Depletion of nonlinearity in space-analytic space-periodic solutions to equations of diffusive magnetohydrodynamics

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Abstract

We consider solenoidal space-periodic space-analytic solutions to the equations of magnetohydrodynamics. An elementary bound shows that due to the special structure of the nonlinear terms in the equations for modified solutions, effectively they lack a half of the spatial gradient, which appears to be a novel mechanism for depletion of nonlinearity. We present a two-phase iterative procedure yielding an expanded bound for the guaranteed time of the space analyticity of the hydrodynamic solutions. Each iteration involves two regimes: In phase 1, the enstrophy of the modified solution and the bound for the radius of the analyticity of the original solution simultaneously increase (the bound is proportional to the elapsed time since the beginning of phase 1). In phase 2, the enstrophy and bound simultaneously decrease. It is straightforward to generalize this construction for the equations of magnetohydrodynamics.

1. Introduction

The simultaneous evolution of the velocity, $\mathbf{V}(\mathbf{x},t)$, of an electrically conducting fluid flow and the magnetic field, $\mathbf{B}(\mathbf{x},t)$, satisfies the Navier–Stokes equation involving the Lorentz force,

$$\partial\mathbf{V}/\partial t=\nu\nabla^2\mathbf{V}+\mathbf{V}\times\boldsymbol{\omega}-\mathbf{B}\times(\nabla\times\mathbf{B})-\nabla P, \tag{1a}$$

and the magnetic induction equation,

$$\partial\mathbf{B}/\partial t=\eta\nabla^2\mathbf{B}+\nabla\times(\mathbf{V}\times\mathbf{B}). \tag{1b}$$

The fluid is supposed to be incompressible and the magnetic field is solenoidal,

$$\nabla\cdot\mathbf{V}=\nabla\cdot\mathbf{B}=0. \tag{1c}$$

Here P denotes the modified pressure, $\boldsymbol{\omega}=\nabla\times\mathbf{V}$ the vorticity, $\mathbf{x}\in\mathbb{R}^3$ are the Eulerian coordinates and t is time. For the sake of simplicity, no external forcing is assumed. We consider solutions in $\mathbb{T}^3=[0,2\pi]^3$, which are 2π-periodic in each Cartesian variable x_i.

The nonlinear terms in the equations (1) involve numerous vector products (including the curl operators, which also reduce to vector products in the Fourier space) resulting in significant cancellations and having implications for the structures developing in the two physical fields. This phenomenon, called the depletion of nonlinearity (see [1]), was studied extensively in the simulations of turbulence in two (e.g., see [2]) and three dimensions. In hydrodynamics, it can be associated with the flow beltramization in the areas of large vorticity, i.e., the velocity and vorticity tend to approximately line up, whereby their vector product can be significantly smaller than on the periphery. In these regions, the depletion can also be caused by the flow self-organization into the structures of reduced dimensionality such as one-dimensional ropes [1]. Furthermore, for a random solenoidal Gaussian vector field $\mathbf{V}$ modelling turbulence, the potential part of the Lamb vector $\mathbf{V}\times\boldsymbol{\omega}$ can exceed twice the solenoidal one (in the sense of the mean-square norm) [3]. In solutions to the full magnetohydrodynamic (MHD) problems, the flow and magnetic field show a tendency to become parallel in the areas of large gradients [4, 5], and $\mathbf{V}\approx\mathbf{B}$ in the extreme case of the Archontis dynamo [6–8].

Also, this cancellation has mathematical consequences. The analysis [9, 10] (see also [11]) of solutions to the Navier–Stokes equation computed with a high spatial resolution (up to 8192^3 Fourier harmonics) revealed an unexpected monotonic growth in the quantities D_m (that are powers of the scaled moments of the vorticity) on increasing m; this was interpreted as a manifestation of the nonlinearity weakening. A higher depletion was found in numerical space-periodic simulations of magnetohydrodynamic (MHD) turbulence [12], where similar quantities $D^{\pm}_m$ computed for the curls of the Elsässer variables $\mathbf{V}\pm\mathbf{B}$ were studied (see also [13]).

