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Convex Ternary Quartics Are SOS-Convex

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Abstract

We show that if a ternary quartic form is convex, then it must be sos-convex; i.e, if the Hessian $H(\mathbf{x})$ of a ternary quartic form is positive semidefinite for all $\mathbf{x}$, then the biquadratic form $\mathbf{y}^T H(\mathbf{x})\mathbf{y}$ in the variables $\mathbf{x} := (x_1, x_2, x_3)^T$ and $\mathbf{y} := (y_1, y_2, y_3)^T$ must be a sum of squares. This result is in a meaningful sense the "convex analogue" of Hilbert's celebrated theorem on ternary quartics. We show that exploiting the structure of the Hessian matrix is crucial in any possible proof of this result by presenting an explicit example of a biquadratic form $b(\mathbf{x}, \mathbf{y})$ that is symmetric in $\mathbf{x}$ and $\mathbf{y}$, nonnegative, but not a sum of squares.

1 Introduction

A form (i.e., homogeneous polynomial) $p : \mathbb{R}^n \to \mathbb{R}$ with real coefficients is said to be *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ and a *sum of squares* (sos) if it can be written as $p(x) = \sum_{i=1}^m q_i^2(x)$ for some forms $q_1(x), \ldots, q_m(x)$. In 1888, Hilbert [21] showed that a ternary quartic form (i.e., a form in 3 variables of degree 4) is nonnegative if and only if it is a sum of squares. Out of the degrees and dimensions in which nonnegative forms can be written as a sum of squares, the case of ternary quartics is the most astonishing. Several new proofs of this result as well as modern expositions of Hilbert's original proof have appeared in recent years; see [14], [27, p. 89-93], [31], [26], [25].

In this paper, we show that interestingly enough an analogous fact holds for convexity and sos-convexity of ternary quartic forms. A polynomial $p := p(\mathbf{x})$ in the variables $\mathbf{x} := (x_1, \ldots, x_n)^T$ is *convex* if its Hessian matrix $H_p := H_p(\mathbf{x})$ is positive semidefinite (i.e., has nonnegative eigenvalues) for all $\mathbf{x} \in \mathbb{R}^n$. It is easy to see that this condition holds if and only if the scalar polynomial $\mathbf{y}^T H_p(\mathbf{x})\mathbf{y}$ in the variables $\mathbf{x}$ and $\mathbf{y} := (y_1, \ldots, y_n)^T$ is nonnegative. A polynomial p is said to be *sos-convex* if the polynomial $\mathbf{y}^T H_p(\mathbf{x})\mathbf{y}$ is a sum of squares. Clearly, sos-convexity is a sufficient condition for convexity of polynomials.

The term "sos-convexity" was introduced by Helton and Nie in [20] in relation to the study of semidefinite representability of convex sets. An alternative definition of sos-convexity that is commonly used (e.g., in [20]) and is equivalent to the definition we gave above is given by the requirement that the Hessian matrix H_p can be factored as $H_p(\mathbf{x}) = M^T(\mathbf{x})M(\mathbf{x})$ for a possibly nonsquare polynomial matrix $M(\mathbf{x})$. There are also other natural sos relaxations for convexity based on the usual definition of convexity (via Jensen's inequality) or its first order characterization.

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However, these sos relaxations are shown in [5], [7] to also be equivalent to sos-convexity. All these equivalence results confirm that sos-convexity is indeed *the* rightful analogue of sum of squares when the notion of convexity instead of nonnegativity of polynomials is of interest.

From a computational viewpoint, the significance of sos-convexity stems from the fact that it can be checked efficiently by solving a single semidefinite program. This is in contrast to deciding convexity which has been shown to be strongly NP-hard already for polynomials of degree four [4]. Motivated in part by its connection to semidefinite programming, sos-convexity has has found a range of applications, e.g., in the study of polynomial norms [1], shape-constrained regression [17], [24], polynomial optimization [22], difference of convex optimization [2], robust multi-objective optimization [16], and dynamics and control [12], [3]. There has also been much interest in the role of convexity in semialgebraic optimization [23], [22], [18], [8], [20], [19], [30], and an understanding of the relationship between convexity and sos-convexity is of direct relevance to this line of research. For example, it is known that the semidefinite relaxation arising from the first level of the so-called sum of squares hierarchy is exact for polynomial optimization problems whose objective and constraints are given by sos-convex polynomials [22].

In [6], the first and third authors gave the first example of a convex polynomial that is not sos-convex. In a subsequent paper [7], they gave a full characterization of the degrees and dimensions in which the set of convex and sos-convex polynomials coincide. Such a characterization is also given in [7] for homogeneous polynomials, except for the case of ternary quartics. The main contribution of the current paper (Theorem 3.1 in Section 3) is to settle this remaining case by showing that all convex ternary quartic forms are sos-convex. The intriguing overall outcome of this research is that convex polynomials (resp. forms) are sos-convex exactly in degrees and dimensions where nonnegative polynomials (resp. forms) are sums of squares, as characterized by Hilbert in [21]. However, neither the results in [7] nor the result of this paper follow (as far as we know) from the characterization of Hilbert. See the discussion in [7, Sec. 5].

Upon dehomogenization (see, e.g., [28, Sec. 2]), the result of Hilbert on ternary quartic forms is equivalent to the statement that all nonnegative bivariate quartic polynomials are sos. The situation, however, is quite different for convexity and sos-convexity. It turns out that the proof of the fact that all convex bivariate quartic polynomials are sos-convex follows from a theorem on factorization of positive semidefinite bivariate and homogeneous polynomial matrices without the need to exploit the additional structure of the Hessian matrix [7, Thm. 5.6]; see also [7, Rmk. 5.1]. By contrast, the result for ternary quartic forms crucially relies on linear relations that are imposed on the entries of a matrix that is a valid Hessian. In Section 2, we make this fact evident for the reader. We present an explicit example of a positive semidefinite polynomial matrix which has dimension, degree, and a special symmetry property in common with the Hessian of ternary quartics, but yet fails to have a sum of squares decomposition since it violates a few of the linear relations imposed by the Hessian structure. Following this result, we give the proof of our main theorem on equivalence of convexity and sos-convexity of ternary quartics in Section 3.

2 Symmetric biquadratic forms and Hessian biquadratic forms

A *biquadratic form* $b(\mathbf{x}, \mathbf{y})$ is a form in two sets of variables $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_m)^T$ that can be written as

$$b(\mathbf{x}, \mathbf{y}) := \sum_{i \leq j,\, k \leq l} \alpha_{ijkl} x_i x_j y_k y_l.$$

Equivalently, a biquadratic form is a quartic form that can be written as

$$\mathbf{y}^T A(\mathbf{x}) \mathbf{y},$$

where $A(\mathbf{x})$ is a polynomial matrix with quadratic forms in $\mathbf{x}$ as its entries. The relation between nonnegative and sum of squares biquadratic forms is a well-studied subject. In particular, it is known that when $nm \leq 6$, all nonnegative biquadratic forms are sos; see, e.g., [15] and references therein.

The question of checking nonnegativity of biquadratic forms arises in the study of convexity of quartic forms. If $H_p(\mathbf{x})$ is the Hessian of a quartic form $p(\mathbf{x})$, then the entries of $H_p(\mathbf{x})$ must be quadratic forms and hence $\mathbf{y}^T H_p(\mathbf{x})\mathbf{y}$ is a biquadratic form. Convexity (resp. sos-convexity) of p is equivalent to this biquadratic form being nonnegative (resp. sos).

