BISECTING MASSES WITH FAMILIES OF PARALLEL HYPERPLANES

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Abstract. We prove a common generalization to several mass partition results using hyperplane arrangements to split $\mathbb{R}^d$ into two sets. Our main result implies the ham-sandwich theorem, the necklace splitting theorem for two thieves, a theorem about chessboard splittings with hyperplanes with fixed directions, and all known cases of Langerman's conjecture about equipartitions with n hyperplanes.

Our main result also confirms an infinite number of previously unknown cases of the following conjecture of Takahashi and Soberón:

For any $d+k-1$ measures in $\mathbb{R}^d$, there exist an arrangement of k parallel hyperplanes that bisects each of the measures.

The general result follows from the case of measures that are supported on a finite set with an odd number of points. The proof for this case is inspired by ideas of differential and algebraic topology, but it is a completely elementary parity argument.

1. Introduction

Understanding how one can split families of measures on real spaces into pieces of the same size is a central topic in topological combinatorics [RPS22, Živ17]. The classic example, known as the ham sandwich theorem, states that *given d smooth probability measures on $\mathbb{R}^d$, there exists a hyperplane simultaneously splitting each measure in half.*

Early popularity of this theorem was likely due to the elegance of its proof and maybe to the culinary reference. Before Tukey and Stone called it the *ham sandwich* theorem (as we slice a ham and two pieces of bread simultaneously) Steinhaus used the motivation of a *leg of pork* (as we slice fat, bone, and meat).

A myriad of related results have appeared since then, many of them recover the foodie references (cakes [ANRCU98], spicy chickens [KHA14], pizzas [BPS19], etc.), most of these results are proven with topological methods, which range from fixed point and Borsuk–Ulam type theorems to more advanced techniques in algebraic topology (cohomological index theory, equivariant obstruction theory, spectral sequences, etc). Even splitting measures with families of few hyperplanes can lead to very rich topological statements (see, for instance, the history of the Hadwiger–Grünbaum–Ramos theorem [BFHZ18]).

Beyond their inherent interest, these results have numerous applications in various fields like computational geometry [Mat94], combinatorial geometry [APP$^+$05, FGL$^+$12]

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convexity [Leh09, FHM$^+$19], geometric analysis [Gro03, Gut10], harmonic analysis [Gut16] and incidence geometry [GK15]. In these applications one may use partitions in which one part avoids a hyperplane [YY85], partitions in which continuous functions on convex bodies are equalized [Gro03, FHM$^+$19], or partitions by the zero set of a polynomial [ST42]. The current paper extends this last type of result for polynomials that are products of affine functions satisfying certain constrains. As in the classical polynomial ham sandwich theorem, the Veronese trick combined with our theorem imply non linear avatars as corollaries.

Results which deal with partitions of $\mathbb{R}^d$ into two sets separated by a possibly singular $(d-1)$-dimensional manifold are known as "chessboard colorings". The most studied family of examples of chessboard colorings are the ones that have an hyperplane arrangement as boundary. Depending on the number of hyperplanes used and conditions on their position, there have been several important results in the area. The backbone of the proof of each of these results is topological, most boiling down to the study of zeros of certain equivariant maps (the group involved changes from theorem to theorem).

1.1. **Our result.** The main result of this manuscript is a common generalization to several known results regarding chessboard colorings bounded by hyperplane arrangements. A surprising aspect of the proof is that, rather than proving a common generalization to a scattered set of Borsuk–Ulam-type theorems, our elementary proof was inspired by one of the proofs of the Borsuk–Ulam theorem. The main idea is to study the parity of a discrete set of partitions as we go through some geometric deformation of the measures. The idea behind this approach, in the language of differential topology was used by Hubard and Karasev to confirm new cases of a conjecture of Langerman [HK20]. Later, Patrick Schnider managed to make the argument elementary in dimension two [Sch21]. We show how this elementary approach can be used in any dimension and involving families of parallel hyperplanes, which increases significantly the number of results we generalize. Let us first describe the main theorem in full generality, and then specify how it implies several known results.

Given two positive integers m, k, let $S(m,k)$ be the *Stirling number of the second kind.* This counts the number of partitions of a set of m elements into k non-empty sets. Given positive integers $m_1, \dots, m_n$ and $M = m_1 + \dots + m_k$, let $\binom{M}{m_1,\dots,m_n}$ be the multinomial coefficient that counts the number of ordered partitions of a set of M elements into n sets $U_1, \dots, U_n$ such that $|U_i| = m_i$ for each i.

An oriented hyperplane h in $\mathbb{R}^d$ defines two closed halfspaces: h^+ and h^-, which we refer to as respectively as the positive and negative halfspace of h. A finite family $\mathcal{F}$ of oriented hyperplanes in $\mathbb{R}^d$ defines a chessboard coloring, which is a pair of set (A,B) defined as:

$$A = \{x \in \mathbb{R}^d : x \text{ is in an even number of positive halfspaces of hyperplanes of } \mathcal{F}\}$$
$$B = \{x \in \mathbb{R}^d : x \text{ is in an odd number of positive halfspaces of hyperplanes of } \mathcal{F}\}.$$

Notice that since a hyperplane is the zero set of an affine function, $a\colon \mathbb{R}^d \to \mathbb{R}$, given a family of hyperplanes, with corresponding affine functions $\{a_1, a_2, \dots a_M\}$ their chessboard coloring can alternatively be succinctly described by

$$A = \{x \in \mathbb{R}^d : p(x) \geq 0\} \text{ and } B = \{x \in \mathbb{R}^d : p(x) \leq 0\}, \text{ where,}$$

$$p(x) := \Pi_{i=1}^{M} a_i(x).$$

Given a hyperplane arrangement, with corresponding chessboard partition (A, B), and a finite Borel measure μ, we say that the arrangement **bisects** the measure if

$$\min(\mu(A), \mu(B)) \geq \frac{\mu(\mathbb{R}^d)}{2}.$$

A central instance of this definition is when a finite measure μ is absolutely continuous with respect to the Lebesgue measure (or more generally, $\mu(h) = 0$ for every affine hyperplane h), in this case the bisecting condition is simply:

$$\mu(A) = \mu(B).$$

Theorem 1.1. *Let n, d be positive integers, and $L_1, L_2 \dots, L_n$ be subspaces of $\mathbb{R}^d$ of positive dimension. Let $l_i = \dim L_i$, and $k_1, \dots, k_n$ be positive integers. Let G be the subgroup of permutations of $[n] = \{1, \dots, n\}$ such that for every $g \in G$, $L_{g(i)} = L_i$ and $k_{g(i)} = k_i$ for all i. Put $m_i := l_i + k_i - 1$, $M := \sum_{i=1}^{n} m_i$. Suppose that*

$$N := \frac{1}{|G|} \binom{M}{m_1, m_2 \dots, m_n} \prod_{i=1}^{n} S(m_i, k_i) \equiv 1 \mod 2.$$

Then, for any M finite measures on $\mathbb{R}^d$, there exist unit vectors $v_1 \in L_1, v_2 \in L_2 \dots$, $v_n \in L_n$, and families of hyperplanes $\mathcal{F}_1, \dots, \mathcal{F}_n$ such that for each i, each hyperplane in $\mathcal{F}_i$ is orthogonal to v_i, $|\mathcal{F}_i| = k_i$, and the chessboard coloring induced by the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ bisects each of the M measures.

