THE ZETA-DETERMINANT OF THE DIRICHLET-TO-NEUMANN OPERATOR OF THE STEKLOV PROBLEM ON FORMS

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Abstract. On a compact Riemannian manifold M with boundary Y, we express the log of the zeta-determinant of the Dirichlet-to-Neumann operator acting on q-forms on Y as the difference of the log of the zeta-determinant of the Laplacian on q-forms on M with absolute boundary conditions and that of the Laplacian with Dirichlet boundary conditions with some additional terms which are expressed by curvature tensors. When the dimension of M is 2 or 3, we compute these terms explicitly. We also discuss the value of the zeta function at zero associated to the Dirichlet-to-Neumann operator by using a conformal rescaling method. As an application, we recover the result of the conformal invariance obtained in [13] when $\dim M = 2$.

1. Introduction

Let $(M, Y; g)$ be an m-dimensional Riemannian manifold with smooth boundary Y and $\Omega^q(M)$ be the space of smooth q-forms. We consider the exterior derivative $d_q : \Omega^q(M) \to \Omega^{q+1}(M)$ and its formal adjoint $\delta_q = (-1)^{mq+1} \star_M d \star_M$, where $\star_M$ is the Hodge star operator. Then the Hodge-De Rham Laplacian Δ_M^q acting on $\Omega^q(M)$ is defined by $\Delta_M^q = \delta_q d_q + d_{q-1}\delta_{q-1}$. If there is no confusion, we will drop the q on d_q and δ_q. We choose a collar neighborhood U of Y which is diffeomorphic to $Y \times [0, 1)$ and denote the canonical inclusion by $i : Y \to M$. We also choose a unit vector field $\frac{\partial}{\partial u}$ which is an inward normal vector to Y and denote the dual by du. We write a q-form ω on U by $\omega = \omega_1 + du \wedge \omega_2$, and define the tangential part $\omega_{\tan}$ and normal part ω_{nor} of ω as follows.

$$\omega_{\tan} := i^*\omega = \omega_1|_Y, \qquad \omega_{\mathrm{nor}} = i^*\left(\iota_{\frac{\partial}{\partial u}}\omega\right) = \omega_2|_Y, \tag{1.1}$$

where $\iota_{\frac{\partial}{\partial u}}\omega$ is the interior product of ω and $\frac{\partial}{\partial u}$. A q-form ω is said to satisfy absolute boundary conditions if $\omega_{\mathrm{nor}} = (d\omega)_{\mathrm{nor}} = 0$, and it is said to satisfy relative boundary conditions if $\omega_{\tan} = (\delta\omega)_{\tan} = 0$. We denote by $\Omega^q_{\mathrm{abs}\,/\,\mathrm{rel}}(M)$ the space of smooth q-forms satisfying absolute/relative boundary conditions, *i.e.*

$$\Omega^q_{\mathrm{abs}}(M) = \{\omega \in \Omega^q(M) \mid \omega_{\mathrm{nor}} = (d\omega)_{\mathrm{nor}} = 0\}, \quad \Omega^q_{\mathrm{rel}}(M) = \{\omega \in \Omega^q(M) \mid \omega_{\tan} = (\delta\omega)_{\tan} = 0\}. \tag{1.2}$$

We also denote by $\Delta^q_{M,\mathrm{abs}\,/\,\mathrm{rel}}$ and $\Delta^q_{M,\mathrm{D}}$ the Laplacian Δ^q_M with absolute/relative and Dirichlet boundary conditions, respectively. Then, $\Delta^q_{M,\mathrm{abs}\,/\,\mathrm{rel}}$ and $\Delta^q_{M,\mathrm{D}}$ are self-adjoint operators having discrete eigenvalues. We note that for $q = 0$, absolute/relative boundary conditions are equal to Neumann/Dirichlet boundary conditions. For $0 \le \lambda \in \mathbb{R}$, we define the Dirichlet-to-Neumann operator $Q^q_{\mathrm{abs}}(\lambda)$ and $Q^q_{\mathrm{rel}}(\lambda)$ acting on $\Omega^q(Y)$ as in [7, 26, 29]. For $\varphi \in \Omega^q(Y)$, we choose arbitrary extensions $\phi \in \Omega^q(M)$ and $\widetilde{\phi} \in \Omega^{q+1}(M)$ of φ and $du \wedge \varphi$ satisfying

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$$i^*\phi = \varphi, \qquad i^*\left(\iota_{\frac{\partial}{\partial u}}\phi\right) = 0, \qquad i^*\widetilde{\phi} = 0, \qquad i^*\left(\iota_{\frac{\partial}{\partial u}}\widetilde{\phi}\right) = \varphi. \tag{1.3}$$

We define the Poisson operators

$$\begin{aligned} &\mathcal{P}^q_{\mathrm{abs}}(\lambda) : \Omega^q(Y) \to \Omega^q(M), \qquad \mathcal{P}^q_{\mathrm{abs}}\varphi := \phi - \left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1}\left(\Delta^q_M + \lambda\right)\phi, \\ &\mathcal{P}^q_{\mathrm{rel}}(\lambda) : \Omega^q(Y) \to \Omega^{q+1}(M), \qquad \mathcal{P}^q_{\mathrm{rel}}\varphi := \widetilde{\phi} - \left(\Delta^{q+1}_{M,\mathrm{D}} + \lambda\right)^{-1}\left(\Delta^{q+1}_M + \lambda\right)\widetilde{\phi}. \end{aligned} \tag{1.4}$$

It is not difficult to see that the definition of $\mathcal{P}^q_{\mathrm{abs}}(\lambda)$ and $\mathcal{P}^q_{\mathrm{rel}}(\lambda)$ do not depend on the choices of the extensions of ϕ and $\widetilde{\phi}$ [7, 23]. $\mathcal{P}^q_{\mathrm{abs}}(\lambda)$ and $\mathcal{P}^q_{\mathrm{rel}}(\lambda)$ satisfy the following relations.

$$\begin{aligned} &\left(\Delta^q_M + \lambda\right)\mathcal{P}^q_{\mathrm{abs}}(\lambda)\varphi = 0, \qquad i^*\mathcal{P}^q_{\mathrm{abs}}(\lambda)\varphi = \varphi, \qquad i^*\left(\iota_{\frac{\partial}{\partial u}}\mathcal{P}^q_{\mathrm{abs}}(\lambda)\varphi\right) = 0, \\ &\left(\Delta^{q+1}_M + \lambda\right)\mathcal{P}^q_{\mathrm{rel}}(\lambda)\varphi = 0, \qquad i^*\mathcal{P}^q_{\mathrm{rel}}(\lambda)\varphi = 0, \qquad i^*\left(\iota_{\frac{\partial}{\partial u}}\mathcal{P}^q_{\mathrm{rel}}(\lambda)\varphi\right) = \varphi. \end{aligned} \tag{1.5}$$

Definition 1.1. *We define two Drichlet-to-Neumann operators $Q^q_{\mathrm{abs}}(\lambda)$ and $Q^q_{\mathrm{rel}}(\lambda)$ as follows* [7, 26, 29].

$$\begin{aligned} &Q^q_{\mathrm{abs}}(\lambda) : \Omega^q(Y) \to \Omega^q(Y), \qquad Q^q_{\mathrm{abs}}(\lambda)(\varphi) = -i^*\left(\iota_{\frac{\partial}{\partial u}}d\mathcal{P}^q_{\mathrm{abs}}(\lambda)\varphi\right), \\ &Q^q_{\mathrm{rel}}(\lambda) : \Omega^q(Y) \to \Omega^q(Y), \qquad Q^q_{\mathrm{rel}}(\lambda)(\varphi) = i^*\left(\delta\mathcal{P}^q_{\mathrm{rel}}(\lambda)\varphi\right). \end{aligned}$$

Remark : (1) When $q = 0$ and $\lambda = 0$, $Q^0_{\mathrm{abs}}(0)$ is the usual Dirichlet-to-Neumann operator on the Steklov problem on the space of smooth functions.
(2) In eq.(3.15) below, $Q^q_{\mathrm{abs}}(\lambda)$ and $Q^{q-1}_{\mathrm{rel}}(\lambda)$ are defined by using a local coordinate system, which is more intuitive.

The Green formula for the Hodge-De Rham Laplacians is given as follows [22, 27]. For ω, $\theta \in \Omega^q(M)$,

$$\langle d\omega, d\theta\rangle_M + \langle \delta\omega, \delta\theta\rangle_M = \langle \Delta^q_M\omega, \theta\rangle_M + \int_Y i^*(\theta \wedge \star_M d\omega - \delta\omega \wedge \star_M\theta), \tag{1.6}$$

where we use the convention that $d\,\mathrm{vol}(M) = -du \wedge d\,\mathrm{vol}(Y)$ on Y. For φ_1, $\varphi_2 \in \Omega^q(Y)$, eq.(1.6) shows that

$$\begin{aligned} \langle Q^q_{\mathrm{abs}}(\lambda)\varphi_1,\ \varphi_2\rangle_Y &= \lambda\langle \mathcal{P}^q_{\mathrm{abs}}\varphi_1,\ \mathcal{P}^q_{\mathrm{abs}}\varphi_2\rangle_M + \langle d\mathcal{P}^q_{\mathrm{abs}}\varphi_1,\ d\mathcal{P}^q_{\mathrm{abs}}\varphi_2\rangle_M + \langle \delta\mathcal{P}^q_{\mathrm{abs}}\varphi_1,\ \delta\mathcal{P}^q_{\mathrm{abs}}\varphi_2\rangle_M, \\ \langle Q^q_{\mathrm{rel}}(\lambda)\varphi_1,\ \varphi_2\rangle_Y &= \lambda\langle \mathcal{P}^q_{\mathrm{rel}}\varphi_1,\ \mathcal{P}^q_{\mathrm{rel}}\varphi_2\rangle_M + \langle d\mathcal{P}^q_{\mathrm{rel}}\varphi_1,\ d\mathcal{P}^q_{\mathrm{rel}}\varphi_2\rangle_M + \langle \delta\mathcal{P}^q_{\mathrm{rel}}\varphi_1,\ \delta\mathcal{P}^q_{\mathrm{rel}}\varphi_2\rangle_M, \end{aligned}$$

which implies that $Q^q_{\mathrm{abs}\,/\,\mathrm{rel}}(\lambda)$ are non-negative self-adjoint operators. Moreover, they are elliptic ΨDO's with parameter λ of order 1 and weight 2 with the principal symbol $\sigma_L(Q^q_{\mathrm{abs}\,/\,\mathrm{rel}}(\lambda))(x,\xi) = \sqrt{|\xi|^2 + \lambda}$ (see [6, 28] for a ΨDO's with parameter). It is well known (Theorem 2.7.3 in [10]) that

$$\begin{aligned} &\ker\Delta^q_{M,\mathrm{abs}} = \{\omega \in \Omega^q_{\mathrm{abs}}(M) \mid d\omega = \delta\omega = 0\} \cong H^q(M), \\ &\ker\Delta^q_{M,\mathrm{rel}} = \{\omega \in \Omega^q_{\mathrm{rel}}(M) \mid d\omega = \delta\omega = 0\} \cong H^q(M, Y), \end{aligned} \tag{1.7}$$

which shows that

$$\ker Q^q_{\rm abs}(0) = \left\{ i^*\omega \mid \omega \in \ker \Delta^q_{M,{\rm abs}} \right\}, \qquad \ker Q^q_{\rm rel}(0) = \left\{ i^* \left(\iota_{\frac{\partial}{\partial u}} \omega \right) \mid \omega \in \ker \Delta^{q+1}_{M,{\rm rel}} \right\}. \tag{1.8}$$