In particular, when $p(\mathbf{x}) = p(x_1, x_2, x_3)$ is a quartic form in three variables, the Hessian $H_p(\mathbf{x})$ is a 3×3 matrix, and we are in the situation where $\mathbf{y}^T H_p(\mathbf{x})\mathbf{y}$ is a biquadratic form in two sets of three variables $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. It is well known that there exist biquadratic forms in two sets of three variables—henceforth referred to as *ternary biquadratic forms*—that are nonnegative but not sos. This fact was originally proven through a nonconstructive argument by Terpstra in [32] and later independently by Choi [13] via an explicit example. In [13], Choi showed that the biquadratic form $\mathbf{y}^T C(\mathbf{x})\mathbf{y}$ with

$$C(\mathbf{x}) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ \\ -x_1x_3 & -x_2x_3 & x_3^2 + 2x_1^2 \end{bmatrix} \tag{1}$$

is nonnegative but not sos. However, the matrix $C(\mathbf{x})$ above is *not* a valid Hessian, i.e., it cannot be the matrix of the second derivatives of any polynomial. If this was the case, the third partial derivatives would commute. But for this matrix, we have, e.g.,

$$\frac{\partial C_{1,1}(\mathbf{x})}{\partial x_3} = 0 \neq -x_3 = \frac{\partial C_{1,3}(\mathbf{x})}{\partial x_1}.$$

Our main result in this paper (Theorem 3.1) can be equivalently phrased as the statement that all nonnegative ternary biquadratic forms that arise from valid Hessians are sos. It turns out that biquadratic forms that arise from valid Hessians have a special symmetry property. To facilitate discussion, let us define three families of biquadratic forms.

Definition 2.1.

- *An* n-ary biquadratic form *is a biquadratic form* $b(\mathbf{x}, \mathbf{y})$ *in two sets of* n *scalar variables* $\mathbf{x} = (x_1, \ldots, x_n)^T$ *and* $\mathbf{y} = (y_1, \ldots, y_n)^T$.
- *A* symmetric biquadratic form *is an (n-ary) biquadratic form that satisfies* $b(\mathbf{y}, \mathbf{x}) = b(\mathbf{x}, \mathbf{y})$.
- *A* Hessian biquadratic form *is a biquadratic form* $b(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T H_p(\mathbf{x})\mathbf{y}$ *where* $H_p(\mathbf{x})$ *is a valid Hessian, i.e., it is the matrix of second derivatives of some quartic form* $p(\mathbf{x})$.

A simple counting argument shows that the dimension of the vector space of n-ary (resp. symmetric) biquadratic forms is $\binom{n+1}{2}^2$ (resp. $\frac{1}{2}\binom{n+1}{2}^2 + \frac{1}{2}\binom{n+1}{2}$). We claim that the dimension of the vector space of Hessian biquadratic forms is the same as that of quartic forms in n variables, i.e., $\binom{n+3}{4}$. This is because there is a linear bijection between these two vector spaces: from any quartic form p, we can obtain a valid Hessian matrix H_p by differentiation; conversely, from any valid Hessian matrix H_p, we can produce the originating quartic form as[1]

$$p(\mathbf{x}) = \frac{1}{12}\mathbf{x}^T H_p(\mathbf{x})\mathbf{x}. \tag{2}$$

Lemma 2.2. *Hessian biquadratic forms are symmetric.*

Proof. Consider a Hessian biquadratic form $\mathbf{y}^T H_p(\mathbf{x})y$, where $H_p(\mathbf{x})$ is the Hessian matrix of a quartic form $p(\mathbf{x})$. Observe that if the quadratic form in the ij-th entry of $H_p(\mathbf{x})$ is denoted by $\mathbf{x}^T S^{ij}\mathbf{x}$, then the kl-the entry S^{ij}_{kl} of the symmetric matrix S^{ij} is given by

$$S^{ij}_{kl} = \frac{1}{2}\partial_{x_i}\partial_{x_j}\partial_{x_k}\partial_{x_l}p(\mathbf{x}).$$

Since partial derivatives commute, we have $S^{ij}_{kl} = S^{kl}_{ij}$, and therefore

$$\mathbf{y}^T H_p(\mathbf{x})\mathbf{y} = \sum_{ij} y_i y_j \mathbf{x}^T S^{ij}\mathbf{x} = \sum_{ij} y_i y_j \sum_{kl} x_k x_l S^{ij}_{kl} = \sum_{ij} x_i x_j \sum_{kl} y_k y_l S^{ij}_{kl} = \sum_{ij} x_i x_j \mathbf{y}^T S^{ij}\mathbf{y} = \mathbf{x}^T H_p(\mathbf{y})\mathbf{x}.$$

□

The symmetry of a biquadratic form in $\mathbf{x}$ and $\mathbf{y}$ is a rather strong condition that is not satisfied e.g. by the Choi biquadratic form $\mathbf{y}^T C(\mathbf{x})\mathbf{y}$ in (1) (since, in particular, there is a $2y_1^2x_2^2$ term but no term of the type $y_2^2x_1^2$). When $n = 3$, the vector spaces of n-ary biquadratic forms, symmetric biquadratic forms, and Hessian biquadratic forms respectively have dimensions 36, 21, and 15. Since the symmetry requirement drops the dimension of the space of ternary biquadratic forms significantly, and since sos polynomials are known to generally cover much larger volume in the set of nonnegative polynomials in presence of symmetries (see, e.g., [11]), one may initially suspect (as we did) that the equivalence between nonnegative and sos ternary Hessian biquadratic forms is an artifact the symmetry property. Our next theorem shows that interestingly enough this is not the case. This makes the result for Hessian biquadratic forms even more striking.

Theorem 2.3. *There exist ternary symmetric biquadratic forms that are nonnegative but not a sum of squares. In particular, the following biquadratic form has the desired properties:*

$$\begin{aligned} b(x_1,x_2,x_3,y_1,y_2,y_3) \;=\;& 12(x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) \\ &+31x_1x_2y_1y_2 - 10x_1x_3y_1y_3 - 5x_2x_3y_2y_3 \\ &+12(x_2^2y_1^2 + y_2^2x_1^2) + 6(x_3^2y_1^2 + y_3^2x_1^2) + 12(x_2^2y_3^2 + y_2^2x_3^2) \\ &+4(x_1x_2y_1^2 + y_1y_2x_1^2) + 9(x_1x_3y_1^2 + y_1y_3x_1^2) - 10(x_2x_3y_1^2 + y_2y_3x_1^2) \\ &+13(x_1x_3y_2^2 + y_1y_3x_2^2) + 13(x_2x_3y_2^2 + y_2y_3x_2^2) + 23(x_1x_2y_2^2 + y_1y_2x_2^2) \\ &+5(x_1x_2y_3^2 + y_1y_2x_3^2) + 3(x_1x_3y_3^2 + y_1y_3x_3^2) + 7(x_2x_3y_3^2 + y_2y_3x_3^2) \\ &+5(x_1x_2y_2y_3 + y_1y_2x_2x_3) - 11(x_1x_3y_2y_3 + y_1y_3x_2x_3) + 3(x_1x_3y_1y_2 + y_1y_3x_1x_2). \end{aligned} \tag{3}$$

The proof of this theorem appears in the appendix.

