\frac{1}{x}\right)\varphi_0\varphi_1^T\right]=D_{2,1}^{(0)}L[\varphi_2\varphi_1^T]+D_{1,1}^{(0)}L[\varphi_1\varphi_1^T]+D_{0,1}^{(0)}L[\varphi_0\varphi_1^T]=D_{1,1}^{(0)}L[\varphi_1\varphi_1^T]$$

and

$$\begin{aligned}L\left[\left(x+\frac{1}{x}\right)\varphi_0\varphi_1^T\right] &= L[\varphi_3]\left(D_{3,1}^{(1)}\right)^T+L[\varphi_2]\left(D_{2,1}^{(1)}\right)^T+L[\varphi_1]\left(D_{1,1}^{(1)}\right)^T+L[\varphi_0]\left(D_{0,1}^{(1)}\right)^T\\ &= \left(D_{0,1}^{(1)}\right)^T=D_{1,1}^{(0)}.\end{aligned}$$

The same argument can be used to prove $L\left[\left(y+\frac{1}{y}\right)\varphi_0\varphi_1^T\right]=D_{1,2}^{(0)}L[\varphi_1\varphi_1^T]=D_{1,2}^{(0)}$, so we get $D_1^{(0)}L\left[\varphi_1\varphi_1^T\right]=D_1^{(0)}$ with $D_1^{(0)}$ introduced in (33). Thus, the proof follows since $D_1^{(0)}$ is regular. □

Let us introduce now the *reproducing kernel*

$$\mathcal{K}_n(x_1,y_1,x_2,y_2)=\sum_{k=0}^{n}\varphi_k^T(x_1,y_1)\varphi_k(x_2,y_2).$$

This definition is clearly independent on the election of the orthonormal family $\{\varphi_n\}_{n\ge0}$ (recall it is uniquely determined up to left multiplication by orthogonal matrices). The name reproducing kernel is justified as in the ordinary polynomial situation because it is easy to verify the reproducing property $\psi(x,y)=\langle\psi(u,v),\mathcal{K}_n^T(x,y,u,v)\rangle$, $\forall\psi\in\mathcal{L}_n$. The extension of the well known Christoffel-Darboux formula for the ordinary polynomial situation (see [14, Section 3.6.1]) is given by the following

Theorem 3.3 (Christoffel-Darboux)**.** *Under the above conditions it holds*

$$\mathcal{K}_n(x_1,y_1,x_2,y_2)=\frac{\Omega_{n,1}+\Lambda_{n,1}+\Lambda_{n-1,1}}{\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)}=\frac{\Omega_{n,2}+\Lambda_{n,2}+\Lambda_{n-1,2}}{\left(y_1+\frac{1}{y_1}\right)-\left(y_2+\frac{1}{y_2}\right)},$$

whenever $x_1+\frac{1}{x_1}\neq x_2+\frac{1}{x_2}$ *in the first equality,* $y_1+\frac{1}{y_1}\neq y_2+\frac{1}{y_2}$ *in the second one, and where for* $i=1,2$ *and* $k\ge0$*,*

$$\begin{aligned}\Lambda_{k,i} &= \varphi_{k+2}^T(x_1,y_1)\left(D_{k+2,i}^{(k)}\right)^T\varphi_k(x_2,y_2)-\varphi_k^T(x_1,y_1)D_{k+2,i}^{(k)}\varphi_{k+2}(x_2,y_2),\\ \Omega_{k,i} &= \varphi_{k+1}^T(x_1,y_1)\left(D_{k+1,i}^{(k)}\right)^T\varphi_k(x_2,y_2)-\varphi_k^T(x_1,y_1)D_{k+1,i}^{(k)}\varphi_{k+1}(x_2,y_2).\end{aligned} \tag{36}$$

Proof. From (19) and (21) we can write for all $k\ge2$ and $x_1+\frac{1}{x_1}\neq x_2+\frac{1}{x_2}$,

$$\left[\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)\right]\varphi_k^T(x_1,y_1)\varphi_k(x_2,y_2)=\Lambda_{k,1}+\Omega_{k,1}-\Lambda_{k-2,1}-\Omega_{k-1,1}.$$

