

# Unified ODE Analysis of Smooth Q-Learning Algorithms

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**Abstract**—Convergence of Q-learning has been the focus of extensive research over the past several decades. Recently, an asymptotic convergence analysis for Q-learning was introduced using a switching system framework. This approach applies the so-called ordinary differential equation (ODE) approach to prove the convergence of the asynchronous Q-learning modeled as a continuous-time switching system, where notions from switching system theory are used to prove its asymptotic stability without using explicit Lyapunov arguments. However, to prove stability, restrictive conditions, such as the quasi-monotonicity, must be satisfied for the underlying switching systems, which makes it hard to easily generalize the analysis method to other reinforcement learning algorithms, such as the smooth Q-learning variants. In this paper, we present a more general and unified convergence analysis that improves upon the switching system approach and can analyze Q-learning and its smooth variants. The proposed analysis is motivated by previous work on the convergence of synchronous Q-learning based on  $p$ -norm serving as a Lyapunov function. However, the proposed analysis addresses more general ODE models that can cover both asynchronous Q-learning and its smooth versions with simpler and more intuitive proofs.

**Index Terms**—Reinforcement learning, Q-learning, convergence, stability, stochastic approximation

## I. INTRODUCTION

Reinforcement learning (RL) solves the optimal sequential decision-making problem in unknown environments using experiences [1]. Recent RL algorithms have excelled beyond human performance in many complex tasks [2], spurring increased interest in both theoretical and experimental RL. Among various algorithms, Q-learning [3] stands out as a foundational and widely used method. Its convergence properties have been thoroughly explored for decades, and classical analysis mostly focuses on asymptotic convergence [4]–[8]. Recently, advances have been made in finite-time convergence analysis [9]–[18], which measures the speed of iterative progress towards a solution.

The main focus of this paper is on the asymptotic convergence analysis of Q-learning and its smooth variants using the ordinary differential equation (ODE) analysis [5]. The fundamental idea of the ODE approach is to approximate stochastic algorithms into the corresponding continuous-time dynamical system models and infer the convergence of the original algorithm from the global asymptotic stability of the dynamical system model. The ODE approach has been widely applied to prove the convergence of RLs [5], [6], [19]–[24]. We note that although finite-time analysis provides stronger results, studying asymptotic convergence based on the ODE method is still meaningful for several reasons: 1) the ODE methods are generally simpler than direct finite-time analysis because it usually uses existing standard lemmas; 2) it gives more intuition from control theoretic perspectives; 3) it provides a preliminary check of convergence before engaging in more sophisticated finite-time analysis.

Given these considerations, the main goal of this paper is to present a unified ODE analysis of Q-learning and its smooth variants [25]–[29] for additional insights and complementary analysis. Here, smooth variants of Q-learning indicate Q-learning algorithms using smooth approximations of the max operator, such as the log-sum-exp (LSE) [25], Boltzmann softmax [27], and mellowmax [28]. It is known that these smooth variants can improve the exploration and

performance [25]–[27] and mitigate the overestimation bias of Q-learning [26], [29]. However, their convergence has not been fully investigated until now. In particular, [25] and [26] only studied deep RL counterparts for the smooth Q-learning algorithms with the LSE and Boltzmann softmax operators, respectively, without rigorous convergence analysis for tabular cases. The result in [28] only considered SARSA and value iteration algorithms using the mellowmax operator. A tabular Q-learning with the Boltzmann softmax operator was investigated in [27], [29], where asymptotic convergence was proved using the stochastic approximation methods in [8], which are different from the ODE approaches. Therefore, the underlying assumptions and conditions for the convergence are different, and are not directly comparable. For instance, we consider a scalar step-size which does not depend on the state-action pair, while the step-size adopted in [27], [29] should depend on the state-action pair. Moreover, to the author's knowledge, convergence of smooth Q-learning with the LSE and mellowmax operators in the tabular setting has not been reported in the literature so far. Besides, the proposed analysis is quite general, and provide a unified framework that can cover many Q-learning variants mentioned above.

The proposed analysis uses a  $p$ -norm with  $p \in (1, \infty]$  as a Lyapunov function to establish the global asymptotic stability of the ODE model. We note that this idea was first introduced in [5], [30]. The main differences lie in the fact that [30] can only address synchronous Q-learning, while the ODE models addressed in this paper are more general so that it can cover asynchronous Q-learning and its variants as well. Moreover, we provide simpler proofs for the asymptotic stability analysis using the  $p$ -norm than those in [5], [30]. On the other hand, it is worth mentioning the recent work [6], which developed a switching system model [31] of asynchronous Q-learning for the ODE analysis, and applied notions from switching system theory to prove its global asymptotic stability without using explicit Lyapunov function arguments. Subsequently, the Borkar and Meyn theorem [5] was applied to prove the convergence of asynchronous Q-learning. However, its main drawback is that to prove the global stability, some restrictive conditions, such as the quasi-monotonicity, must be satisfied for the underlying switching systems, which makes it hard to easily generalize the analysis method to other RL algorithms, such as the smooth Q-learning variants. On the other hand, the proposed method can more widely cover various algorithms, such as the smooth Q-learning variants and standard Q-learning, in a unified way. Therefore, it provides an additional option for convergence analysis of asynchronous Q-learning other than [6].

In summary, the main contributions of this paper can be listed as follows:

- 1) The proposed approach complements the analysis in [5], [30] by considering more general ODE models. Therefore, it can more generally address asynchronous Q-learning. Moreover, the proposed analysis in general provides simpler and more intuitive proofs.
- 2) The proposed method improves upon the switching system approach in [6] by removing restrictive conditions, providing an additional option for the ODE analysis asynchronous Q-learning.
- 3) We provide new convergence analyses of smooth Q-learning variants [25]–[29], which complement the existing analyses given in [25]–[29] with additional insights.

## II. PRELIMINARIES

### A. Notation

The adopted notation is as follows:  $\mathbb{R}$ : set of real numbers;  $\mathbb{R}^n$ :  $n$ -dimensional Euclidean space;  $\mathbb{R}^{n \times m}$ : set of all  $n \times m$  real matrices;  $A^T$ : transpose of matrix  $A$ ;  $I_n$ : identity matrix with dimension  $n$ ;  $\exp(x)$  or  $e^x$ : exponential function (they will be used interchangeably in this paper);  $\ln(x)$ : natural log function;  $|\mathcal{S}|$ : cardinality of a finite set  $\mathcal{S}$ ;  $A \otimes B$ : Kronecker product of matrices  $A$  and  $B$ ;  $e_i \in \mathbb{R}^n$ :  $i$ -th basis vector of  $\mathbb{R}^n$  (all components are 0 except for the  $i$ -th component which is 1).

### B. Markov decision problem

We consider the infinite-horizon discounted Markov decision problem (MDP) and Markov decision process, where the agent sequentially takes actions to maximize cumulative discounted rewards. In a Markov decision process with the state-space  $\mathcal{S} := \{1, 2, \dots, |\mathcal{S}|\}$  and action-space  $\mathcal{A} := \{1, 2, \dots, |\mathcal{A}|\}$ , the decision maker selects an action  $a \in \mathcal{A}$  at the current state  $s \in \mathcal{S}$ , then the state transits to the next state  $s' \in \mathcal{S}$  with probability  $P(s'|s, a)$ , and the transition incurs a reward  $r(s, a, s') \in \mathbb{R}$ , where  $P(s'|s, a)$  is the state transition probability from the current state  $s \in \mathcal{S}$  to the next state  $s' \in \mathcal{S}$  under action  $a \in \mathcal{A}$ , and  $r(s, a, s')$  is the reward function. For convenience, we consider a deterministic reward function and simply write  $r(s_k, a_k, s_{k+1}) =: r_k, k \in \{0, 1, \dots\}$ .