In particular, it follows that

$$\begin{aligned} \dim \ker Q^q_{\rm abs}(0) &= \dim \ker \Delta^q_{M,{\rm abs}} = \dim H^q(M), \\ \dim \ker Q^q_{\rm rel}(0) &= \dim \ker \Delta^{q+1}_{M,{\rm rel}} = \dim H^{q+1}(M,Y). \end{aligned} \tag{1.9}$$

For $\mathcal{D} = \Delta^q_{M,{\rm abs}\,/\,{\rm rel}} + \lambda$ or $Q^q(\lambda)_{{\rm abs}\,/\,{\rm rel}}$, we define the zeta function $\zeta_{\mathcal{D}}(s)$ by

$$\zeta_{\mathcal{D}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\operatorname{Tr} e^{-t\mathcal{D}} - \dim \ker \mathcal{D} \right) dt. \tag{1.10}$$

It is well known that $\zeta_{\mathcal{D}}(s)$ is analytic for $\Re s > \frac{m}{\operatorname{ord}(\mathcal{D})}$ and has a meromorphic continuation to the whole complex plane having a regular value at $s = 0$. When $\ker \mathcal{D} = \{0\}$, we define the zeta-determinant of $\mathcal{D}$ by $\operatorname{Det} \mathcal{D} = e^{-\zeta'_{\mathcal{D}}(0)}$. If $\dim \ker \mathcal{D} \geq 1$, we define the modified zeta-determinant by the same formula, which we denote by $\operatorname{Det}^* \mathcal{D} = e^{-\zeta'_{\mathcal{D}}(0)}$. In this paper, we are going to discuss

$$\begin{aligned} &\ln \operatorname{Det}^* \Delta^q_{M,{\rm abs}} - \ln \operatorname{Det} \Delta^q_{M,{\rm D}} - \ln \operatorname{Det}^* Q^q_{\rm abs}(0) \quad \text{and} \\ &\ln \operatorname{Det}^* \Delta^{q+1}_{M,{\rm rel}} - \ln \operatorname{Det} \Delta^{q+1}_{M,{\rm D}} - \ln \operatorname{Det}^* Q^q_{\rm rel}(0), \end{aligned} \tag{1.11}$$

and their applications. However, for the Hodge star operators $\star_M$ and $\star_Y$ of M and Y, simple computation shows that

$$\star_M^{-1} \Delta^{m-q}_{M,{\rm rel}} \star_M = \Delta^q_{M,{\rm abs}}, \qquad \star_M^{-1} \Delta^{m-q}_{M,{\rm D}} \star_M = \Delta^q_{M,{\rm D}}, \quad \text{and} \quad \star_Y^{-1} Q^{m-1-q}_{\rm rel}(\lambda) \star_Y = Q^q_{\rm abs}(\lambda), \tag{1.12}$$

which shows that

$$\begin{aligned} &\ln \operatorname{Det}^* \Delta^q_{M,{\rm abs}} - \ln \operatorname{Det} \Delta^q_{M,{\rm D}} - \ln \operatorname{Det}^* Q^q_{\rm abs}(0) \\ = \; &\ln \operatorname{Det}^* \Delta^{m-q}_{M,{\rm rel}} - \ln \operatorname{Det} \Delta^{m-q}_{M,{\rm D}} - \ln \operatorname{Det}^* Q^{m-1-q}_{\rm rel}(0). \end{aligned} \tag{1.13}$$

Hence, it is enough to consider $\ln \operatorname{Det}^* \Delta^q_{M,{\rm abs}} - \ln \operatorname{Det} \Delta^q_{M,{\rm D}} - \ln \operatorname{Det}^* Q^q_{\rm abs}(0)$.

In this paper, we use the method of proving the BFK-gluing formula for zeta-determinants of Laplacians [6, 7, 9] to show that $\ln \operatorname{Det}^* \Delta^q_{M,{\rm abs}} - \ln \operatorname{Det} \Delta^q_{M,{\rm D}} - \ln \operatorname{Det}^* Q^q_{\rm abs}(0)$ is expressed by some curvature tensors on Y. We compute it explicitly when $\dim M = 2$ and 3. We also discuss the value of the zeta function $\zeta_{Q^q_{\rm abs}}(s)$ at $s = 0$ by using the conformal metric rescaling method. Finally, when M is a 2-dimensional smooth Riemannian manifold with smooth boundary Y and $\ell(Y)$ is the length of Y, we show that $\frac{1}{\ell(Y)} \operatorname{Det}^* Q^0_{\rm abs}(0)$ is a conformal invariant, which was proved earlier by Guillarmou and Guillopé in [13] (see also [8]).

2. Relation between $\ln \operatorname{Det}^* \Delta^q_{M,\mathrm{abs}} - \ln \operatorname{Det} \Delta^q_{M,\mathrm{D}}$ and $\ln \operatorname{Det}^* Q^q_{\mathrm{abs}}(0)$

In this section, we are going to discuss the relation between $\ln \operatorname{Det}^* \Delta^q_{M,\mathrm{abs}} - \ln \operatorname{Det} \Delta^q_{M,\mathrm{D}}$ and $\ln \operatorname{Det}^* Q^q_{\mathrm{abs}}(0)$. We first recall that for $\lambda \neq 0$,

$$(\Delta^q_M + \lambda) \cdot \mathcal{P}^q_{\mathrm{abs}}(\lambda) = 0, \qquad i^* \mathcal{P}^q_{\mathrm{abs}}(\lambda) = \mathrm{Id}, \qquad i^* \iota_{\frac{\partial}{\partial u}} \mathcal{P}^q_{\mathrm{abs}}(\lambda) = 0, \tag{2.1}$$

which shows that

$$\left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} - \left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1} \;=\; \mathcal{P}^q_{\mathrm{abs}}(\lambda) i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1}. \tag{2.2}$$

Lemma 2.1. *For $\lambda \neq 0$ we have*

$$\frac{d}{d\lambda} \mathcal{P}^q_{\mathrm{abs}}(\lambda) \;=\; -\left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1} \mathcal{P}^q_{\mathrm{abs}}(\lambda).$$

Proof. Taking the derivative of (2.1), we obtain the following equalities.

$$\mathcal{P}^q_{\mathrm{abs}}(\lambda) + (\Delta^q_M + \lambda) \cdot \frac{d}{d\lambda} \mathcal{P}^q_{\mathrm{abs}}(\lambda) = 0, \qquad i^* \frac{d}{d\lambda} \mathcal{P}^q_{\mathrm{abs}}(\lambda) = 0, \qquad i^* \iota_{\frac{\partial}{\partial u}} \frac{d}{d\lambda} \mathcal{P}^q_{\mathrm{abs}}(\lambda) = 0,$$

which yields the conclusion. $\square$

From Definition 1.1 we note that

$$\begin{aligned}
\frac{d}{d\lambda} Q^q_{\mathrm{abs}}(\lambda) &= -i^* \iota_{\frac{\partial}{\partial u}} \, d \, \frac{d}{d\lambda} \mathcal{P}^q_{\mathrm{abs}}(\lambda) \;=\; -i^* \iota_{\frac{\partial}{\partial u}} \, d \, \left(- \left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1} \right) \mathcal{P}^q_{\mathrm{abs}}(\lambda) \\
&= -i^* \iota_{\frac{\partial}{\partial u}} \, d \, \left(\left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} - \left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1} \right) \mathcal{P}^q_{\mathrm{abs}}(\lambda) \\
&= -i^* \iota_{\frac{\partial}{\partial u}} \, d \, \mathcal{P}^q_{\mathrm{abs}}(\lambda) i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} \mathcal{P}^q_{\mathrm{abs}}(\lambda) \\
&= Q^q_{\mathrm{abs}}(\lambda) i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} \mathcal{P}^q_{\mathrm{abs}}(\lambda),
\end{aligned} \tag{2.3}$$

where in the third equality we used the fact that $\omega_{\mathrm{nor}} = (d\omega)_{\mathrm{nor}} = 0$ for $\omega \in \Omega^q_{\mathrm{abs}}(M)$. This yields

$$Q^q_{\mathrm{abs}}(\lambda)^{-1} \frac{d}{d\lambda} Q^q_{\mathrm{abs}}(\lambda) \;=\; i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} \mathcal{P}^q_{\mathrm{abs}}(\lambda). \tag{2.4}$$

For $\nu = [\frac{m-1}{2}] + 1$, we also note that

$$\begin{aligned}
& \frac{d^\nu}{d\lambda^\nu} \left\{ \log \operatorname{Det} \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right) - \log \operatorname{Det} \left(\Delta^q_{M,\mathrm{D}} + \lambda\right) \right\} \\
= \;& \operatorname{Tr} \left\{ \frac{d^{\nu-1}}{d\lambda^{\nu-1}} \left(\left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} - \left(\Delta^q_{M,\mathrm{D}} + \lambda\right)^{-1} \right) \right\} \\
= \;& \operatorname{Tr} \left\{ \frac{d^{\nu-1}}{d\lambda^{\nu-1}} \left(\mathcal{P}^q_{\mathrm{abs}}(\lambda) i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} \right) \right\} \\
= \;& \operatorname{Tr} \left\{ \frac{d^{\nu-1}}{d\lambda^{\nu-1}} \left(i^* \left(\Delta^q_{M,\mathrm{abs}} + \lambda\right)^{-1} \mathcal{P}^q_{\mathrm{abs}}(\lambda) \right) \right\} \;=\; \operatorname{Tr} \left\{ \frac{d^{\nu-1}}{d\lambda^{\nu-1}} \left(Q^q_{\mathrm{abs}}(\lambda)^{-1} \frac{d}{d\lambda} Q^q_{\mathrm{abs}}(\lambda) \right) \right\} \\
= \;& \frac{d^\nu}{d\lambda^\nu} \log \operatorname{Det} Q^q_{\mathrm{abs}}(\lambda).
\end{aligned} \tag{2.5}$$

This equality leads to the following result, which has been already proved in Theorem 6.2 of [7] by a different method.

Lemma 2.2. *There exists a polynomial* $P^q(\lambda) = \sum_{k=0}^{[\frac{m-1}{2}]} a_k \lambda^k$ *such that*

$$\log \operatorname{Det}\left(\Delta^q_{M,\mathrm{abs}} + \lambda\right) - \log \operatorname{Det}\left(\Delta^q_{M,\mathrm{D}} + \lambda\right) \;=\; \sum_{k=0}^{[\frac{m-1}{2}]} a_k \lambda^k + \ln \operatorname{Det} Q^q_{\mathrm{abs}}(\lambda).$$

To determine the coefficients a_k, we are going to consider the asymptotic expansion of each term in Lemma 2.2 for $\lambda \to \infty$. When $t \to 0^+$, it is well known [10] that for some $\mathfrak{a}_j$, $\mathfrak{b}_j \in \mathbb{R}$,

$$\operatorname{Tr} e^{-t\Delta^q_{M,\mathrm{abs}}} \;\sim\; \sum_{j=0}^{\infty} \mathfrak{a}_j t^{-\frac{m-j}{2}}, \qquad \operatorname{Tr} e^{-t\Delta^q_{M,\mathrm{D}}} \;\sim\; \sum_{j=0}^{\infty} \mathfrak{b}_j t^{-\frac{m-j}{2}}. \tag{2.6}$$

