New point of view about optical activity in helically-coiled fiber

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Abstract

The optical activity in a helically-coiled optical fiber is reexamined. It is proven that not only is there no circular birefringence in the fiber but the polarization relative to the laboratory reference frame is not rotated along the fiber. The reason for this is that in contrast with the polarization vector, the Jones vector does not give a complete description of the polarization. As a mathematical entity in some local reference frame that depends on the instantaneous propagation direction, it can only describe the state of polarization relative to that reference frame. With the new implication of the Jones vector, the results of the experiment reported by Papp and Harms in 1977 are explained satisfactorily. In particular, it is shown that the state of polarization relative to the Tang frame remains unchanged along the fiber. The optical activity appears only relative to the Serret-Frenet frame.

Keywords: Nonexistence of circular birefringence, Optical activity, Helically-coiled optical fiber, Polarization state of light, Jones vector, Laboratory reference frame, Local reference frame

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I. INTRODUCTION

Optical activity is one of the most fundamental optical phenomena in nature. It refers to the rotation of the polarization plane of linearly-polarized light as it travels through optically active media [1]. Both chiral media [2–4] and helically-coiled optical fibers [5–7] show optical activity. Allogyric birefringence [8] or circular birefringence [9], which means different propagation velocities or different refraction indices for right- and left-handed circularly-polarized waves, was commonly believed [10, 11] to underlie the optical activity. In a previous paper [12], we proved through a logical analysis that there is no circular birefringence in a chiral medium. Meanwhile, we found that the conventional description of the polarization of light in terms of the Jones vector [13] is incomplete. We will further prove in the present paper that there is no circular birefringence in a helically-coiled fiber either. More importantly, we will show that the reason for the Jones vector not to be able to completely describe the polarization is that it can only describe the polarization relative to the local reference frame depending on the propagation direction of light.

It should be noted that the optical activity in the helically-coiled fiber is distinguished from that in the chiral medium. About half a century ago, Papp and Harms [5] observed in their experiment that "the plane of polarization is not really rotated along the fiber." This observation has actually ruled out the possibility of circular birefringence in the helically-coiled fiber. Unfortunately, such an important finding has not received the attention it deserves. Of course, due to the nonzero curvature of the fiber, the polarization relative to the laboratory reference frame cannot remain unchanged along the fiber. One needs also to explain what on earth Papp and Harms observed. The purpose of this paper is to address these issues. The contents are arranged as follows.

For the sake of clarity, Section II gives a review of the conventional interpretation of the optical activity in the helically-coiled fiber. Eqs. (10) were interpreted as expressing the circular birefringence, which was considered to be responsible for the optical activity expressed by Eq. (7) for the Jones vector $\tilde{A}$ or its solution (12). The nonexistence of circular birefringence is proven in Section III in one particular case, the adiabatic limit [14, 15]. Eqs. (10) are reinterpreted to reflect the rotation of the Serret-Frenet frame [16] about the instantaneous direction of the fiber. This is achieved by making use of the Tang frame [17], a different local reference frame that does not rotate about the instantaneous direction of the fiber. The reason is argued in Section IV to be that in contrast with the polarization vector, the Jones vector is not a complete description of the polarization. What it describes is merely the state of polarization relative to the local reference frame. Based on this, we show in Section V that it is relative to the Tang frame that the state of polarization does not change along the fiber. Eq. (12) just means that the optical activity appears relative to the Serret-Frenet frame. Section VI concludes the paper with remarks. In particular, we point out that in the general non-adiabatic case, one cannot have a Jones vector to describe the polarization relative to the laboratory reference frame.

II. REVIEW OF CONVENTIONAL INTERPRETATION

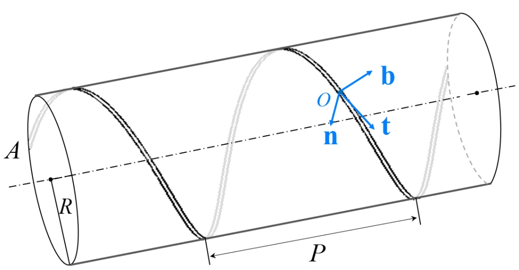

FIG. 1. Helically-coiled optical fiber showing the Serret-Frenet frame **tnb** at a point O.