The finite-time spatial analyticity of space-periodic solutions to the Navier–Stokes equation was demonstrated in [14] by deriving bounds for their Gevrey class norms (see [15] for a review of results on the analyticity of solutions to the hydrodynamic problem). Moreover, if a three-dimensional space-periodic solution to the MHD equations (1) belongs initially to the Sobolev space $H_{1/2}(\mathbb{T}^3)$, it instantly acquires space analyticity [16] (see also [17]) and subsequently it is almost always space-analytic [16]. The proofs [16] relied on using the modified solutions

$$\mathbf{v}=\sum_{\mathbf{n}\neq0}\mathbf{v_n}(t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},\quad \mathbf{V_n}=\mathbf{v_n}\mathrm{e}^{-\Gamma|\mathbf{n}|},\tag{2a}$$

$$\mathbf{b}=\sum_{\mathbf{n}\neq0}\mathbf{b_n}(t)\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}},\quad \mathbf{B_n}=\mathbf{b_n}\mathrm{e}^{-\Gamma|\mathbf{n}|},\tag{2b}$$

where $\mathbf{V_n}$ and $\mathbf{B_n}$ are the Fourier coefficients of $\mathbf{V}$ and $\mathbf{B}$, respectively, and $\Gamma\geq0$ is independent of the three-dimensional wave vector $\mathbf{n}$ (see also [18, 19]). If the modified solution is smooth, Γ is a lower bound for the radius of the space analyticity of the original solution. The time integrals of certain powers of high-index Sobolev norms of hydrodynamic solutions were shown in [20] to be finite. This was generalized [16] to encompass MHD solutions for space-analytic initial conditions by employing

$$\Gamma=\delta(1+\|\mathbf{v}\|^2_{3/2}+\|\mathbf{b}\|^2_{3/2})^{-1/2}$$

in (2). Here $\|\cdot\|_p$ denotes the norm in the Sobolev space $H_p(\mathbb{T}^3)$ and $\delta>0$ is a constant.

The identities

$$\mathbf{c}\times(\nabla\times\mathbf{c})=-(\mathbf{c}\cdot\nabla)\mathbf{c}+\frac{1}{2}\nabla|\mathbf{c}|^2,\qquad \nabla\times(\mathbf{c}\times\mathbf{d})=(\mathbf{d}\cdot\nabla)\mathbf{c}-(\mathbf{c}\cdot\nabla)\mathbf{d},$$

which hold true for any solenoidal fields $\mathbf{c}$ and $\mathbf{d}$, attest that all the nonlinear terms in (1) can be expressed as the linear combinations of directional derivatives (of the type $(\mathbf{d}\cdot\nabla)\mathbf{c}$). The Fourier coefficients $\mathbf{v_n}$ and $\mathbf{b_n}$ are governed by the ordinary differential equations

$$\frac{\mathrm{d}\mathbf{v_n}}{\mathrm{d}t}+\nu|\mathbf{n}|^2\mathbf{v_n}-|\mathbf{n}|\mathbf{v_n}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}=\mathrm{i}\sum_{\mathbf{k}}\mathrm{e}^{\Gamma(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)}\mathfrak{P}_\mathbf{n}\big((\mathbf{b_k}\cdot\mathbf{n})\mathbf{b_{n-k}}-(\mathbf{v_k}\cdot\mathbf{n})\mathbf{v_{n-k}}\big),\tag{3a}$$

$$\frac{\mathrm{d}\mathbf{b_n}}{\mathrm{d}t}+\eta|\mathbf{n}|^2\mathbf{b_n}-|\mathbf{n}|\mathbf{b_n}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}=\mathrm{i}\sum_{\mathbf{k}}\mathrm{e}^{\Gamma(|\mathbf{n}|-|\mathbf{k}|-|\mathbf{n}-\mathbf{k}|)}\big((\mathbf{b_k}\cdot\mathbf{n})\mathbf{v_{n-k}}-(\mathbf{v_k}\cdot\mathbf{n})\mathbf{b_{n-k}}\big),\tag{3b}$$

where $\mathfrak{P}_{\mathbf{n}}$ is the projection of a three-dimensional vector on the plane, orthogonal to $\mathbf{n}$.

We present here an alternative mechanism of the depletion of nonlinearity. In the next section we prove an elementary estimate, implying that when majorizing the nonlinear terms in (3), the factor $|\mathbf{n}|$ can be replaced by $(|\mathbf{n}-\mathbf{k}|^{1/2}+|\mathbf{n}-\mathbf{k}||\mathbf{k}|^{-1/2})\Gamma^{-1/2}$, i.e., although formally the gradient is linear in the wave number, effectively a half of it disappears in the bound. Thus, it is shown that the nonlinearity depletes, inter alia, due to the reduction in the effective order of the differential operators involved. By contrast, the factor $\Gamma^{-1/2}$ increases the advective terms, when the bound Γ is small. To the best of our knowledge, this mechanism of the nonlinearity depletion was never considered in the literature previously. We apply it to estimate the interval of the guaranteed existence of the space-analytic solution in section 3.