A deterministic policy,  $\pi : \mathcal{S} \rightarrow \mathcal{A}$ , maps a state  $s \in \mathcal{S}$  to an action  $\pi(s) \in \mathcal{A}$ . The objective of the Markov decision problem (MDP) is to find a deterministic (or stochastic) optimal policy,  $\pi^*$ , such that the cumulative discounted rewards over infinite time horizons is maximized, i.e.,

$$\pi^* := \arg \max_{\pi \in \Theta} \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k r_k \middle| \pi \right],$$

where  $\gamma \in [0, 1)$  is the discount factor,  $\Theta$  is the set of all deterministic policies,  $(s_0, a_0, s_1, a_1, \dots)$  is a state-action trajectory generated by the Markov chain under policy  $\pi$ , and  $\mathbb{E}[\cdot | \pi]$  is an expectation conditioned on the policy  $\pi$ . Moreover, Q-function under policy  $\pi$  is defined as

$$Q^\pi(s, a) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k r_k \middle| s_0 = s, a_0 = a, \pi \right], \quad (s, a) \in \mathcal{S} \times \mathcal{A},$$

and the optimal Q-function is defined as  $Q^*(s, a) = Q^{\pi^*}(s, a)$  for all  $(s, a) \in \mathcal{S} \times \mathcal{A}$ . Once  $Q^*$  is known, then an optimal policy can be retrieved by the greedy policy  $\pi^*(s) = \arg \max_{a \in \mathcal{A}} Q^*(s, a)$ . Throughout, we assume that the MDP is ergodic so that the stationary state distribution exists and the Markov decision problem is well posed.

### C. Basics of nonlinear system theory

In this paper, we will frequently encounter several notions from nonlinear system theory for the ODE analysis. Let us consider the general nonlinear system

$$\frac{d}{dt} x_t = f(x_t), \quad t \geq 0, \quad x_0 \in \mathbb{R}^n, \quad (1)$$

where  $x_t \in \mathbb{R}^n$  is the state at time  $t \geq 0$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a nonlinear mapping. For simplicity, we assume that the solution to (1) exists and is unique. This holds true if  $f$  is globally Lipschitz continuous.

**Lemma 1** ([32, Theorem 3.2]). *Let us consider the nonlinear system (1) and assume that  $f$  is globally Lipschitz continuous, i.e.,  $\|f(x) - f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^n$ , for some real number*

*$L > 0$  and norm  $\|\cdot\|$ , then it has a unique solution  $x_t$  for all  $t \geq 0$  and  $x_0 \in \mathbb{R}^n$ .*

An important concept in dealing with the nonlinear system is the equilibrium point. A point  $x = x^e$  in the state space is said to be an equilibrium point of (1) if it has the property that whenever the state of the system starts at  $x^e$ , it will remain at  $x^e$  [32]. For (1), the equilibrium points are the real roots of the equation  $f(x) = 0$ . The equilibrium point  $x^e$  is said to be globally asymptotically stable if for any initial state  $x_0 \in \mathbb{R}^n$ ,  $x_t \rightarrow x^e$  as  $t \rightarrow \infty$ .

### D. ODE-based stochastic approximation

Because of its generality, the convergence analyses of many RL algorithms rely on the so-called ODE approach [24], [33]. It analyzes convergence of general stochastic recursions by examining stability of the associated ODE model based on the fact that the stochastic recursions with diminishing step-sizes approximate the corresponding ODE in the limit. One of the most popular approach is based on the Borkar and Meyn theorem [5]. We now briefly introduce the Borkar and Meyn's ODE approach for analyzing convergence of the general stochastic recursions.

$$\theta_{k+1} = \theta_k + \alpha_k (f(\theta_k) + \varepsilon_{k+1}) \quad (2)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a nonlinear mapping and the integer  $k \geq 0$  is the iteration step. Basic technical assumptions are given below.

#### Assumption 1.

- 1) The mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is globally Lipschitz continuous.
- 2) There exists a unique globally asymptotically stable equilibrium  $\theta^e \in \mathbb{R}^n$  for the ODE  $\dot{x}_t = f(x_t)$ , i.e.,  $x_t \rightarrow \theta^e$  as  $t \rightarrow \infty$ .
- 3) There exists a function  $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\lim_{c \rightarrow \infty} \frac{f(cx)}{c} = f_\infty(x), \forall x \in \mathbb{R}^n$ .
- 4) The origin in  $\mathbb{R}^n$  is an asymptotically stable equilibrium for the ODE  $\dot{x}_t = f_\infty(x_t)$ .
- 5) The sequence  $\{\varepsilon_k, \mathcal{G}_k, k \geq 1\}$  with  $\mathcal{G}_k = \sigma(\theta_i, \varepsilon_i, i \leq k)$  is a martingale difference sequence. In addition, there exists a constant  $C_0 < \infty$  such that for any initial  $\theta_0 \in \mathbb{R}^n$ , we have  $\mathbb{E}[\|\varepsilon_{k+1}\|_2^2 | \mathcal{G}_k] \leq C_0(1 + \|\theta_k\|_2^2), \forall k \geq 0$ .
- 6) The step-sizes satisfy

$$\alpha_k > 0, \quad \sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty. \quad (3)$$

**Lemma 2** ([5, Borkar and Meyn theorem]). *Under Assumption 1, for any initial  $\theta_0 \in \mathbb{R}^n$ ,  $\sup_{k \geq 0} \|\theta_k\|_2 < \infty$  with probability one. In addition,  $\theta_k \rightarrow \theta^e$  as  $k \rightarrow \infty$  with probability one.*

The above O.D.E approach Lemma 2 has been widely used to prove convergence of RLs, such as synchronous Q-learning algorithm [5], synchronous TD-learning [5], asynchronous Q-learning [6], gradient TD-learning algorithms [19]–[22], Q-learning with linear function approximation [23], and other algorithms [24] to name just a few.

**Remark 1.** *As noted in [34, Chap. 2.1], the above result holds for the stochastic recursion*

$$\theta_{k+1} = \theta_k + \alpha_k (f(\theta_k) + \varepsilon_{k+1} + w_k), \quad (4)$$

where  $w_k$  is an additional deterministic or stochastic bounded sequence such that  $w_k \rightarrow 0$  with probability one as  $k \rightarrow \infty$ .

### E. Definitions and lemmas

In this subsection, several essential definitions and lemmas will be presented. In this paper, we mainly focus on the weighted  $p$ -norm for our analysis, defined by

$$\|x\|_{p,w} = \left( \sum_{i=1}^n w_i |x_i|^p \right)^{1/p}$$

where the real numbers  $w_i > 0$  for all  $i \in \{1, 2, \dots, n\}$  are weights. The weighted  $p$ -norm above is more general than the standard  $p$ -norm, i.e., when  $w_i = 1$  for all  $i \in \{1, 2, \dots, n\}$ , then the standard  $p$ -norm is recovered. In the case  $p \rightarrow \infty$ , we consider the weighted  $\infty$ -norm defined by

$$\|x\|_{\infty,w} := \max_{i \in \{1, 2, \dots, n\}} w_i |x_i|.$$

It is easily proved that  $\lim_{p \rightarrow \infty} \|x\|_{p,w} = \|x\|_{\infty,w}$ , and the convergence is uniform. This property is crucial in our main analysis, and therefore, we present the related convergence results in the sequel.