It is straightforward ((5.1) in [31], Lemma 2.1 in [16]) that for $\lambda \to \infty$,

$$\begin{aligned} \ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{abs}} + \lambda\right) - \ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{D}} + \lambda\right) \;&\sim\; -\sum_{\substack{j=0\\ j\neq m}}^{N} (\mathfrak{a}_j - \mathfrak{b}_j) \frac{d}{ds}\left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)}\right)\Bigg|_{s=0} \lambda^{\frac{m-j}{2}} \\ + (\mathfrak{a}_m - \mathfrak{b}_m)\ln\lambda \;+\; \sum_{j=0}^{m-1} (\mathfrak{a}_j - \mathfrak{b}_j) &\left(\frac{\Gamma(s - \frac{m-j}{2})}{\Gamma(s)}\right)\Bigg|_{s=0} \lambda^{\frac{m-j}{2}} \ln\lambda \;+\; O(\lambda^{-\frac{N+1-m}{2}}), \end{aligned} \tag{2.7}$$

where we note that the constant term does not appear. Since $Q^q_{\mathrm{abs}}(\lambda)$ is an elliptic ΨDO of order 1 with parameter of weight 2, it is shown in the Appendix of [6] that for $\lambda \to \infty$, $\ln \operatorname{Det} Q^q_{\mathrm{abs}}(\lambda)$ has the following asymptotic expansion,

$$\ln \operatorname{Det} Q^q_{\mathrm{abs}}(\lambda) \;\sim\; \sum_{j=0}^{\infty} \pi_j \lambda^{\frac{m-1-j}{2}} + \sum_{j=0}^{m-1} q_j \lambda^{\frac{m-1-j}{2}} \ln\lambda, \tag{2.8}$$

where π_j and q_j are locally computable as follows. For a fixed local coordinate system we denote the homogeneous symbols of $Q^q_{\mathrm{abs}}(\lambda)$ and its resolvent $(\mu - Q^q_{\mathrm{abs}}(\lambda))^{-1}$ by

$$\begin{aligned} \sigma(Q^q_{\mathrm{abs}}(\lambda))(y,\xi,\lambda) \;&\sim\; \sum_{j=0}^{\infty} \widetilde{\alpha}_{1-j}(y,\xi,\lambda), \\ \sigma\left((\mu - Q^q_{\mathrm{abs}}(\lambda))^{-1}\right)(y,\xi,\lambda,\mu) \;&\sim\; \sum_{j=0}^{\infty} \widetilde{r}_{-1-j}(y,\xi,\lambda,\mu). \end{aligned} \tag{2.9}$$

The densities $\pi_j(y)$ and $q_j(y)$ are computed as follows (Appendix of [6]).

$$\pi_j(y) \;=\; -\frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^{m-1}}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\,\mathrm{Tr}\,\widetilde{r}_{-1-j}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right)d\mu d\xi\right), \tag{2.10}$$
$$q_j(y) \;=\; \frac{1}{2}\left(\frac{1}{(2\pi)^{m-1}}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\,\mathrm{Tr}\,\widetilde{r}_{-1-j}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right)d\mu d\xi\right)\Big|_{s=0},$$
$$\pi_j \;=\; \int_Y \pi_j(y)\;dy, \qquad q_j \;=\; \int_Y q_j(y)\;dy,$$

where dy is the volume form on Y and γ is a contour enclosing the poles of $\mathrm{Tr}\,\widetilde{r}_{-1-j}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right)$ counterclockwise. Comparing the coefficients of λ^j, we have the following result.

Lemma 2.3.

$$a_0 \;=\; -\pi_{m-1}, \qquad a_k \;=\; -\pi_{m-1-2k} - (\mathfrak{a}_{m-2k}-\mathfrak{b}_{m-2k})\left(\frac{d}{ds}\frac{\Gamma(s-k)}{\Gamma(s)}\right)\Big|_{s=0}, \quad 1 \le k \le [(m-1)/2],$$
$$q_{m-1} \;=\; \mathfrak{a}_m - \mathfrak{b}_m, \qquad q_k \;=\; (\mathfrak{a}_{k+1}-\mathfrak{b}_{k+1})\left(\frac{\Gamma(s-\frac{m-k-1}{2})}{\Gamma(s)}\right)\Big|_{s=0}, \quad 0 \le k \le m-2.$$

Since the heat coefficients are quite well known [11, 15], we are going to concentrate on computing the π_k's to determine the coefficients a_k. We note that the coefficients π_k are expressed by some curvature tensors including the scalar curvatures and principal curvatures of Y in M like heat coefficients (cf.[25]). In Section 3 we are going to compute π_1 and π_2 along these lines when $\dim M = 2,\ 3$.

Before going further, we make one observation. If M has a product structure near Y so that Δ_M^q is $-\partial_{y_m}^2 + \Delta_Y$ on a collar neighborhood of Y, it is known that $Q_{\mathrm{abs}}^q(\lambda) = \sqrt{\Delta_Y + \lambda}$ + *a smoothing operator* (cf. [18, 24]). In this case, $\ln\mathrm{Det}\,Q_{\mathrm{abs}}^q(\lambda)$ and $\ln\mathrm{Det}\sqrt{\Delta_Y+\lambda}$ have the same asymptotic expansions for $\lambda \to \infty$, which is shown in the Appendix of [6]. Since $\ln\mathrm{Det}\sqrt{\Delta_Y+\lambda} = \frac{1}{2}\ln\mathrm{Det}(\Delta_Y+\lambda)$, the constant term in the asymptotic expansion of $\ln\mathrm{Det}\,Q_{\mathrm{abs}}^q(\lambda)$ is zero (cf.(2.8)), which shows that $a_0 = 0$.

We next discuss the asymptotic behavior of each term in Lemma 2.2 for $\lambda \to 0$. We first note that

$$\ln\mathrm{Det}(\Delta_{M,\mathrm{D}}^q + \lambda) \;=\; \ln\mathrm{Det}\,\Delta_{M,\mathrm{D}}^q + O(\lambda). \tag{2.11}$$

In view of (1.9), we let $\dim\ker\Delta_{M,\mathrm{abs}}^q = \dim\ker Q_{\mathrm{abs}}^q(0) = \ell_q$ and $\{\psi_1(0),\cdots,\psi_{\ell_q}(0)\}$ be an orthonormal basis for $\ker\Delta_{M,\mathrm{abs}}^q$. Considering $\lambda \in \mathbb{C} - (-\infty, 0)$, $Q_{\mathrm{abs}}^q(\lambda)$ is a self-adjoint holomorphic family of type (A) in the sense of T. Kato (for the definition see p. 375 of [14]) and Theorem 3.9 on p. 392 of [14] shows that there exist holomorphic families $\{\theta_j(\lambda) \mid j = 1,2,\cdots\}$ and $\{\phi_j(\lambda) \mid j = 1,2,\cdots\}$ of eigenvalues and corresponding orthonormal eigensections of $Q_{\mathrm{abs}}^q(\lambda)$ such that $0 < \theta_1(\lambda) \le \cdots \le \theta_{\ell_q}(\lambda) < \theta_{\ell_q+1}(\lambda) \le \cdots$ and

$$\|\,\phi_j(\lambda)\,\| = 1, \qquad \lim_{\lambda\to 0}\theta_j(\lambda) = 0 \quad \text{for} \quad 1 \le j \le \ell_q. \tag{2.12}$$

This leads to

$$\ln\mathrm{Det}(\Delta_{M,\mathrm{abs}}^q + \lambda) \;=\; \ell_q\ln\lambda \;+\; \ln\mathrm{Det}^*\,\Delta_{M,\mathrm{abs}}^q \;+\; O(\lambda), \tag{2.13}$$
$$\ln\mathrm{Det}\,Q_{\mathrm{abs}}^q(\lambda) \;=\; \ln\theta_1(\lambda)\cdots\theta_{\ell_q}(\lambda) \;+\; \ln\mathrm{Det}^*\,Q_{\mathrm{abs}}^q(0) \;+\; O(\lambda).$$

For each $\phi_j(\lambda)$, we denote $\Phi_j(\lambda) := \mathcal{P}^q_{\mathrm{abs}}(\lambda)\phi_j(\lambda)$. Then,

$$Q^q_{\mathrm{abs}}(\lambda)(\phi_j(\lambda)) \quad = \quad -i^* \left(\iota_{\frac{\partial}{\partial u}} d\Phi_j(\lambda) \right) \; = \; \theta_j(\lambda)\phi_j(\lambda), \tag{2.14}$$

and hence $\{\Phi_1(0), \cdots, \Phi_{\ell_q}(0)\}$ is also a basis for $\ker \Delta^q_{M,\mathrm{abs}}$. The Green Theorem (1.6) shows that

$$\begin{aligned} 0 \quad &= \quad \langle (\Delta^q_M + \lambda)\Phi_j(\lambda), \Phi_k(0)\rangle = \langle \Delta^q_M \Phi_j(\lambda), \Phi_k(0)\rangle + \lambda \langle \Phi_j(\lambda), \Phi_k(0)\rangle \\ &= \quad \langle d\Phi_j(\lambda), d\Phi_k(0)\rangle + \langle \delta\Phi_j(\lambda), \delta\Phi_k(0)\rangle - \int_Y i^* \left(\Phi_k(0) \wedge \star_M d\Phi_j(\lambda) - \delta\Phi_j(\lambda) \wedge \star_M \Phi_k(0) \right) \\ &\qquad + \lambda \, \langle \Phi_j(\lambda), \Phi_k(0)\rangle \\ &= \quad -\langle \phi_k(0), \; Q^q_{\mathrm{abs}}(\lambda)\phi_j(\lambda)\rangle_Y + \lambda \langle \Phi_j(\lambda), \Phi_k(0)\rangle_M \\ &= \quad -\theta_j(\lambda)\langle \phi_k(0), \phi_j(\lambda)\rangle_Y + \lambda \langle \Phi_j(\lambda), \Phi_k(0)\rangle_M, \end{aligned}$$

which leads to

$$\lim_{\lambda \to 0} \frac{\theta_j(\lambda)}{\lambda} \delta_{jk} \quad = \quad \langle \Phi_j(0), \; \Phi_k(0)\rangle_M. \tag{2.15}$$

We define $\ell_q \times \ell_q$ matrices $\mathcal{R} = (r_{ij})$ and $\mathcal{S} = (s_{ij})$ by

$$r_{ij} \quad = \quad \langle \Phi_i(0), \; \psi_j(0)\rangle_M, \qquad s_{ij} \; = \; \langle \psi_i(0)|_Y, \; \psi_j(0)|_Y\rangle_Y, \tag{2.16}$$

where $\psi_i(0)|_Y$ is equal to $i^*\psi_i(0)$ since $\psi_i(0) \in \Omega^q_{\mathrm{abs}}(M)$. Since $\Phi_i(0) = \sum_k r_{ik}\psi_k(0)$, we have

$$\langle \Phi_i(0), \; \Phi_j(0)\rangle_M \quad = \quad \sum_{a,b=1}^{\ell_q} \langle r_{ia}\psi_a(0), \; r_{jb}\psi_b(0)\rangle \; = \; \left(\mathcal{R}\mathcal{R}^T \right)_{ij} \; = \; \lim_{\lambda \to 0} \frac{\theta_i(\lambda)}{\lambda} \delta_{ij}, \tag{2.17}$$

which shows that $\mathcal{R}\mathcal{R}^T$ is a diagonal matrix. The above equalities show that

$$\lim_{\lambda \to 0} \frac{\theta_1(\lambda) \cdots \theta_{\ell_q}(\lambda)}{\lambda^{\ell_q}} \quad = \quad \Pi_{j=1}^{\ell_q} \left(\mathcal{R}\mathcal{R}^T \right)_{jj} \; = \; \det(\mathcal{R}\mathcal{R}^T) \; = \; \det(\mathcal{R}^2). \tag{2.18}$$

Since $\phi_i(0) = \Phi_i(0)|_Y = \sum_{k=1}^{\ell_q} r_{ik}\psi_k(0)|_Y$, we have

$$\begin{aligned} \psi_i(0)|_Y \quad &= \quad \sum_{k=1}^{\ell_q} \langle \psi_i(0)|_Y, \; \phi_k(0)\rangle_Y \; \phi_k(0) \; = \; \sum_{k,a,b=1}^{\ell_q} \langle \psi_i(0)|_Y, \; r_{ka}\psi_a(0)|_Y\rangle_Y \; r_{kb}\psi_b(0)|_Y \\ &= \quad \sum_{b=1}^{\ell_q} \left(\mathcal{S}\mathcal{R}^T\mathcal{R} \right)_{ib} \; \psi_b(0)|_Y, \end{aligned}$$

which shows that $\mathcal{S}\mathcal{R}^T\mathcal{R} = \mathrm{Id}$. Hence,

$$\det \mathcal{R}^2 = \frac{1}{\det \mathcal{S}}. \tag{2.19}$$

From (2.18) and (2.19), we obtain the following result.