In one case this condition was known to be necessary for the theorem to hold. The case is $n = 2, l_1 = l_2 = 1, k_1 = k_2 = 1$, and a counterexample was constructed by Karasev et al. [KRPS16]. In Section 4 we extend this known counterexample to many instances with $l_i = 1$ for all i.

Question 1.2. *Is the numeric condition condition of Theorem 1.1 necessary.*

We now make a few remarks to shed some light on this condition. Firstly, the group G is a direct product of symmetric groups. A subset $I \subset \{1, 2, \dots n\}$ gives rise to a factor S_I, if and only if, for all $i, j \in I$, $L_i = L_j$ and $k_i = k_j$. Secondly, the multinomial coefficient $\binom{M}{m_1, m_2 \dots, m_n}$ is odd when the expressions of the numbers $\{m_1, m_2, \dots m_n\}$ written in base two don't share any non zero term (for instance if for each i, $m_i = 2^i$, then the multinomial coefficient is odd, in contrast if e.g. $m_1 = 1, m_2 = 3$, then it is even, since in binary $m_1 = 01$ and $m_2 = 11$). Finally, it is known that:

$$S(m, k) \equiv \binom{m - \lceil \frac{k+1}{2} \rceil}{\lfloor \frac{k-1}{2} \rfloor} \mod 2.$$

Combining the previous remarks one can precisely understand in which cases N is odd. Let us specify some instances of the theorem above that give interesting known results.

- The case $n = 1, L_1 = \mathbb{R}^d, k_1 = 1$ is the ham sandwich theorem [Ste38]. Since $S(d, 1) = 1$, we don't impose any additional conditions.

- The case $d = 1$, $n = 1$ and any k is the "necklace splitting theorem". This was originally proved by Hobby and Rice [HR65] and then rediscovered in the 80's with two new proofs [GW85, AW86]. As $S(k, k) = 1$, we don't impose any new requirement on k.
- The case $n = 1, L_1 = \mathbb{R}^d, k_1 = 2$ is a recent result of Soberón and Takahashi [ST22], which says that *for any $d + 1$ measures in $\mathbb{R}^d$, there are two parallel hyperplane whose chessboard coloring bisects each measure.* Notice that since $S(d + 1, 2)$ is always odd for $d \geq 1$, we don't impose additional conditions on d.
- The case $l_1 = \cdots = l_n = 1$ is a result of Karasev, Roldán-Pensado, and Soberón [KRPS16], which describes chessboard colorings by families of hyperplanes with fixed directions. The condition of parity of a multinomial coefficient is the same in the theorem above as the one found by Karasev et al.
- The case $L_1 = \cdots = L_n = \mathbb{R}^d$ and $k_1 = \cdots = k_n = 1$ is recent result of Karasev and Hubard [HK20] which solved an infinite number of instances of a conjecture of Langerman [BPS19] that states the following. *For any dn measures in $\mathbb{R}^d$ there exist n hyperplanes whose chessboard coloring bisects each of them.* The condition of parity of the multinomial coefficient boils down to the dimension being a power of 2, like in the aforementioned paper [HK20]. Blagojević et al showed how the same cases can be confirmed with a different approach [BDBKK22].

The case $n = 1, L_1 = \mathbb{R}^d$ and any k confirms an infinite number of cases of the conjecture below, due to Takahashi and Soberón [ST22]. This is one of many generalizations of the necklace splitting theorem to higher dimensions, see [LŽ08, BS18] for other generalizations.

Conjecture 1.3. *For any $d + k - 1$ measures in $\mathbb{R}^d$, there exist k parallel hyperplanes whose induced chessboard coloring simultaneously bisects every measure.*

We confirm Conjecture 1.3 when $S(d + k - 1, k)$ is odd. Our theorem provides for each d an infinite set of positive integers k, and for each k an infinite number of ds, for which the pair (d, k) satisfies this conjecture. In fact, if we arrange the pairs (d, k) for which Theorem 1.1 implies the Takahashi–Soberón conjecture in a triangular array we obtain a Sierpinski triangle pattern.

1.2. **Veronese trick.** Since the regions are bounded by hyperplanes, we can replace them by sets defined on the zeroes of a polynomial. This is better illustrated with an example, as in the following corollary, in which we consider a line as a degenerate case of a circle with center at infinity.

Corollary 1.4. *Let $\mu_1, \dots, \mu_7$ be seven finite measures on $\mathbb{R}^2$, each absolutely continuous with respect to the Lebesgue measure. There exist two concentric circles, a line and a vertical line whose union induces a chessboard coloring that bisects each measure.*

Proof. Consider the map $(x, y) \to (x, y, x^2 + y^2)$. This lifts each of the measures to a regular paraboloid P in $\mathbb{R}^3$. Moreover, if H is a plane in $\mathbb{R}^3$, the projection of $H \cap P$ onto the xy-plane is a circle. If H has a normal vector in the xy-plane, then $H \cap P$ projects down to a line. Apply Theorem 1.1 to the lifted measures using $n = 3$ and:

- $L_1 = \mathbb{R}^3, k_1 = 2$ (these two parallel planes will give the two concentric circles)

- L_2 equal to the the xy-plane, $k_2 = 1$ (this will give us the line without conditions), and
- $L_3 = \mathrm{span}\{(1,0,0)\}$, $k_3 = 1$.

Since $\binom{7}{4,2,1} = 105$, $S(4,2)$ is odd, and $S(3,1) = S(2,1) = 1$, we meet the conditions of Theorem 1.1. □

The idea of using this trick for ham sandwich type results is already implicit in the seminal paper [ST42] of Stone and Tukey. We can combine the Veronese trick with careful choices for the spaces L_1 to get chessboard partitions induced by curves of seemingly very different polynomials.