Taking $n=0,1$ in (19) it follows that the relation also holds for $k=0$ and $k=1$ respectively (recall $\varphi_{-1}\equiv\varphi_{-2}\equiv0$), if we define $\Lambda_{-2,1}=\Lambda_{-1,1}=\Omega_{-1,1}=0$. So,

$$\sum_{k=0}^{n}\left[\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)\right]\varphi_k^T(x_1,y_1)\varphi_k(x_2,y_2)=\Omega_{n,1}+\Lambda_{n,1}+\Lambda_{n-1,1}$$

and the first equality of the statement is deduced. The second equality follows in a similar way from (27) and (30). □

Corollary 3.4 (Confluent formula)**.** *Under the above conditions it holds*

$$\mathcal{K}_n(x,y,x,y) = \frac{x^2}{x^2-1}\left[\tilde{\Omega}_{n,1} + \tilde{\Lambda}_{n,1} + \tilde{\Lambda}_{n-1,1}\right] = \frac{y^2}{y^2-1}\left[\hat{\Omega}_{n,2} + \hat{\Lambda}_{n,2} + \hat{\Lambda}_{n-1,2}\right]$$

whenever $x^2 \neq 1$ in the first equality, $y^2 \neq 1$ in the second one, and

$$\begin{aligned}
\tilde{\Lambda}_{k,1} &= \varphi_{k+2}^T(x,y)\left(D_{k+2,1}^{(k)}\right)^T \tfrac{\partial}{\partial x}\varphi_k(x,y) - \varphi_k^T(x,y)D_{k+2,1}^{(k)}\tfrac{\partial}{\partial x}\varphi_{k+2}(x,y),\\
\tilde{\Lambda}_{k,2} &= \varphi_{k+2}^T(x,y)\left(D_{k+2,2}^{(k)}\right)^T \tfrac{\partial}{\partial y}\varphi_k(x,y) - \varphi_k^T(x,y)D_{k+2,2}^{(k)}\tfrac{\partial}{\partial y}\varphi_{k+2}(x,y),\\
\tilde{\Omega}_{k,1} &= \varphi_{k+1}^T(x,y)\left(D_{k+1,1}^{(k)}\right)^T \tfrac{\partial}{\partial x}\varphi_k(x,y) - \varphi_k^T(x,y)D_{k+1,1}^{(k)}\tfrac{\partial}{\partial x}\varphi_{k+1}(x,y),\\
\tilde{\Omega}_{k,2} &= \varphi_{k+1}^T(x,y)\left(D_{k+1,2}^{(k)}\right)^T \tfrac{\partial}{\partial y}\varphi_k(x,y) - \varphi_k^T(x,y)D_{k+1,2}^{(k)}\tfrac{\partial}{\partial y}\varphi_{k+1}(x,y).
\end{aligned}$$

Proof. Since $\varphi_s^T(x_1,y_1)\left(D_{s,i}^{(k)}\right)^T\varphi_k(x_1,y_1)$ is a scalar function for $s \in \{k+1,k+2\}$ and $i \in \{1,2\}$, we can write (compare with (36))

$$\begin{aligned}
\Lambda_{k,i} &= \varphi_{k+2}^T(x_1,y_1)\left(D_{k+2,i}^{(k)}\right)^T\left[\varphi_k(x_2,y_2) - \varphi_k(x_1,y_1)\right]\\
&\quad -\varphi_k^T(x_1,y_1)D_{k+2,i}^{(k)}\left[\varphi_{k+2}(x_2,y_2) - \varphi_{k+2}(x_1,y_1)\right],\\[1ex]
\Omega_{k,i} &= \varphi_{k+1}^T(x_1,y_1)\left(D_{k+1,i}^{(k)}\right)^T\left[\varphi_k(x_2,y_2) - \varphi_k(x_1,y_1)\right]\\
&\quad -\varphi_k^T(x_1,y_1)D_{k+1,i}^{(k)}\left[\varphi_{k+1}(x_2,y_2) - \varphi_{k+1}(x_1,y_1)\right].
\end{aligned}$$