Theorem 2.4. *Let M be an oriented m-dimensional compact Riemannian manifold with boundary Y. Then, for $0 \leq q \leq m-1$,*

$$\ln \operatorname{Det}^* \Delta^q_{M,\mathrm{abs}} - \ln \operatorname{Det} \Delta^q_{M,\mathrm{D}} \;=\; a_0 - \ln \det \mathcal{S} + \ln \operatorname{Det}^* Q^q_{\mathrm{abs}}(0).$$

Here a_0 is the constant term in the asymptotic expansion of $-\ln \operatorname{Det} Q^q_{\mathrm{abs}}(\lambda)$ for $\lambda \to \infty$. If M has a product structure near Y so that $\Delta^q_{M,\mathrm{abs}}$ is $-\partial^2_{y_m} + \Delta_Y$ on a collar neighborhood of Y, then $a_0 = 0$.

Corollary 2.5. *When $q = 0$, $\Delta^0_{M,\mathrm{abs}}$ is the Laplacian acting on smooth functions with the Neumann boundary condition on Y. Then $\ell_0 = \dim \ker \Delta^0_{M,\mathrm{abs}} = 1$ and $\mathcal{S} = \left(\frac{\ell(Y)}{V(M)}\right)$, where $V(M)$ and $\ell(Y)$ are volumes of M and Y, respectively. In this case, Theorem 2.4 is rewritten as*

$$\ln \operatorname{Det}^* \Delta^0_{M,\mathrm{abs}} - \ln \operatorname{Det} \Delta^0_{M,\mathrm{D}} \;=\; a_0 + \ln \frac{V(M)}{\ell(Y)} + \ln \operatorname{Det}^* Q^0_{\mathrm{abs}}(0).$$

We should mention that the constant term corresponding to a_0 in the BFK-gluing formula of zeta-determinants is zero when M is an even dimensional manifold since the density is an odd function with respect to ξ. However, the density for a_0 in Theorem 2.4 need not be an odd function so that a_0 may not be zero even though M is an even dimensional manifold. In the next section we are going to compute a_0 precisely when the dimension of M is 2 and 3.

In the remaining part of this section, we are going to discuss the values of zeta functions at zero by considering the metric rescaling from g to $c^2 g$ for $c > 0$ on M. It is well known [3] that

$$\Delta^q_M(c^2 g) \;=\; c^{-2} \Delta^q_M(g), \qquad \Delta^q_M(c^2 g) + \lambda \;=\; c^{-2}\left(\Delta^q_M(g) + c^2 \lambda\right). \tag{2.20}$$

Lemma 2.2 is rewritten as

$$\begin{aligned} \ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{abs}}(c^2 g) + \lambda\right) - \ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{D}}(c^2 g) + \lambda\right) \;&=\; P^q_{c^2 g}(\lambda) + \ln \operatorname{Det} Q^q_{\mathrm{abs},c^2 g}(\lambda) \\ &=\; \sum_{j=0}^{[(m-1)/2]} a_j(c^2 g)\lambda^j \;+\; \ln \operatorname{Det} Q^q_{\mathrm{abs},c^2 g}(\lambda), \end{aligned} \tag{2.21}$$

where $P^q_g(\lambda) = \sum_{j=0}^{[\frac{m-1}{2}]} a_j(g)\lambda^j$ and $P^q_{c^2 g}(\lambda) = \sum_{j=0}^{[\frac{m-1}{2}]} a_j(c^2 g)\lambda^j$. The Dirichlet-to-Neumann operator $Q^q_{\mathrm{abs},c^2 g}(\lambda)$ is described as follows. For $\varphi \in \Omega^q(Y)$, we choose $\phi \in \Omega^q(M)$ such that

$$\left(\Delta^q_M(c^2 g) + \lambda\right)\phi \;=\; c^{-2}\left(\Delta^q_M(g) + c^2 \lambda\right)\phi \;=\; 0, \qquad i^*\phi = \varphi, \qquad i^*\left(\iota_{\frac{1}{c}\frac{\partial}{\partial u}} \phi\right) = 0.$$

Then, $Q^q_{\mathrm{abs},c^2 g}(\lambda) f$ is defined by

$$Q^q_{\mathrm{abs},c^2 g}(\lambda)\varphi \;=\; -i^*\left(\iota_{\frac{1}{c}\frac{\partial}{\partial u}} d\phi\right) \;=\; -\frac{1}{c}\, i^*\left(\iota_{\frac{\partial}{\partial u}} d\phi\right) \;=\; \frac{1}{c} Q^q_{\mathrm{abs},g}(c^2\lambda)\varphi,$$

which shows that

$$Q^q_{\mathrm{abs},c^2 g}(\lambda) \;=\; \frac{1}{c} Q^q_{\mathrm{abs},g}(c^2 \lambda). \tag{2.22}$$

From (2.20) and (2.22), it follows that

$$\begin{aligned}
\ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{abs}\,/\,\mathrm{D}}(c^2 g)+\lambda\right) \;&=\; -2\ln c\cdot \zeta_{(\Delta^q_{M,\mathrm{abs}\,/\,\mathrm{D}}(g)+c^2\lambda)}(0) \;+\; \ln \operatorname{Det}\left(\Delta^q_{M,\mathrm{abs}\,/\,\mathrm{D}}(g)+c^2\lambda\right),\\
\ln \operatorname{Det} Q^q_{\mathrm{abs},c^2 g}(\lambda) \;&=\; -\ln c\cdot \zeta_{Q^q_{\mathrm{abs},g}(c^2\lambda)}(0) \;+\; \ln \operatorname{Det} Q^q_{\mathrm{abs},g}(c^2\lambda). \qquad (2.23)
\end{aligned}$$

We use the above equalities to rewrite (2.21) as

$$\begin{aligned}
&-2\ln c\cdot\left\{\zeta_{(\Delta^q_{M,\mathrm{abs}}(g)+c^2\lambda)}(0)-\zeta_{(\Delta^q_{M,\mathrm{D}}(g)+c^2\lambda)}(0)\right\}\\
&\qquad\qquad\qquad +\left\{\ln \operatorname{Det}(\Delta^q_{M,\mathrm{abs}}(g)+c^2\lambda)-\ln \operatorname{Det}(\Delta^q_{M,\mathrm{D}}(g)+c^2\lambda)\right\}\\
&= -2\ln c\cdot\left\{\zeta_{\left(\Delta^q_{M,\mathrm{abs}}(g)+c^2\lambda\right)}(0)-\zeta_{\left(\Delta^q_{M,\mathrm{D}}(g)+c^2\lambda\right)}(0)\right\} \;+\; P^q_g(c^2\lambda) \;+\; \ln \operatorname{Det} Q^q_{\mathrm{abs},g}(c^2\lambda)\\
&= P^q_{c^2 g}(\lambda)-\ln c\cdot\zeta_{Q^q_{\mathrm{abs},g}(c^2\lambda)}(0) \;+\; \ln \operatorname{Det} Q^q_{\mathrm{abs},g}(c^2\lambda), \qquad (2.24)
\end{aligned}$$

which leads to the following result.

Lemma 2.6.

$$\begin{aligned}
-\ln c\cdot\left\{2\left(\zeta_{\left(\Delta^q_{M,\mathrm{abs}}(g)+c^2\lambda\right)}(0)-\zeta_{\left(\Delta^q_{M,\mathrm{D}}(g)+c^2\lambda\right)}(0)\right)-\zeta_{Q^q_{\mathrm{abs},g}(c^2\lambda)}(0)\right\} \;&=\; P^q_{c^2 g}(\lambda) \;-\; P^q_g(c^2\lambda)\\
&= \sum_{j=0}^{[(m-1)/2]}\left(a_j(c^2 g)-a_j(g)c^{2j}\right)\lambda^j.
\end{aligned}$$

From (2.9) we have the following.

$$\sigma\left(\left(\mu-Q^q_{\mathrm{abs},c^2 g}(\lambda)\right)^{-1}\right) \;\sim\; \sum_{j=0}^{\infty}\widetilde{r}_{-1-j}(y,\xi,\lambda,\mu;c^2 g), \qquad (2.25)$$

$$\begin{aligned}
&\sigma\left(\left(\mu-Q^q_{\mathrm{abs},c^2 g}(\lambda)\right)^{-1}\right) \;=\; \sigma\left(\left(\mu-\frac{1}{c}Q^q_{\mathrm{abs},g}(c^2\lambda)\right)^{-1}\right) \;=\; \sigma\left(c\left(c\mu-Q^q_{\mathrm{abs},g}(c^2\lambda)\right)^{-1}\right)\\
\sim\;& c\sum_{j=0}^{\infty}\widetilde{r}_{-1-j}(y,\xi,c^2\lambda,c\mu;g) \;=\; c\sum_{j=0}^{\infty}\widetilde{r}_{-1-j}\left(y,c\frac{1}{c}\xi,c^2\lambda,c\mu;g\right) \;=\; c\sum_{j=0}^{\infty}c^{-1-j}\widetilde{r}_{-1-j}\left(y,\frac{1}{c}\xi,\lambda,\mu;g\right)\\
=\;& \sum_{j=0}^{\infty}c^{-j}\widetilde{r}_{-1-j}\left(y,\frac{1}{c}\xi,\lambda,\mu;g\right),
\end{aligned}$$

which shows that

$$\widetilde{r}_{-1-j}(y,\xi,\lambda,\mu;c^2 g) \;=\; c^{-j}\widetilde{r}_{-1-j}\left(y,\frac{1}{c}\xi,\lambda,\mu;g\right). \qquad (2.26)$$

By (2.10) the density $\pi_j(y;c^2 g)$ for $\pi_j(c^2 g)$ is given by

$$\begin{aligned}
\pi_j(y;c^2 g) \;&=\; -\frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^{m-1}}\int_{T^*_y Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\operatorname{Tr}\widetilde{r}_{-1-j}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu;c^2 g\right)\,d\mu\,d\xi(c^2 g)\right),\\
\pi_j(c^2 g) \;&=\; \int_Y \pi_j(y;c^2 g)\,d\operatorname{vol}(Y;c^2 g). \qquad (2.27)
\end{aligned}$$

Using (2.27) with the following relation

$$d\xi(c^2 g) \;=\; c^{-(m-1)} d\xi(g), \qquad d\,\mathrm{vol}(Y; c^2 g) \;=\; c^{m-1} d\,\mathrm{vol}(Y; g), \tag{2.28}$$

we have the following result.

Lemma 2.7.

$$\pi_k(y; c^2 g) \;=\; c^{-k} \pi_k(y; g), \qquad \pi_k(c^2 g) \;=\; c^{m-1-k} \pi_k(g).$$

Lemma 2.3 shows that the coefficient $a_k(c^2 g)$ of the polynomial $P^q_{c^2 g}(\lambda)$ is given by

$$\begin{aligned} a_k(c^2 g) &= -\pi_{m-1-2k}(c^2 g) - (\mathfrak{a}_{m-2k}(c^2 g) - \mathfrak{b}_{m-2k}(c^2 g)) \frac{d}{ds} \left(\frac{\Gamma(s-k)}{\Gamma(s)} \right) \Big|_{s=0} \\ &= c^{2k} a_k(g), \end{aligned} \tag{2.29}$$

where we used the fact that $\mathfrak{a}_k(c^2 g) = c^{m-k} \mathfrak{a}_k(g)$ and $\mathfrak{b}_k(c^2 g) = c^{m-k} \mathfrak{b}_k(g)$ (Theorem 3.1.9 in [11] or (4.2.5) in [15]). Hence, $P^q_{c^2 g}(\lambda) - P^q_g(c^2 \lambda) = 0$ in Lemma 2.6. Replacing λ with $\frac{1}{c^2}\lambda$, we obtain the following result.