2. A bound for nonlinear terms

We demonstrate here the inequality

$$|(\mathbf{c}_{\mathbf{k}}\cdot\mathbf{n})\mathfrak{P}_{\mathbf{n}}\mathbf{d}_{\mathbf{n}-\mathbf{k}}|\mathrm{e}^{\Gamma(|\mathbf{n}|-|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|)}\le|\mathbf{c}_{\mathbf{k}}||\mathbf{d}_{\mathbf{n}-\mathbf{k}}|\sqrt{\frac{|\mathbf{n}-\mathbf{k}|}{\Gamma}}\left(1+\sqrt{\frac{|\mathbf{n}-\mathbf{k}|}{|\mathbf{k}|}}\right) \tag{4}$$

for the nonlinear terms in the sums in the r.h.s. of Eqs. (3); $\mathbf{c}$ and $\mathbf{d}$ are "metavariables" standing for $\mathbf{v}_{\mathbf{m}}$ and/or $\mathbf{b}_{\mathbf{m}}$ (the projection may be unused in the case of (3b)).

Since $\mathbf{c}$ is solenoidal, $\mathbf{c}_{\mathbf{k}}=|\mathbf{k}|^{-2}\mathbf{k}\times(\mathbf{c}_{\mathbf{k}}\times\mathbf{k})$ implying

$$|\mathbf{c}_{\mathbf{k}}\cdot(\mathbf{n}-\mathbf{k})|\mathrm{e}^{\Gamma(|\mathbf{n}|-|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|)}\le|\varphi(\theta)||\mathbf{c}_{\mathbf{k}}||\mathbf{n}-\mathbf{k}|,$$

where

$$\varphi(\theta)=\sin\theta\,\exp(\Gamma(|\mathbf{n}|-|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|))$$

and θ is the angle between the vectors $\mathbf{n}-\mathbf{k}$ and $\mathbf{k}$. At the point $\theta=\theta_{\max}$ of the maximum of $|\varphi(\theta)|$, the necessary condition for an extremum, $\mathrm{d}\varphi/\mathrm{d}\theta=0$, holds true, which is equivalent to the relation

$$\begin{aligned}\sin^2\theta_{\max}\,\Gamma|\mathbf{n}-\mathbf{k}||\mathbf{k}|&=-\cos\theta_{\max}|\mathbf{n}|\,\Big|_{\theta=\theta_{\max}}\\&\le|\mathbf{n}-\mathbf{k}|+|\mathbf{k}|.\end{aligned} \tag{5}$$

By the triangle inequality, the exponent in the definition of $\varphi(\theta)$ is non-positive and hence the exponential is majorized by the unity; now, by (5),

$$|\varphi(\theta)|\le|\varphi(\theta_{\max})|\le|\sin\theta_{\max}|\le\left(\frac{|\mathbf{n}-\mathbf{k}|+|\mathbf{k}|}{\Gamma|\mathbf{n}-\mathbf{k}||\mathbf{k}|}\right)^{1/2}, \tag{6}$$

which proves (4).

Remark. Similar arguments prove the inequality

$$|(\mathbf{c}_{\mathbf{k}}\cdot\mathbf{n})\mathfrak{P}_{\mathbf{n}}\mathbf{d}_{\mathbf{n}-\mathbf{k}}|\mathrm{e}^{\Gamma(|\mathbf{n}|-|\mathbf{n}-\mathbf{k}|-|\mathbf{k}|)}\le|\mathbf{c}_{\mathbf{k}}||\mathbf{d}_{\mathbf{n}-\mathbf{k}}|\left(|\mathbf{n}|^2+|\mathbf{n}|^4/|\mathbf{k}|^2\right)^{1/4}\Gamma^{-1/2}.$$

3. Interval of the guaranteed existence of a space-analytic solution

For simplicity, we consider in this section exclusively the hydrodynamic problem for $\mathbf{B}=0$. Since all nonlinear terms in (3) have the same structure, it is easy to obtain similar bounds for the MHD problem following the same approach. We assume $\|\mathbf{V}\|_1<\infty$ at $t=0$.