[1]The identity $p(\mathbf{x}) = \frac{1}{d(d-1)}\mathbf{x}^T H_p(\mathbf{x})\mathbf{x}$ holds for any form p of degree d and can be derived from Euler's identity for homogeneous functions. This identity also shows that convex forms are nonnegative, and that sos-convex forms are sos.

3 Equivalence of convexity and sos-convexity for ternary quartics

Let us denote the set of convex (resp. sos-convex) ternary quartic forms by $C_{3,4}$ (resp. $\Sigma C_{3,4}$). These sets are both closed convex cones and we have the obvious inclusion $\Sigma C_{3,4} \subseteq C_{3,4}$. Our main result is to show the reverse inclusion.

Theorem 3.1. $\Sigma C_{3,4} = C_{3,4}$.

The proof of this theorem is done in two steps. We first show that it suffices to consider convex forms that have a specific set of zeroes (Lemma 3.4). We then show that all such convex forms are sos-convex (Theorem 3.5). Throughout this section, we use the notation H_p to denote the Hessian matrix of a form p, and $h_p(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T H_p \mathbf{y}$ to denote the *Hessian form* of p. We recall that when p is a quartic, h_p is a biquadratic form and satisfies the symmetry relation $h_p(\mathbf{x}, \mathbf{y}) = h_p(\mathbf{y}, \mathbf{x})$; see Lemma 2.2. Zeroes of h_p are treated as points in $\mathbb{RP}^2 \times \mathbb{RP}^2$; for evaluation, we may choose an affine representative, which will usually be taken to lie in the bi-sphere $\mathbb{S}^2 \times \mathbb{S}^2$.

3.1 Reduction

For a point $\mathbf{u} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{RP}^2 \times \mathbb{RP}^2$, let $F_{\mathbf{u}}$ be the face of $C_{3,4}$ consisting of all convex forms f for which $h_f(\mathbf{u}) = 0$. Let $\mathbf{e}_i$ denote the i-th standard basis vector, $\mathbf{u}_1 = (\mathbf{e}_1, \mathbf{e}_2)^T$, and $\mathbf{u}_2 = (\mathbf{e_3}, \mathbf{d})^T$ with $\mathbf{d} = [a, b, 1]^T$ for some scalars a, b. Let $L_{a,b}$ be the linear subpsace of ternary quartics consisting of forms g such that $H_g(\mathbf{e_1}) \cdot \mathbf{e}_2 = H_g(\mathbf{e_3}) \cdot \mathbf{d} = 0$. Let $T_{a,b}$ be the face of $C_{3,4}$ consisting of convex ternary quartics g such that $h_g(\mathbf{u}_1) = h_g(\mathbf{u}_2) = 0$. Note that $T_{a,b} = F_{\mathbf{u}_1} \cap F_{\mathbf{u}_2}$. In Theorem 3.5, we will derive an explicit description of the face $T_{a,b}$, and using it we will show that all forms in $T_{a,b}$ are sos-convex. Our main task for now is to show that if a convex and non-sos-convex ternary quartic exists, then there exists one in $T_{a,b}$. We start with some preparatory lemmas.

Lemma 3.2. *The face $F_{\mathbf{u_1}}$ has dimension* 10*.*

Proof. We first show that the dimension of $F_{\mathbf{u_1}}$ is at most 10. Let g be a convex ternary quartic such that $h_g(\mathbf{e}_1, \mathbf{e_2}) = 0$. Since the Hessian matrices $H_g(\mathbf{e}_1)$ and $H_g(\mathbf{e}_2)$ are positive semidefinite, it follows that $H_g(\mathbf{e}_1) \cdot \mathbf{e}_2 = H_g(\mathbf{e}_2) \cdot \mathbf{e}_1 = 0$. These two conditions imply that g is missing monomials $x_1^3x_2, x_1^2x_2^2, x_1^2x_2x_3, x_2^3x_1, x_2^2x_1x_3$, which shows that $F_{\mathbf{u_1}}$ lies in a 10-dimensional subspace.

We now show that dimension of $F_{\mathbf{u_1}}$ is at least 10. Let

$$f = (x_1 + x_3)^4 + (x_2 + x_3)^4 + (2x_1 + x_3)^4 + (2x_2 + x_3)^4.$$

The Hessian form h_f is equal to

$$12((x_1 + x_3)^2(y_1 + y_3)^2 + (x_2 + x_3)^2(y_2 + y_3)^2 + (2x_1 + x_3)^2(2y_1 + y_3)^2 + (2x_2 + x_3)^2(2y_2 + y_3)^2).$$

We see that $f \in F_{\mathbf{u_1}}$ and h_f is strictly positive on $\mathbb{S}^2 \times \mathbb{S}^2$ outside of $\mathbf{u_1}$. A straightforward computation shows that the Hessian of h_f at $\mathbf{u_1}$ is positive definite on the tangent space to $\mathbb{S}^2 \times \mathbb{S}^2$ at $\mathbf{u_1}$. Therefore, a perturbation argument shows that for any g in the span of the 10 degree-4 monomials that are different from $x_1^3x_2, x_1^2x_2^2, x_1^2x_2x_3, x_2^3x_1, x_2^2x_1x_3$, we have $f + \varepsilon g \in F_{\mathbf{u_1}}$ for all sufficiently small ε. It follows that $F_{\mathbf{u_1}}$ is 10-dimensional.

□

Lemma 3.3. *Whenever at least one of a or b is non-zero, the dimension of the vector space $L_{a,b}$ is* 5 *and the dimension of $T_{a,b}$ is at most 5.*

Proof. We begin by establishing a slightly more general version of the argument in Lemma 3.2. For a vector $\mathbf{v} \in \mathbb{R}^3$, let $D_{\mathbf{v}}$ denote the directional derivative in direction $\mathbf{v}$: $D_{\mathbf{v}}(f) = \langle \mathbf{v}, \nabla f \rangle$. It will be convenient to consider differential operators associated to polynomials: for a polynomial f, let ∂f denote the differential operator obtained by replacing x_i with $\frac{\partial}{\partial x_i}$.

Let g be a convex ternary quartic such that $h_g(\mathbf{u}, \mathbf{v}) = 0$, with non-zero $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \neq \mathbf{v}$. Since the Hessian matrices $H_g(\mathbf{u})$ and $H_g(\mathbf{v})$ are positive semidefinite, it follows that $H_g(\mathbf{u}) \cdot \mathbf{v} = H_g(\mathbf{v}) \cdot \mathbf{u} = 0$, i.e., $\mathbf{u}$ is in the kernel of the Hessian of g evaluated at $\mathbf{v}$, and $\mathbf{u}$ is in the kernel of the Hessian of g evaluated at $\mathbf{u}$.