**Lemma 3.** For any  $x \in \mathbb{R}^n$  and any  $p \in (1, \infty)$ , we have

- 1)  $\|x\|_{\infty,w} \leq \|x\|_{p,w} \leq n^{1/p} \|x\|_{\infty,w}$
- 2)  $w_{\min}^{1/p} \|x\|_p \leq \|x\|_{p,w} \leq w_{\max}^{1/p} \|x\|_p$
- 3)  $w_{\min} \|x\|_{\infty} \leq \|x\|_{\infty,w} \leq w_{\max} \|x\|_{\infty}$
- 4)  $\|x\|_{\infty} \leq \|x\|_p \leq n^{1/p} \|x\|_{\infty}$

where  $w_{\min} := \min_{i \in \{1, 2, \dots, n\}} w_i$  and  $w_{\max} := \max_{i \in \{1, 2, \dots, n\}} w_i$ .

*Proof.* For the first statement, we have  $\|x\|_{\infty,w} = \left( \|x\|_{\infty,w}^p \right)^{1/p} \leq \left( \sum_{i=1}^n w_i |x_i|^p \right)^{1/p} \leq \left( n \|x\|_{\infty,w}^p \right)^{1/p} = n^{1/p} \|x\|_{\infty,w}$ . The second statement can be proved by noting the following relations  $\|x\|_{p,w} = \left( \sum_{i=1}^n w_i |x_i|^p \right)^{1/p} \leq \left( w_{\max} \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq w_{\max}^{1/p} \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq w_{\max}^{1/p} \|x\|_p$  and  $\|x\|_{p,w} = \left( \sum_{i=1}^n w_i |x_i|^p \right)^{1/p} \geq \left( w_{\min} \sum_{i=1}^n |x_i|^p \right)^{1/p} \geq w_{\min}^{1/p} \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \geq w_{\min}^{1/p} \|x\|_p$ . The third statement is trivial, and the last statement is proved by letting  $w_i = 1$  for all  $i \in \{1, 2, \dots, n\}$ . This completes the proof.  $\square$

Let us define the mappings  $h_{\max} : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $h_{\text{mm}} : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $h_{\text{lse}} : \mathbb{R}^n \rightarrow \mathbb{R}$ , and  $h_{\text{bz}} : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$\begin{aligned} h_{\max}(x) &:= \max_{i \in \{1, 2, \dots, n\}} x_i \\ h_{\text{lse}}^\lambda(x) &:= \frac{1}{\lambda} \ln \left( \sum_{i \in \{1, 2, \dots, n\}} e^{x_i \lambda} \right) \\ h_{\text{mm}}^\lambda(x) &:= \frac{1}{\lambda} \ln \left( \frac{1}{n} \sum_{i \in \{1, 2, \dots, n\}} e^{\lambda x_i} \right) \\ h_{\text{bz}}^\lambda(x) &:= \frac{\sum_{i \in \{1, 2, \dots, n\}} x_i e^{\lambda x_i}}{\sum_{i \in \{1, 2, \dots, n\}} e^{\lambda x_i}} \end{aligned}$$

where  $\lambda > 0$  is called the temperature parameter,  $h_{\max}$  is the standard max operator,  $h_{\text{lse}}^\lambda$  is called the log-sum-exp (LSE) operator (or smooth max operator) that is widely used in machine learning and RL [25], [35],  $h_{\text{mm}}^\lambda$  is called the mellowmax operator first suggested in [28] in order to overcome some drawbacks of the so-called Boltzmann softmax operator  $h_{\text{bz}}^\lambda$ , which is widely used in

RL [27], [29], [36] to approximate a probability distribution or the max operator as well. In this paper, we will consider smooth variants of Q-learning using these smooth approximations of the max operator, and analyze their convergence in a unified manner based on the ODE analysis.

The following relations are essential to address the smooth Q-learning algorithms. Although their proofs are given in [27] and other literatures, they are presented here for completeness and convenience. Especially, the proof for the bound on the Boltzmann softmax operator is simpler than that in [27].

**Lemma 4.** For any  $x_1, x_2, \dots, x_n \in \mathbb{R}$ , we have

- 1)  $h_{\max}(x) \leq h_{\text{lse}}^\lambda(x) \leq h_{\max}(x) + \frac{\ln(n)}{\lambda}$
- 2)  $\frac{1}{\lambda} \ln \left( \frac{1}{n} \right) + h_{\max}(x) \leq h_{\text{mm}}^\lambda(x) \leq h_{\max}(x)$
- 3)  $h_{\max}(x) - \frac{\ln(n)}{\lambda} \leq h_{\text{bz}}^\lambda(x) \leq h_{\max}(x)$
- 4)  $h_{\text{bz}}^\lambda(x) \leq h_{\text{lse}}^\lambda(x) \leq h_{\text{bz}}^\lambda(x) + \frac{1}{\lambda} \ln(n)$

*Proof.* For convenience, let us define  $\mathcal{I}_n := \{1, 2, \dots, n\}$ . Then, noting

$$\exp(\max_{i \in \mathcal{I}_n} x_i) \leq \sum_{i \in \mathcal{I}_n} e^{x_i} \leq n \exp(\max_{i \in \mathcal{I}_n} x_i), \quad (5)$$

we have

$$\begin{aligned} & \ln(\exp(\max_{i \in \mathcal{I}_n} x_i)) \\ & \leq \ln \left( \sum_{i \in \mathcal{I}_n} e^{x_i} \right) \\ & \leq \ln(n \exp(\max_{i \in \mathcal{I}_n} x_i)). \end{aligned}$$

Replacing  $x$  with  $\lambda x$  and dividing by  $\lambda$ , the first statement follows. The second statement follows similar steps: we can divide the terms in (5) by  $n$  and take the natural log function to get

$$\ln \left( \frac{1}{n} \right) + \max_{i \in \mathcal{I}_n} x_i \leq \ln \left( \frac{1}{n} \sum_{i \in \mathcal{I}_n} e^{x_i} \right) \leq \max_{i \in \mathcal{I}_n} x_i$$

Replacing  $x$  with  $\lambda x$  and dividing by  $\lambda$  yields the second statement. For the third statement, the upper bound is obtained through

$$\frac{\sum_{i \in \mathcal{I}_n} x_i e^{x_i/\lambda}}{\sum_{i \in \mathcal{I}_n} e^{x_i/\lambda}} \leq \frac{\max_{i \in \mathcal{I}_n} x_i \sum_{i \in \mathcal{I}_n} e^{x_i/\lambda}}{\sum_{i \in \mathcal{I}_n} e^{x_i/\lambda}} = \max_{i \in \mathcal{I}_n} x_i$$

For the lower bound, first note that by using the first and second statements, we obtain

$$h_{\text{bz}}^\lambda(x) \leq h_{\max}(x) \leq h_{\text{lse}}^\lambda(x) \quad (6)$$

Next, we use the fact that  $\nabla_x h_{\text{lse}}^\lambda(x)^T x = h_{\text{bz}}^\lambda(x)$  and  $h_{\text{lse}}^\lambda(x)$  is convex in  $x$  from [36]. From the property of a convex function, it follows that

$$h_{\text{lse}}^\lambda(y) \geq h_{\text{lse}}^\lambda(x) + \nabla_x h_{\text{lse}}^\lambda(x)^T (y - x), \quad \forall x, y \in \mathbb{R}^n.$$

Letting  $y = 0$  leads to  $h_{\text{lse}}^\lambda(0) \geq h_{\text{lse}}^\lambda(x) - \nabla_x h_{\text{lse}}^\lambda(x)^T x, \forall x \in \mathbb{R}^n$ , where  $h_{\text{lse}}^\lambda(0) = \frac{1}{\lambda} \ln \left( \sum_{i \in \mathcal{I}_n} e^0 \right) = \frac{1}{\lambda} \ln(n)$ . Therefore, we have

$$h_{\text{lse}}^\lambda(x) \leq h_{\text{bz}}^\lambda(x) + \frac{1}{\lambda} \ln(n).$$

Combining the above inequality with (6) leads to the third statement. The final statement is a byproduct of the above proof.  $\square$

Furthermore, the following lemma will be used in the analysis.