1.3. **Notation.** To shorten some statements in the rest of the paper use the notation of Theorem 1.1 throughout, we also use the following notation and terms:

- $k := (k_1, \ldots, k_n), l := (l_1, \ldots, l_n)$ are positive integer vectors with n entries each (in a lemma we will take $n = 1$).
- $B_{L_1}, \ldots, B_{L_n}$ are basis of the orthogonal subspaces $L_1^\perp, \ldots, L_n^\perp$
- $\mathcal{H} = \cup_{i=1}^n \mathcal{H}_i$, where for each i, $\mathcal{H}_i := \{h_{i,1}, h_{i,2} \ldots h_{i,k_i}\}$ is an arrangement of k_i parallel hyperplanes.
- $\mu^- = \{\mu_1, \mu_2, \ldots, \mu_M\}$ is a family measures on $\mathbb{R}^d$.

1.4. **Summary.** The proof of Theorem 1.1 has three steps. First, we construct a particular family of measures supported in sets of an odd number of points for which the number of bisecting arrangements is exactly N. Second, we show that for any deformation of these point sets the parity of the number of bisecting hyperplane arrangements is invariant.

Finally, we show that the result for measures supported on an odd set of points implies the general result.

2. The ham sandwich theorem revisited

In this section we show how our method give a new and elementary proof of the ham sandwich theorem. This is a particular case of the general proof described in the next sections, but may help the reader build intuition about this approach. Due to the equivalence of the Borsuk–Ulam theorem and the ham sandwich theorem [KS17], this can be used as one of the ingredients of a new convoluted proof of the Borsuk–Ulam theorem.

First, we state the version of the ham sandwich we will prove.

Theorem 2.1. *If $U_1, \ldots, U_d$ are d sets of points in $\mathbb{R}^d$, each with an odd cardinality and such that their union is in general position. Then, there exist points $p_1 \in U_1, \ldots, p_d \in U_d$ such that the hyperplane spanned by $p_1, \ldots, p_d$ halves each of the sets $U_1, \ldots, U_d$.*

Proof. We will show that the number of such halving hyperplanes is odd, so it cannot be zero. It is not difficult to show (see [BHJ08]) that if the sets $M_1, \ldots, M_d$ are well separated (the convex hull of the union of any collection is disjoint from the convex hull of the union of the rest), then there is exactly one halving hyperplane. We consider each of $U_1, \ldots, U_d$ as a different color.

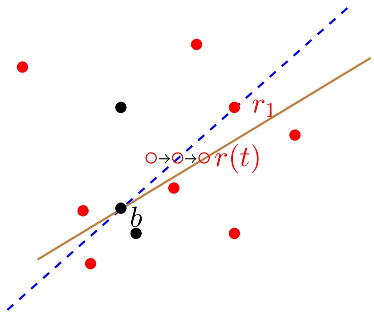

FIGURE 1. Here $U(t) = U_1(t) \cup U_2(t) \subset \mathbb{R}^d$, represented by red and black points. The red hollowed point $r(t)$ moves right along the arrows line. It is represented at three different times. Before the movement the dashed blue line h containing r_1 and b bisects the point sets. When $r(t)$ is incident to h, the line $h'(t)$ spanned by $r(t)$ and b is almost bisecting and coincides with h. When $r(t)$ passes to the right of h, the line $h'(t)$ becomes bisecting, and h stops being bisecting.

Suppose we have another configuration of points $U_1', U_2', \ldots, U_d'$ such that $|U_i| = |U_i'|$ for each i. Consider a bijection between U_i and U_i' for each i. We are going to move continuously the points of $(U_1, \ldots, U_d)$ to their corresponding points in $(U_1', \ldots, U_d')$. For each point $p \in \bigcup_i U_i$ this gives us a function $p : [0,1] \to \mathbb{R}^d$ such that $p(0) = p$ and $p(1)$ is the corresponding point in $\bigcup_i U_i'$. Let $(U_1(t), \ldots, U_d(t))$ be the induced configurations of points at time t. We may assume without loss of generality that only for a finite number exceptional times the configuration is not in general position, and when the configuration is not in general position, there is a unique $(d+1)$-tupe of points contained in a hyperplane.

Each halving hyperplane at a time when $(U_1(t), \ldots, U_d(t))$ is in general position goes through one point of each color. For each colorful d-tuple $X(t)$ at time t, let $h_X(t)$ be the hyperplane they span. Consider a colorful d-tuple $A(t)$, corresponding to a halving hyperplane. If A stops being a halving hyperplane at time t_0, it means that there is an $\varepsilon > 0$ such that $h_A(t_0 - \varepsilon)$ is a halving hyperplane, and $h_A(t_0 + \varepsilon)$ is not. Moreover, $h_A(t_0)$ must contain a point p of the configuration that is not in A. Suppose that the point is of color i, and let B be the colorful d-tuple formed by replacing the point p_i of color i from A by p. Note that $h_B(t_0) = h_A(t_0)$, so it halves all colors except i, which is almost balanced. Moreover, p_i is on different sides of $h_B(t)$ for $t = t_0 - \varepsilon$ and $t = t_0 + \varepsilon$, so exactly at one of those times $h_B(t)$ is a halving hyperplane.

If $h_B(t + \varepsilon)$ is a halving hyperplane, then the total number of halving hyperplanes did not change. If $h_B(t - \varepsilon)$ is a halving hyperplane, then the total number of halving hyperplanes decreased exactly by two. An analogous analysis shows that when a hyperplane starts being a halving hyperplane, then the total number of halving hyperplanes either stays constant or increases by two. Therefore, the parity of halving hyperplanes does not change. Since there is a configuration with exactly one halving hyperplane, we are done. $\square$

Once we involve families of parallel hyperplanes, there will be more nuance for the parity analysis. The key ideas remain the same.

3. Proof

We will say that a measure is **oddly supported** if it is the sum of an odd number of delta masses. Generic families of oddly supported measures are central to our approach, because in this case, one of the points of each measure lies in the hyperplane arrangement, and each of the open chessboard regions contains exactly half of the remaining points. In consequence, the number of bisecting arrangements for generic oddly supported measures is finite.

3.1. **Points in generic position.** We say that a finite set of points S in $\mathbb{R}^d$ is in **generic position** if the set of all $d|S|$ coordinates of the points is algebraically independent. In other words, the only multinomial with integer coefficients that has a zero if we evaluate it using only values of the $d|S|$ coordinates is the constant zero. In particular, if S is in generic position any d vectors formed by the differences of d different pairs of S that don't form any cycles is a linearly independent set. Notice that this condition is much stronger than general position, in which we require that no $d+1$ points of S contained by a hyperplane. Indeed, if we have $x_0, \dots, x_d$ points in a hyperplane, the d vectors of the for $x_i - x_0$ would be linearly dependent. It is easy to show that if $\{x_1, x_2, \dots x_m\}$ is a point set and $\{\xi_1, \xi_2, \dots \xi_m\}$ is a set of independent random vectors, chosen uniformly from the ball $B(0, \delta)$ of radius $\delta > 0$, then $\{x_1 + \xi_1, x_2 + \xi_2, \dots x_m + \xi_m\}$ is in generic position almost surely.