The condition $H_g(\mathbf{u}) \cdot \mathbf{v} = 0$ implies that at the point $\mathbf{u}$, for any vector $\mathbf{w}$, the directional derivative $D_{\mathbf{w}} D_{\mathbf{v}}(g)$ is equal to 0, i.e., we have $(D_{\mathbf{w}} D_{\mathbf{v}}(g))(\mathbf{u}) = 0$. This is equivalent to the equation $D_{\mathbf{u}} D_{\mathbf{u}} D_{\mathbf{w}} D_{\mathbf{v}}(g) = D_{\mathbf{w}} D_{\mathbf{u}} D_{\mathbf{u}} D_{\mathbf{v}}(g) = 0$ for any vector $\mathbf{w} \in \mathbb{R}^3$.

We therefore see that the conditions on the Hessian H_g of g are equivalent to g satisfying the following linear conditions: $D_{\mathbf{u}} D_{\mathbf{u}} D_{\mathbf{v}}(g) = 0$ and $D_{\mathbf{v}} D_{\mathbf{v}} D_{\mathbf{u}}(g) = 0$. Using differential operators, we can write the above conditions as

$$\partial(\tilde{u}^2 \tilde{v})[g] = \partial(\tilde{v}^2 \tilde{u})[g] = 0,$$

where $\tilde{u} = u_1 x_1 + u_2 x_2 + u_3 x_3$ and $\tilde{v} = v_1 x_1 + v_2 x_2 + v_3 x_3$ are the associated linear forms. We can rephrase this in the following way: for any ternary quartic q which is a multiple of $\tilde{u}^2 \tilde{v}$ or $\tilde{v}^2 \tilde{u}$, we have $\partial(q)[f] = 0$. In other words, the number of conditions imposed on g is equal to the number of (linearly independent) quartics generated by $\tilde{u}^2 \tilde{v}$ and $\tilde{v}^2 \tilde{u}$. We see that there are 5 conditions, since both $\tilde{u}^2 \tilde{v}$ and $\tilde{v}^2 \tilde{u}$ generate $\tilde{u}^2 \tilde{v}^2$, and this is the only intersection between the ideals generated by $\tilde{u}^2 \tilde{v}$ and $\tilde{v}^2 \tilde{u}$ in degree 4.

Consider zeroes at the two points $\mathbf{u_1} = (\mathbf{e}_1, \mathbf{e}_2)^T$ and $\mathbf{u_2} = (\mathbf{e_3}, [a, b, 1])^T$. From the first zero we get cubics $x_1^2 x_2$ and $x_2^2 x_1$, which generate the span of the following five degree 4 monomials: $x_1^3 x_2, x_1^2 x_2^2, x_1^2 x_2 x_3, x_2^3 x_1, x_2^2 x_1 x_3$. Note that only $x_1^2 x_2 x_3$ and $x_2^2 x_1 x_3$ are divisible by x_3.

From the second zero we get two distinct cubics: $x_3^2(ax_1 + bx_2 + x_3)$ and $(ax_1 + bx_2 + x_3)^2 x_3$. Any quartic that these cubics generate has to be divisible by $x_3(ax_1 + bx_2 + x_3)$. So the only possible intersection with the quartics from the first zero comes as linear combination $\alpha x_1^2 x_2 x_3 + \beta x_2^2 x_1 x_3 = x_1 x_2 x_3(\alpha x_1 + \beta x_2)$. But this linear combination is not divisible by $x_3(ax_1 + bx_2 + x_3)$. So we see that two zeroes impose 10 linearly independent conditions, and therefore the dimension of $L_{a,b}$ is $15 - 10 = 5$. Since $T_{a,b}$ is contained in $L_{a,b}$, the dimension of $T_{a,b}$ is at most 5. ☐

Lemma 3.4 (Reduction Lemma)**.** *Suppose that $\Sigma C_{3,4} \neq C_{3,4}$. Then there exists a form $p \in T_{a,b}$ with both a and b non-zero, such that $p \notin \Sigma C_{3,4}$.*

Proof. If $\Sigma C_{3,4} \neq C_{3,4}$ then there exists a form p on the boundary of $C_{3,4}$ that is not in $\Sigma C_{3,4}$. The boundary of $C_{3,4}$ consists of convex forms whose Hessian forms have a nontrivial zero in $\mathbb{RP}^2 \times \mathbb{RP}^2$. Therefore, we may assume that $h_p(\mathbf{c}, \mathbf{d}) = 0$ for some point $(\mathbf{c}, \mathbf{d}) \in \mathbb{RP}^2 \times \mathbb{RP}^2$.

We first observe that $\mathbf{c} \neq \mathbf{d}$. Suppose not and $h_p(\mathbf{c}, \mathbf{c}) = 0$. In view of (2), this implies that $p(\mathbf{c}) = 0$, and hence p is a convex ternary form with a nontrivial zero. By an observation of Reznick [29, Prop. 4.1], it follows that p is a bivariate form defined on the orthogonal complement of $\mathbf{c}$ in $\mathbb{R}^3$. But then p is sos-convex [7, Thm. 5.4], which is a contradiction.

We can apply a nonsingular linear change of coordinates and move the zero of h_p to $\mathbf{u}_1$. The new form will still be convex but not sos-convex. Therefore, we may assume that h_p has a zero at $\mathbf{u}_1$. Recall that $F_{\mathbf{u}_1}$ is the face of $C_{3,4}$ consisting of all forms f for which $h_f(\mathbf{u}_1) = 0$. It follows that there exists a form $\hat{p}$ in the relative interior of $F_{\mathbf{u}_1}$ such that $\hat{p} \notin \Sigma C_{3,4}$. Note that this implies that $h_{\hat{p}}(\mathbf{x}, \mathbf{y})$ has a single zero at $\mathbf{u}_1$, and moreover the Hessian of $h_{\hat{p}}$ is positive definite on the tangent space to $\mathbb{S}^2 \times \mathbb{S}^2$ at $\mathbf{u_1}$.

Let $q = \frac{1}{12}x_3^4$, then we have $h_q(\mathbf{x}, \mathbf{y}) = x_3^2 y_3^2$. We observe that the Hessian matrix of h_q is identically zero at the point $\mathbf{u}_1$. Therefore for small enough ε, we have that $\hat{p} - \varepsilon q$ is still convex. Now let $\varepsilon > 0$ be such that $\bar{p} = \hat{p} - \varepsilon q$ is on the relative boundary of $F_{\mathbf{u_1}}$. We note that $\bar{p}$ is convex but not sos-convex. Since the Hessian matrix of h_q is identically zero at the point $\mathbf{u}_1$, it follows that the Hessian of $h_{\bar{p}}$ is positive definite on the tangent space to $\mathbb{S}^2 \times \mathbb{S}^2$ at $\mathbf{u_1}$. Therefore $h_{\bar{p}}$ must acquire an additional zero at a point $\mathbf{v} = (\bar{\mathbf{c}}, \bar{\mathbf{d}})$, $\bar{\mathbf{c}}, \bar{\mathbf{d}} \in \mathbb{S}^2$. Since $\bar{p}$ is not sos-convex, we see that $\bar{\mathbf{c}} \neq \bar{\mathbf{d}}$. Furthermore, we must have $\bar{c}_3$, $\bar{d}_3 \neq 0$. Otherwise, $x_3^2 y_3^2$ is zero at the point $\bar{\mathbf{c}}, \bar{\mathbf{d}}$ and thus $h_{\bar{p}}$ has the same value at $(\bar{\mathbf{c}}, \bar{\mathbf{d}})$ as $h_{\hat{p}}$, while $h_{\hat{p}}$ had no zeroes outside of $\mathbf{u}_1$. In other words, $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$ are not in the span of $[1,0,0]^T$, $[0,1,0]^T$.