**Lemma 5.** We have  $\lim_{c \rightarrow \infty} \frac{h_{\max}(cx)}{c} = \lim_{c \rightarrow \infty} \frac{h_{\text{lse}}^\lambda(cx)}{c} = \lim_{c \rightarrow \infty} \frac{h_{\text{mm}}^\lambda(cx)}{c} = \lim_{c \rightarrow \infty} \frac{h_{\text{bz}}^\lambda(cx)}{c} = h_{\max}(x)$ .

*Proof.* For convenience, let us define  $\mathcal{I}_n := \{1, 2, \dots, n\}$ . For the max operator, we have  $\lim_{c \rightarrow \infty} \frac{h_{\max}(cx)}{c} = \lim_{c \rightarrow \infty} \frac{\max_{i \in \mathcal{I}_n} cx_i}{c} = h_{\max}(x)$ . For the LSE operator, one gets  $\lim_{c \rightarrow \infty} \frac{h_{\text{lse}}(cx)}{c} = \lim_{c \rightarrow \infty} \frac{1}{c\lambda} \ln \left( \sum_{i \in \mathcal{I}_n} e^{cx_i \lambda} \right) = \lim_{\lambda \rightarrow \infty} h_{\text{lse}}^\lambda(x) = h_{\max}(x)$ , while for the mellowmax operator, it follows that  $\lim_{c \rightarrow \infty} \frac{h_{\text{mm}}^\lambda(cx)}{c} = \lim_{c \rightarrow \infty} \frac{1}{c\lambda} \ln \left( \frac{1}{n} \sum_{i \in \mathcal{I}_n} e^{cx_i \lambda} \right) = \lim_{\lambda \rightarrow \infty} h_{\text{mm}}^\lambda(x) = h_{\max}(x)$ . Finally, we have  $\lim_{c \rightarrow \infty} \frac{h_{\text{bz}}^\lambda(cx)}{c} = \lim_{c \rightarrow \infty} \frac{\sum_{i \in \mathcal{I}_n} cx_i e^{c\lambda x_i}}{c \sum_{i \in \mathcal{I}_n} e^{c\lambda x_i}} = \lim_{\lambda \rightarrow \infty} h_{\text{bz}}^\lambda(x) = h_{\max}(x)$ .  $\square$

Before closing this subsection, the classical version of the Grönwall's inequality in differential form is presented as follows.

**Lemma 6** ([37]). *Let  $\psi_t : [0, \infty) \rightarrow [0, \infty)$  be an absolutely continuous function satisfying almost everywhere the differential inequality*

$$\frac{d}{dt} \psi_t \leq -a\psi_t + b, \quad \forall t \geq 0$$

where  $a > 0, b \geq 0$ . Then,

$$\psi_t \leq \psi_0 e^{-at} + \frac{b}{a}$$

### III. STABILITY OF NONLINEAR ODE MODELS UNDER CONTRACTION

In this paper, we will consider the following ODE form:

$$\frac{d}{dt} x_t = DF(x_t) - Dx_t, \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n \quad (7)$$

where  $t \geq 0$  is the continuous time,  $x_t \in \mathbb{R}^n$  is the state at time  $t$ ,  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a mapping that will be specified later, and  $D \in \mathbb{R}^{n \times n}$  is a positive definite diagonal matrix with strictly positive diagonal elements  $d_i > 0, i \in \{1, 2, \dots, n\}$ . This nonlinear system can be used to describe Q-learning and its variants in the remaining parts of this paper. A similar ODE form has been originally considered in [30], and the difference is the existence of the matrix  $D$ , i.e., when  $D = I_n$ , then (7) becomes identical to the ODE considered in [30]. To address the diagonal scaling due to  $D$ , we need to consider the weighted  $p$ -norm and weighted  $\infty$ -norm in the stability analysis of (7). To proceed, the following lemma needs to be addressed first.

**Lemma 7.** *Let us consider the system in (7), let  $x_t \in \mathbb{R}^n, t \geq 0$  be its unique solution, and suppose that  $x^*$  is a unique fixed point of  $F$ , i.e.,  $x^* = F(x^*)$ . Then, for all finite  $p \in (1, \infty)$ , we have*

$$\begin{aligned} \frac{d}{dt} \|x_t - x^*\|_{p,w} &\leq -\frac{1}{w_{\min}^{(p-1)/p}} \|x - x^*\|_p \\ &\quad + \frac{1}{w_{\min}^{(p-1)/p}} \|F(x_t) - F(x^*)\|_p, \end{aligned}$$

where  $w_{\min} := \min_{i \in \{1, 2, \dots, n\}} w_i$ ,  $w_{\max} := \max_{i \in \{1, 2, \dots, n\}} w_i$ , and  $w_i = \frac{1}{d_i}, \forall i \in \{1, 2, \dots, n\}$ .

*Proof.* By using chain rule and  $\frac{\partial}{\partial x_i} \left( \sum_{j=1}^n w_j |x_j|^p \right) = \frac{w_i x_i |x_i|^{p-2}}{\|x\|_{p,w}^{p-1}}$ , one can show

$$\begin{aligned} &\frac{d}{dt} \|x_t - x^*\|_{p,w} \\ &= \frac{d}{dt} \left( \sum_{j=1}^n w_j |x_{t,j} - x_{j}^*|^p \right)^{1/p} \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^n \frac{\partial}{\partial x_i} \left( \sum_{i=1}^n w_j |x_j - x_j^*|^p \right)^{1/p} \bigg|_{x_i = x_{t,i}} \frac{dx_{t,i}}{dt} \\ &= \frac{1}{\|x_t - x^*\|_{p,w}^{p-1}} \begin{bmatrix} w_1 (x_{t,1} - x_1^*) |x_{t,1} - x_1^*|^{p-2} \\ w_2 (x_{t,2} - x_2^*) |x_{t,2} - x_2^*|^{p-2} \\ \vdots \\ w_n (x_{t,n} - x_n^*) |x_{t,n} - x_n^*|^{p-2} \end{bmatrix}^T \\ &\quad \times (DF(x_t) - Dx_t + Dx^* - DF(x^*)) \\ &= T_1 + T_2, \end{aligned}$$

where  $T_1$  and  $T_2$  are defined below and  $x^* = F(x^*)$  is used in the last line. For  $T_1$ , one gets

$$\begin{aligned} T_1 &= \frac{1}{\|x_t - x^*\|_{p,w}^{p-1}} \begin{bmatrix} w_1 (x_{t,1} - x_1^*) |x_{t,1} - x_1^*|^{p-2} \\ w_2 (x_{t,2} - x_2^*) |x_{t,2} - x_2^*|^{p-2} \\ \vdots \\ w_n (x_{t,n} - x_n^*) |x_{t,n} - x_n^*|^{p-2} \end{bmatrix}^T \\ &\quad \times (Dx^* - Dx_t) \\ &\leq \frac{1}{w_{\min}^{(p-1)/p} \|x_t - x^*\|_p^{p-1}} \begin{bmatrix} (x_{t,1} - x_1^*) |x_{t,1} - x_1^*|^{p-2} \\ (x_{t,2} - x_2^*) |x_{t,2} - x_2^*|^{p-2} \\ \vdots \\ (x_{t,n} - x_n^*) |x_{t,n} - x_n^*|^{p-2} \end{bmatrix}^T \\ &\quad \times (x^* - x_t) \\ &= -\frac{\sum_{i=1}^n |x_{t,i} - x_i^*|^2 |x_{t,i} - x_i^*|^{p-2}}{w_{\min}^{(p-1)/p} \|x_t - x^*\|_p^{p-1}} \\ &= -\frac{\|x - x^*\|_p}{w_{\min}^{(p-1)/p}} \end{aligned}$$