Whenever we have a partition of a set, its **core** consists of all the subsets of the partition that have more than one element. In the following discussion, we intentionally use language loosely and conflate a partition of a point set with the corresponding partition of the finite set that indexes the point set.

Lemma 3.1. *Let L be a subspace of $\mathbb{R}^d$ of dimension l, B_L a basis of $L^\perp$, and P a set of $m = l - 1 + k$ points such that $P \cup B_L$ is in generic position in $\mathbb{R}^d$. The set k-tuples of parallel hyperplanes whose union contain P and whose normal vector is contained in L is in bijection with the partitions of P into k non-empty sets.*

Proof. Suppose that $h_1, \dots, h_k$ are parallel hyperplanes whose union contains P and whose normal vector v is contained in L. We claim that none of the sets of the partition $\chi = (h_1 \cap P, \dots, h_k \cap P)$ is empty. Let r be the number of non-empty sets in this partition. For each $i \le k$ such that $h_i \cap P$ is in χ choose some element $p_i \in h_i \cap P$. Put $k_i := |h_i \cap P|$. Let V be the union $\cup_i V_i$, where V_i is the set of vectors of the form $q - p_i$ with $q \in h_i \cap (P \setminus \{p_i\})$. Then

$$|V| = \sum_{j=1}^{r} (k_j - 1) = (l + k - 1) - r.$$

If $r \le k-1$, this means that we have constructed a set V of at least l vector differences, contained in $v^\perp$. Since B_L contains $d-l$ vectors, and $B_L \cup V \subset v^\perp$, then the set $B_L \cup V$ is linearly dependent, contradicting the generic position assumption. Hence $r \ge k$, showing that each hyperplane contains at least one point.

Now, consider a partition χ of P into k non-empty sets $P_1, \dots, P_k$. For any P_i, consider a point $p_i \in P_i$, and then take the $|P_i|-1$ vectors $q-p_i$ for $q \in P_i \setminus \{p_i\}$. We have formed a total of $\sum_{i=1}^k (|P_i|-1) = l-1$ vectors. These vectors together with B_L are linearly independent, so they form a basis of a hyperplane h. Then, consider h_i the translate of h through p_i. By construction, $P_i \subset h_i \cap P$. By the arguments above, it is not possible to cover P with fewer than k translates of h, so $P_i = h_i \cap P$. This means that for every partition of P into k non-empty subsets, there exists a k-tuple of translates of a hyperplane that induces that partition with perpendicular vector in L.

In fact the two functions we described from partitions to arrangements, and from arrangements to partitions are inverses of each other. □

Before generalizing this lemma to arrangements with several sub-arrangements of parallel hyperplanes, we prove a similar lemma that will be used in Section 3.3.

Lemma 3.2. *Let L be a subspace of $\mathbb{R}^d$ of dimension l, B_L a basis of $L^\perp$, and P a set of $m = l-2+k$ points such that $P \cup B_L$ is in generic position in $\mathbb{R}^d$. Let X be a fixed partition of P into k non-empty subsets. There is a unique subspace K of dimension $d-2$ such that there exists a vector $v \in L$ orthogonal to K and there are k translates of K that cover P inducing X as a partition of P.*

Proof. The proof is analogous to the one for Lemma 3.1. Let $X = X_1 \cup \dots \cup X_k$. Pick a point $p_i \in X_i$ for each $i = 1, \dots, k$, and consider the $l-2$ vectors of the form $p - p_i$ for $p \in X_i \setminus \{p_i\}$. These, together with B_L, form $d-2$ linearly independent vectors, which gives us the basis for K. The k translates of K containing each of $p_1, \dots, p_k$ are the affine spaces we were looking for. □

Let l and k be two non negative integer vectors of length n. A **valid** (l,k)**-partition** of a finite set $[M]$, consists of a labeled partition of $[M]$ into n subsets, each of which is refined to an unlabeled partition of non-empty subsets. The i-th set of the first partition must have cardinality $l_i - 1 + k_i$, which is refined into k_i non-empty subsets in the second partition.

For example if $M = 7$, and $l = (2,2)$, $k = (3,2)$, then a valid (l,k) partition of the set $[7] = \{1,2,3,4,5,6,7\}$, can be represented by

$$14|3|7||25|6,$$

where the first ordered partition is $1347|256$ and the subset 1347 is partitioned to $14|3|7$, while the set 256 to $25|6$. As a valid partition, it is equivalent to

$$3|41|7||6|52,$$

since the order of the refined partition is irrelevant, but different from,

$$25|6||14|3|7.$$

since the order of the first partition is relevant.

Given a point set P in $\mathbb{R}^d$ with cardinality M, we say that a hyperplane arrangement $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_n$ is $(\mathcal{L}, k, P)$**-valid**, if each hyperplane contains at least one point of P, $\mathcal{H}_i$ consists of k_i parallel hyperplanes all of which are perpendicular to some $v_i \in L_i$. We also say that two such arrangements $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_n$ and

$\mathcal{H}' := \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \ldots \cup \mathcal{H}'_n$ are equivalent if there is a permutation $g : [n] \to [n]$ such that $\mathcal{H}_i = \mathcal{H}_{g(i)}$, $L_{g(i)} = L_i$, and $k_{g(i)} = k_i$ for all $i \in [n]$.

Lemma 3.3. *Let n, d be positive integers, and $L_1, L_2 \ldots, L_n$ be subspaces of $\mathbb{R}^d$ of positive dimension. Let $l_i = \dim L_i$, and $k_1, \ldots, k_n$ be positive integers. Let G be the subgroup of permutations of $[n] = \{1, \ldots, n\}$ such that for every $g \in G$, $L_{g(i)} = L_i$ and $k_{g(i)} = k_i$ for all i. Put $m_i := l_i + k_i - 1$, $M := \sum_{i=1}^n m_i$. Let $P := \{p_1, p_2 \ldots p_M\}$ be a set of points, such that $P \cup B_{L_i}$ is generic for $i = 1, \ldots, n$. For every (l, k)-valid partition χ of P, there exists a unique $(\mathcal{L}, k, P)$-valid hyperplane arrangement $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ such that the induced refined partition $\{h \cap P : h \in \mathcal{F}\}$ is the valid partition χ. Additionally, the number of equivalence classes of $(\mathcal{L}, k, P)$-valid partitions is N.*

Proof. By Lemma 3.1, if a hyperplane arrangement is $(\mathcal{L}, k, P)$-valid, the condition on each $\mathcal{H}_i$ imply that $|P \cap \mathcal{H}_i| = l_i + k_i - 1$ for all i. So $\mathcal{H}$ corresponds to an (l, k)-valid partition of P: we must first find an ordered partition of P into sets of size $l_1 + k_1 - 1, \ldots, l_n + k_n - 1$ to determine the sets $\mathcal{H}_i \cap P$. Then, to find the hyperplanes of $\mathcal{H}_i \cap P$ we must find a partition of $\mathcal{H}_i$ into k_i non-empty parts.