For any $\mathbf{v} = (\mathbf{c}, \mathbf{d})$ such that $\mathbf{c}$ and $\mathbf{d}$ are not in the span of $[1,0,0]^T$, $[0,1,0]^T$, we claim that the face $F_{\mathbf{c},\mathbf{d}}$ of $C_{3,4}$ consisting of convex ternary quartics with zeroes at $\mathbf{u_1}$ and $\mathbf{v}$ is at most 5-dimensional. Apply an invertible linear change of coordinates that fixes $[1,0,0]^T$, $[0,1,0]^T$ and maps $\bar{\mathbf{c}}$ to $[0,0,1]^T$. Since $\mathbf{d}$ is not in the span of $\mathbf{e_1}$ and $\mathbf{e_2}$ and $\mathbf{c} \neq \mathbf{d}$, the point $\mathbf{d}$ will be taken to (a non-zero multiple of) $[a,b,1]^T$ for some $a, b \in \mathbb{R}$ and a, b not both zero. We apply Lemma 3.3 to see that the dimension of $F_{\mathbf{c},\mathbf{d}}$ is at most 5.

Note that there is an open ball B of forms around $\hat{p}$ that are convex and not sos-convex. We can push any form in B to the boundary of $F_{\mathbf{u_1}}$ by subtracting an appropriate multiple of q. It follows that we cover an open neighborhood B' of $\bar{p}$ in the boundary of F_{u_1} by convex ternary quartics that acquire a second zero at some (not necessarily same) point $\mathbf{v} = (\bar{\mathbf{c}}, \bar{\mathbf{d}})$, with $\bar{\mathbf{c}} \neq \bar{\mathbf{d}}$ and c_3, d_3 not equal to 0. It remains to show that there exists a point in B' for which the additional zero $\mathbf{v} = (\bar{\mathbf{c}}, \bar{\mathbf{d}})$ is such that the four points $[1,0,0]^T$, $[0,1,0]^T$, $\mathbf{c}$ and $\mathbf{d}$ are in general linear position in $\mathbb{R}^3$. If this is the case, then an invertible linear change of coordinates can map $\mathbf{c}$ to $[0,0,1]^T$ and $\mathbf{d}$ to $[a,b,1]^T$ with both a, b non-zero.

By Lemma 3.2, the face $F_{\mathbf{u_1}}$ is 10-dimensional, and therefore its boundary is 9-dimensional. We now derive a contradiction to all forms in B' acquiring a zero $\mathbf{v} = (\mathbf{c}, \mathbf{d})$, where $[1,0,0]^T$, $[0,1,0]^T$, $\mathbf{c}$ and $\mathbf{d}$ are not in general linear position via a dimension counting argument. As we saw above, the face $T_{\mathbf{c},\mathbf{d}}$ is at most 5-dimensional. The pairs $(\mathbf{c}, \mathbf{d}) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $[1,0,0]^T$, $[0,1,0]^T$, $\mathbf{c}$ and $\mathbf{d}$ are not in general linear position form a 3-dimensional family. Therefore, all together such faces $T_{\mathbf{c},\mathbf{d}}$ cover at most an 8-dimensional subset of the boundary of $F_{\mathbf{u}_1}$, and hence they cannot cover all of B'. □

3.2 Cone Description

We now derive an explicit desciption of the face $T_{a,b}$ with both a and b non-zero. Let

$$q_1 = x_1^4, \quad q_2 = x_2^4, \quad q_3 = (x_1 - a x_3)^4, \quad q_4 = (x_2 - b x_3)^4, \quad q_5 = x_3^2 (b x_1 - a x_2)^2.$$

We have the following:

Theorem 3.5. *The face $T_{a,b}$ consists of all forms $\alpha_1 q_1 + \cdots + \alpha_5 q_5$ such that $\alpha_1, \ldots, \alpha_4 \geq 0$ and*

$$-\frac{4a^2b^2}{\frac{b^4}{\alpha_1} + \frac{a^4}{\alpha_2} + \frac{b^4}{\alpha_3} + \frac{a^4}{\alpha_4}} \leq \alpha_5 \leq 0.$$

Furthermore, all forms in $T_{a,b}$ are sos-convex.

In view of Lemma 3.4, we note that a proof of Theorem 3.5 would complete the proof of Theorem 3.1. We start by proving the latter claim of Theorem 3.5. Let $S_{a,b}$ be the cone of all forms $p = \alpha_1 q_1 + \cdots + \alpha_5 q_5$ such that $\alpha_1, \ldots, \alpha_4 \geq 0$ and

$$-\frac{4a^2b^2}{\frac{b^4}{\alpha_1} + \frac{a^4}{\alpha_2} + \frac{b^4}{\alpha_3} + \frac{a^4}{\alpha_4}} \leq \alpha_5 \leq 0. \tag{4}$$

Lemma 3.6. *Suppose that $p \in S_{a,b}$. Then p is sos-convex.*

Proof. Let $p = \alpha_1 q_1 + \cdots + \alpha_5 q_5 \in S_{a,b}$. First we observe that if for some $i \in \{1, \ldots, 4\}$, $\alpha_i = 0$, then condition (4) implies that $\alpha_5 = 0$ and then p is a sum of fourth powers of linear forms and hence sos-convex. Therefore we may restrict ourselves to the case of strictly positive $\alpha_1, \ldots, \alpha_4$. We establish that p is sos-convex by showing that $h_p(\mathbf{x}, \mathbf{y})$ is a sum of squares.

We will use an explicit set of squares in our decomposition. For this we need a basis of the linear subspace of bilinear forms in $\mathbf{x}$ and $\mathbf{y}$ with zeroes at $\mathbf{u}_1$ and $\mathbf{u}_2$. Let

$$s_1 = x_1 y_1,\ \ s_2 = x_2 y_2,\ \ s_3 = (x_1 - a x_3)(y_1 - a y_3),\ \ s_4 = (x_2 - b x_3)(y_2 - b y_3),\ \ s_5 = x_3(b y_1 - a y_2). \tag{5}$$

The forms s_i were chosen so that for $i \in \{1, \ldots, 4\}$ we have $h_{q_i}(\mathbf{x}, \mathbf{y}) = s_i^2$. Let $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5)^T$ and let $c = \frac{a}{b}$. Consider the matrix:

$$M = \begin{pmatrix} 12\alpha_1 + 2\alpha_5 c^{-2} & -2\alpha_5 & -2\alpha_5 c^{-2} & 2\alpha_5 & 2\alpha_5 c \\ -2\alpha_5 & 12\alpha_2 + 2\alpha_5 c^2 & 2\alpha_5 & -2\alpha_5 c^2 & -2\alpha_5 c \\ -2\alpha_5 c^{-2} & 2\alpha_5 & 12\alpha_3 + 2\alpha_5 c^{-2} & -2\alpha_5 & -2\alpha_5 c^{-1} \\ 2\alpha_5 & -2\alpha_5 c^2 & -2\alpha_5 & 12\alpha_4 + 2\alpha_5 c^2 & 2\alpha_5 c \\ 2\alpha_5 c & -2\alpha_5 c & -2\alpha_5 c^{-1} & 2\alpha_5 c & -4\alpha_5 \end{pmatrix}. \tag{6}$$

The matrix M is the *Gram matrix* of $h_p(\mathbf{x}, \mathbf{y})$ with respect to the basis $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5)^T$. This means that $h_p(\mathbf{x}, \mathbf{y}) = \mathbf{s}^T M \mathbf{s}$. To show that p is sos-convex, it suffices to show that M is a positive semidefinite matrix for all α_i allowed in $S_{a,b}$.