Also, $\left(x_1 + \frac{1}{x_1}\right) - \left(x_2 + \frac{1}{x_2}\right) = (x_1 - x_2)\,\frac{x_1x_2-1}{x_1x_2}$, so if $x_1 + \frac{1}{x_1} \neq x_2 + \frac{1}{x_2}$,

$$\mathcal{K}_n(x_1,y_1,x_2,y_2) = \frac{\Omega_{n,1}+\Lambda_{n,1}+\Lambda_{n-1,1}}{\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)} = \frac{x_1x_2}{x_1x_2-1}\times\Bigg[$$

$$\varphi_{n+1}^T(x_1,y_1)\left(D_{n+1,1}^{(n)}\right)^T\cdot\tfrac{\varphi_n(x_2,y_2)-\varphi_n(x_1,y_1)}{x_1-x_2} - \varphi_n^T(x_1,y_1)D_{n+1,1}^{(n)}\cdot\tfrac{\varphi_{n+1}(x_2,y_2)-\varphi_{n+1}(x_1,y_1)}{x_1-x_2} +$$

$$\varphi_{n+2}^T(x_1,y_1)\left(D_{n+2,1}^{(n)}\right)^T\cdot\tfrac{\varphi_n(x_2,y_2)-\varphi_n(x_1,y_1)}{x_1-x_2} - \varphi_n^T(x_1,y_1)D_{n+2,1}^{(n)}\cdot\tfrac{\varphi_{n+2}(x_2,y_2)-\varphi_{n+2}(x_1,y_1)}{x_1-x_2} +$$

$$\varphi_{n+1}^T(x_1,y_1)\left(D_{n+1,1}^{(n-1)}\right)^T\cdot\tfrac{\varphi_{n-1}(x_2,y_2)-\varphi_{n-1}(x_1,y_1)}{x_1-x_2} - \varphi_{n-1}^T(x_1,y_1)D_{n+1,1}^{(n-1)}\cdot\tfrac{\varphi_{n+1}(x_2,y_2)-\varphi_{n+1}(x_1,y_1)}{x_1-x_2}\Bigg].$$

The first equality follows letting $(x_2,y_2) \to (x_1,y_1) = (x,y)$ and the second one is obtained in a similar way. □

4. A connection with the one variable case

Consider the rectangle $\mathcal{R} = [a,b]\times[c,d]$, $0 < a < b < \infty$, $0 < c < d < \infty$, and a positive Borel measure on $\mathcal{R}$ that can be factorized in the form $d\mu(x,y) = d\mu_1(x)d\mu_2(y)$. Let $\langle\cdot,\cdot\rangle_\mu$ be the inner product given by (13) and $\langle\cdot,\cdot\rangle_{\mu_i}$ the corresponding inner products for the measures $d\mu_i$, $i = 1,2$:

$$\begin{aligned}
\langle f,g\rangle_{\mu_1} &= \textstyle\int_a^b f(x)g(x)d\mu_1(x), &\quad f,g \in L_2^{\mu_1} &= \left\{h:[a,b]\to\mathbb{R}\;:\;\textstyle\int_a^b h^2(x)d\mu_1(x) < \infty\right\},\\
\langle f,g\rangle_{\mu_2} &= \textstyle\int_c^d f(y)g(y)d\mu_2(y), &\quad f,g \in L_2^{\mu_2} &= \left\{h:[c,d]\to\mathbb{R}\;:\;\textstyle\int_c^d h^2(y)d\mu_2(y) < \infty\right\}.
\end{aligned}$$

Notice that the corresponding moments $m_k^{(i)}$ are strictly positive, for all $k \in \mathbb{Z}$ and $i \in \{1,2\}$. Let us denote by $\left\{\psi_n^{(i)}\right\}_{n\geq 0}$ for $i = 1,2$ the families of orthogonal Laurent polynomials in one variable with respect to me measures μ_i and the "balanced" ordering (1). Thus we can prove how from these two families we can construct an orthonormal basis of Laurent polynomials in two variables.

Proposition 4.1. *Under the above conditions, let $\varphi_{n,k}$ be given by*

$$\varphi_{n,k}(x,y) = \psi_{n-k}^{(1)}(x)\psi_k^{(2)}(y), \quad \textit{for all} \quad n \geq 0 \quad \textit{and} \quad k \in \{0,1,\ldots,n\}.$$

Then the set $\{\varphi_{n,k} : n \geq 0,\ k = 0,\ldots,n\}$ is an orthonormal basis of $\mathcal{L}$.