Theorem 2.8. *For $\lambda > 0$, we obtain the following equality:*

$$\zeta_{Q^q_{\mathrm{abs},g}(\lambda)}(0) \;=\; 2 \left\{ \zeta_{\left(\Delta^q_{M,\mathrm{abs}}(g) + \lambda\right)}(0) - \zeta_{\left(\Delta^q_{M,\mathrm{D}}(g) + \lambda\right)}(0) \right\}.$$

If $\dim \ker \Delta^q_{M,\mathrm{abs}}(g) = \dim \ker Q^q_{\mathrm{abs},g}(0) = \ell_q$, we obtain the following equality by taking $\lambda \to 0$:

$$\zeta_{Q^q_{\mathrm{abs},g}(0)}(0) + \ell_q \;=\; 2 \left\{ \left(\zeta_{\Delta^q_{M,\mathrm{abs}}(g)}(0) + \ell_q \right) - \zeta_{\Delta^q_{M,\mathrm{D}}(g)}(0) \right\}.$$

The following heat trace asymptotic expansion is well known [12, 19, 25].

$$\mathrm{Tr}\, e^{-t Q^q_{\mathrm{abs}}(0)} \sim \sum_{j=0}^{\infty} v_j t^{-(m-1)-j} + \sum_{j=1}^{\infty} (w_j \ln t + z_j) t^j, \tag{2.30}$$

where the v_j's and w_j's are locally computed and the z_j's are not. The second statement of Theorem 2.8 can be rewritten as follows.

Corollary 2.9.

$$v_{m-1} \;=\; 2\left(\mathfrak{a}_m - \mathfrak{b}_m\right).$$

Let $\kappa_1(y), \cdots, \kappa_{m-1}(y)$ be the principal curvatures of Y in M at $y \in Y$. We define the r-mean curvature H_r by

$$H_r(y) \;=\; \frac{1}{\binom{m-1}{r}} \sigma_r(\kappa_1, \cdots, \kappa_{m-1}) \;=\; \frac{r!(m-1-r)!}{(m-1)!} \sigma_r(\kappa_1, \cdots, \kappa_{m-1}), \tag{2.31}$$

where $\sigma_r : \mathbb{R}^{m-1} \to \mathbb{R}$ is the r-th elementary symmetric polynomial defined by $\sigma_r(u_1, \cdots .u_{m-1}) = \sum_{1 \le i_1 < \cdots < i_r \le m-1} u_{i_1} \cdots u_{i_r}$ [1]. For example, for $m \ge 3$, $H_1(y)$ and $H_2(y)$ are

$$H_1(y) \;=\; \frac{1}{m-1}\sum_{\alpha=1}^{m-1}\kappa_\alpha(y), \qquad H_2(y) \;=\; \frac{2}{(m-1)(m-2)}\sum_{1\le\alpha<\beta\le m-1}\kappa_\alpha(y)\kappa_\beta(y). \tag{2.32}$$

When $m=3$, the following equality was proved in Lemma 3.2 of [17].

$$\sum_{\alpha=1}^{2} R^M_{\alpha 3\alpha 3}(y) \;=\; -\operatorname{Ric}^M_{33} \;=\; -\frac{1}{2}\tau_M(y)+\frac{1}{2}\tau_Y(y)-H_2(y), \tag{2.33}$$

where $R^M_{\alpha 3\alpha 3}(y)$ and $\operatorname{Ric}^M_{33}$ are defined in (3.6) below. For $m=2,\ 3$, $\mathfrak{a}_m-\mathfrak{b}_m$ can be computed concretely by using Theorem 3.4.1 and Theorem 3.6.1 in [11] or Section 4.2 and 4.5 in [15], which together with (2.33) and (3.35) below yields the following result.

Corollary 2.10. *Let $(M,Y;g)$ be an m-dimensional compact oriented Riemannian manifold with boundary Y. We define $Q^q_{\mathrm{abs}}(0)$ on $\Omega^q(Y)$ as above and denote by τ_M and τ_Y the scalar curvatures of M and Y, respectively. If $m=2$, then*

$$\zeta_{Q^q_{\mathrm{abs}}(0)}(0)+\ell_q \;=\; \begin{cases} 0 & \text{if}\quad q=0\\ -\frac{1}{\pi}\int_Y \kappa(y)\ dy & \text{if}\quad q=1.\end{cases}$$

If $m=3$, then

$$\zeta_{Q^q_{\mathrm{abs}}(0)}(0)+\ell_q \;=\; \begin{cases} \frac{1}{4\pi}\int_Y\left\{\frac{1}{8}\tau_M+\frac{1}{24}\tau_Y+\frac{1}{4}H_1^2\right\}dy & \text{if}\quad q=0\\ \frac{1}{4\pi}\int_Y\left\{-\frac{1}{4}\tau_M-\frac{5}{12}\tau_Y+\frac{1}{2}H_1^2\right\}dy & \text{if}\quad q=1\\ \frac{1}{4\pi}\int_Y\left\{-\frac{3}{8}\tau_M+\frac{13}{24}\tau_Y+\frac{1}{4}H_1^2\right\}dy & \text{if}\quad q=2.\end{cases}$$

Remark : When $q=0$, the above result is obtained in Theorem 1.5 of [25] or Theorem 5.1 of [19].

Example 2.11 : For a closed Riemannian manifold N, we consider a Riemannian product $M=[0,\ a]\times N$. Let $\Delta^q_{M,\mathrm{abs}}$ and $\Delta^q_{M,\mathrm{D}}$ be the Laplacian $-\frac{\partial^2}{\partial u^2}+\begin{pmatrix}\Delta^q_N & \\ & \Delta^{q-1}_N\end{pmatrix}$ on M acting on smooth q-forms with the absolute and Dirichlet boundary conditions on $Y:=\{0,a\}\times N$, respectively. Let $\mathbb{N}=\{1,2,3,\cdots\}$ be the set of all positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The spectra of $\Delta^q_{M,\mathrm{abs}}$ and $\Delta^q_{M,\mathrm{D}}$ are given by

$$\operatorname{Spec}\left(\Delta^q_{M,\mathrm{abs}}\right) \;=\; \left\{\lambda_n+\left(\frac{k\pi}{a}\right)^2,\ \mu_s+\left(\frac{l\pi}{a}\right)^2\ \middle|\ \lambda_n\in\operatorname{Spec}(\Delta^q_N),\ \mu_s\in\operatorname{Spec}(\Delta^{q-1}_N),\ k\in\mathbb{N}_0,\ l\in\mathbb{N}\right\},$$
$$\operatorname{Spec}\left(\Delta^q_{M,\mathrm{D}}\right) \;=\; \left\{\lambda_n+\left(\frac{k\pi}{a}\right)^2,\ \mu_s+\left(\frac{l\pi}{a}\right)^2\ \middle|\ \lambda_n\in\operatorname{Spec}(\Delta^q_N),\ \mu_s\in\operatorname{Spec}(\Delta^{q-1}_N),\ k\in\mathbb{N},\ l\in\mathbb{N}\right\},$$

which shows that $\zeta_{\Delta^q_{M,\mathrm{abs}}}(s)-\zeta_{\Delta^q_{M,\mathrm{D}}}(s)=\zeta_{\Delta^q_N}(s)$ and hence

$$\ln\operatorname{Det}^*\Delta^q_{M,\mathrm{abs}}\;-\;\ln\operatorname{Det}\Delta^q_{M,\mathrm{D}}\;=\;\ln\operatorname{Det}^*\Delta^q_N, \qquad \zeta_{\Delta^q_{M,\mathrm{abs}}}(0)\;-\;\zeta_{\Delta^q_{M,\mathrm{D}}}(0)\;=\;\zeta_{\Delta^q_N}(0).$$

Simple computation shows that the spectrum of $Q^q_{\mathrm{abs}}(0):\Omega^q(N\times\{0\})\oplus\Omega^q(N\times\{a\})\to\Omega^q(N\times\{0\})\oplus\Omega^q(N\times\{a\})$ is given by

$$\begin{aligned}&\operatorname{Spec}(Q^q_{\rm abs}(0))\\ =\;& \{0\} \;\cup\; \left\{\frac{2}{a}\right\} \;\cup\; \left\{\sqrt{\lambda_n}\left(1+\frac{2}{e^{a\sqrt{\lambda_n}}-1}\right),\; \sqrt{\lambda_n}\left(1-\frac{2}{e^{a\sqrt{\lambda_n}}+1}\right) \mid 0<\lambda_n \in \operatorname{Spec}(\Delta^q_N)\right\},\end{aligned}$$

where the multiplicities of 0 and $\frac{2}{a}$ are $\ell_q := \dim\ker H^q(N) := \dim\ker H^q(M)$. Hence,

$$\begin{aligned}\ln \operatorname{Det}^* Q^q_{\rm abs}(0) &= \ell_q \ln\frac{2}{a} + \ln\operatorname{Det}^*\Delta^q_N + \sum_{0<\lambda_n\in\operatorname{Spec}(\Delta^q_N)} \left\{\ln\left(1+\frac{2}{e^{a\sqrt{\lambda_n}}-1}\right)+\ln\left(1-\frac{2}{e^{a\sqrt{\lambda_n}}+1}\right)\right\}\\ &= \ell_q \ln\frac{2}{a} + \ln\operatorname{Det}^*\Delta^q_N,\\ \zeta_{Q^q_{\rm abs}(0)}(0) &= \ell_q + \zeta_{\Delta^q_N}(0) + \zeta_{\Delta^q_N}(0) \;=\; \ell_q + 2\zeta_{\Delta^q_N}(0).\end{aligned}$$

Let $\{\psi_1, \cdots, \psi_{\ell_q}\}$ be an orthonormal basis of $\ker\Delta^q_N$. Then $\{\frac{1}{\sqrt{a}}\psi_1, \cdots, \frac{1}{\sqrt{a}}\psi_{\ell_q}\}$ is an orthonormal basis of $\ker\Delta^q_{M,\rm abs}$. Hence,

$$\langle \frac{1}{\sqrt{a}}\psi_i,\; \frac{1}{\sqrt{a}}\psi_j\rangle_Y \;=\; \frac{1}{a}\langle\psi_i,\;\psi_j\rangle_{\{0\}\times N} \;+\; \frac{1}{a}\langle\psi_i,\;\psi_j\rangle_{\{a\}\times N} \;=\; \frac{2}{a}\delta_{ij}.$$

Since $\ln\det\mathcal{S} = \ell_q \ln\frac{2}{a}$ and $a_0 = 0$, this result agrees with Theorem 2.4 and Theorem 2.8.