The axis of a helically-coiled optical fiber is described mathematically as a twisted curve with a constant curvature and a constant torsion [16]. The geometry of the fiber is schematically shown in Fig. 1, where P is the pitch and R is the radius of the helix. The radius of the fiber core is assumed to be much smaller than R so that the linear birefringence induced by bending the fiber is negligibly small [18]. The Serret-Frenet frame at a point O on the fiber axis is an orthogona1 trihedron of unit vectors denoted by $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$, which obey

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1, \quad \mathbf{t} = \mathbf{n} \times \mathbf{b}, \tag{1}$$

where $\mathbf{t}$ is the tangent representing the instantaneous direction of the fiber, $\mathbf{n}$ is the principal normal, and $\mathbf{b}$ is the binormal. They satisfy the Serret-Frenet formulae,

$$\frac{d\mathbf{t}}{ds} = \chi \mathbf{n}, \tag{2a}$$

$$\frac{d\mathbf{n}}{ds} = -\chi \mathbf{t} + \tau \mathbf{b}, \tag{2b}$$

$$\frac{d\mathbf{b}}{ds} = -\tau \mathbf{n}, \tag{2c}$$

where s is the directed arc measured along the fiber from a fixed point A to the variable point O, χ is the curvature, and τ is the torsion.

Consider a monochromatic wave traveling down the fiber. In the plane-wave approximation, its electric field at a point of distance s vibrates according to [5, 6, 19]

$$\mathbf{E}(s,t) = \mathbf{A}(s) \exp[i(ks - \omega t)], \tag{3}$$

where the polarization vector $\mathbf{A}$ denotes the state of polarization, ω is the angular frequency, $k = nk_0$, n is the refraction index of the fiber core, and k_0 is the wavenumber in free space. As the vector amplitude, $\mathbf{A}$ is transverse to the local propagation direction, the tangent $\mathbf{t}$,

$$\mathbf{t} \cdot \mathbf{A} = 0, \tag{4}$$

and therefore can be mathematically expressed as

$$\mathbf{A}(s) = A_n \mathbf{n} + A_b \mathbf{b}, \tag{5}$$

where A_n and A_b are the projections onto $\mathbf{n}$ and $\mathbf{b}$, respectively. To simplify the analysis, we assume no linear birefringence in the fiber. In this case, A_n and A_b satisfy the following coupled equations [6, 10, 19],

$$\frac{dA_n}{ds} = \tau A_b, \quad \frac{dA_b}{ds} = -\tau A_n. \tag{6}$$

Denoting $\tilde{A} = \begin{pmatrix} A_n \\ A_b \end{pmatrix}$, which is the well-known Jones vector, these two equations can be integrated into a single one [20],

$$\frac{d\tilde{A}}{ds} = i\tau \hat{\sigma}_3 \tilde{A}, \tag{7}$$

where $\hat{\sigma}_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. This equation was conventionally interpreted [6] as expressing the rotation of the polarization along the fiber. Letting

$$A_r = \frac{1}{\sqrt{2}}(A_n - iA_b), \quad A_l = \frac{1}{\sqrt{2}}(A_n + iA_b), \tag{8}$$

where A_r and A_l were called [10] the amplitudes of right- and left-handed circularly-polarized modes, respectively, it can be obtained from Eqs. (6) that

$$\frac{dA_r}{ds} = i\tau A_r, \quad \frac{dA_l}{ds} = -i\tau A_l. \tag{9}$$

They have solutions of the form

$$A_r = \alpha_r \exp(i\tau s), \tag{10a}$$

$$A_l = \alpha_l \exp(-i\tau s), \tag{10b}$$

under the "initial" conditions $A_r|_{s=0} = \alpha_r$ and $A_l|_{s=0} = \alpha_l$. These two equations were conventionally interpreted as expressing the circular birefringence, with $k+\tau$ and $k-\tau$ being interpreted [10] as the propagation constants of the right- and left-handed circularly-polarized modes, respectively. Taking Eqs. (10) into account, it can be readily found from Eqs. (8) that

$$\begin{aligned} A_n &= \alpha_n \cos\tau s + \alpha_b \sin\tau s, \\ A_b &= \alpha_b \cos\tau s - \alpha_n \sin\tau s, \end{aligned}$$

where

$$\alpha_n = \frac{1}{\sqrt{2}}(\alpha_r + \alpha_l), \quad \alpha_b = \frac{i}{\sqrt{2}}(\alpha_r - \alpha_l). \tag{11}$$