Notice that for each maximal subset of indices I such that for all $i, j \in I$, $L_i = L_j$ and $k_i = k_j$, we are over-counting the corresponding arrangements $|I|!$ times. The number of equivalence classes follows directly.

□

In what follows, l, k, $\mathcal{L}$ and P are always fixed in advance, so to unclutter notation we just talk about **valid partitions**, and **valid arrangements** but we insist that we are actually talking about a partition and a refinement of it, on the one hand, and about a hyperplane arrangement with certain constrains on their directions coming from $\mathcal{L}$ and k.

3.2. **Well separated families of almost symmetric measures.** Now we construct a particular family of generic oddly supported measures. Suppose we are given the family $\mathcal{L}$ of subspaces and the vectors k, l. We say that a family of sets $\{K_1, K_2, \ldots K_M\}$ in $\mathbb{R}^d$ for $M > d$ is **well separated** if for each $i = 1, \ldots, n$ a set of k_i parallel hyperplanes, whose normal vector is in L_i, can intersect at most $k_i + l_i - 1$ sets of the form $conv(K_j)$.

We say that a point set X is symmetric around x if for any point $x' \in X$, the point $2x - x'$ is also in X. It follows directly that a linear projection of a point set which is symmetric around x, is itself symmetric around the image of x. We will say that a pointset X is **roughly symmetric** around x if for any linear projection $\pi \colon \mathbb{R}^d \to \mathbb{R}$, $\pi(x)$ is the median of $\pi(X)$.

Notice that given a set of point sets in generic position P, it is easy to construct a colored point set, with one color for each point in P, such that the color classes are well separated and each color class is roughly symmetric around a point in P. One chooses ϵ small enough so that the balls $\{B(p, \epsilon) : p \in P\}$ are well separated, choosing for each such ball $B(p, \epsilon)$ a sample of random points together with their antipodals with respect to p, yields a set symmetric with respect to p. Perturbing the whole construction, each point independently at random, without moving any of them more than a very small constant $\delta > 0$, makes each color class roughly symmetric around a point in the set and the union with some fixed subset in generic position.

Lemma 3.4. *Let $\mathcal{L}$ be an ordered set of n sub-spaces of positive dimension of $\mathbb{R}^d$, put $l_i = \dim(L_i)$, $(k_1, k_2, \dots k_n) \in \mathbb{N}^n$, and P a set of M points such that for each i, $P \cup B_{L_i}$ is in generic position. Let $\mathcal{U}$ be a well separated M-colored point set, such that for each $j \in [M]$, U_j is roughly symmetric around p_j, $U_j \subset B(p_j, \epsilon)$, and for each $i \in [n]$, $U \cup B_{L_i}$ is generic. The set of labelled valid arrangements that bisect each measure equals the set of (l, k)-valid partitions of $[M]$.*

Proof. By Lemma 3.3,there exists a bijection between valid arrangements and pairs (χ, S) where χ is a valid partition of $[M]$ and $S \subset U$ a colorful subset. We claim a valid arrangement is bisecting, if and only if, the corresponding pair (χ, S) has $S = P$. Indeed, the well separated condition implies that the partiton of U_j induced by the cheesboard regions depends only on the unique hyperplane $h \in \mathcal{H}$, for which $h \cap U_j \neq \emptyset$. Since U_j is roughly symmetric, h bisects U_j if and only if $h \cap U_j = p_j$. □

In particular, the number of equivalence classes of valid arrangements that bisect each set of points is N.

3.3. **Parity under deformation.** This section contains the key step in the proof of Theorem 1.1. We refer to a smooth family of colored point sets parameterized by $[0, 1]$ as a **path of point sets**, which we denote by $U(t) := \{U_i(t)\}_{i=1}^{M}$. As before, we assume that each color class has an odd number of (paths of) points. We say that a path of point sets is **generic** with respect to an n-tuple of sets of vectors $\{B_{L_i}\}_{i \in [n]}$ if for at most a finite set of times t, which we call **exceptional times** there is a unique i, such that $U(t) \cup B_{L_i}$ is not generic. Moreover we assume that at an exceptional time t_j, the pointset $U(t_j)$ is **almost generic**, in the following sense. There exists a colorful set S of M points and a valid partition P of S such that the hyperplane arragement they induce contains exactly $M + 1$ points. Moreover, this set of $M + 1$ points and the partition generated is uniquely determined in the sense that there exist a unique index $i \leq M$, and a set of vectors $V(t_j) \subset U(t_j)$ such that for every subset $U'(t_j) \subset U(t_j)$, such that $U(t_j)' \cup B_{L_i}$ is not generic, then $V(t_j) \subset U'(t_j)$, and $V(t_j) \cup B_{L_i}$ is non generic and $|V(t_j)| = l_i + k_i$. We say that H_i is the **oversaturated** family of $\mathcal{H}$.

Lemma 3.5. *Let $\mathcal{L}$ be a family of n subspaces, and $\mathcal{U}(t)$ a generic path of odd point sets of $\mathbb{R}^d$. The parity of the number of valid bisecting arrangements does not depend on t.*

Assuming this lemma we are ready to give a proof of Theorem 1.1 for generic oddly supported measures.

Proof of Theorem 1.1. It is easy to see that any two generic families of point sets

$$\{U_1, U_2, \dots U_M\}$$

and

$$\{U'_1, U'_2, \dots U'_M\}$$

such that $|U_i| = |U'_i|$ can be connected by a generic path of point sets. One might achieve this by choosing arbitrary bijections and moving one point at the time on a path that avoids the $(d-2)$-flats that contain more than $d-1$ points of U. So if $U(0)$ is an arbitrary generic point set, with M color classes, with an odd number of points in each, there exists a generic path of odd point sets $U(t)$ such that the colored point set $U(1)$ is generic and the color classes are very well separated. By Lemma 3.5 the number of valid bisecting arrangements of $U(0)$ has the same parity as the number of such arrangements of $U(1)$. By Lemma 3.3 the latter equals N. In particular if N is odd, at least one valid bisecting arrangement should exist for $U(0)$. □

If at an exceptional time there exists a hyperplane arrangement $\mathcal{H}$, that is valid except from the fact that it intersects two points of the same color, we call it, **almost valid.** For non exceptional times we defined a bijection that takes each valid partition of a colorful point set to a valid hyperplane arrangement. At exceptional times, this function is still well defined but cannot be inverted. When we refer to a path of valid arrangements $\mathcal{H}(t)$, we assume that the valid partition, and the colorful subset are fixed.