3. The homogeneous symbol of $Q^q_{\rm abs}(\lambda)$

In this section we are going to compute the homogeneous symbol of $Q^q_{\rm abs}(\lambda)$ in the boundary normal coordinate system defined below. For $y_0\in Y$ and a small open neighborhood V of y_0 in Y, we choose a normal coordinate system on V with $y=(y_1,\cdots,y_{m-1})$ and $y_0=(0,\cdots,0)$. For $y\in Y$, we denote by $\gamma_y(u)$ the unit speed geodesic such that $\gamma'_y(0)$ is an inward normal vector to Y. Then, $(y,u) = (y_1,\cdots,y_{m-1},u)$ gives a local coordinate system. We will write $u=y_m$ for notational convenience. For $1\leq\alpha,\;\beta,\;\gamma\leq m-1$, the metric satisfies

$$g_{\alpha\beta}(y_0)=\delta_{\alpha\beta}, \qquad g_{\alpha\beta;\gamma}(y_0)=0, \qquad g_{\alpha m}(y)=0, \qquad g_{mm}(y)=1, \tag{3.1}$$

where $g_{\alpha\beta;k} := \frac{\partial}{\partial y_k} g_{\alpha\beta}$, $1\leq k\leq m$. Moreover, we may choose the coordinate system such that

$$g^{\alpha\beta;m}(y_0) \;=\; -\,g_{\alpha\beta;m}(y_0) \;=\; \begin{cases} 2\kappa_\alpha & \text{for}\quad \alpha=\beta\\ 0 & \text{for}\quad \alpha\neq\beta,\end{cases} \tag{3.2}$$

where the κ_α's $(1\leq\alpha\leq m-1)$ are the principal curvatures of Y in M. For simplicity, we are going to write $\frac{\partial}{\partial y_k}$ by ∂_{y_k} for $1\leq k\leq m$. We denote by ∇^M the Levi-Civita connection on M associated to g and denote by ω the connection form for ∇^M with respect to $\{\partial_{y_1},\cdots,\partial_{y_m}\}$ and put $\omega_k=\omega(\partial_{y_k})$. For some endomorphism E_q acting on $\wedge^q T^*M$, $\Delta^q_M+\lambda$ is expressed as follows [17, 25]:

$$\begin{aligned}&\Delta^q_M+\lambda \;=\; -\operatorname{Tr}\left(\left(\nabla^M\right)^2\right) - E_q \qquad (3.3)\\ =\;& -\partial^2_{y_m}\operatorname{Id} \;+\; \left(A(y,y_m) -\; 2\omega_m\right)\partial_{y_m} \;+\; D\big(y,y_m,\frac{\partial}{\partial y},\lambda\big) - \left(\partial_{y_m}\omega_m + \omega_m\omega_m - A(x,y_m)\omega_m\right),\end{aligned}$$

where Id is an $\binom{m}{q} \times \binom{m}{q}$ identity matrix and

$$A(y, y_m) = \left\{ -\frac{1}{2} \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m)\ g_{\alpha\beta;m}(y, y_m) \right\} \mathrm{Id}, \tag{3.4}$$

$$\begin{aligned} D\left(y, y_m, \frac{\partial}{\partial y}, \lambda\right) = & \left\{ \left(- \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m) \partial_{y_\alpha} \partial_{y_\beta} + \lambda \right) \right. \\ & \left. - \sum_{\alpha,\beta=1}^{m-1} \left(\frac{1}{2} g^{\alpha\beta}(y, y_m)(\partial_{y_\alpha} \ln |g|(y, y_m)) + g^{\alpha\beta;\alpha}(y, y_m)) \right) \partial_{y_\beta} \right\} \mathrm{Id} \\ & -2 \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m) \omega_\alpha \partial_{y_\beta} - \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m) \left(\partial_{y_\alpha} \omega_\beta + \omega_\alpha \omega_\beta - \sum_{\gamma=1}^{m-1} \Gamma^\gamma_{\alpha\beta} \omega_\gamma \right) - E_q, \end{aligned} \tag{3.5}$$

We use the Weitzenböck formula (for example, Lemma 4.1.2 in [10]) to describe E_q explicitly. It is known ([11]) that $E_0 = 0$. Let $\{e_1, \cdots, e_m\}$ and $\{e^1, \cdots, e^m\}$ be local orthonormal bases of $TM|_U$ and $T^*M|_U$ for some open set U in M, respectively. We denote by R^M_{ijkl} and Ric^M_{ij} the Riemann curvature tensor and Ricci tensor on M defined by

$$R^M_{ijkl} = \langle \nabla^M_{e_i} \nabla^M_{e_j} e_k - \nabla^M_{e_j} \nabla^M_{e_i} e_k - \nabla^M_{[e_i, e_j]} e_k, e_l \rangle, \qquad \mathrm{Ric}^M_{ij} = \sum_{k=1}^{m} R^M_{ikkj}. \tag{3.6}$$

Then,

$$E_1 = \Big(- \mathrm{Ric}^M_{ij} \Big)_{1 \le i,j \le m}. \tag{3.7}$$

For later use, we compute E_2 for $m = 3$ with respect to a local orthonormal basis $\{e^1 \wedge e^2, e^3 \wedge e^1, e^3 \wedge e^2\}$, which is given by

$$E_2 = \begin{pmatrix} -\mathrm{Ric}^M_{33} & -R^M_{2113} & R^M_{1223} \\ -R^M_{2113} & -\mathrm{Ric}^M_{22} & R^M_{1332} \\ R^M_{1223} & R^M_{1332} & -\mathrm{Ric}^M_{11} \end{pmatrix}. \tag{3.8}$$

Since Y is compact, we can choose a uniform constant $\epsilon_0 > 0$ such that $\gamma_x(u)$ is well defined for $0 \le u \le \epsilon_0$. Then,

$$U_{\epsilon_0} \ := \ \{(y, y_m) \mid y \in Y,\ 0 \le y_m < \epsilon_0\} \tag{3.9}$$

is a collar neighborhood of Y. We note that for a fixed y_m in $[0, \epsilon_0)$,

$$Y_{y_m} \ := \ \{(y, y_m) \mid y \in Y\} \tag{3.10}$$

is a submanifold of M diffeomorphic to Y, and it is the y_m-level of Y. For $0 < y_m < \epsilon$, we denote

$$M_{y_m} \quad := \quad M - \cup_{0 \leq u < y_m} Y_u, \tag{3.11}$$

and denote by $i_{y_m} : Y_{y_m} \to M_{y_m}$ the natural inclusion. We also denote by $\Delta^q_{M_{y_m}}$ the Hodge-De Rham Laplacian Δ^q_M restricted to M_{y_m}. For each $0 \leq y_m < \epsilon_0$, we define $Q^q_{\mathrm{abs},y_m}(\lambda) : \Omega^q(Y_{y_m}) \to \Omega^q(Y_{y_m})$ and $Q^{q-1}_{\mathrm{rel},y_m}(\lambda) : \Omega^{q-1}(Y_{y_m}) \to \Omega^{q-1}(Y_{y_m})$ in the same way as $Q^q_{\mathrm{abs}}(\lambda)$ and $Q^{q-1}_{\mathrm{rel}}(\lambda)$. Indeed, for $\alpha_{y_m}(y) \in \Omega^q(Y_{y_m})$ and $\beta_{y_m}(y) \in \Omega^{q-1}(Y_{y_m})$, we choose $\phi_{y_m} \in \Omega^q(M_{y_m})$, $\psi_{y_m} \in \Omega^q(M_{y_m})$ satisfying

$$\begin{aligned}
&(\Delta^q_{M_{y_m}} + \lambda)\phi_{y_m} = 0, \qquad i^*_{y_m}\phi_{y_m} = \alpha_{y_m}, \qquad i^*\iota_{\partial_{y_m}}\phi_{y_m} = 0, \\
&(\Delta^q_{M_{y_m}} + \lambda)\psi_{y_m} = 0, \qquad i^*_{y_m}\psi_{y_m} = 0, \qquad i^*\iota_{\partial_{y_m}}\psi_{y_m} = \beta_{y_m}.
\end{aligned} \tag{3.12}$$

We define

$$Q^q_{\mathrm{abs},y_m}(\lambda)(\varphi_{y_m}) \;:=\; -i^*_{y_m}\iota_{\partial_{y_m}} d\phi_{y_m}, \qquad Q^q_{\mathrm{rel},y_m}(\lambda)(\varphi_{y_m}) \;:=\; i^*_{y_m}\left(\delta\psi_{y_m}\right). \tag{3.13}$$

Using local coordinates on U_{ϵ_0}, with multi-indices $i = (i_1, ..., i_q)$, $j = (j_1, ..., j_{q-1})$, $k = (k_1, ..., k_q)$, and $l = (l_1, ..., l_{q-1})$, we write $\phi_{y_m}(y, y_m)$ and $\psi_{y_m}(y, y_m)$ as

$$\begin{aligned}
\phi_{y_m}(y, y_m) &= \sum_i \phi_{1,i}(y, y_m)\, dy_{i_1} \wedge \cdots \wedge dy_{i_q} + \sum_j \phi_{2,j}(y, y_m)\, dy_m \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_{q-1}}, \\
\psi_{y_m}(y, y_m) &= \sum_k \psi_{1,k}(y, y_m)\, dy_{k_1} \wedge \cdots \wedge dy_{k_q} + \sum_l \psi_{2,l}(y, y_m)\, dy_m \wedge dy_{l_1} \wedge \cdots \wedge dy_{l_{q-1}},
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\alpha_{y_m}(y) &= \sum_i \phi_{1,i}(y, y_m)\big|_{Y_{y_m}}\, dy_{i_1} \wedge \cdots \wedge dy_{i_q}, \\
\beta_{y_m}(y) &= \sum_l \psi_{2,l}(y, y_m)\big|_{Y_{y_m}}\, dy_{l_1} \wedge \cdots \wedge dy_{l_{q-1}}, \\
\phi_{2,j}\big|_{Y_{y_m}} &= \psi_{1,k}\big|_{Y_{y_m}} = 0.
\end{aligned}$$

In this local coordinate system, $Q^q_{\mathrm{abs},y_m}(\lambda)(\varphi_{y_m})$ and $Q^q_{y_m,\mathrm{rel}}(\lambda)(\varphi_{y_m})$ can be rewritten as follows (cf. Definition 1.1).

$$\begin{aligned}
Q^q_{\mathrm{abs},y_m}(\lambda)(\alpha_{y_m}(y)) &= -\sum_i \left(\partial_{y_m}\phi_{1,i}(y, y_m)\right)\big|_{Y_{y_m}}\, dy_{i_1} \wedge \cdots \wedge dy_{i_q}, \\
Q^{q-1}_{\mathrm{rel},y_m}(\lambda)(\beta_{y_m}(y)) &= -\sum_l \left(\partial_{y_m}\psi_{2,l}(y, y_m)\right)\big|_{Y_{y_m}}\, dy_{l_1} \wedge \cdots \wedge dy_{l_{q-1}}.
\end{aligned} \tag{3.15}$$

When $y_m = 0$, $Q^q_{\mathrm{abs},0}(\lambda)$ and $Q^{q-1}_{\mathrm{rel},0}(\lambda)$ are equal to $Q^q_{\mathrm{abs}}(\lambda)$ and $Q^{q-1}_{\mathrm{rel}}(\lambda)$, respectively.