In terms of the Jones vector $\tilde{A}$, the above two equations can be converted into

$$\tilde{A} = \exp(i\hat{\sigma}_3 \tau s)\tilde{\alpha}, \tag{12}$$

where $\tilde{\alpha} = \begin{pmatrix} \alpha_n \\ \alpha_b \end{pmatrix}$. It can be viewed as the solution of Eq. (7) under the initial condition $\tilde{A}|_{s=0} = \tilde{\alpha}$.

III. NO CIRCULAR BIREFRINGENCE EXISTS

Let us first show that Eqs. (10) cannot be interpreted as expressing the circular birefringence. As is known, the torsion τ and the curvature χ are two independent parameters of a helix. But it is noticed that Eqs. (10) depend only on τ. They have nothing to do with χ. So it is enough to prove the assertion in one particular case, the adiabatic limit $\chi \to 0$ [14, 15].

A. A proof in the adiabatic limit

Since the polarization is essentially the property denoted by the polarization vector, we pay attention to expression (5). Although Eqs. (6) for A_n and A_b do not depend on χ, Eqs.

(2b) and (2c) for the rotation of the Serret-Frenet frame along the fiber do. In the adiabatic limit $\chi \to 0$, they reduce to

$$\frac{d\mathbf{n}}{ds} = \tau\mathbf{b}, \quad \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}. \tag{13}$$

With the help of Eqs. (6) and (13), it is not difficult to find from expression (5) that

$$\frac{d}{ds}\mathbf{A}(s) = 0. \tag{14}$$

This is the equation for the variation of the polarization vector $\mathbf{A}$ with respect to s in the adiabatic limit. Its solution is simply a constant vector,

$$\mathbf{A}(s) = \mathbf{a}.$$

With this in mind, one can see from expression (3) that the propagation constant of the wave is always k no matter what the concrete state of its polarization is. As a corollary, the right- and left-handed circularly-polarized modes propagate at the same velocity. We thus prove the assertion that there is no circular birefringence. This is in fact not difficult to understand, because the helically-coiled fiber tends to be a straight one when the curvature approaches zero [16].

If this is the case, how do we understand Eqs. (10)? To address this issue, we note, as is shown by Eq. (14), that the polarization vector $\mathbf{A}$ in the adiabatic limit remains unchanged along the fiber regardless of the concrete state of the polarization. As a result, the polarization vectors of circularly-polarized modes should have the same property. To see this, we make use of Eqs. (8) to rewrite expression (5) as $\mathbf{A}(s) = \mathbf{A}_r(s) + \mathbf{A}_l(s)$, where

$$\begin{aligned} \mathbf{A}_r(s) &= \frac{1}{\sqrt{2}}A_r(\mathbf{n} + i\mathbf{b}) \\ \mathbf{A}_l(s) &= \frac{1}{\sqrt{2}}A_l(\mathbf{n} - i\mathbf{b}) \end{aligned} \tag{15}$$

are the polarization vectors of right- and left-handed circularly-polarized components, respectively, and A_r and A_l are given by Eqs. (10). With the help of Eqs. (9) and (13), a straightforward calculation gives

$$\frac{d}{ds}\mathbf{A}_r(s) = 0, \quad \frac{d}{ds}\mathbf{A}_l(s) = 0.$$

As expected, they are indeed in the same form as Eq. (14). Their solutions are thus constant vectors, too,

$$\mathbf{A}_r(s) = \mathbf{a}_r, \quad \mathbf{A}_l(s) = \mathbf{a}_l. \tag{16}$$

What is noteworthy is that it is only in the sense of Eqs. (15) that A_r and A_l stand for the amplitudes of right- and left-handed circularly polarized modes, respectively. The point, however, is that $\mathbf{n}$ and $\mathbf{b}$ are not fixed unit vectors. As is indicated by Eqs. (13), the Serret-Frenet frame in the adiabatic limit still rotates along the fiber. The torsion τ is the per length rate of rotation about the tangent $\mathbf{t}$. A comparison of Eqs. (15) with Eqs. (16) might lead us to a conclusion that Eqs. (10) reflect the rotation of the Serret-Frenet frame about $\mathbf{t}$. Considering that the torsion is independent of the curvature, this conclusion should hold true in the general non-adiabatic case. Let us show this in detail below.