We will say that a valid arrangement is **almost bisecting** if it is bisecting for every color class except for one that we call the **unbalanced color**. This color class is assumed to contain two points that are incident to the arrangement and the number of points of this color in the interiors of the chessboard regions differ by exactly one.

Claim 3.6. Let $\mathcal{L}$ be a family of n subspaces, and $\mathcal{U}(t)$ a generic path of M-colored odd point sets of $\mathbb{R}^d$, let t_* be an exceptional time, so that $\mathcal{H}(t_*)$ is an almost bisecting arrangement of $U(t_*)$. If a point of the unbalanced color $q(t_*)$ is contained in $U(t_*) \cap h(t_*)$ for some $h(t_*) \in \mathcal{H}(t_*)$, but for $\epsilon > 0$ small enough, $q(t_* - \epsilon) \notin h(t_* - \epsilon)$, then exactly one of the arrangements $\mathcal{H}(t_* - \epsilon)$, $\mathcal{H}(t_* + \epsilon)$ is a valid bisecting arrangement.

Proof. Since the problem is local, we might parameterize in a neighbourhood around t_* so that all the points of $U(t)$ are fixed except for $q(t)$ with a trajectory that is transversal to $h(t_*)$. From the definition of almost bisecting it follows that exactly one of the two directions balances the color of q □

Let $\mathcal{H}$ be a valid or almost valid arrangement, and $\mathcal{H}_i$ a sub-arrangement with common perpendicular direction v in L_i. The **core** of $\mathcal{H}_i$, is the set of points in U that determine the direction v, in other words a point in $U \cap H_i$ is in the core, if it is contained in a hyperplane $h \in H_i$, such that $|h \cap U| > 1$.

Lemma 3.7. *If $\mathcal{U}(t)$ is a generic point set path, t_* an exceptional time, and $\mathcal{H}(t_*)$ an almost bisecting arrangement, then there exists a unique valid arrangement path $\mathcal{F}(t) \neq \mathcal{H}(t)$, such that $\mathcal{F}(t_*)$ is almost bisecting, and the oversaturated family of $\mathcal{F}(t_*)$ has the same core as the oversaturated family of $\mathcal{H}(t_*)$.*

Before going into the proof of this key lemma, let us show how combining it with Claim 3.6 we obtain Lemma 3.5.

Proof. Again we might think of $q(t)$ moving and the rest of the points being fixed. Since the path is generic any bisecting arrangement at time $t_j - \epsilon$ that is not almost bisecting at time t_j, is bisecting in an interval that contains the interval $[t_{j-1}, t_{j+1}]$. So we only need to analyze the paths of arrangements that are almost bisecting at t_j. By Lemma 3.7 if there are any such arrangements, then there are exaclty two, $\mathcal{H}(t_j + \epsilon)$ and $\mathcal{F}(t_j + \epsilon)$. Now we use Claim 3.6 to analyse the four cases: if both are bisecting, the number of bisecting arrangements increases by two at t_j, if both are non bisecting then number of bisecting arrangements decreases by two at t_j. If one of them is bisecting and the other one is not (this can happen in two ways), then the number of bisecting arrangements stays invariant at t_j. □

Our final step is proving Lemma 3.7.

Proof of Lemma 3.7. Assume that $\mathcal{H}(t_j)$ is almost bisecting. We are going to stop the movement of the points, and move continuously $\mathcal{H}(t_j)$ along a path of arrangements until we find another valid almost bisecting arrangement, $\mathcal{F}(t_j)$.

We denote the resulting path of valid almost bisecting hyperplane arrangements by $\mathcal{A}(s)$. So $\mathcal{A}(0) := \mathcal{H}(t_*)$, $\mathcal{A}(1) := \mathcal{F}(t_*)$ and for every $s \in [0,1]$, the point set $\mathcal{U}(t_*)$ is fixed.

Denote by (S_1, P_1) the colored set of points and valid partition that induced $\mathcal{H}(t_j - \varepsilon)$. Find the unique hyperplane $h^1(t_* - \epsilon) \in \mathcal{H}(t_* - \epsilon)$ that contains the point p_1 of the unbalanced color not in S_1 at time t_j. This is the only point contained by $H(t_*)$ that is not in S_1. We denote by $S = S_1 \cap \{p_1\}$ the set of $M+1$ points contained by $H(t_*)$ and by P the partition induced by $\mathcal{H}(t_*)$ on S, which consists of adding p_1 to one of the sets in the second layer of P_1.

We call this h^1, the **moving hyperplane**. It will remain the moving hyperplane until we reach a point of $U(t_*)$ and we might switch moving hyperplane. Denote by $H(t_*)$ the group of parallel hyperplanes that contains $h^1(t_*)$. Note that $H(t_*)$ is the oversaturated family of $\mathcal{H}(t_*)$.

Let $h(t_*)$ the hyperplane of $\mathcal{H}(t_*)$ that contains the point p_0 of S_1 of the same color as p_1. It might be that $h(t_*) = h^1(t_*)$, this is what happens in the ham-sandwich theorem. In this case, we can put $\mathcal{F}(t_*) = \mathcal{H}(t_*)$. The arrangement $\mathcal{F}(t)$ will be induced by replacing p_0 by p_1 in both S_1 and P_1. In general, if p_0 is in the core of $H(t_*)$, then we can simply replace p_0 by p_1 in both S and P to obtain $\mathcal{F}(t_*)$. Otherwise, we will start moving $h(t_*)$, with the process described below, until we reach a new point of $U(t_*)$.

We now explain how to define a path $\mathcal{A}(s)$ moving $h^1(s)$ (and sometimes the hyperplanes parallel to it) until we reach a new point that defines a new unbalanced color, which defines a new moving hyperplane $h^2(s)$. We repeat this process until we reach a point that is in the core of $H(t_*)$, at which point the process terminates. We first describe the movement process, than then show that it always terminates.

We will define a sequence of hyperplanes $h^1, \dots, h^r, \dots$ and time intervals

$$[s_1, s_2], \dots, [s_r, s_{r+1}]$$

such that h^r is the moving hyperplane exactly in $[s_r, s_{r+1}]$. The path of arrangements $\mathcal{A}(s)$ is the concatenation of all these movements. Below, we explain how $h^r(s)$ moves when $s \in [s_r, s_{r+1}]$. We denote by $j(r)$ the color that is unbalanced at time r. Throughout this process

If h^{r-1} at time s_r reaches a point whose color is not in the core of $H(t_*)$:

Let $h^r(s_r)$ be the hyperplane containing the other point of color $j(r)$

Case 1 : $h^r(s_r)$ contains a single point.