We next define auxiliary operators $\mathcal{T}^q_{\mathrm{abs},y_m}(\lambda) : \Omega^q(Y_{y_m}) \to \Omega^{q-1}(Y_{y_m})$ and $\mathcal{T}^{q-1}_{\mathrm{rel},y_m}(\lambda) : \Omega^{q-1}(Y_{y_m}) \to \Omega^q(Y_{y_m})$ by

$$\begin{aligned} \mathcal{T}^{q}_{\mathrm{abs},y_m}(\lambda)(\alpha_{y_m}(y)) &= -\sum_j \big(\partial_{y_m}\phi_{2,j}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{j_1}\wedge\cdots\wedge dy_{j_{q-1}}, \\ \mathcal{T}^{q-1}_{\mathrm{rel},y_m}(\lambda)(\beta_{y_m}(y)) &= -\sum_k \big(\partial_{y_m}\psi_{1,k}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{k_1}\wedge\cdots\wedge dy_{k_q}. \end{aligned} \tag{3.16}$$

We finally define $\mathcal{R}^q_{y_m}(\lambda) : \Omega^q(Y_{y_m})\oplus\Omega^{q-1}(Y_{y_m}) \to \Omega^q(Y_{y_m})\oplus\Omega^{q-1}(Y_{y_m})$ by

$$\mathcal{R}^q_{y_m}(\lambda) \;=\; \begin{pmatrix} Q^{q}_{\mathrm{abs},y_m}(\lambda) & \mathcal{T}^{q-1}_{\mathrm{rel},y_m}(\lambda) \\ \mathcal{T}^{q}_{\mathrm{abs},y_m}(\lambda) & Q^{q-1}_{\mathrm{rel},y_m}(\lambda) \end{pmatrix}. \tag{3.17}$$

Using local coordinates on U_{ϵ_0}, we write

$$\begin{aligned} & \mathcal{R}^q_{y_m}(\lambda)(\alpha_{y_m},\ \beta_{y_m}) \\ =\; & \Big(Q^{q}_{\mathrm{abs},y_m}(\lambda)\alpha_{y_m} + \mathcal{T}^{q-1}_{\mathrm{rel},y_m}(\lambda)\beta_{y_m},\ \mathcal{T}^{q}_{\mathrm{abs},y_m}(\lambda)\alpha_{y_m} + Q^{q-1}_{\mathrm{rel},y_m}(\lambda)\beta_{y_m} \Big) \\ =\; & \Big(-\sum_i \big(\partial_{y_m}\phi_{1,i}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{i_1}\wedge\cdots\wedge dy_{i_q} \; - \; \sum_k \big(\partial_{y_m}\psi_{1,k}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{k_1}\wedge\cdots\wedge dy_{k_q}, \\ & - \sum_j \big(\partial_{y_m}\phi_{2,j}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{j_1}\wedge\cdots\wedge dy_{j_{q-1}} - \sum_l \big(\partial_{y_m}\psi_{2,l}(y,y_m)\big)\big|_{Y_{y_m}} \; dy_{l_1}\wedge\cdots\wedge dy_{l_{q-1}} \Big), \end{aligned} \tag{3.18}$$

where $\phi_{y_m}+\psi_{y_m}\in\Omega^q(M_{y_m})$ satisfies

$$\left(\Delta^q_M+\lambda\right)\left(\phi_{y_m}+\psi_{y_m}\right)=0, \quad i^*_{y_m}\left(\phi_{y_m}+\psi_{y_m}\right)=\alpha_{y_m}, \quad i^*_{y_m}\big(\iota_{\partial_{y_m}}\left(\phi_{y_m}+\psi_{y_m}\right)\big)=\beta_{y_m}. \tag{3.19}$$

Then, $\mathcal{R}^q_{y_m}(\lambda)$ is an elliptic pseudodifferential operator of order 1.

We can identify Y_{y_m} with $Y := Y_0$ by the geodesic $\gamma_y(u)$ and regard $\mathcal{R}^q_{y_m}(\lambda)$ to be a one parameter family of operators defined on $\Omega^q(Y)\oplus\Omega^{q-1}(Y)$. We are going to take the derivative of $\mathcal{R}^q_{y_m}(\lambda)$ with respect to y_m to obtain a Riccati type equation for $\mathcal{R}^q_{y_m}(\lambda)$, from which we can compute the homogeneous symbol of $\mathcal{R}^q_{y_m}(\lambda)$. This idea goes back to I. M. Gelfand. The symbol of $Q^q_{\mathrm{abs}}(\lambda)$ is obtained from the symbol of $\mathcal{R}^q_{y_m}(\lambda)$.

We start from $\phi_{y_m}(y,y_m)+\psi_{y_m}(y,y_m)\in\Omega^q(M_{y_m})$. We note that

$$\partial_{y_m}\Big(\phi_{y_m}(y,y_m)+\psi_{y_m}(y,y_m)\Big)\big|_{Y_{y_m}} \;=\; -\mathcal{R}^q_{y_m}(\lambda)\left(\phi_{y_m}(y,y_m)+\psi_{y_m}(y,y_m)\right)\big|_{Y_{y_m}}. \tag{3.20}$$

We take the derivative with respect to y_m again to obtain

$$\begin{aligned} & \partial^2_{y_m}\Big(\phi_{y_m}(y,y_m)+\psi_{x_m}(y,y_m)\Big)\big|_{Y_{y_m}} \\ =\; & -\Big(\partial_{y_m}\mathcal{R}^q_{y_m}(\lambda)\Big)\left(\phi_{y_m}(y,y_m)+\psi_{y_m}(y,y_m)\right)\big|_{Y_{y_m}} \;+\; \mathcal{R}^q_{y_m}(\lambda)^2\left(\phi_{y_m}(y,y_m)+\psi_{y_m}(y,y_m)\right)\big|_{Y_{y_m}}, \end{aligned} \tag{3.21}$$

which together with (3.3) leads to the following equality.

$$\begin{aligned}
& \left\{-\partial_{y_m}\mathcal{R}^q_{y_m}(\lambda) + \mathcal{R}^q_{y_m}(\lambda)^2\right\}\left(\phi_{y_m}(y,y_m) + \psi_{y_m}(y,y_m)\right)\big|_{Y_{y_m}} \qquad (3.22)\\
= \;& \Big\{(A(y,y_m) - \ 2\omega_m)\,\partial_{y_m} \ + \ D\big(y,y_m,\partial_y,\lambda\big) - (\partial_{y_m}\omega_m + \omega_m\omega_m - A(y,y_m)\omega_m)\Big\}\left(\phi_{y_m}(y,y_m) + \right.\\
& \left.\psi_{y_m}(y,y_m)\right)\big|_{Y_{y_m}}.
\end{aligned}$$

Using (3.20) again, we obtain the following result.

Lemma 3.1.

$$\begin{aligned}
& \mathcal{R}^q_{y_m}(\lambda)^2\\
= \;& D\big(y,y_m,\partial_y,\lambda\big) - (A(y,y_m) - \ 2\omega_m)\,\mathcal{R}^q_{y_m}(\lambda) + \partial_{y_m}\mathcal{R}^q_{y_m}(\lambda) - (\partial_{y_m}\omega_m + \omega_m\omega_m - A(y,y_m)\omega_m)\,.
\end{aligned}$$

We now compute the homogeneous symbol in this coordinate system using the above lemma. We denote the homogeneous symbol of $\mathcal{R}^q_{y_m}(\lambda)$ and $D\big(y,y_m,\partial_y,\lambda\big)$ by

$$\begin{aligned}
\sigma\left(\mathcal{R}^q_{y_m}(\lambda)\right)(y,y_m,\xi,\lambda) \;&\sim\; \alpha_1(y,y_m,\xi,\lambda) + \alpha_0(y,y_m,\xi,\lambda) + \alpha_{-1}(y,y_m,\xi,\lambda) + \cdots, \qquad (3.23)\\
\sigma\left(D\big(y,y_m,\partial_y,\lambda\big)\right) \;&=\; p_2(y,y_m,\xi,\lambda) + p_1(y,y_m,\xi) + p_0(y,y_m,\xi),
\end{aligned}$$

where for an $\binom{m}{q}\times\binom{m}{q}$ identity matrix Id, (3.5) shows that

$$\begin{aligned}
p_2(y,y_m,\xi,\lambda) \;&=\; \left(\sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y,y_m)\xi_\alpha\xi_\beta + \lambda\right)\mathrm{Id} \;=\; \big(|\xi|^2 + \lambda\big)\,\mathrm{Id}, \qquad (3.24)\\
p_1(y,y_m,\xi) \;&=\; -i\sum_{\alpha,\beta=1}^{m-1}\left(\frac{1}{2}g^{\alpha\beta}(y,y_m)\partial_{y_\alpha}\ln|g|(y,y_m) + g^{\alpha\beta;\alpha}(y,y_m)\right)\xi_\beta\,\mathrm{Id} - 2i\sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}\omega_\alpha\xi_\beta,\\
p_0(y,y_m,\xi) \;&=\; -\sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}\left(\partial_{y_\alpha}\omega_\beta + \omega_\alpha\omega_\beta - \sum_{\gamma=1}^{m-1}\Gamma^\gamma_{\alpha\beta}\omega_\gamma\right) - E_q.
\end{aligned}$$

The symbol of $\partial_{y_m}\mathcal{R}^q_{y_m}(\lambda)$ is given by

$$\sigma\left(\partial_{y_m}\mathcal{R}^q_{y_m}(\lambda)\right)(y,y_m,\xi,\lambda) \sim \partial_{y_m}\alpha_1(y,y_m,\xi,\lambda) + \partial_{y_m}\alpha_0(y,y_m,\xi,\lambda) + \partial_{y_m}\alpha_{-1}(y,y_m,\xi,\lambda) + \cdots. \qquad (3.25)$$

It is well known [10, 28] that for $D_y = \frac{1}{i}\partial_y$,

$$\begin{aligned}
\sigma\left(\mathcal{R}_{y_m}(\lambda)^2\right) \;&\sim\; \sum_{k=0}^{\infty}\sum_{\substack{|\omega|+i+j=k\\ i,j\geq 0}}\frac{1}{\omega!}\partial_\xi^\omega\alpha_{1-i}(y,y_m,\xi,\lambda)\cdot D_y^\omega\alpha_{1-j}(y,y_m,\xi,\lambda) \qquad (3.26)\\
&=\; \alpha_1^2 + (\partial_\xi\alpha_1\cdot D_y\alpha_1 + 2\alpha_1\cdot\alpha_0)\\
&\quad + \left(2\alpha_1\alpha_{-1} + \alpha_0^2 - i(\partial_\xi\alpha_0)(\partial_y\alpha_1) - i(\partial_\xi\alpha_1)(\partial_y\alpha_0) - \sum_{|\omega|=2}\frac{1}{\omega!}(\partial_\xi^\omega\alpha_1)(\partial_y^\omega\alpha_1)\right) + \cdots.
\end{aligned}$$

Using Lemma 3.1 with (3.24) - (3.26), we can compute the homogeneous symbol of $\mathcal{R}^q_{y_m}(\lambda)$. For example, the first three terms are given as follows.

$$\begin{aligned}
\alpha_1(y, y_m, \xi, \lambda) &= \sqrt{|\xi|^2 + \lambda}\ \mathrm{Id}, \\
\alpha_0(y, y_m, \xi, \lambda) &= \frac{1}{2\sqrt{|\xi|^2+\lambda}} \left\{ -\partial_\xi \alpha_1 \cdot D_y \alpha_1 + p_1 - \left(A(y, y_m) - 2\omega_m\right) \alpha_1 + \partial_{y_m} \alpha_1 \right\}, \\
\alpha_{-1}(y, y_m, \xi, \lambda) &= \frac{1}{2\sqrt{|\xi|^2+\lambda}} \Bigg\{ \sum_{|\omega|=2} \frac{1}{\omega!} (\partial_\xi^\omega \alpha_1)(\partial_y^\omega \alpha_1) + i(\partial_\xi \alpha_0)(\partial_y \alpha_1) + i(\partial_\xi \alpha_1)(\partial_y \alpha_0) - \alpha_0^2 \\
&\quad + p_0 - (A(y, y_m) - 2\omega_m)\alpha_0 + \partial_{y_m}\alpha_0 - (\partial_{y_m}\omega_m + \omega_m\omega_m - A(y, y_m)\omega_m) \Bigg\}.
\end{aligned} \tag{3.27}$$

Let $\mathcal{F}_{y_m} : \Omega^q(Y_{y_m}) \to \Omega^q(Y_{y_m}) \oplus \Omega^{q-1}(Y_{y_m})$ and $\mathcal{G}_{y_m} : \Omega^q(Y_{y_m}) \oplus \Omega^{q-1}(Y_{y_m}) \to \Omega^q(Y_{y_m})$ be the natural inclusion and projection, respectively, *i.e.* $\mathcal{F}_{y_m}(\phi) = (\phi, 0)$ and $\mathcal{G}_{y_m}(\phi, \psi) = \phi$. Then, by (3.17) it follows that

$$Q^q_{\mathrm{abs}, y_m}(\lambda) \quad = \quad \mathcal{G}_{y_m} \cdot \mathcal{R}^q_{y_m}(\lambda) \cdot \mathcal{F}_{y_m}, \tag{3.28}$$

which shows that the symbol of $Q^q_{\mathrm{abs}, y_m}(\lambda)$ is given by

$$\sigma\left(Q^q_{\mathrm{abs}, y_m}(\lambda)\right) \quad = \quad (\mathrm{I},\ \mathrm{O}) \left\{\sigma\left(\mathcal{R}^q_{y_m}(\lambda)\right)\right\} (\mathrm{I},\ \mathrm{O})^T, \tag{3.29}$$

where I is the $\binom{m-1}{q} \times \binom{m-1}{q}$ identity matrix and O is the $\binom{m-1}{q} \times \binom{m-1}{q-1}$ zero matrix.