where we set  $w_i = \frac{1}{d_i}$  for all  $i \in \{1, 2, \dots, n\}$  in the second line. Moreover,  $T_2$  can be bounded as

$$\begin{aligned} T_2 &= \frac{1}{\|x_t - x^*\|_{p,w}^{p-1}} \begin{bmatrix} (x_{t,1} - x_1^*) |x_1 - x_1^*|^{p-2} \\ (x_{t,2} - x_2^*) |x_2 - x_2^*|^{p-2} \\ \vdots \\ (x_{t,n} - x_n^*) |x_n - x_n^*|^{p-2} \end{bmatrix}^T \\ &\quad \times (F(x_t) - F(x^*)) \\ &\leq \frac{1}{w_{\min}^{(p-1)/p} \|x - x^*\|_p^{p-1}} \left\| \begin{bmatrix} (x_1 - x_1^*) |x_1 - x_1^*|^{p-2} \\ (x_2 - x_2^*) |x_2 - x_2^*|^{p-2} \\ \vdots \\ (x_n - x_n^*) |x_n - x_n^*|^{p-2} \end{bmatrix} \right\|_q \\ &\quad \times \|F(x_t) - F(x^*)\|_p \\ &= \frac{1}{w_{\min}^{(p-1)/p} \|x - x^*\|_p^{p-1}} \left( \sum_{i=1}^n |x_i - x_i^*|^q |x_i - x_i^*|^{q(p-2)} \right)^{1/q} \\ &\quad \times \|F(x_t) - F(x^*)\|_p \\ &= \frac{\|F(x_t) - F(x^*)\|_p}{w_{\min}^{(p-1)/p}}, \end{aligned}$$

where Hölder's inequality is used in the first inequality, and  $1/p + 1/q = 1$  is used in the remaining parts.  $\square$

It is important to note that when  $p$  is odd, then  $\|x_t - x^*\|_{p,w}$  is not differentiable when  $x_t$  is in the set  $U := \{x \in \mathbb{R}^n : \exists i \in \{1, 2, \dots, n\} \text{ such that } x_i = 0\}$ , while when  $p$  is even, it is differentiable in  $\mathbb{R}^n$ . Therefore, to avoid the potential issue of the non-differentiability, in the sequel, we will simply assume that  $p$  is

even for convenience of analysis. Based on Lemma 7, we are now ready to present some results on the global asymptotic stability of (7).

**Theorem 1.** *Let us consider the system in (7) and let  $x_t \in \mathbb{R}^n, t \geq 0$  be its unique solution. Suppose that  $p \in (1, \infty)$  is an even and finite positive integer and the mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a contraction with respect to  $\|\cdot\|_p$ , i.e.,  $\|F(x) - F(y)\|_p \leq \alpha\|x - y\|_p, \forall x, y \in \mathbb{R}^n$  for some  $\alpha \in (0, 1)$  so that it admits the unique fixed point  $F(x^*) = x^*$ . Then, we have*

$$\frac{d}{dt}\|x_t - x^*\|_{p,w} \leq \frac{(\alpha - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} \|x - x^*\|_{p,w}, \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n \quad (8)$$

and

$$\|x_t - x^*\|_{p,w} \leq \|x_0 - x^*\|_{p,w} \exp\left(\frac{(\alpha - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} t\right), \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n,$$

where  $w_{\min} := \min_{i \in \{1, 2, \dots, n\}} w_i$ ,  $w_{\max} := \max_{i \in \{1, 2, \dots, n\}} w_i$ , and  $w_i = \frac{1}{d_i}, \forall i \in \{1, 2, \dots, n\}$ . Therefore,  $x^*$  is the unique globally asymptotically (and exponentially) stable equilibrium point.

*Proof.* Using Lemma 7, we have

$$\begin{aligned} & \frac{d}{dt}\|x_t - x^*\|_{p,w} \\ & \leq -\frac{1}{w_{\min}^{(p-1)/p}} \|x_t - x^*\|_p + \frac{1}{w_{\min}^{(p-1)/p}} \|F(x_t) - F(x^*)\|_p \\ & \leq -\frac{1}{w_{\min}^{(p-1)/p}} \|x_t - x^*\|_p + \frac{\alpha}{w_{\min}^{(p-1)/p}} \|x_t - x^*\|_p \\ & \leq \frac{(\alpha - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} \|x_t - x^*\|_{p,w} \end{aligned}$$

where Lemma 3 is used in the last line. Next, using Grönwall's inequality in Lemma 6 yields the desired conclusion.  $\square$

The result in Theorem 1 holds when  $p \in (1, \infty)$  is even and finite. Moreover, (8) implies that indeed  $V(x) := \|x_t - x^*\|_{p,w}$  plays the role of a Lyapunov function [32]. A similar stability result for the case  $p \rightarrow \infty$  is given in the sequel. Note that when  $p = \infty$ ,  $\|x - x^*\|_p$  becomes non-differentiable in  $x$ . In order to bypass this tricky issue, we will use the approximation property in Lemma 3 instead of directly dealing with  $\infty$ -norm. This approach provides simpler stability analysis compared to the approach in [30].

**Theorem 2.** *Let us consider the system in (7) and let  $x_t \in \mathbb{R}^n, t \geq 0$  be its unique solution. Suppose that the mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is contraction with respect to  $\|\cdot\|_\infty$ , i.e.,  $\|F(x) - F(y)\|_\infty \leq \alpha\|x - y\|_\infty, \forall x, y \in \mathbb{R}^n$  for some  $\alpha \in (0, 1)$  so that it admits the unique fixed point  $F(x^*) = x^*$ . Then, for any  $\left[\frac{\ln(n)}{\ln(\alpha^{-1})}\right] < p \in (1, \infty)$ , where  $p$  is an even number, we have*

$$\frac{d}{dt}\|x_t - x^*\|_{p,w} \leq \frac{(\alpha n^{1/p} - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} \|x - x^*\|_{p,w}, \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n \quad (9)$$

and

$$\|x_t - x^*\|_{p,w} \leq \|x_0 - x^*\|_{p,w} \exp\left(\frac{(\alpha n^{1/p} - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} t\right), \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n, \quad (10)$$

where  $w_{\min} := \min_{i \in \{1, 2, \dots, n\}} w_i$ ,  $w_{\max} := \max_{i \in \{1, 2, \dots, n\}} w_i$ , and  $w_i = \frac{1}{d_i}, \forall i \in \{1, 2, \dots, n\}$ . Moreover, we have

$$\|x_t - x^*\|_\infty \leq \|x_0 - x^*\|_\infty \left(\frac{nw_{\max}}{w_{\min}}\right)^{1/p} \exp\left(\frac{(\alpha n^{1/p} - 1)}{w_{\min}} t\right),$$

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#### Algorithm 1 Q-learning variants

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1: Initialize  $Q_0 \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}$ 
2: Sample  $s_0 \sim p$ 
3: for iteration  $k = 0, 1, \dots$  do
4:   Sample  $a_k \sim \beta(\cdot|s_k)$  and  $s_k \sim p(\cdot)$ 
5:   Sample  $s'_k \sim P(\cdot|s_k, a_k)$  and  $r_k = r(s_k, a_k, s'_k)$ 
6:   Update  $Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \alpha_k \{r_k + \gamma h(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k)\}$ 
7: end for

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$$\forall t \geq 0, \quad x_0 \in \mathbb{R}^n. \quad (11)$$

Therefore,  $x^*$  is the unique globally asymptotically (and exponentially) stable equilibrium point.