B. Reinterpretation of Eqs. (10)

To this end, it is beneficial to introduce a local reference frame that does not rotate about the tangent $\mathbf{t}$. Now that the Serret-Frenet frame rotates about $\mathbf{t}$ with a per length rate τ, the result of reversely rotating the Serret-Frenet frame about $\mathbf{t}$ with a rotation rate $-\tau$ will meet our need. Denoting such a reference frame by unit vectors $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$, which constitute an orthogonal trihedron, we have

$$\mathbf{u} = \exp[-i(\mathbf{\Sigma} \cdot \mathbf{t})\psi]\mathbf{n}, \quad \mathbf{v} = \exp[-i(\mathbf{\Sigma} \cdot \mathbf{t})\psi]\mathbf{b},$$

where $(\Sigma_k)_{ij} = -i\epsilon_{ijk}$ with ϵ_{ijk} the Levi-Civitá pseudotensor, $\psi = \delta - \tau s$, and the constant δ represents an initial angle of the rotation. With the help of Eqs. (1) and the formula [21]

$$\exp[-i(\mathbf{\Sigma} \cdot \mathbf{x})\phi]\mathbf{y} = \mathbf{y}\cos\phi + \mathbf{x} \times \mathbf{y}\sin\phi + \mathbf{x}(\mathbf{x} \cdot \mathbf{y})(1 - \cos\phi),$$

where $\mathbf{x}$ and $\mathbf{y}$ are any two vectors, the above two equations can be rewritten as

$$\mathbf{u} = \mathbf{n}\cos\psi + \mathbf{b}\sin\psi, \quad \mathbf{v} = -\mathbf{n}\sin\psi + \mathbf{b}\cos\psi. \tag{17}$$

It is seen that so introduced reference frame is precisely the Tang frame [17]. It should be emphasized that the Tang frame shares the same longitudinal axis $\mathbf{t}$, the propagation direction of the wave, with the Serret-Frenet frame. It differs from the Serret-Frenet frame only in the transverse axes. The variations of $\mathbf{u}$ and $\mathbf{v}$ along the fiber satisfy

$$\frac{d\mathbf{u}}{ds} = -\chi\mathbf{t}\cos\psi, \quad \frac{d\mathbf{v}}{ds} = \chi\mathbf{t}\sin\psi, \tag{18}$$

by virtue of Eqs. (2b) and (2c). The inverse transformation of Eqs. (17) is given by

$$\mathbf{n} = \mathbf{u}\cos\psi - \mathbf{v}\sin\psi, \quad \mathbf{b} = \mathbf{u}\sin\psi + \mathbf{v}\cos\psi, \tag{19}$$

meaning that the Serret-Frenet frame is the result of the rotation of the Tang frame about $\mathbf{t}$ with a rotation rate τ. This in turn illustrates that the Tang frame is a reference frame with zero rate of rotation about $\mathbf{t}$ [19].

From Eqs. (19) one readily finds

$$\begin{aligned} \mathbf{n}+i\mathbf{b} &= \exp(i\delta)(\mathbf{u}+i\mathbf{v})\exp(-i\tau s), \\ \mathbf{n}-i\mathbf{b} &= \exp(-i\delta)(\mathbf{u}-i\mathbf{v})\exp(i\tau s). \end{aligned} \tag{20}$$

Now that the Tang frame does not rotate about $\mathbf{t}$, the phase factors $\exp(-i\tau s)$ and $\exp(i\tau s)$ in these two equations result only from the rotation of the Serret-Frenet frame about $\mathbf{t}$ with the rotation rate τ. They have nothing to do with the propagation of the wave down the fiber. Furthermore, as can be seen from Eqs. (15), the former is canceled out by the phase factor $\exp(i\tau s)$ in Eq. (10a) and the latter is canceled out by the phase factor $\exp(-i\tau s)$ in Eq. (10b). So we are indeed convinced that the two different phase factors $\exp(i\tau s)$ and $\exp(-i\tau s)$ in Eqs. (10) reflect the rotation of the Serret-Frenet frame about $\mathbf{t}$, without meaning the circular birefringence.