A variant of the bound implying the flow space analyticity upon its spontaneous emergence [14] can be obtained for $\Gamma=\nu^{-1}t$. Applying the embedding theorem, we deduce from (3a) for $\mathbf{B}=0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_1^2+\nu\|\mathbf{v}\|_2^2-\nu^{-1}\|\mathbf{v}\|_{3/2}^2\leq\widetilde{C}\|\mathbf{v}\|_2\|\mathbf{v}\|_{3/2}\|\mathbf{v}\|_1,$$

and hence, by the Hölder and Young inequalities,

$$\frac{\mathrm{d}y^2}{\mathrm{d}t}\leq C\nu^{-3}y^2(1+y^2)\quad\Rightarrow\quad y^2\leq\left(\left(1+\frac{1}{y^2(0)}\right)\mathrm{e}^{-C\nu^{-3}t}-1\right)^{-1},\tag{7}$$

where $\widetilde{C}$ and C are the relevant constants and $y=\|\mathbf{v}\|_1^2$. This inequality guarantees the existence of the space-analytic solution until

$$T_*=\nu^3C^{-1}\ln(1+1/y^2(0)).$$

Let us consider a process, which is in some sense reciprocal. Scalar multiplying (3a) by $\mathbf{v}_{-\mathbf{n}}|\mathbf{n}|^2$, summing the results over $\mathbf{n}$, transforming identically the term involving $\mathrm{d}\Gamma/\mathrm{d}t$ and applying (4) to the sum in the r.h.s., we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_1^2+\nu\|\mathbf{v}\|_2^2 \\ \leq&\|\mathbf{v}\|_{3/2}^2\,\Gamma^{-1/2}\left(\frac{2}{3}\frac{\mathrm{d}\Gamma^{3/2}}{\mathrm{d}t}+\|\mathbf{v}\|_{3/2}^{-2}\sum_{\mathbf{n},\mathbf{k}}|\mathbf{v}_\mathbf{k}||\mathbf{v}_{\mathbf{n}-\mathbf{k}}||\mathbf{v}_{-\mathbf{n}}||\mathbf{n}|^2\left(|\mathbf{n}-\mathbf{k}|^{1/2}+\frac{|\mathbf{n}-\mathbf{k}|}{|\mathbf{k}|^{1/2}}\right)\right).\end{aligned}\tag{8}$$

The sum, Σ, in the r.h.s. can be estimated applying the embedding theorem and the Hölder inequality as follows:

$$\begin{aligned}\Sigma&\leq\sum_{\mathbf{n},\mathbf{k}}|\mathbf{v}_\mathbf{k}||\mathbf{v}_{\mathbf{n}-\mathbf{k}}||\mathbf{v}_{-\mathbf{n}}||\mathbf{n}|^{3/2}\Big(2|\mathbf{n}-\mathbf{k}|+|\mathbf{n}-\mathbf{k}|^{1/2}|\mathbf{k}|^{1/2}+|\mathbf{n}-\mathbf{k}|^{3/2}|\mathbf{k}|^{-1/2}\Big)\\&=(2\pi)^{-3}\int_{\mathbb{T}^3}\big(2u_{3/2}u_1u_0+u_{3/2}u_{1/2}^2+u_{3/2}^2u_{-1/2}\big)\mathrm{d}\mathbf{x}\\&\leq(2\pi)^{-3}\big(2|u_{3/2}|_2|u_1|_3|u_0|_6+|u_{3/2}|_2|u_{1/2}|_3|u_{1/2}|_6+|u_{3/2}|_2|u_{3/2}|_{6/(3-\alpha)}|u_{-1/2}|_{6/\alpha}\big)\\&\leq C'\|\mathbf{v}\|_{3/2}^2\|\mathbf{v}\|_1+C'_\alpha\|\mathbf{v}\|_{3/2}\|\mathbf{v}\|_{(3+\alpha)/2}\|\mathbf{v}\|_{1-\alpha/2}\\&\leq C_\alpha\|\mathbf{v}\|_2^\alpha\|\mathbf{v}\|_{3/2}^{2-\alpha}\|\mathbf{v}\|_1\\&\qquad\qquad\Rightarrow\quad\|\mathbf{v}\|_{3/2}^{-2}\Sigma\leq C_\alpha\|\mathbf{v}\|_2^\alpha\|\mathbf{v}\|_1^{1-\alpha}.\end{aligned}\tag{9}$$

Here

$$u_p=\sum_\mathbf{m}|\mathbf{v}_\mathbf{m}||\mathbf{m}|^p\mathrm{e}^{\mathrm{i}\mathbf{m}\cdot\mathbf{x}},$$

$|\cdot|_q$ is the norm in the Lebesgue space $L_q(\mathbb{T}^3)$, C',C'_α and C_α are the relevant constants, α is confined to the interval $1>\alpha>0$ (we note $C_\alpha\to\infty$ when $\alpha\to0$).