*Proof.* Using Lemma 3, we have  $\frac{1}{n^{1/p}} \|F(x) - F(y)\|_p \leq \|F(x) - F(y)\|_\infty \leq \alpha\|x - y\|_\infty \leq \alpha\|x - y\|_p$  so that

$$\|F(x) - F(y)\|_p \leq \alpha n^{1/p} \|x - y\|_p$$

where  $\alpha n^{1/p} \in (0, 1)$  holds if  $\left[\frac{\ln(n)}{\ln(\alpha^{-1})}\right] < p \in (1, \infty)$ . Then, the proofs of (9) and (10) follow those in the proof of Theorem 1. To prove (11), Lemma 3 is applied to (11) in order to derive

$$\begin{aligned} \|x_t - x^*\|_\infty & \leq \|x_0 - x^*\|_\infty n^{1/p} \frac{w_{\max}^{1/p}}{w_{\min}^{1/p}} \exp\left(\frac{(\alpha n^{1/p} - 1)}{w_{\max}^{1/p} w_{\min}^{(p-1)/p}} t\right) \\ & \leq \|x_0 - x^*\|_\infty \left(\frac{nw_{\max}}{w_{\min}}\right)^{1/p} \exp\left(\frac{(\alpha n^{1/p} - 1)}{w_{\min}} t\right) \\ & \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n. \end{aligned}$$

This completes the proof.  $\square$

#### IV. CONVERGENCE OF Q-LEARNING AND ITS SMOOTH VARIANTS

In the previous section, global asymptotic stability of general nonlinear systems of the form (7) has been studied, which will be then used for ODE analysis of various Q-learning algorithms in this section. In this paper, we consider RL algorithms given

$$\text{in Algorithm 1, where } Q_k(s'_k, \cdot) := \begin{bmatrix} Q_k(s'_k, 1) \\ Q_k(s'_k, 2) \\ \vdots \\ Q_k(s'_k, |\mathcal{A}|) \end{bmatrix} \in \mathbb{R}^{|\mathcal{A}|},$$

and  $h : \mathbb{R}^{|\mathcal{A}|} \rightarrow \mathbb{R}$  is a general operator which can cover the standard Q-learning and its smooth variations using the LSE operator, mellowmax operator [28], and Boltzmann softmax operator [27].

In particular, Algorithm 1 becomes the standard Q-learning [3] if  $h(Q(s, \cdot)) = h_{\max}(Q(s, \cdot))$ ; the smooth Q-learning with the LSE operator [25] if  $h(Q(s, \cdot)) = h_{\text{lse}}^\lambda(Q(s, \cdot))$ ; the smooth Q-learning with the mellowmax operator [28] if  $h(Q(s, \cdot)) = h_{\text{mm}}^\lambda(Q(s, \cdot))$ ; the smooth Q-learning with the Boltzmann softmax operator [26], [27] if  $h(Q(s, \cdot)) = h_{\text{bz}}^\lambda(Q(s, \cdot))$ . We note that in [25] and [26], deep RL counterparts have been studied for the Q-learning with LSE and Boltzmann softmax operators, respectively. The result in [28] has only considered a SARSA version and value iteration algorithms using the mellowmax operator. A tabular Q-learning with the Boltzmann softmax operator was considered in [27], [29], where an asymptotic convergence was proved using the stochastic approximation method in [8], which is different from the ODE approach in [5]. Therefore, the underlying assumptions and conditions to ensure the convergence are different, meaning that they are not directly comparable. The main differences of the Q-learning in [27], [29] and ours lie in the step-size: in our approach, we consider a scalar step-size which does not depend on the state-action pair, while the step-size adopted in [27]

should depend on the state-action pair. To the author's knowledge, convergence of smooth Q-learning with the LSE and mellowmax operators in the tabular setting has not been theoretically studied in the literature so far. Besides, the proposed analysis is quite general, and provide a unified framework that can cover many Q-learning variants mentioned above.

In this paper, we focus on the following setting:  $\{(s_k, a_k, s'_k)\}_{k=0}^\infty$  are i.i.d. samples under the behavior policy  $\beta$ , where the time-invariant behavior policy is the policy by which the RL agent actually behaves to collect experiences. Note that the notation  $s'_k$  implies the next state sampled at the time step  $k$ , which is used instead of  $s_{k+1}$  in order to distinguish  $s'_k$  from  $s_{k+1}$ . In this paper, the notation  $s_{k+1}$  indicate the current state at the iteration step  $k+1$ , while it does not depend on  $s_k$ . For simplicity, we assume that the state at each time is sampled from the stationary state distribution  $p$ , and in this case, the joint state-action distribution at each time is identically given by

$$d(s, a) = p(s)\beta(a|s), \quad (s, a) \in \mathcal{S} \times \mathcal{A}. \quad (12)$$

Throughout, we make the following assumptions for convenience.

**Assumption 2.**  $d(s, a) > 0$  holds for all  $(s, a) \in \mathcal{S} \times \mathcal{A}$ .

Assumption 2 guarantees that every state-action pair is visited infinitely often for sufficient exploration. This assumption is used when the state-action occupation frequency is given. It has been also considered in [15] and [16]. The work in [12] considers another exploration condition, called the cover time condition, which states that there is a certain time period, within which every state-action pair is expected to be visited at least once. Slightly different cover time conditions have been used in [11] and [15] for convergence rate analysis.

The update in (1) can expressed as

$$Q_{k+1} = Q_k + \alpha_k(f(Q_k) + \varepsilon_{k+1})$$

with the stochastic error

$$\begin{aligned} \varepsilon_{k+1} = & (e_{s_k} \otimes e_{a_k})(r(s_k, a_k, s'_k) + \gamma h(Q_k(s'_k, \cdot)) \\ & - Q_k(s_k, a_k)) - f(Q_k), \end{aligned}$$

and

$$f(Q_k) := DR + \gamma DPH(Q_k) - DQ_k$$

where the mapping  $H : \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  will be defined soon,  $P \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}| \times |\mathcal{S}|}$  is the state-action pair to state transition probability matrix,  $Q_k \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}$ ,  $R \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}$  is an enumeration of the expected reward  $R(s, a) := \mathbb{E}[r(s_k, a_k, s'_k) | s_k = s, a_k = a]$  for all  $(s, a) \in \mathcal{S} \times \mathcal{A}$ ,  $D \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}| \times |\mathcal{S} \times \mathcal{A}|}$  is a diagonal matrix whose diagonal elements are an enumeration of (12). Note that under Assumption 2,  $D$  is a nonsingular diagonal matrix with strictly positive diagonal elements. Note also that in all the definitions, the order of elements corresponding to  $(s, a) \in \mathcal{S} \times \mathcal{A}$  can be arbitrary, but should be compatible with each other. In this paper, for convenience, we will follow the ordering: our Q-function or its estimate is encoded as a single vector  $Q \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}$ , which enumerates  $Q(s, a)$  for all  $s \in \mathcal{S}$  and  $a \in \mathcal{A}$  such that the single value  $Q(s, a)$  can be written as  $Q(s, a) = (e_a \otimes e_s)^T Q$ , where  $e_s \in \mathbb{R}^{|\mathcal{S}|}$  and  $e_a \in \mathbb{R}^{|\mathcal{A}|}$  are  $s$ -th basis vector (all components are 0 except for the  $s$ -th component which is 1) and  $a$ -th basis vector, respectively. Similarly,  $R(s, a) = (e_a \otimes e_s)^T R$ ,  $d(s, a) = p(s)\beta(a|s) = (e_a \otimes e_s)^T D(e_a \otimes e_s)$ , and  $P(s'|s, a) = (e_a \otimes e_s)^T P e_{s'}$ .