IV. DISTINGUISHING JONES VECTOR FROM POLARIZATION VECTOR

Frankly speaking, the reason to traditionally misinterpret Eqs. (10) as the circular birefringence was to implicitly assume [1, 9] that the polarization vector $\mathbf{A}$ in expression (3) could be fully replaced with the Jones vector in Eq. (12). Under that assumption, expression (3) can be rewritten as [6]

$$\mathbf{E}(s) = \tilde{A}(s)\exp[i(ks-\omega t)]. \tag{21}$$

In addition, with the help of Eqs. (11), the Jones vector in (12) can be split into two parts, $\tilde{A} = \tilde{A}_r + \tilde{A}_l$, where

$$\tilde{A}_r = \tilde{\alpha}_r \exp(i\tau s) = \frac{A_r}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \tag{22a}$$

$$\tilde{A}_l = \tilde{\alpha}_l \exp(-i\tau s) = \frac{A_l}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{22b}$$

are the Jones vectors of right- and left-handed circularly polarized modes, respectively, $\tilde{\alpha}_r = \frac{\alpha_r}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ is the eigen vector of $\hat{\sigma}_3$ with eigenvalue $+1$, $\tilde{\alpha}_l = \frac{\alpha_l}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigen vector of $\hat{\sigma}_3$ with eigenvalue -1, and A_r and A_l are given by Eqs. (10). Taking Eqs. (22) into consideration, one will see from Eq. (21) that the propagation constants of the right- and left-handed circularly polarized modes are $k+\tau$ and $k-\tau$, respectively. Now that Eqs. (10) do not convey the circular birefringence as we showed above, it is unreasonable to replace the polarization vector $\mathbf{A}$ with the Jones vector $\tilde{A}$ to describe the polarization. As a matter of fact, the distinction between Eq. (14) for the polarization vector in the adiabatic limit and Eq. (7) has illustrated the point. Let us expound further on it in more detail.

A. Jones vector cannot completely describe polarization

We turn attention back to expression (5). With the help of Eqs. (2b), (2c), and (6), it is not difficult to find that

$$\frac{d\mathbf{A}}{ds} = -\chi A_n \mathbf{t}. \tag{23}$$

The reason for this is that the polarization vector $\mathbf{A}$ should satisfy Eq. (4). As a matter of fact, upon taking Eqs. (1) and expression (5) into consideration, it can be shown from Eqs. (2a) and (23) that

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{A}) = 0,$$

in agreement with Eq. (4). In the adiabatic limit $\chi \to 0$, Eq. (23) reduces to Eq. (14). Since the polarization vector denotes the state of polarization, Eq. (23) determines the way in which the polarization varies along the fiber. The problem is that even if $\mathbf{A}$ is replaced with $\tilde{A}$, it does not have the same meaning as Eq. (7). Of course, this can be seen by comparing the solution of Eq. (23) with Eq. (12). For the purpose of the present paper, here we introduce a new Jones vector to illustrate the problem.

In the Tang frame, the polarization vector of the wave (3) can be expressed as follows,

$$\mathbf{A}(s) = A_u \mathbf{u}(s) + A_v \mathbf{v}(s), \tag{24}$$

where A_u and A_v are the projections onto $\mathbf{u}$ and $\mathbf{v}$, respectively. Like A_n and A_b in expression (5), A_u and A_v here also make up a Jones vector, denoted by $\tilde{A}' = \begin{pmatrix} A_u \\ A_v \end{pmatrix}$. Substituting Eqs. (19) into expression (5) and comparing the result with expression (24), we have

$$\begin{aligned} A_u &= A_n \cos\psi + A_b \sin\psi, \\ A_v &= -A_n \sin\psi + A_b \cos\psi, \end{aligned}$$

which can be rewritten in terms of $\tilde{A}'$ and $\tilde{A}$ as

$$\tilde{A}' = \exp(i\hat{\sigma}_3\psi)\tilde{A}. \tag{25}$$

With the help of Eq. (7), it is easy to obtain from Eq. (25) that

$$\frac{d\tilde{A}'}{ds} = 0, \tag{26}$$

meaning that the Jones vector $\tilde{A}'$ does not change along the fiber. If the Jones vector could take the place of the polarization vector, Eq. (26) would mean that the polarization remains unchanged along the fiber. This is obviously in contradiction with the prediction of Eq. (23). We thus have to confess that the Jones vector cannot take the place of the polarization vector to describe the polarization. In other words, the Jones vector cannot describe the polarization itself. For this reason, Eq. (7) does not mean the rotation of the polarization along the fiber, just like that Eq. (26) does not mean the invariance of the polarization along the fiber.