By (9), the norms in the l.h.s. of (8) are controlling the sum in the r.h.s. of (8). This suggests an iterative procedure for bounding the modified solution. Each iteration consists of two phases. We use the following notation to describe the procedure: the start time of the jth iteration is denoted by T_j, the durations of the two phases comprising it by $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$; the second phase begins at $T'_j = T_j + \lambda_j^{(1)}$; hence, $T_{j+1} = T'_j + \lambda_j^{(2)}$. Finally, we denote by $\Lambda_j^{(m)}$ the lower bounds that we derive for $\lambda_j^{(m)}$.

Phase 1. The simultaneous growth of the enstrophy of the modified solution and of the bound Γ *for the radius of analyticity of the original solution.* At $t = T_j$ we switch on the regime of the spontaneous emergence of the analyticity until the enstrophy $\|\mathbf{v}\|_1^2$ increases at time $T'_j = T_j + \lambda_j^{(1)}$ by a factor $a_j > 1$. By (7), the duration of this phase satisfies the inequality

$$\lambda_j^{(1)} \geq \Lambda_j^{(1)} = \nu^3 C^{-1} \ln\left(a_j^2(1+y_j^2)/(1+a_j^2 y_j^2)\right), \tag{10}$$

where $y_j = \|\mathbf{v}(T_j)\|_1^2$. Consequently, $\Gamma \geq \nu^{-1}\Lambda_j^{(1)}$ at $t = T'_j$.

Phase 2. Simultaneous decrease in the enstrophy and bound Γ. At $t = T'_j$ we switch on the reciprocal regime of the bound decrease according to the equation

$$-\frac{2}{3}\frac{\mathrm{d}\Gamma^{3/2}}{\mathrm{d}t} = C_\alpha \|\mathbf{v}\|_2^\alpha \|\mathbf{v}\|_1^{1-\alpha} \tag{11}$$

until $\Gamma = 0$ at time $T_{j+1} = T'_j + \lambda_j^{(2)}$. By (8) and (9),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_1^2 + \nu\|\mathbf{v}\|_2^2 \leq 0,$$

implying

$$\int_{T'_j}^{t} \|\mathbf{v}\|_2^2\,\mathrm{d}t \leq a_j y_j/\nu; \qquad \|\mathbf{v}\|_1^2 \leq a_j y_j \mathrm{e}^{-2\nu(t-T'_j)} \tag{12}$$

(the second inequality holds true because $\|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_2$).

To estimate $\lambda_j^{(2)}$, we integrate (11) from T'_j to T_{j+1} exploiting the relations $\Gamma(T_{j+1}) = 0$ ($T_{j+1} = \infty$, if this value is unreachable) and (12), and the Hölder inequality:

$$\begin{aligned}
\frac{2}{3}\left(\nu^{-1}\Lambda_j^{(1)}\right)^{3/2} \leq \frac{2}{3}\Gamma^{3/2}(T'_j) &= \int_{T'_j}^{T_{j+1}} C_\alpha \|\mathbf{v}\|_2^\alpha \|\mathbf{v}\|_1^{1-\alpha}\mathrm{d}t \\
&\leq C_\alpha \left(\int_{T'_j}^{T_{j+1}} \|\mathbf{v}\|_2^2\,\mathrm{d}t\right)^{\alpha/2} \left(\int_{T'_j}^{T_{j+1}} \|\mathbf{v}\|_1^\mu \mathrm{d}t\right)^{1-\alpha/2} \\
&\leq C_\alpha (a_j y_j)^{1/2} \nu^{-\alpha/2} \left(\int_{T'_j}^{T_{j+1}} \mathrm{e}^{-\nu\mu t}\mathrm{d}t\right)^{1-\alpha/2} \\
&= C_\alpha (a_j y_j)^{1/2} \nu^{-1} \mu^{\alpha/2-1} \left(1 - \mathrm{e}^{-\nu\mu\lambda_j^{(2)}}\right)^{1-\alpha/2},
\end{aligned}$$

where $\mu = (1-\alpha)/(1-\alpha/2)$. Thus, the duration of phase 2 satisfies the inequality

$$\lambda_j^{(2)} \geq \Lambda_j^{(2)} = -(\nu\mu)^{-1} \ln Q_j, \tag{13a}$$

where

$$Q_j = 1 - \mu\left(\frac{4\nu^8 \ln^3\left(a_j^2(1+y_j^2)/(1+a_j^2 y_j^2)\right)}{9C^3 C_\alpha^2 a_j y_j}\right)^{1/(2-\alpha)}; \tag{13b}$$

$\lambda_j^{(2)} = T_{j+1} = \infty$, if $Q_j \leq 0$.