We note that with $\alpha_i > 0$ for $i \in \{1, \ldots, 4\}$ and $\alpha_5 = 0$, the matrix M is diagonal with four positive entries and therefore it is positive semidefinite, with a single zero eigenvalue. Now we look at what happens if α_5 is allowed to be negative.

A direct computation shows that

$$\det M = -\frac{20736\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\left(4a^2b^2 + \alpha_5\left(\frac{b^4}{\alpha_1} + \frac{a^4}{\alpha_2} + \frac{b^4}{\alpha_3} + \frac{a^4}{\alpha_4}\right)\right)}{a^2b^2}.$$

Therefore, we see that for $-\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}} \leq \alpha_5 \leq 0$, the determinant of M is nonnegative and strictly positive for α_5 strictly between these bounds. It follows that M is positive semidefinite with $\alpha_5 = -\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$ since it started with 4 positive eigenvalues at $\alpha_5 = 0$ and the product of the eigenvalues is positive as α_5 moves from 0 to $-\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$.

□

Now we show that if $p \in S_{a,b}$ and α_5 is at its lower bound, then $h_p(\mathbf{x}, \mathbf{y})$ has an additional zero different from $\mathbf{u}_1$ and $\mathbf{u}_2$.

Lemma 3.7. *Let $p \in S_{a,b}$ with $\alpha_1, \ldots, \alpha_4 > 0$ and $\alpha_5 = -\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$. Then $h_p(\mathbf{x}, \mathbf{y})$ has an additional zero with $x_1, x_2, y_1, y_2 \neq 0$.*

Proof. When $\alpha_5 = -\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$ the matrix M from (6) is singular and the nullspace of M is spanned by the vector

$$\mathbf{v} = \left(\frac{2ab^3}{\alpha_1}, -\frac{2a^3b}{\alpha_2}, -\frac{2ab^3}{\alpha_3}, \frac{2a^3b}{\alpha_4}, \frac{b^4}{\alpha_1} + \frac{a^4}{\alpha_2} + \frac{b^4}{\alpha_3} + \frac{a^4}{\alpha_4}\right)^T.$$

We would like to show that $h_p(\mathbf{x},\mathbf{y}) = \mathbf{s}^T M \mathbf{s}$ has a nontrivial zero. Therefore we need to show that the system of equations:

$$s_1 = v_1,\ s_2 = v_2,\ s_3 = v_3, s_4 = v_4,\ s_5 = v_5,$$

where s_i are defined in (5), has a real solution in $\mathbf{x}$ and $\mathbf{y}$. Note that since $a, b, \alpha_i \neq 0$, it follows that this solution must have $x_1, x_2, y_1, y_2 \neq 0$.

Using the fact that $y_3(ax_2 - bx_1) = s_5 + ab\left(\frac{s_3-s_1}{a^2} - \frac{s_4-s_2}{b^2}\right)$ and equations $s_1 = v_1,\ s_2 = v_2,\ s_5 = v_5$, we can express y_1, y_2, y_3 and x_3 in terms of x_1 and x_2 and substitute these into the equation $s_3 = v_3$. After simplification, we get a quadratic equation in x_1 and x_2:

$$\begin{aligned}
&-2a^3b\alpha_1\alpha_3\alpha_4(a^4\alpha_1\alpha_2\alpha_3 + a^4\alpha_1\alpha_3\alpha_4 - b^4\alpha_1\alpha_2\alpha_4 - b^4\alpha_2\alpha_3\alpha_4)x_1^2\\
&+(2\alpha_4^2b^8\alpha_3\alpha_2^2\alpha_1 + 2\alpha_1^2a^8\alpha_4\alpha_3^2\alpha_2 - 6a^4\alpha_2b^4\alpha_3^2\alpha_4^2\alpha_1 - 2\alpha_4b^4\alpha_3^2\alpha_2^2\alpha_1a^4 - 2\alpha_1^2a^4\alpha_4^2\alpha_3\alpha_2b^4\\
&+2\alpha_1^2a^4\alpha_2^2\alpha_3b^4\alpha_4 + \alpha_4^2b^8\alpha_3^2\alpha_2^2 + \alpha_1^2a^8\alpha_4^2\alpha_3^2 + \alpha_1^2a^8\alpha_2^2\alpha_3^2 + \alpha_1^2\alpha_2^2b^8\alpha_4^2)x_1x_2\\
&+2ab^3\alpha_2\alpha_3\alpha_4(a^4\alpha_1\alpha_2\alpha_3 + a^4\alpha_1\alpha_3\alpha_4 - b^4\alpha_1\alpha_2\alpha_4 - b^4\alpha_2\alpha_3\alpha_4)x_2^2 = 0.
\end{aligned}$$

We note that the coefficients of x_1^2 is almost the negative of the coefficient of x_2^2. With positive α_i it follows that the discriminant of this equation is positive and therefore it always has a real solution. $\square$

We now finish the proof of Theorem 3.5.

Proof of the first claim in Theorem 3.5. First we claim that $T_{a,b}$ is contained in the linear span of $q_1, \ldots, q_5$. Observe that these polynomials are linearly independent ternary quartics, and furthermore they are all contained in the vector space $L_{a,b}$. By Lemma 3.3, we know that $\dim L_{a,b} = 5$ and therefore $q_1, \ldots, q_5$ are a basis of $L_{a,b}$. The claim now follows.

Let $p = \alpha_1 q_1 + \ldots + \alpha_5 q_5$ and suppose that $p \in T_{a,b}$. We now show that $\alpha_1, \ldots, \alpha_4 \geq 0$ and $\alpha_5 \leq 0$. Let $\mathbf{v} = ([0, b, 1], [a, 0, 1])^T$. Then $h_{q_i}(\mathbf{v}) = 0$ for $i \in \{1, \ldots, 4\}$ while $h_{q_5}(\mathbf{v}) = -4a^2b^2$. Since h_p is a nonnegative biquadratic form it follows that $\alpha_5 \leq 0$.

Similarly, we can find a common zero $\mathbf{v}$ for any four h_{q_i} with the 5-th h_{q_j} not equal to zero at $\mathbf{v}$, which determines the sign of α_j. For example, let $\mathbf{v} = ([0, (2+\sqrt{3})b, 1)], [a, 0, 1])^T$. Then $h_{q_i}(\mathbf{v}) = 0$ for $i \neq 4$ and $h_{q_4}(\mathbf{v}) = (48 + 24\sqrt{3})b^4$ and therefore $\alpha_4 \geq 0$.