Thus far, we have shown that the Jones vector cannot give a complete description of the polarization. But, as is known, it does describe the polarization somehow. So, in order to clearly and unambiguously explain the meaning of Eq. (7) as well as of Eq. (26), it is required to make clear how the Jones vector is distinguished from the polarization vector and in what way the Jones vector describes the polarization.

B. Jones vector describes state of polarization relative to local reference frame

To do this, the first thing to note is that the state of polarization denoted by the polarization vector $\mathbf{A}$ ultimately means the state of polarization relative to the laboratory reference frame. That is to say, $\mathbf{A}$ needs to be thought of as a mathematical entity in the laboratory reference frame. However, the fact that Eq. (26) does not mean the invariance of the polarization along the fiber indicates that the Jones vector $\tilde{A}'$ is not a mathematical entity in the laboratory reference frame. Likewise, the fact that Eq. (7) does not mean the rotation of the polarization along the fiber indicates that the Jones vector $\tilde{A}$ is not a mathematical entity in the laboratory reference frame, either. Furthermore, that Eq. (26) is different from Eq. (7) demonstrates that $\tilde{A}'$ and $\tilde{A}$ are not quantities in the same reference frame. As a matter of fact, the Jones vector $\tilde{A}$ is introduced when the polarization vector is expanded in the Serret-Frenet frame via expression (5) and the Jones vector $\tilde{A}'$ is introduced when the polarization vector is expanded in the Tang frame via expression (24). It is therefore reasonable to think that the former is a quantity in the Serret-Frenet frame and the latter is a quantity in the Tang frame. In view of this, we can say that the Jones vector plays the role of describing the state of polarization relative to some local reference frame. Specifically, the Jones vector $\tilde{A}$ describes the state of polarization relative to the Serret-Frenet frame; the Jones vector $\tilde{A}'$ describes the state of polarization relative to the Tang frame.

To explicitly express that the Jones vector $\tilde{A}$ is a quantity in the Serret-Frenet frame, we introduce the 2-by-3 matrix $\varpi = (\, \mathbf{n} \;\; \mathbf{b} \,)^\dagger$ and rewrite expression (5) as [22]

$$\mathbf{A} = \varpi^\dagger \tilde{A}, \tag{27}$$

where the unit vectors $\mathbf{n}$ and $\mathbf{b}$ are regarded as column matrices of three elements, which are their Cartesian components in the laboratory reference frame, the superscript $\dagger$ denotes conjugate transpose, and the rule of matrix multiplication is used. By virtue of Eqs. (1), the matrix ϖ has the property

$$\varpi \varpi^\dagger = I_2,$$

where I_2 is the 2-by-2 unit matrix. Multiplying both sides of Eq. (27) from the left with ϖ and taking this property into account, we have

$$\tilde{A} = \varpi \mathbf{A}. \tag{28}$$

It is worth noting that the matrix ϖ has included the unit vectors of the transverse axes of the Serret-Frenet frame as its two rows $\mathbf{n}^\dagger$ and $\mathbf{b}^\dagger$. Considering that $\mathbf{n}$ and $\mathbf{b}$ are always perpendicular to $\mathbf{t}$, the matrix ϖ can also be said to represent the Serret-Frenet frame itself. In this sense, Eq. (28) means that $\tilde{A}$ is the projection of the polarization vector $\mathbf{A}$ onto the Serret-Frenet frame. Similarly, to explicitly express that the Jones vector $\tilde{A}'$ is a quantity in the Tang frame, we introduce $\varpi' = (\, \mathbf{u} \;\; \mathbf{v} \,)^\dagger$ to represent the Tang frame, the relation of which with ϖ is given by

$$\varpi' = \exp(i\hat{\sigma}_3 \psi)\varpi, \tag{29}$$

by virtue of Eqs. (17). It also has the property

$$\varpi' \varpi'^\dagger = I_2.$$

With the help of ϖ', $\tilde{A}'$ is related to the polarization vector $\mathbf{A}$ by

$$\tilde{A}' = \varpi' \mathbf{A}, \tag{30}$$

meaning that $\tilde{A}'$ is the projection of the polarization vector $\mathbf{A}$ onto the Tang frame. It is noted, by the way, that Eq. (29) is in consistency with relation (25) as can be seen from Eqs. (30) and (28).