In this case we translate $h^r(s)$ along its perpendicular direction. Among the two opposite possible directions there is one along which the balance of color $j(r)$ is immediately fixed. We continue translating $h^r(s)$ in the same direction until we hit another point not covered by the arrangement. We are allowed to pass the translation through infinity and come back on the parallel hyperplane which arrives to convex hull of the points from the opposite side. So eventually we hit a new point. The new point defines the new unbalanced color $j(r+1)$, we let $h^{r+1}(s)$ be the hyperplane containing the other point of the new unbalanced color $j(r+1)$. We update S and P and reiterate this procedure. See Fig. 2.

Case 2 : $h^r(s_r)$ contains more than one point.

Let p be the point of color $j(r)$ in $h^r(s_r)$. Let H' be the family of hyperplanes parallel to $h^r(s_r)$. Note that p is in the core of H'. Since p is not in the core of $H(t_*)$, this is a different group of parallel hyperplanes. Denote by L_{ir} the space that determines the possible normal vectors for $h^r(s_r)$, and $k_{i(r)}$ be the number of hypeplanes of H'. Let X be the set of points covered by H'.

Consider the set $X \setminus \{p\}$. By Lemma 3.2, there is a unique $(d-2)$-dimensional space K and k translates of K that cover $X \setminus \{p\}$ inducing the same partition as H' on them. Let K_1 be the translate that covers $X \cap h^r(s_r)$. Note that $K^\perp$ is a two-dimensional space. Under the orthogonal projection to $K^\perp$, $X \setminus \{p\}$ is mapped to k distinct points $p_1, \ldots, p_k$. The way we form H' is by joining p_1 with the projection of p, forming a line ℓ, and then take the inverse of the orthogonal projection of the k translates of ℓ that contain each of $p_1, \ldots, p_k$.

To rotate h^r, we simply rotate ℓ around p_1, doing the same for its $k-1$ translates, and then take the inverse image. As before, of the two possible directions to start the rotation, there is a unique direction that balances color $j(r)$. We continue this rotation until the time s_{r+1} that a hyperplane of H' contains a point not previously contained by $\mathcal{A}(s_r)$ for $s > s_r$. We update P, S and repeat this process. See Fig. 3 for an illustration.

This finishes the description of the algorithm to construct $\mathcal{A}(s)$.

Correctness

We now finish the proof of Lemma 3.7 Notice that every time we switch the moving hyperplane, we obtain a new almost bisecting arrangement. We claim that the process terminates by eventually reaching a new point in a color of the core of $H(T_*)$.

Assume the contrary in search for a contradiction. The sequence of moving hyperplanes is finite, since they are all almost valid, eventually we have to repeat a pair made by a configuration and an unbalanced color, forcing us into a cycle.

However, the path $\mathcal{A}(s)$ is one dimensional and can be reversed all the way to $\mathcal{H}(t_*)$, so there cannot be a cycle. $\square$

3.4. **From oddly supported to general measures.** Our theorem follows for the particular case in which each of the measures is of the form $\frac{1}{|X|}\sum_{x\in X}\delta_x$, and $X \subset \mathbb{R}^d$ is a finite set with an odd number of points. An advantage of this case, which we call a *oddly supported measure*, is that every family of parallel hyperplanes that bisects a set with an odd number of points, must have one of the hyperplanes passing though one of the points. Any affine hyperplane is the zero set of an affine function of the form

$$(x_1, x_2, \ldots x_d) \to \langle (x_1, x_2, \ldots x_d), (v_1, v_2, \ldots v_d)\rangle + v_{d+1},$$

where $(v_1, v_2, \ldots v_d, v_{d+1}) \in \mathbb{S}^d$. The cases where the first d coordinates of v are zero corresponds to the the empty set when $v_{d+1} = -1$ and the whole space when $v_{d+1} = 1$.

Now for any Borel measure μ, if X is a sample of $2r+1$ points independently at random from μ and form the measure $\mu^r := \frac{1}{|X|}\sum_{x\in X}\delta_x$, then, as r tends to infinity, μ-almost surely μ^r converges weakly to μ. That is, for every closed set C, $\limsup_{r\to\infty}\mu^r(C) \leq$

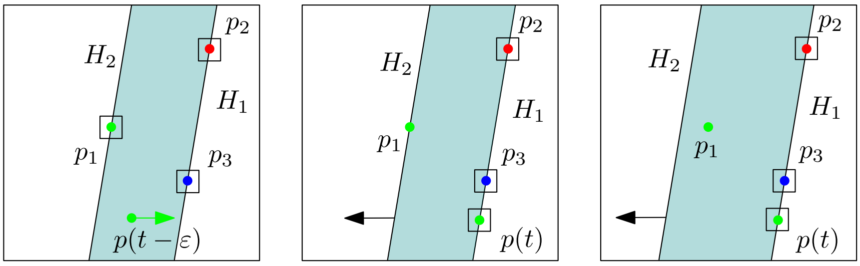

FIGURE 2. Example of a case when we a translation is needed. In this case, p belongs to $\operatorname{supp}\mu_1$ and will cross the hyperplane H_2 of the arrangement, which contains p_2, p_3. If p_1, the point in the arrangement from $\operatorname{supp}\mu_1$, is contained in a hyperplanes H_2 that contains no other point of the arrangement, we need to translate H_2 until it hits a new point of $U(t)$ not already contained in $A(t)$. The direction we translate H_2 is uniquely determined by which region lost $p(t-\varepsilon)$, as p_1 should replace $p(t-\varepsilon)$ in that region.
This figure exemplifies the case when H_2 is in the class of hyperplanes parallel to H_1. The same process follows if this is not the case. The square boxes indicate the candidates for the points generating the new arrangement $B(t)$. If the first point H_2 arrives to is in the support of $\mu_1(t), \mu_2(t), \mu_3(t)$, we would replace $p(t), p_2$, or p_3 by it, as needed. If it is in the support of another measure, further translations or rotations are needed to arrive to $B(t)$.

$\mu(C)$. Assume that we have shown the theorem for oddly supported measures, let A_r, B_r be the corresponding (closed) chessboard regions.