Moreover, the mapping  $H : \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  in (13) is defined as

$$H(Q) := \begin{bmatrix} h(Q(1, \cdot)) \\ h(Q(2, \cdot)) \\ \vdots \\ h(Q(|\mathcal{S}|, \cdot)) \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}|}$$

where  $h \in \{h_{\max}, h_{\text{lse}}^\lambda, h_{\text{mm}}^\lambda, h_{\text{bz}}^\lambda\}$ . In particular, we define  $H = H_{\max}$  if  $h = h_{\max}$ ;  $H = H_{\text{lse}}^\lambda$  if  $h = h_{\text{lse}}^\lambda$ ;  $H = H_{\text{mm}}^\lambda$  if  $h = h_{\text{mm}}^\lambda$ ;  $H = H_{\text{bz}}^\lambda$  if  $h = h_{\text{bz}}^\lambda$ . The corresponding ODE model of the Q-learning algorithms in Algorithm 1 can be written as

$$\begin{aligned} \frac{d}{dt} Q_t &= DR + \gamma DPH(Q_t) - DQ_t, \quad \forall t \geq 0, \\ Q_0 &\in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}, \end{aligned} \quad (13)$$

where  $H \in \{H_{\max}, H_{\text{lse}}^\lambda, H_{\text{mm}}^\lambda, H_{\text{bz}}^\lambda\}$ . Defining the mapping (Bellman operator)  $F(Q) := R + \gamma PH(Q)$ , the system in (13) can be rewritten as

$$\frac{d}{dt} Q_t = DF(Q_t) - DQ_t, \quad \forall t \geq 0, \quad Q_0 \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}, \quad (14)$$

which matches with the form in (7). The Bellman operator  $F = R + \gamma PH(Q)$  can be one of the following four cases:

$$\begin{aligned} F_{\max}(Q) &:= R + \gamma PH_{\max}(Q), \quad F_{\text{bz}}^\lambda(Q) := R + \gamma PH_{\text{bz}}^\lambda(Q) \\ F_{\text{lse}}^\lambda(Q) &:= R + \gamma PH_{\text{lse}}^\lambda(Q), \quad F_{\text{mm}}^\lambda(Q) := R + \gamma PH_{\text{mm}}^\lambda(Q). \end{aligned}$$

We note that the ODE analysis in [5], [30] only considers the case  $D = I_{|\mathcal{S} \times \mathcal{A}|}$ , and hence, it can only address synchronous Q-learning. Recently, to deal with this issue, [6] developed a switching system model [31] of asynchronous Q-learning, and applied notions in switching system theory to prove its global asymptotic stability without using an explicit Lyapunov arguments. Therefore, the Borkar and Meyn theorem can be applied to prove convergence of the asynchronous Q-learning. However, the main drawback of the approach in [6] is that to prove the global stability, some restrict conditions, such as the quasi-monotonicity, should be satisfied for the underlying switching system, which makes it hard to easily generalize the analysis method to other Q-learning variants, such as the smooth Q-learning variants, and other RL algorithms. On the other hand, the approach developed in this paper can more widely cover various algorithms, such as the smooth Q-learning variants and standard Q-learning, in a unified way. Therefore, it complements the switching system method in [6] and provide an additional option for the ODE analysis of asynchronous Q-learning. In this section, we will analyze the convergence of Algorithm 1 for the four cases in a unified way using the ODE analysis and the stability results obtained in the previous sections.

#### A. Convergence under the max, mellowmax, and LSE operators

Next, we prove convergence of Algorithm 1 with the max, mellowmax, Boltzmann softmax, and LSE operators using the Borkar and Meyn theorem. To this end, we will establish the global asymptotic stability of the corresponding ODE model in (14) using Theorem 2. It is known that  $F(Q) = R + \gamma PH(Q)$  is a contraction mapping when  $H$  is a non-expansive mapping. Moreover, it is known that the max, mellowmax, and LSE operators are non-expansive [28], [35]. Therefore, the corresponding  $F$  is a contraction mapping. For convenience and completeness, the results are formally stated in the following lemma.

**Lemma 8.** *The mapping  $F \in \{F_{\max}, F_{\text{lse}}^\lambda, F_{\text{mm}}^\lambda\}$  is a contraction with respect to  $\|\cdot\|_\infty$*

$$\|F(x) - F(y)\|_\infty \leq \gamma \|x - y\|_\infty, \quad \forall x, y \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}.$$

*Proof.* First of all, the max and mellowmax operators are known to be non-expansive [28]:  $\|H_{\max}(x) - H_{\max}(y)\|_{\infty} \leq \gamma\|x - y\|_{\infty}$ ,  $\|H_{\text{mm}}^{\lambda}(x) - H_{\text{mm}}^{\lambda}(y)\|_{\infty} \leq \gamma\|x - y\|_{\infty} \forall x, y \in \mathbb{R}^{|\mathcal{S} \times \mathcal{A}|}$ . Moreover, the LSE operator is also known to be non-expansive in [35]. Keeping this in mind, we have

$$\begin{aligned} \|F(x) - F(y)\|_{\infty} &= \|\gamma PH(x) - \gamma PH(y)\|_{\infty} \\ &\leq \gamma\|P\|_{\infty}\|H(x) - H(y)\|_{\infty} \\ &= \gamma\|H(x) - H(y)\|_{\infty} \\ &\leq \gamma\|x - y\|_{\infty}, \end{aligned}$$

where the last line comes from the non-expansive mapping property. This completes the proof.  $\square$

Since  $F \in \{F_{\max}, F_{\text{lse}}^{\lambda}, F_{\text{mm}}^{\lambda}\}$  is a contraction, we can define the corresponding unique fixed point as

$$\begin{aligned} Q_{\max}^* &= F_{\max}(Q_{\max}^*), \quad Q_{\text{lse}}^{\lambda} = F_{\text{lse}}^{\lambda}(Q_{\text{lse}}^{\lambda}), \\ Q_{\text{mm}}^{\lambda} &= F_{\text{mm}}^{\lambda}(Q_{\text{mm}}^{\lambda}). \end{aligned} \quad (15)$$

Unfortunately, it is known that the Boltzmann softmax operator is in general not a non-expansive mapping [28]. However, we can also derive some positive convergence results for the Boltzmann softmax operator as well. Before delving into the Boltzmann softmax operator case, we first focus on the other three cases. In particular, the asymptotic convergence of Algorithm 1 for the max, mellowmax, and LSE operators is given below based on the Borkar and Meyn theorem in [5]. To this end, we first present the following technical lemma.