Proof. Recall $\mathcal{L}_k = \text{span}\{\phi_0,\dots,\phi_k\}$. It is clear from the construction of the ordering (2) and (3)-(5) that $\psi^{(1)}_{n-k}(x)\in\mathcal{L}_{n-k}\backslash\mathcal{L}_{n-k-1}$, $\psi^{(2)}_k(y)\in\mathcal{L}_k\backslash\mathcal{L}_{k-1}$ and $\psi^{(1)}_{n-k}(x)\psi^{(2)}_k(y)\in\mathcal{L}_n\backslash\mathcal{L}_{n-1}$. Thus, for fixed $n\geq 0$ and $k,l\in\{0,\dots,n\}$, it is clear from Fubini's theorem that

$$\langle\psi^{(1)}_{n-k}(x)\psi^{(2)}_k(y),\psi^{(1)}_{n-l}(x)\psi^{(2)}_l(y)\rangle_\mu = \langle\psi^{(1)}_{n-k}(x),\psi^{(1)}_{n-l}(x)\rangle_{\mu_1}\cdot\langle\psi^{(2)}_k(y),\psi^{(2)}_l(y)\rangle_{\mu_2}=\delta_{k,l}.$$

Also for $n\neq m$, $n,m\geq 0$, $k\in\{0\dots,n\}$ and $l\in\{0\dots,m\}$, we get by the same reason

$$\langle\psi^{(1)}_{n-k}(x)\psi^{(2)}_k(y),\psi^{(1)}_{m-l}(x)\psi^{(2)}_l(y)\rangle_\mu = \langle\psi^{(1)}_{n-k}(x),\psi^{(1)}_{m-l}(x)\rangle_{\mu_1}\cdot\langle\psi^{(2)}_k(y),\psi^{(2)}_l(y)\rangle_{\mu_2}=0.$$

This concludes the proof. □

The aim of this section is to make use of Theorem 1.1 and Proposition 4.1 to obtain explicitly the relations (32) in this particular situation. We start with the following

Lemma 4.2. *Let $\{\Omega^{(2)}_n\}_{n\geq0}$ and $\{C^{(2)}_n\}_{n\geq0}$ be the sequences of positive real numbers appearing in Theorem 1.1, associated with the measure $d\mu_2$. Then, under the above conditions the family $\{\varphi_n\}_{n\geq0}$ satisfies the recurrence realations*

$$\begin{aligned}
C^{(2)}_{2m}\varphi_{n,2m+1}(x,y) &= \left(\Omega^{(2)}_{2m}y-1\right)\varphi_{n-1,2m}(x,y)-C^{(2)}_{2m-1}\varphi_{n-2,2m-1}(x,y),\\
&\quad \textit{for } 0\leq 2m-1\leq n-2,\\
C^{(2)}_{2m+1}\varphi_{n+1,2m+2}(x,y) &= \left(1-\frac{\Omega^{(2)}_{2m+1}}{y}\right)\varphi_{n,2m+1}(x,y)-C^{(2)}_{2m}\varphi_{n-1,2m}(x,y),\\
&\quad \textit{for } 0\leq 2m\leq n-1,\\
C^{(2)}_0\varphi_{1,1}(x,y) &= \left(\Omega^{(2)}_0y-1\right)\varphi_{0,0}(x,y).
\end{aligned}\tag{37}$$

Proof. From Theorem 1.1 we have

$$C^{(2)}_{2m}\psi^{(2)}_{2m+1}(y)=\left(\Omega^{(2)}_{2m}y-1\right)\psi^{(2)}_{2m}(y)-C^{(2)}_{2m-1}\psi^{(2)}_{2m-1}(y),$$

and multiplying in both sides of this equality by $\psi^{(1)}_{n-(2m+1)}(x)$ we get from Proposition 4.1 the first relation in (37). We can prove the second and third relations in (37) proceeding in a similar way. □