Careful readers may have realized that the Tang frame is not uniquely determined due to the arbitrary parameter δ in $\psi = \delta - \tau s$. This is indeed the case. As mentioned before, the Tang frame is introduced by rotating the Serret-Frenet frame about the local propagation direction $\mathbf{t}$. Because the geometric meanings of $\mathbf{n}$ and $\mathbf{b}$ are clearly defined, the Serret-Frenet frame ϖ is uniquely determined by the Serret-Frenet formulae (2) if the initial conditions are given. But on the other hand, as an initial angle of rotation, δ can take on any real value in the interval $[0, 2\pi]$. Hence, different values of δ denote different Tang frames as can be seen from Eq. (29). Substituting Eq. (12) into Eq. (25) gives

$$\tilde{A}' = \exp(i\hat{\sigma}_3 \delta)\tilde{\alpha}.$$

This again illustrates that the Jones vector is a quantity in the local reference frame specified by the parameter δ.

In concluding this section, we summarize our main results. In contrast with the polarization vector, the Jones vector cannot completely describe the polarization. As a mathematical entity in some local reference frame, it can only describe the state of polarization relative to that reference frame. After clarifying the meaning of the Jones vector, we are prepared to show that Eqs. (7) and (26) serve well to explain Papp and Harms' experimental results.

V. EXPLANATION OF PAPP AND HARMS' EXPERIMENTAL RESULTS

Let us first look at Eq. (26). If the Jones vector $\tilde{A}'$ describes the state of polarization relative to the Tang frame, Eq. (26) reveals that the state of polarization, when viewed in the Tang frame, remains unchanged along the fiber. This should be nothing but Papp and Harms' observation that "the plane of polarization is not really rotated along the fiber." [5]. After all, as Eq. (23) shows, the polarization relative to the laboratory reference frame does not remain fixed along the fiber.

Then we turn attention to Eq. (7) or its solution (12). Now that the Jones vector $\tilde{A}$ describes the state of polarization relative to the Serret-Frenet frame, Eq. (12) shows that the Serret-Frenet frame is such a reference frame relative to which the state of polarization is rotated along the fiber. This is what Papp and Harms exactly meant by saying "the optical activity exists only with respect to the coordinate system along the fiber." In the cases of right- and left-handed circular polarization, Eq. (12) becomes (22a) and (22b), respectively. So, the torsion-dependent phase factor $\exp(i\tau s)$ or $\exp(-i\tau s)$ in Eqs. (10) can also be interpreted as the result of the rotation of the circularly-polarized state relative to the Serret-Frenet frame.

We stress that although the polarization relative to the laboratory reference frame does not remain fixed along the fiber, it is not rotated in accordance with Eq. (12). The relation between the polarization vector $\mathbf{A}$ and the Jones vector $\tilde{A}$ in (12) is given by Eq. (27). So, in order to know how the polarization relative to the laboratory reference frame varies along the fiber, it is only required to find the solutions to Eqs. (2b) and (2c) under certain initial conditions. A discussion of this issue is beyond the scope of the present paper.

VI. CONCLUSIONS AND REMARKS

In conclusion, we not only proved that Eqs. (10) do not mean the circular birefringence but also showed that Eq. (12) does not mean the rotation of the polarization relative to the laboratory reference frame. We argued that the Jones vector of a plane light wave is mathematically different from its polarization vector and therefore cannot give a complete description of its polarization. We demonstrated that the Jones vector defined via Eq. (28) is a quantity in the Serret-Frenet frame represented by the matrix ϖ and the Jones vector defined via Eq. (30) is a quantity in the Tang frame represented by the matrix ϖ'. Each describes the state of polarization relative to the relevant local reference frame. On this basis, we finally explained what Papp and Harms had really observed in their experiment. We found that it is relative to the Tang frame that the state of polarization remains unchanged along the fiber. The optical activity appears only relative to the Serret-Frenet frame.