Lemma 3.8. *Let $L_1, L_2, \dots L_n$ be subsapces of $\mathbb{R}^d$, and μ be a Borel measure in $\mathbb{R}^d$. Let μ^r a sequence of measures converging weakly to μ. If for each r, (A_r, B_r) is a valid chessboard partition of $\mathbb{R}^d$ that bisects μ^r, then there exists a subsequence of chessboard partitions that converges to a chessboard partition (A, B) that bisects μ.*

Proof. The chessboard partition (A_r, B_r) is determined by a valid hyperplane arrangement $\mathcal{H}^r$. Valid hyperplane arrangements are parametrised by a closed subset of $(\mathbb{S}^d)^{|\mathcal{H}^r|}$, which is a compact space. Hence, a subsequence of arrangements (and of chessboard regions) converges to a hyperplane arrangement $\mathcal{H}$.

Let A and B be the corresponding limiting chessboard regions. Then for any $\delta > 0$ there exists r_0 large enough, so that for every $r > r_0$,

$$\mu(A) \geq \mu^r(A) - \delta \geq \mu^r(A_r) - \mu^r(A_r \setminus A) - \delta \geq$$

$$\mu^r(A_r) - \mu(A_r \setminus A) - 2\delta \geq \mu^r(A_r) - 3\delta \geq \frac{1}{2} - 3\delta,$$

hence $\mu(A) \geq \frac{1}{2}$, and similary for B. $\square$

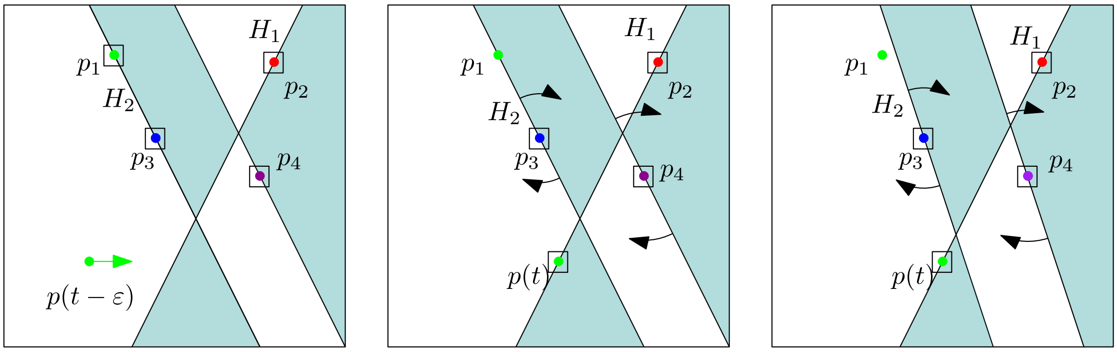

FIGURE 3. A similar setting as Fig. 2, but now p_1, the point of $A(t)$ in the same measure as $p(t)$ is a hyperplane which has additional point of the support (p_3 in the figure above). In this case, we rotate the class of hyperplanes parallel to H_2. The rotation of H_2 is around p_3, and every hyperplane parallel to H_2 has a unique rotation point. The direction of the rotation is uniquely determined, as p_1 must be at time t in the same region as $p(t-\varepsilon)$ was at time $t-\varepsilon$. The rotation continues until the class of hyperplanes parallel to H_2 contain as new point of $U(t)$ that is not contained in $A(t)$.

To derive Theorem 1.1 as claimed, we apply this lemma to each of the measures μ_i, refining the converging subsequence of cheesboard regions. Since we do this a finite number of times, the final subsequence converges. Notice that we are allowing for one or more hyperplanes to escape to infinity.

4. Necessity of conditions

We show that for the case $l_1 = \cdots = l_n = 1$, if $L_1, \ldots, L_n$ are all different and at least two of the numbers $k_1, \ldots, k_n$ are odd, then Theorem 1.1 may fail. Note that the parity condition for Theorem 1.1 in that instance is that the mulitnomial coefficient

$$\binom{M}{k_1, \ldots, k_n}$$

must be odd. This happens if and only iff the numbers $k_1, \ldots, k_n$ do not share 1's in the same position when written in base two. If two of them are odd, then we are breaking this property in the first instance.

Claim 4.1. Let $l_1 = \cdots = l_n = 1$, and $L_1, \ldots, L_n$ be n different lines in $\mathbb{R}^d$ through the origin. Let $k_1, \ldots, k_n$ such that at k_1, k_2 are odd. Then, there is a set of $M = k_1 + \cdots + k_n$ measures on $\mathbb{R}^d$ such that no set of M hyperplanes such that for each $i \in [n]$, exactly k_i of the hyperplanes are orthogonal to l_i induces a chessboard coloring that bisects each measure.

Proof. We will divide the proof into two cases, when M is even and when M is odd.

Case 1. M is even.

In this case, consider a set of $M/2$ segments in $\mathbb{R}^d$ such that none of them is parallel nor orthogonal to an L_i, and their midpoints are in general position. Assume that the segments are sufficiently short so that no hyperplane orthogonal to an L_i can intersect two of the constructed segments.

Finally duplicate each segment and translate their copy orthogonally in a direction that is neither parallel nor orthogonal to any L_i. This gives us in total $M/2$ small pairs of parallel hyperplanes. We distribute each measure uniformly in one of these M little segments.

For any two directions L_i, L_j, if we project a pair of parallel segments onto $\Pi = \operatorname{span}(L_i, L_j)$, we obtain exactly the two-dimensional example constructed by Karasev, Roldá-Pensado, and Soberón [KRPS16]. It is impossible to bisect both projected segments using a line orthogonal to L_1 and a line orthogonal to L_2.

Now we go back to $\mathbb{R}^d$, and assume we have a valid hyperplane arrangement. If any pair of segment is intersected by three or more hyperplanes, since we have $M/2$ pairs of segments and M hyperplanes, there mus be another pair of segments that is intersected by 1 or less hyperplane. Using just one hyperplane it is impossible to bisect the pair, and we would be done. If every pair of segments is intersected by exactly two hyperplanes, by construction we can only bisect all of them if each pair is intersected by a pair of parallel hyperplanes. Since there is an odd number of hyperplanes orthogonal to L_1, at least one of them will be paired with a hyperplane with another fixed direction, so there will be a pair of parallel segments we are not bisecting.

Case 1. M is odd.

In this case, we construct $(M + 1)/2$ pairs of parallel segments as before, and then remove one segment of the final pair. Assume that we have a valid arrangement of hyperplanes that bisects this last segment. If at least two hyperplanes cut this last segment, then we do not have enough hyperplanes to bisect the remaining $(M - 1)/2$ pairs of tiny parallel segments. If exactly one hyperplane cuts the last segment, we may assume without loss of generality that is is not one of the hyperplanes orthogonal to L_1 (otherwise, replace the roles of L_1 and L_2 in the rest of the argument). Then, since we still have to use an odd number of hyperplanes orthogonal to L_1, the argument follows as in case 1. □

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