Remark 4.3. *A similar result can be proved involving only the coefficients $\{\Omega^{(1)}_n\}_{n\geq0}$ and $\{C^{(1)}_n\}_{n\geq0}$ related to the measure $d\mu_1$, we omit these details. It should be clear to the reader that despite the recurrence in Lemma 4.2 only involves the coefficients related to the measure $d\mu_2$, there is no relation between the families $\{\varphi_n\}_{n\geq0}$ and $\{\tilde\varphi_n\}_{n\geq0}$ associated with two measures of the form $d\mu(x,y)=d\mu_1(x)d\mu_2(y)$ and $d\tilde\mu(x,y)=d\tilde\mu_1(x)d\mu_2(y)$ respectively, since the influence of the first measure is due to*

$$\begin{aligned}
&\varphi_{0,0}\equiv\frac{1}{\sqrt{m^{(1)}_0m^{(2)}_0}},\\
&\varphi_{n,0}(x,y)=\frac{1}{\sqrt{m^{(2)}_0}}\psi^{(1)}_n(x),\qquad \varphi_{n,1}(x,y)=\frac{1}{C^{(2)}_0\sqrt{m^{(2)}_0}}\left(\Omega^{(2)}_0y-1\right)\psi^{(1)}_{n-1}(x),\quad\forall n\geq1.
\end{aligned}\tag{38}$$

From Lemma 4.2 we see actually how the combination of (38) and the relations in (37) let us compute the full sequence $\{\varphi_n\}_{n\geq0}$.

Next, let us see how explicit expressions for the matrices $\mathcal{F}_1$ and $\mathcal{F}_2$ in (24)-(25) and (31) respectively, can be found from Lemma 4.2. We present a proof for $\mathcal{F}_2$, the corresponding for $\mathcal{F}_1$ follows in a similar way.

Theorem 4.4. *Under the above conditions, let us introduce the constants*

$$\Gamma_l=\frac{C^{(2)}_lC^{(2)}_{l+1}}{\Omega^{(2)}_{l+1}}>0,\qquad \Delta_l=(-1)^lC^{(2)}_l\left(\frac{1}{\Omega^{(2)}_l}-\frac{1}{\Omega^{(2)}_{l+1}}\right),$$

$$\Xi_l=\Omega^{(2)}_l+\frac{1}{\Omega^{(2)}_l}+\frac{\left(C^{(2)}_l\right)^2}{\Omega^{(2)}_{l+1}}+\frac{\left(C^{(2)}_{l-1}\right)^2}{\Omega^{(2)}_{l-1}}>0,\qquad\forall l\geq0,$$

with $C_{-1}^{(2)} = 0$ and $\Omega_{-1}^{(2)}$ an arbitrary nonzero constant. Then, the matrix $\mathcal{F}_2$ in (31) is explicitly given for all $n \geq 1$ by

$$\begin{aligned}
D_{n+1,2}^{(n-1)} &= (\mathcal{O}_{n,2} \mid diag(\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1})),\\
D_{n,2}^{(n-1)} &= (\mathcal{O}_{n,1} \mid diag(\Delta_0, \Delta_1, \ldots, \Delta_{n-1})),\\
D_{n-1,2}^{(n-1)} &= diag(\Xi_0, \ldots, \Xi_{n-1}).
\end{aligned}$$

The matrices $D_{n-2,2}^{(n-1)}$ ($n \geq 2$) and $D_{n-3,2}^{(n-1)}$ ($n \geq 3$) are the transpose of $D_{n-1,2}^{(n-2)}$ and $D_{n-1,2}^{(n-3)}$.

Proof. The initial conditions $D_{s,2}^{(n)}$ with $n = 0$ ($s = 0,1,2$) and $n = 1$ ($s = 0,1,2,3$) are deduced by direct computations derived from Theorem 1.1 and Proposition 4.1, in an analog procedure as in (22)-(23).

For $n \geq 2$ we have to consider separately the cases that involves the two-term relation for $\psi_0^{(2)}$ and $\psi_1^{(2)}$ in Theorem 1.1. So, from Theorem 1.1 and Proposition 4.1 we can write

$$C_0^{(2)} \varphi_{n,1} = \left(\Omega_0^{(2)} y - 1\right) \varphi_{n-1,0} \quad \Rightarrow \quad \frac{1}{y}\varphi_{n-1,0} = \Omega_0^{(2)} \varphi_{n-1,0} - C_0^{(2)} \frac{1}{y}\varphi_{n,1} \tag{39}$$

and

$$\begin{aligned}
C_1^{(2)} \varphi_{n+1,2} &= \left(1 - \frac{\Omega_1^{(2)}}{y}\right)\varphi_{n,1} - C_0^{(2)}\varphi_{n-1,0}\\
&\Rightarrow \quad \frac{1}{y}\varphi_{n,1} = \frac{1}{\Omega_1^{(2)}}\varphi_{n,1} - \frac{C_1^{(2)}}{\Omega_1^{(2)}}\varphi_{n+1,2} - \frac{C_0^{(2)}}{\Omega_1^{(2)}}\varphi_{n-1,0}.
\end{aligned} \tag{40}$$