Finally, we claim that if any $\alpha_i = 0$ for $i \in \{1, \ldots, 4\}$, then $\alpha_5 = 0$. For example, suppose that $\alpha_1 = 0$. We already know that $\alpha_5 \leq 0$. We need to exhibit a point $\mathbf{v}$ for which $h_{q_5}(\mathbf{v}) > 0$ and $h_{q_2}(\mathbf{v}) = h_{q_3}(\mathbf{v}) = h_{q_4}(\mathbf{v}) = 0$, as this will imply that $\alpha_5 \geq 0$ and we will be done. This occurs for $\mathbf{v} = ([a, b, 1], [1, 0, 1])^T$. It is easy to construct similar examples for $i = 2, 3, 4$ as well.

Therefore we may restrict ourselves to the case of strictly positive $\alpha_1, \ldots, \alpha_4$. Let $\bar{p} = \alpha_1 q_1 + \ldots + \alpha_4 q_4$. We know that $\bar{p}$ is convex and since $\alpha_5 \leq 0$, we just need to know the lowest value of α_5 so that $p = \bar{p} + \alpha_5 q_5$ is convex.

We note that

$$h_{\bar{p}}(\mathbf{x},\mathbf{y}) = 12(\alpha_1 x_1^2 y_1^2 + \alpha_2 x_2^2 y_2^2 + \alpha_3(x_1 - ax_3)^2(y_1 - ay_3)^2 + \alpha_4(x_2 - bx_3)^2(y_2 - by_3)^2).$$

Therefore $h_{\bar{p}}(\mathbf{x},\mathbf{y})$ has zeroes only at the points where either $x_1 = 0$ or $y_1 = 0$. Now, for $\alpha_5 = -\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$, we know that $p = \bar{p} + \alpha_5 q_5$ is sos-convex by Lemma 3.6. However, by Lemma 3.7, the biquadratic form $h_p(\mathbf{x},\mathbf{y})$ has a zero at a point where $h_{\bar{p}}(\mathbf{x},\mathbf{y})$ is strictly positive. It follows that $-\frac{4a^2b^2}{\frac{b^4}{\alpha_1}+\frac{a^4}{\alpha_2}+\frac{b^4}{\alpha_3}+\frac{a^4}{\alpha_4}}$ is the smallest α_5 can be for $p = \bar{p} + \alpha_5 q_5$ to be convex. $\square$

We end by a few remarks around Theorem 3.1. In [15], Choi, Lam, and Reznick show that a nonnegative quartic form in 4 variables that has more than 11 nontrivial zeroes, or a nonnegative sextic form in 3 variables that has more than 10 nontrivial zeroes, must be a sum of squares. The bound for the first claim was improved to 10 in [9]. The next corollary is of the same spirit, but in relation to convex forms.

Corollary 3.8. *Let p be a convex form in 4 variables of degree 4, or a convex form in 3 variables (of any degree). If p vanishes at a nonzero point, then p is sos-convex (and sos).*

Proof. By [29, Prop. 4.1], a convex form that vanishes at a nonzero point can be written, after a nonsingular linear change of coordinates, as a convex form in one fewer variable. The claim for convex forms in 4 variables of degree 4 then follows from our Theorem 3.1. Similarly, the claim for convex forms in 3 variables follows from the fact that bivariate convex forms are sos-convex [7, Thm. 5.4].

□

Together with the results in [7], Theorem 3.1 characterizes all dimensions and degrees for which one can have convex forms that are not sos-convex. As mentioned earlier, these turns out to be the same dimensions and degrees for which there are nonnegative forms that are not sums of squares [21], though for very different reasons. An interesting remaining question is to characterize dimensions and degrees for which one can have convex forms that are not sums of squares. Existence of such forms was shown in [8] (see also [10, Chapter 4]) by Blekherman when the degree is 4 or larger and the dimension is large enough. El Khadir has shown that such a quartic form does not exist in dimension 4 [19], and Saunderson has constructed an explicit example in dimension 272 [30].

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A Proofs for Section 2 (Symmetric biquadratic forms and Hessian biquadratic forms)

In this appendix, we present the proof of Theorem 2.3.

Proof. For the convenience of the reader, let us recall the biquadratic form in (3):

$$\begin{aligned} b(x_1,x_2,x_3,y_1,y_2,y_3) \;=\;& 12(x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2) \\ &+31x_1x_2y_1y_2-10x_1x_3y_1y_3-5x_2x_3y_2y_3 \\ &+12(x_2^2y_1^2+y_2^2x_1^2)+6(x_3^2y_1^2+y_3^2x_1^2)+12(x_2^2y_3^2+y_2^2x_3^2) \\ &+4(x_1x_2y_1^2+y_1y_2x_1^2)+9(x_1x_3y_1^2+y_1y_3x_1^2)-10(x_2x_3y_1^2+y_2y_3x_1^2) \\ &+13(x_1x_3y_2^2+y_1y_3x_2^2)+13(x_2x_3y_2^2+y_2y_3x_2^2)+23(x_1x_2y_2^2+y_1y_2x_2^2) \\ &+5(x_1x_2y_3^2+y_1y_2x_3^2)+3(x_1x_3y_3^2+y_1y_3x_3^2)+7(x_2x_3y_3^2+y_2y_3x_3^2) \\ &+5(x_1x_2y_2y_3+y_1y_2x_2x_3)-11(x_1x_3y_2y_3+y_1y_3x_2x_3)+3(x_1x_3y_1y_2+y_1y_3x_1x_2). \end{aligned}$$

The fact that $b(\mathbf{x},\mathbf{y})=b(\mathbf{y},\mathbf{x})$ can readily be seen from the order in which we have written the monomials. To prove that $b(\mathbf{x},\mathbf{y})$ is nonnegative, we show that

$$b(\mathbf{x},\mathbf{y})(x_1^2+x_2^2) \tag{7}$$

is sos. This, together with nonnegativity of $(x_1^2+x_2^2)$ and continuity of $b(\mathbf{x},\mathbf{y})$, implies that $b(\mathbf{x},\mathbf{y})$ is nonnegative. A rational sum of squares certificate for (7), which we have obtained from the software package SOSTOOLS [PPP05], is as follows:

$$b(\mathbf{x},\mathbf{y})(x_1^2+x_2^2)=\frac{1}{384}\mathbf{z}^TQ\mathbf{z},$$

where $\mathbf{z}$ is the vector of monomials

$$\begin{aligned}\mathbf{z} \;=\;& [x_2x_3y_3, x_2x_3y_2, x_2x_3y_1, x_2^2y_3, x_2^2y_2, x_2^2y_1, x_1x_3y_3, \\ & x_1x_3y_2, x_1x_3y_1, x_1x_2y_3, x_1x_2y_2, x_1x_2y_1, x_1^2y_3, x_1^2y_2, x_1^2y_1]^T,\end{aligned}$$