It is noted, as is expressed by Eqs. (28) and (30), that the two-element Jones vector of a plane wave is defined in such a local reference frame that depends on its propagation direction. Because the propagation direction of the wave traveling down a helically-coiled fiber is not fixed along the fiber, it is impossible to define in the laboratory reference frame a Jones vector for the polarization of the wave. That is to say, one cannot have a Jones vector that describes the polarization relative to the laboratory reference frame.

In a word, we showed, through reexamining the optical activity in a helically-coiled fiber, that the Jones vector cannot completely take the place of the polarization vector to describe the polarization of a plane wave. In order to make use of the Jones vector to do so, the local reference frame in which the Jones vector is defined must be specified simultaneously. It is hoped that the findings presented here will deepen understanding of the phenomenon of optical polarization.

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[1] E. Hecht, *Optics*, 4th ed. (Addison-Wesley, San Francisco, 2001).

[2] L. D. Barron, *Molecular Light Scattering and Optical Activity*, 2nd ed. (Cambridge University Press, Cambridge, 2004).

[3] A. Lakhtakia, V. V. Varadan, and V. K. Varadan, A parametric study of microwave reflection characteristics of a planar achiral-chiral interface, IEEE Trans. Electromagn. Compat. **28**, 90 (1986).

[4] J. Lekner, Optical properties of isotropic chiral media, Pure Appl. Opt. **5**, 417–443 (1996).

[5] A. Papp and H. Harms, Polarization optics of liquid-core optical fibers, Appl. Opt. **16**, 1315 (1977).

[6] J. N. Ross, The rotation of the polarization in low birefringence monomode optical fibres due to geometric effects, Opt Quant Electron **16**, 455 (1984).

[7] Y. Chen and P. St. J. Russell, Serret-Frenet analysis of helical Bloch modes in N-fold rotationally symmetric rings of coupled spiraling optical waveguides, J. Opt. Soc. Am. B **38**, 1173 (2021).

[8] R. W. Ditchburn, *Light* (Blackie & Son, London, 1958).

[9] D. H. Goldstein, *Polarized Light*, 3rd ed. (Taylor and Francis, New York, 2011).

[10] J. Qian and C. D. Hussey, Circular birefringence in helical-core fibre, Electron. Lett. **22**, 515 (1986).

[11] A. Ghosh and P. Fischer, Chiral molecules split light: reflection and refraction in a chiral liquid, Phys. Rev. Lett. **97**, 173002 (2006).

[12] Z.-J. Hu and C.-F. Li, No circular birefringence in a chiral medium: an analysis of single-mode refraction, New J. Phys. **25**, 023007 (2023).

[13] R. C. Jones, A new calculus for the treatment of optical systems I. Description and discussion of the calculus, J. Opt. Soc. Am. **31**, 488–493 (1941).

[14] R. Y. Chiao and Y.-S. Wu, Manifestation of Berry's topological phase for the photon, Phys. Rev. Lett. **57**, 933–936 (1986).

[15] J. Segert, Photon Berry's phase as a classical topological effect, Phys. Rev. A **36**, 10–15 (1987).

[16] H. S. M. Coxeter, *Introduction to Geometry*, 3rd ed. (John Wiley & Sons, New York, 1963).

[17] C. H. Tang, An orthogonal coordinate system for curved pipes, IEEE Trans. Microwave Theory Tech. **18**, 69–69 (1970).

[18] R. Ulrich, S. C. Rashleigh, and W. Eickhoff, Bending-induced birefringence in single-mode fibers, Opt. Lett. **5**, 273–275 (1980).

[19] X.-S. Fang and Z.-Q. Lin, Field in single-mode helically-wound optical fibres, IEEE Trans. Microwave Theory Tech. **33**, 1150–1154 (1985).

[20] M. V. Berry, Interpreting the anholonomy of coiled light, Nature **326**, 277–278 (1987).

[21] J.-M. Normand, *A Lie Group: Rotations in Quantum Mechanics* (North-Holland, Amsterdam, 1980).

[22] C.-F. Li, Unified theory for Goos-Hänchen and Imbert-Fedorov effects, Phys. Rev. A **76**, 013811 (2007).