By (12), we can now set

$$y_{j+1} = a_j \mathrm{e}^{-2\nu\Lambda_j^{(2)}} y_j. \tag{14}$$

4. Conclusions

We have presented a new mechanism of the nonlinearity depletion (weakening) for space-analytic solutions to the MHD equations. It is manifested by the inequality (4), whereby the directional derivative $(\mathbf{c}\cdot\nabla)\mathbf{d}$ for solenoidal vector fields $\mathbf{c}$ is bounded by a homogeneous function of wave vector lengths of degree 1/2. The inequality involves the lower bound Γ for the radius of the analyticity of the solution. A qualified reduction of Γ has been shown to cause a decrease in the enstrophy of the modified solution. We have designed a two-phase iterative procedure for estimating the guaranteed time of the existence of space-analytic solutions that exploits the two mechanisms. It involves two alternating regimes: a simultaneous increase in phase 1 in the enstrophy of the modified solution with a linear in time growth of the bound Γ for the radius of the analyticity of the original solution (the regime discovered in [14]), and a simultaneous decrease in the enstrophy and Γ in phase 2.

We have established that when the initial flow enstrophy is finite, a solution to the Navier–Stokes equation remains space-analytic at least up to $T_{**}=\sum_j(\Lambda_j^{(1)}+\Lambda_j^{(2)})$, see (10) and (13). Formally, in our construction Γ is allowed to decrease to zero when phase 2 terminates; however, the enstrophy of the flow (which does not exceed the enstrophy of the modified solution) is uniformly bounded on the intervals $[0,T_{**}-\varepsilon]$ for any $\varepsilon>0$ and this guarantees the space analyticity of the flow on any open interval $(0,T_{**})$. This expands the guaranteed life-span T_* [14] of the analytical solution. In particular, $T_*<\sum_j\Lambda_j^{(1)}$: while T_* is the sum of times, during which a solution to the ODE

$$\mathrm{d}y^2/\mathrm{d}t=C\nu^{-3}y^2(1+y^2)$$

(cf. (7)) grows monotonically from y_j to y_{j+1}, each $\Lambda_j^{(1)}$ is the time it takes it to grow from y_j to a larger value $\mathrm{e}^{2\nu\lambda_j^{(2)}}y_{j+1}$. For small ν, $\Lambda_j^{(1)}=\mathrm{O}(\nu^3)$ and $\Lambda_j^{(2)}=\mathrm{O}(\nu^{8/(2-\alpha)-1})$, i.e., in terms of the orders of ν the newly introduced phases 2 contribute almost as much to the length of the guaranteed interval of the space analyticity as phases 1 (since any value of the parameter α from the interval $1>\alpha>0$ is permitted).

By (14), the enstrophy of the modified solution (2a) changes during the jth iteration by a factor equal to or below $a_j\mathrm{e}^{-2\nu\Lambda_j^{(2)}}$. This product does not exceed the unity (enabling the iterations to have fixed or increasing durations, thus implying the existence of a global analytical solution), if and only if the initial enstrophy $y_j=\|\mathbf{v}(T_j)\|_1^2$ is sufficiently small or the viscosity ν is sufficiently large.

Some intriguing questions remain open: What is the optimal choice of the factors a_j maximizing the guaranteed life-span of the solution T_{**}? Is letting Γ decay in phase 2 according to (12) an optimal strategy? If Γ decreases faster, then phase 2 is shorter, resulting in the larger factor $\mathrm{e}^{-2\nu\Lambda_j^{(2)}}$, but the decrease in the enstrophy is enhanced by the negative r.h.s. of (8), involving two factors: $\|\mathbf{v}\|_{3/2}^2$, which can be large and $\Gamma^{-1/2}$, which is large towards the ends of phases 2. So, what is the optimal control of Γ in phase 2? Furthermore, is there a better strategy than implementing the alternating Γ growth from 0 and decay to 0 as we have assumed here?

For the full system of the equations of the diffusive magnetohydrodynamics, the derivations are similar, since all the nonlinear terms have the same structure of the directional derivatives along solenoidal fields.

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