and Q is the positive definite matrix given by

$$Q = \begin{pmatrix}
4608 & 1344 & 576 & 1344 & -504 & -264 & 0 & -900 & -264 & 1392 & -612 & -303 & 972 & -612 & 576 \\
1344 & 4608 & 960 & -456 & 2496 & 2340 & 900 & 0 & 864 & 264 & 3072 & 1572 & 396 & 672 & 240 \\
576 & 960 & 2304 & -1848 & -1380 & -1920 & 264 & -864 & 0 & -483 & -1440 & 1200 & -888 & 240 & -1164 \\
1344 & -456 & -1848 & 4608 & 2496 & 2496 & -816 & -1548 & -1047 & 960 & 840 & 24 & 1896 & -1944 & 1452 \\
-504 & 2496 & -1380 & 2496 & 4608 & 4416 & -216 & -576 & 312 & 120 & 4416 & 1356 & 1380 & -284 & 1620 \\
-264 & 2340 & -1920 & 2496 & 4416 & 4608 & -87 & 132 & 528 & 552 & 4596 & 768 & 1512 & -24 & 1892 \\
0 & 900 & 264 & -816 & -216 & -87 & 4608 & 1344 & 576 & 372 & -2400 & -1980 & 576 & -1572 & -2400 \\
-900 & 0 & -864 & -1548 & -576 & 132 & 1344 & 4608 & 960 & 1656 & 1824 & -828 & -540 & 2496 & -84 \\
-264 & 864 & 0 & -1047 & 312 & 528 & 576 & 960 & 2304 & 180 & 1308 & -756 & 480 & 660 & 1728 \\
1392 & 264 & -483 & 960 & 120 & 552 & 372 & 1656 & 180 & 3120 & 1140 & 1260 & 960 & 876 & 1152 \\
-612 & 3072 & -1440 & 840 & 4416 & 4596 & -2400 & 1824 & 1308 & 1140 & 9784 & 3588 & 84 & 4416 & 3660 \\
-303 & 1572 & 1200 & 24 & 1356 & 768 & -1980 & -828 & -756 & 1260 & 3588 & 5432 & -576 & 2292 & 768 \\
972 & 396 & -888 & 1896 & 1380 & 1512 & 576 & -540 & 480 & 960 & 84 & -576 & 2304 & -1920 & 1728 \\
-612 & 672 & 240 & -1944 & -284 & -24 & -1572 & 2496 & 660 & 876 & 4416 & 2292 & -1920 & 4608 & 768 \\
576 & 240 & -1164 & 1452 & 1620 & 1892 & -2400 & -84 & 1728 & 1152 & 3660 & 768 & 1728 & 768 & 4608
\end{pmatrix}.$$

Let us now prove that b is not sos. If we denote the cone of sos ternary biquadratic forms by $\Sigma_{B,3}$ and its dual cone by $\Sigma_{B,3}^*$, our proof will simply proceed by presenting a dual functional $\xi \in \Sigma_{B,3}^*$ that takes a negative value on the polynomial b. Let us fix the following ordering for monomials of ternary biquadratic forms:

$$\begin{aligned}
&\{x_3^2y_3^2, x_3^2y_2y_3, x_3^2y_2^2, x_3^2y_1y_3, x_3^2y_1y_2, x_3^2y_1^2, x_2x_3y_3^2, x_2x_3y_2y_3, x_2x_3y_2^2, x_2x_3y_1y_3, x_2x_3y_1y_2, x_2x_3y_1^2,\\
&x_2^2y_3^2, x_2^2y_2y_3, x_2^2y_2^2, x_2^2y_1y_3, x_2^2y_1y_2, x_2^2y_1^2, x_1x_3y_3^2, x_1x_3y_2y_3, x_1x_3y_2^2, x_1x_3y_1y_3, x_1x_3y_1y_2, x_1x_3y_1^2,\\
&x_1x_2y_3^2, x_1x_2y_2y_3, x_1x_2y_2^2, x_1x_2y_1y_3, x_1x_2y_1y_2, x_1x_2y_1^2, x_1^2y_3^2, x_1^2y_2y_3, x_1^2y_2^2, x_1^2y_1y_3, x_1^2y_1y_2, x_1^2y_1^2\}.
\end{aligned} \tag{8}$$

With this ordering, the vector of coefficients $\vec{\mathbf{b}}$ of the biquadratic form b in (3) is given by

$$\begin{aligned}
\vec{\mathbf{b}} = \quad &[12, 7, 12, 3, 5, 6, 7, -5, 13, -11, 5, -10, 12, 13, 12, 13, 23,\\
&12, 3, -11, 13, -10, 3, 9, 5, 5, 23, 3, 31, 4, 6, -10, 12, 9, 4, 12]^T.
\end{aligned}$$

Using the same ordering, we can represent our dual functional ξ with the vector

$$\begin{aligned}
\mathbf{c} = \quad &[37, -18, 18, -23, -1, 66, -18, 12, -15, 1, -1, 35, 18, -15, 96, -5, -37,\\
&64, -23, 1, -5, 34, -7, -48, -1, -1, -37, -7, -15, 0, 66, 35, 64, -48, 0, 61]^T.
\end{aligned}$$

We have

$$\langle \xi, b\rangle = \mathbf{c}^T\vec{\mathbf{b}} = -37 < 0.$$

On the other hand, we claim that $\xi \in \Sigma_{B,3}^*$; i.e., for any form $w \in \Sigma_{B,3}$, we should have

$$\langle \xi, w\rangle = \mathbf{c}^T\vec{\mathbf{w}} \geq 0, \tag{9}$$

where $\vec{\mathbf{w}}$ here denotes the coefficients of w listed according to the ordering in (8). Indeed, if w is sos, then it can be written in the form

$$w(x) = \tilde{\mathbf{z}}^T\tilde{Q}\tilde{\mathbf{z}} = \mathrm{Tr}\ \tilde{Q}\cdot\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T,$$

for some symmetric positive semidefinite matrix $\tilde{Q}$, and a vector of monomials

$$\tilde{\mathbf{z}} = [x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3]^T.$$

It is not difficult to see that

$$\mathbf{c}^T\vec{\mathbf{w}} = \operatorname{Tr}\tilde{Q}\cdot(\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T)|_{\mathbf{c}}, \tag{10}$$

where by $(\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T)|_{\mathbf{c}}$ we mean a matrix where each monomial in $\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T$ is replaced with the corresponding element of the vector $\mathbf{c}$. This yields the matrix

$$(\tilde{\mathbf{z}}\tilde{\mathbf{z}}^T)|_{\mathbf{c}} = \begin{pmatrix} 61 & 0 & -48 & 0 & -15 & -7 & -48 & -7 & 34 \\ 0 & 64 & 35 & -15 & -37 & -1 & -7 & -5 & 1 \\ -48 & 35 & 66 & -7 & -1 & -1 & 34 & 1 & -23 \\ 0 & -15 & -7 & 64 & -37 & -5 & 35 & -1 & 1 \\ -15 & -37 & -1 & -37 & 96 & -15 & -1 & -15 & 12 \\ -7 & -1 & -1 & -5 & -15 & 18 & 1 & 12 & -18 \\ -48 & -7 & 34 & 35 & -1 & 1 & 66 & -1 & -23 \\ -7 & -5 & 1 & -1 & -15 & 12 & -1 & 18 & -18 \\ 34 & 1 & -23 & 1 & 12 & -18 & -23 & -18 & 37 \end{pmatrix},$$

which can easily be checked to be positive definite. Therefore, equation (10) along with the fact that Q is positive semidefinite implies that (9) holds. This completes the proof.[2] □

References for the Appendix

[PPP05] S. Prajna, A. Papachristodoulou, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2002-05. Available from `http://www.cds.caltech.edu/sostools` and `http://www.mit.edu/~parrilo/sostools`.

[2]For verification purposes, we have made the content of this proof available in electronic form at `http://aaa.princeton.edu/software`. In particular, whenever we state that a matrix is positive definite, this claim is certified by a rational LDL^T factorization.