If we substitute in (39) the term $\frac{1}{y}\varphi_{n,1}$ in (40) and we add $y\varphi_{n-1,0}$ we get

$$\left(y + \frac{1}{y}\right)\varphi_{n-1,0} = \Xi_0 \varphi_{n-1,0} + \Delta_0 \varphi_{n,1} + \Gamma_0 \varphi_{n+1,2}.$$

In a similar way, we can write (40) with n replaced by $n-1$ as

$$y\varphi_{n-1,1} = C_1^{(2)} y\varphi_{n,2} + \Omega_1^{(2)} \varphi_{n-1,1} + C_0^{(2)} y\varphi_{n-2,0} \tag{41}$$

and we also have from Theorem 1.1 and Proposition 4.1 the relation

$$y\varphi_{n,2} = \frac{C_2^{(2)}}{\Omega_2^{(2)}}\varphi_{n+1,3} + \frac{1}{\Omega_2^{(2)}}\varphi_{n,2} + \frac{C_1^{(2)}}{\Omega_2^{(2)}}\varphi_{n-1,1}. \tag{42}$$

If we substitute in (41) the terms $y\varphi_{n,2}$ in (42) and $y\varphi_{n-2,0}$ in (39) with n replaced by $n-1$ we get

$$\left(y + \frac{1}{y}\right)\varphi_{n-1,1} = \Gamma_1 \varphi_{n+1,3} + \Delta_1 \varphi_{n,2} + \Xi_1 \varphi_{n-1,1} + \Delta_0 \varphi_{n-2,0}.$$

For the general case $n \geq 3$ and $l = 2, 3, \ldots, n-1$ we start writing the first equation of (37) as

$$\frac{1}{y}\varphi_{n-1,2m}(x,y) = -C_{2m}^{(2)} \frac{1}{y}\varphi_{n,2m+1}(x,y) + \Omega_{2m}^{(2)} \varphi_{n-1,2m}(x,y) - C_{2m-1}^{(2)} \frac{1}{y}\varphi_{n-2,2m-1}(x,y) \tag{43}$$

and the second equation of (37) as

$$\begin{aligned}
\tfrac{1}{y}\varphi_{n,2m+1}(x,y) &= \tfrac{1}{\Omega_{2m+1}^{(2)}} \Big[-C_{2m+1}^{(2)} \varphi_{n+1,2m+2}(x,y) + \varphi_{n,2m+1}(x,y)\\
&\qquad - C_{2m}^{(2)} \varphi_{n-1,2m}(x,y)\Big],\\
\tfrac{1}{y}\varphi_{n-2,2m-1}(x,y) &= \tfrac{1}{\Omega_{2m-1}^{(2)}} \Big[-C_{2m-1}^{(2)} \varphi_{n-1,2m}(x,y)\\
&\qquad + \varphi_{n-2,2m-1}(x,y) - C_{2m-2}^{(2)} \varphi_{n-3,2m-2}(x,y)\Big].
\end{aligned} \tag{44}$$

Thus, if we substitute (44) in (43) and we add the term $y\varphi_{n-1,2m}(x,y)$ from the first equation of (37) we get

$$\begin{aligned}\left(y+\frac{1}{y}\right)\varphi_{n-1,2m}(x,y) &= \Gamma_{2m}\varphi_{n+1,2m+2}(x,y)+\Delta_{2m}\varphi_{n,2m+1}(x,y)+\Xi_{2m}\varphi_{n-1,2m}(x,y)\\ &\quad +\Delta_{2m-1}\varphi_{n-2,2m-1}(x,y)+\Gamma_{2m-2}\varphi_{n-3,2m-2}(x,y).\end{aligned}$$

A similar relation is obtained for $\left(y+\frac{1}{y}\right)\varphi_{n-1,2m-1}(x,y)$, yielding for all $l=2,3,\ldots,n-1$ and $n\geq 3$,

$$\begin{aligned}\left(y+\frac{1}{y}\right)\varphi_{n-1,l}(x,y) &= \Gamma_{l}\varphi_{n+1,l+2}(x,y)+\Delta_{l}\varphi_{n,l+1}(x,y)+\Xi_{l}\varphi_{n-1,l}(x,y)\\ &\quad +\Delta_{l-1}\varphi_{n-2,l-1}(x,y)+\Gamma_{l-2}\varphi_{n-3,l-2}(x,y).\end{aligned}$$