**Theorem 3.** *Let us assume that the step-sizes satisfy (3). Moreover, let us consider the LSE, mellowmax, and max operators in Algorithm 1. Then, Algorithm 1 converge to the corresponding fixed point defined in (15) with probability one.*

*Proof.* For the proof, it suffices to verify the statements in Assumption 1 for the Borkar and Meyn theorem. Let us consider the system  $\frac{dQ_t}{dt} = f(Q_t)$ , where  $f(Q) = DF(Q) - DQ$  and  $F \in \{F_{\max}^{\lambda}, F_{\text{lse}}^{\lambda}, F_{\text{mm}}^{\lambda}\}$ . For the first statement, one can prove that  $f$  is Lipschitz continuous because

$$\begin{aligned} \|f(x) - f(y)\|_{\infty} &\leq \|DF(x) - DF(y)\|_{\infty} + \|D(x - y)\|_{\infty} \\ &\leq \|D\|_{\infty}\|F(x) - F(y)\|_{\infty} + \|D\|_{\infty}\|x - y\|_{\infty} \\ &\leq \gamma\|D\|_{\infty}\|x - y\|_{\infty} + \|D\|_{\infty}\|x - y\|_{\infty} \\ &= (\gamma + 1)\|D\|_{\infty}\|x - y\|_{\infty} \end{aligned}$$

where Lemma 8 is used in the third line. For the second statement, we note that by Lemma 8,  $F$  is a contraction mapping with respect to  $\|\cdot\|_{\infty}$ , and by Theorem 2, the fixed point is the unique globally asymptotically stable equilibrium point. For the third statement, it follows from Lemma 5 that

$$\begin{aligned} f_{\infty}(Q) &= \lim_{c \rightarrow \infty} \frac{f(cQ)}{c} \\ &= \lim_{c \rightarrow \infty} \frac{DR + \gamma DPH(cQ) - cDQ}{c} \\ &= \gamma DPH_{\max}(Q) - DQ. \end{aligned}$$

For the forth statement, let us consider the system  $\frac{dQ_t}{dt} = f_{\infty}(Q_t)$ , where  $f_{\infty}(Q) = D\bar{F}(Q) - DQ$  and  $\bar{F}(Q) = \gamma PH_{\max}(Q)$ . Similar to Lemma 8, it can be easily proved that  $\bar{F}$  is a contraction mapping with respect to  $\|\cdot\|_{\infty}$ . Moreover, the fixed point is the origin, which is the unique globally asymptotically stable equilibrium point by Theorem 2. For the sixth statement, define the history  $\mathcal{G}_k := (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1, Q_k, Q_{k-1}, \dots, Q_0)$ , and the

process  $(M_k)_{k=0}^{\infty}$  with  $M_k := \sum_{i=1}^k \varepsilon_i$ . Then, we can prove that  $(M_k)_{k=0}^{\infty}$  is Martingale. To do so, we have

$$\begin{aligned} \mathbb{E}[M_{k+1} | \mathcal{G}_k] &= \mathbb{E}\left[\sum_{i=1}^{k+1} \varepsilon_i \middle| \mathcal{G}_k\right] \\ &= \mathbb{E}[\varepsilon_{k+1} | \mathcal{G}_k] + \mathbb{E}\left[\sum_{i=1}^k \varepsilon_i \middle| \mathcal{G}_k\right] \\ &= \mathbb{E}\left[\sum_{i=1}^k \varepsilon_i \middle| \mathcal{G}_k\right] \\ &= M_k, \end{aligned}$$

where the third line comes from the i.i.d. sampling assumption. Therefore,  $(M_k)_{k=0}^{\infty}$  is a Martingale sequence, and  $\varepsilon_{k+1} = M_{k+1} - M_k$  is a Martingale difference. Moreover, for the fourth condition of Assumption 1, we have

$$\begin{aligned} &\mathbb{E}[\|\varepsilon_{k+1}\|_2^2 | \mathcal{G}_k] \\ &= \mathbb{E}[\|(e_{s_k} \otimes e_{a_k})(r(s_k, a_k, s'_k) + \gamma h(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k)) - f(Q_k)\|_2^2 | \mathcal{G}_k] \\ &= \mathbb{E}[\|(e_{s_k} \otimes e_{a_k})(r(s_k, a_k, s'_k) + \gamma h(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k))\|_2^2 | \mathcal{G}_k] - \mathbb{E}[\|f(Q_k)\|_2^2 | \mathcal{G}_k] \\ &\leq \mathbb{E}[\|(e_{s_k} \otimes e_{a_k})(r(s_k, a_k, s'_k) + \gamma h(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k))\|_2^2 | \mathcal{G}_k] \\ &= \mathbb{E}[(r(s_k, a_k, s'_k) + \gamma h(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k))^2 | \mathcal{G}_k] \\ &\leq 3\mathbb{E}[r(s_k, a_k, s'_k)^2 | \mathcal{G}_k] + 3\gamma^2\mathbb{E}[h(Q_k(s'_k, \cdot))^2 | \mathcal{G}_k] \\ &\quad + 3\mathbb{E}[Q_k(s_k, a_k)^2 | \mathcal{G}_k] \\ &\leq 3R_{\max}^2 + 3\gamma^2\mathbb{E}[h_{\max}(Q_k(s'_k, \cdot))^2 | \mathcal{G}_k] \\ &\quad + C + 3\mathbb{E}[\|Q_k\|_{\infty}^2 | \mathcal{G}_k] \\ &\leq 3R_{\max}^2 + C + 3\gamma^2\mathbb{E}[\|Q_k\|_{\infty}^2 | \mathcal{G}_k] + 3\mathbb{E}[\|Q_k\|_{\infty}^2 | \mathcal{G}_k] \\ &\leq 3R_{\max}^2 + C + 3(\gamma^2 + 1)\mathbb{E}[\|Q_k\|_2^2 | \mathcal{G}_k] \end{aligned}$$

where  $R_{\max} := \max_{(s,a,s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}} |r(s, a, s')|$ , the second inequality is due to  $\|a + b + c\|_2^2 \leq 3\|a\|_2^2 + 3\|b\|_2^2 + 3\|c\|_2^2$  for any  $a, b, c \in \mathbb{R}^n$ , the third inequality uses Lemma 4,  $C = 0$  when  $h \in \{h_{\max}, h_{\text{mm}}^{\lambda}\}$  and  $C = \ln(|\mathcal{A}|)/\lambda$  when  $h = h_{\text{lse}}^{\lambda}$ , the last inequality is due to  $\|\cdot\|_{\infty} \leq \|\cdot\|_2$ . This completes the proof.  $\square$

Next, we consider Algorithm 1 with the Q-updated replaced with

$$Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \alpha_k \{r_k + \gamma h_{\text{bz}}^{\lambda_k}(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k)\},$$

where  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ . In this case, the update can be rewritten by

$$Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \alpha_k \{r_k + \gamma h_{\max}(Q_k(s'_k, \cdot)) - Q_k(s_k, a_k) + w_k\},$$

where  $w_k = \gamma h_{\text{bz}}^{\lambda_k}(Q_k(s'_k, \cdot)) - \gamma h_{\max}(Q_k(s'_k, \cdot))$ , which is bounded by Lemma 4 as

$$\begin{aligned} |w_k| &= |\gamma h_{\text{bz}}^{\lambda_k}(Q_k(s'_k, \cdot)) - \gamma h_{\max}(Q_k(s'_k, \cdot))| \\ &\leq \gamma \frac{\ln(|\mathcal{A}|)}{\lambda_k}, \end{aligned}$$

which converges to zero with probability one as  $k \rightarrow \infty$ . Therefore, as noted in Remark 1, Lemma 2 can still be applied in this case.

**Corollary 1.** *Let us assume that the step-sizes satisfy (3). Moreover, let us consider the Boltzmann softmax operator in Algorithm 1 with  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Then, Algorithm 1 converge to  $Q_{\max}^*$  with probability one.*

## CONCLUSION

In this paper, we present a general and unified ODE framework for the convergence analysis of Q-learning and its smooth variants. The proposed analysis is motivated by previous work on the convergence of synchronous Q-learning based on  $p$ -norm serving as a Lyapunov function. However, the proposed analysis addresses more general ODE models that can cover both asynchronous Q-learning and its smooth versions with simpler frameworks. The proposed method complements the recently developed ODE analysis of asynchronous Q-learning using switching system models by removing the need for restrictive conditions on the ODE model.

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