□

As it is indicated in [13, Section 2], the families $\{\psi_k\}_{k\geq 0}$ of orthogonal Laurent polynomials in the one variable case computed from Theorem 1.1 are related to the families of ordinary polynomials satisfying Laurent orthogonal conditions (that have been considered in the literature, e.g. by A. Sri Ranga and collaborators, see [27, 28, 29, 30]). So, we can make use of some of the results available in those references to get explicit expressions for the coefficients $\{\Omega_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ related to some particular absolutely continuous measures, like

$$d\omega_1(x)=\frac{dx}{\sqrt{(b-x)(x-a)}},\qquad d\omega_2(x)=\frac{dx}{\sqrt{x}},\qquad d\omega_3^{\mu}(x)=\frac{[(b-x)(x-a)]^{\mu-\frac{1}{2}}}{\left(\sqrt{b}-\sqrt{a}\right)x^{\mu}}dx,$$

$$d\omega_4(x)=\frac{x\left(1+\frac{\sqrt{ab}}{x}\right)^2}{\sqrt{(b-x)(x-a)}}dx,\quad d\omega_5(x)=\frac{dx}{\left(x+\sqrt{ab}\right)\sqrt{(b-x)(x-a)}},\quad d\omega_6^{\kappa}(t)=\frac{1}{2\kappa\sqrt{\pi}}\left(1+\frac{1}{t}\right)e^{-\left(\frac{\log(t)}{2\kappa}\right)^2}dt,$$

with $x\in(a,b)$, $0<a<b<\infty$, $t>0$, $\kappa>0$ and $\mu>-1/2$. Thus, if the measure $d\mu(x,y)=d\mu_1(x)d\mu_2(y)$ defined on the rectangle $\mathcal{R}$ is of the form $\mu_1,\mu_2\in\{\omega_i\}_{i=1}^{6}$ we can recover directly from the results of this section the recurrence relations for $\{\varphi_n\}_{n\geq 0}$ explicitly.

5. Conclusions

We have introduced for the first time in the literature the theory of sequences of orthogonal Laurent polynomials in two real variables (x,y) (for the sake of simplicity, but it can be generalized to several variables) with respect to a positive Borel measure μ defined on $\mathbb{R}^2$ such that $\{x=0\}\cup\{y=0\}\notin\operatorname{supp}(\mu)$. We have considered an appropriate ordering for the Laurent monomials x^iy^j, $i,j\in\mathbb{Z}$ that let us to obtain five term relations involving multiplication by $x+\frac{1}{x}$ and $y+\frac{1}{y}$. The corresponding matrices representations of these operators are block symmetric five diagonals. Our approach enables us to extend some known results for the ordinary polynomials to the Laurent case. In this respect, we have included a Favard's theorem and Christoffel-Darboux and confluent formulas. Also, a connection with the one variable is done when the measure μ is a product measure of separate variables defined on the rectangle $\mathcal{R}=[a,b]\times[c,d]$, $0<a<b<\infty$, $0<c<d<\infty$.

In the one variable case, there are very few measures that gives rise to explicit expressions for sequences of orthogonal Laurent polynomials. We could almost say that the only ones are practically the weight functions $\{\omega_i\}_{i=1}^{6}$ mentioned at the end of Section 4. In general, these families are computed making use of Theorem 1.1, under the knowledge of the corresponding moments. In the several variables case, there is not any sequence of orthogonal Laurent polynomials explicitly known, except in the situations described in Section 4. However, these families can always be obtained from the five-term relations obtained in Section 2 as long as the moments (14) exist and are computable. Also, unlike the situation in the one variable case, there are not known applications in the literature for the moment of orthogonal Laurent polynomials in two or more real variables. All these questions will be considered in a forthcoming paper, mainly focused in the applications of orthogonal Laurent polynomials of two real variables in the construction and characterization of cubature formulas for the numerical estimation of integrals of the form $\iint_D f(x,y)d\mu(x,y)$.

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