We consider the boundary normal coordinate system $(y, y_m) = (y_1, \cdots, y_{m-1}, y_m)$ on a collar neighborhood of Y introduced at the beginning of this section. For $y_0 \in Y$, we denote $e_i := \partial_{x_i}(y_0)$ and $e^i := dx_i(y_0)$ for $1 \le i \le m$. Eq.(3.1) and (3.2) show that $\{e_1, \cdots, e_m\}$ and $\{e^1, \cdots, e^m\}$ at $y_0 \in Y$ satisfy the following relations.

$$\nabla^M_{e_\alpha} e^\beta = \omega_\alpha(e^\beta) = \kappa_\alpha \delta_{\alpha\beta} e^m, \qquad \nabla^M_{e_\alpha} e^m = -\kappa_\alpha e^\alpha, \qquad \nabla^M_{e_m} e^\alpha = \kappa_\alpha e^\alpha, \qquad \nabla^M_{e_m} e^m = 0, \tag{3.30}$$

The following result is straightforward (cf. Lemma 1.5.4 of [11]).

Lemma 3.2. *For $1 \le \alpha \le m-1$ and $1 \le i_1 < \cdots < i_q \le m-1$, the following equalities hold.*

$$\begin{aligned}
\omega_\alpha\left(e^{i_1} \wedge \cdots \wedge e^{i_q}\right) &= \kappa_\alpha\ e^m \wedge \left(\iota_{e_\alpha} e^{i_1} \wedge \cdots \wedge e^{i_q}\right), \\
\omega_\alpha\left(e^m \wedge e^{j_1} \wedge \cdots \wedge e^{j_{q-1}}\right) &= -\kappa_\alpha\ e^\alpha \wedge e^{j_1} \wedge \cdots \wedge e^{j_{q-1}}, \\
\omega_m\left(e^{i_1} \wedge \cdots \wedge e^{i_q}\right) &= \left(\kappa_{i_1} + \cdots + \kappa_{i_q}\right) e^{i_1} \wedge \cdots \wedge e^{i_q}, \\
\omega_m\left(e^m \wedge e^{j_1} \wedge \cdots \wedge e^{j_{q-1}}\right) &= \left(\kappa_{j_1} + \cdots + \kappa_{j_{q-1}}\right) e^m \wedge e^{j_1} \wedge \cdots \wedge e^{j_{q-1}}.
\end{aligned}$$

When $\dim M = 3$, Lemma 3.2 is reduced to the following result.

Corollary 3.3. *Let $\dim M = 3$. For $p = 1$ and an ordered basis $\{e^1, e^2, e^3\}$ of $T^*M\big|_U$, we can write ω_1, ω_2 and ω_m $(m = 3)$ by*

$$\omega_1 = \begin{pmatrix} 0 & 0 & -\kappa_1 \\ 0 & 0 & 0 \\ \kappa_1 & 0 & 0 \end{pmatrix}, \qquad \omega_1\,\omega_1 = -\kappa_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.31}$$
$$\omega_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}, \qquad \omega_2\,\omega_2 = -\kappa_2^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\omega_m = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \omega_m\,\omega_m = \begin{pmatrix} \kappa_1^2 & 0 & 0 \\ 0 & \kappa_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $p = 2$ and an ordered basis $\{e^1 \wedge e^2,\ e^3 \wedge e^1,\ e^3 \wedge e^2\}$ of $\wedge^2 T^*M\big|_U$, we can write ω_1, ω_2 and ω_m ($m = 3$) by

$$\omega_1 = \begin{pmatrix} 0 & 0 & -\kappa_1 \\ 0 & 0 & 0 \\ \kappa_1 & 0 & 0 \end{pmatrix}, \qquad \omega_1\,\omega_1 = -\kappa_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.32}$$
$$\omega_2 = \begin{pmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \omega_2\,\omega_2 = -\kappa_2^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\omega_m = \begin{pmatrix} \kappa_1+\kappa_2 & 0 & 0 \\ 0 & \kappa_1 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix}, \qquad \omega_m\,\omega_m = \begin{pmatrix} (\kappa_1+\kappa_2)^2 & 0 & 0 \\ 0 & \kappa_1^2 & 0 \\ 0 & 0 & \kappa_2^2 \end{pmatrix}.$$

We denote

$$\begin{aligned} &(\mathrm{I},\ \mathrm{O})\ E_q\ (\mathrm{I},\ \mathrm{O})^T = \widetilde{E}_q, \qquad (\mathrm{I},\ \mathrm{O})\ \omega_m(y,y_m)\ (\mathrm{I},\ \mathrm{O})^T = \widetilde{\omega}_m(y,y_m), \\ &(\mathrm{I},\ \mathrm{O})\ \omega_\alpha(y,y_m)\ (\mathrm{I},\ \mathrm{O})^T = \widetilde{\omega}_\alpha(y,y_m), \qquad (\mathrm{I},\ \mathrm{O})\ \omega_\alpha(y,y_m)\omega_\beta(y,y_m)\ (\mathrm{I},\ \mathrm{O})^T = \widetilde{\omega_\alpha\omega_\beta}(y,y_m). \end{aligned} \tag{3.33}$$

When $m = 3$, $\widetilde{E}_1$ and $\widetilde{E}_2$ are given by (3.7) and (3.8) as follows.

$$\widetilde{E}_1 = \begin{pmatrix} -\operatorname{Ric}^M_{11} & -\operatorname{Ric}^M_{12} \\ -\operatorname{Ric}^M_{12} & -\operatorname{Ric}^M_{22} \end{pmatrix}, \qquad \widetilde{E}_2 = \begin{pmatrix} -\operatorname{Ric}^M_{33} \end{pmatrix}. \tag{3.34}$$

Moreover, we use (2.33) to obtain the following equalities.

$$\begin{aligned} \operatorname{Tr} \widetilde{E}_1 &= -\tau_M + \operatorname{Ric}^M_{33} = -\frac{1}{2}\big(\tau_M + \tau_Y\big) + H_2, \\ \operatorname{Tr} \widetilde{E}_2 &= -\operatorname{Ric}^M_{33} = -\frac{1}{2}\big(\tau_M - \tau_Y\big) - H_2. \end{aligned} \tag{3.35}$$

We also denote

$$\begin{aligned}
\widetilde{p}_2(y, y_m, \xi, \lambda) &= \left(\sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m) \xi_\alpha \xi_\beta + \lambda \right) \widetilde{\mathrm{Id}} \;=\; \left(|\xi|^2 + \lambda \right) \widetilde{\mathrm{Id}}, \qquad (3.36) \\
\widetilde{p}_1(y, y_m, \xi) &= -i \sum_{\alpha,\beta=1}^{m-1} \left(\frac{1}{2} g^{\alpha\beta}(y, y_m) \partial_{y_\alpha} \ln |g|(y, y_m) + g^{\alpha\beta;\alpha}(y, y_m) \right) \xi_\beta \widetilde{\mathrm{Id}} - 2i \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta} \widetilde{\omega}_\alpha \xi_\beta, \\
\widetilde{p}_0(y, y_m, \xi) &= - \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta} \left(\partial_{y_\alpha} \widetilde{\omega}_\beta + \widetilde{\omega_\alpha \omega_\beta} - \sum_{\gamma=1}^{m-1} \Gamma_{\alpha\beta}^{\gamma} \widetilde{\omega}_\gamma \right) - \widetilde{E}_q, \\
\widetilde{A}(y, y_m) &= \left\{ -\frac{1}{2} \sum_{\alpha,\beta=1}^{m-1} g^{\alpha\beta}(y, y_m) \; g_{\alpha\beta;m}(y, y_m) \right\} \widetilde{\mathrm{Id}},
\end{aligned}$$

where $\widetilde{\mathrm{Id}}$ is the $\binom{m-1}{q} \times \binom{m-1}{q}$ identity matrix.

Remark : At $(y_0, 0) \in Y$, Lemma 3.2 (or Corollary 3.3) shows that $\widetilde{\omega}_\alpha(y_0, 0) = 0$, and hence $\widetilde{p}_1(y_0, 0) = 0$ by (3.1). Since $\widetilde{\omega_\alpha \omega_\alpha} \neq 0$ as shown in (3.31) and (3.32), it follows that

$$\left(\widetilde{p}_1(y_0, 0) \right)^2 = 0, \qquad (\mathrm{I},\ \mathrm{O}) \; p_1^2(y_0, 0, \xi) \; (\mathrm{I},\ \mathrm{O})^T = -4 \sum_{\alpha,\beta=1}^{m-1} \widetilde{\omega_\alpha \omega_\beta}(y_0, 0) \; \xi_\alpha \xi_\beta. \qquad (3.37)$$

Using Lemma 3.1 with (3.29), we can compute the homogeneous symbol of $Q^q_{\mathrm{abs}, y_m}(\lambda)$, whose first three terms are given as follows (cf. (1.7)-(1.9) in [20], (2.2)-(2.3) in [25] for $q = 0$).

Theorem 3.4. *In the boundary normal coordinate system given at the beginning of this section, we denote the homogeneous symbol of* $Q^q_{\mathrm{abs}, y_m}(\lambda)$ *by*

$$\sigma \left(Q^q_{\mathrm{abs}, y_m}(\lambda) \right) (y, y_m, \xi, \lambda) \;\sim\; \widetilde{\alpha}_1(y, y_m, \xi, \lambda) + \widetilde{\alpha}_0(y, y_m, \xi, \lambda) + \widetilde{\alpha}_{-1}(y, y_m, \xi, \lambda) + \cdots .$$

Then,

$$\begin{aligned}
\widetilde{\alpha}_1(y, y_m, \xi, \lambda) &= (\mathrm{I},\ \mathrm{O}) \; \alpha_1 \; (\mathrm{I},\ \mathrm{O})^T \;=\; \sqrt{|\xi|^2 + \lambda} \; \widetilde{\mathrm{Id}}, \\
\widetilde{\alpha}_0(y, y_m, \xi, \lambda) &= (\mathrm{I},\ \mathrm{O}) \; \alpha_0 \; (\mathrm{I},\ \mathrm{O})^T \\
&= \frac{1}{2\sqrt{|\xi|^2 + \lambda}} \left\{ -\partial_\xi \widetilde{\alpha}_1 \cdot D_y \widetilde{\alpha}_1 + \widetilde{p}_1 - \left(\widetilde{A}(y, y_m) - \; 2\widetilde{\omega}_m \right) \widetilde{\alpha}_1 + \partial_{y_m} \widetilde{\alpha}_1 \right\}, \\
\widetilde{\alpha}_{-1}(y, y_m, \xi, \lambda) &= (\mathrm{I},\ \mathrm{O}) \; \alpha_{-1} \; (\mathrm{I},\ \mathrm{O})^T \\
&= \frac{1}{2\sqrt{|\xi|^2 + \lambda}} \Bigg\{ \sum_{|\omega|=2} \frac{1}{\omega!} (\partial_\xi^\omega \widetilde{\alpha}_1)(\partial_y^\omega \widetilde{\alpha}_1) + i(\partial_\xi \widetilde{\alpha}_0)(\partial_y \widetilde{\alpha}_1) + i(\partial_\xi \widetilde{\alpha}_1)(\partial_y \widetilde{\alpha}_0) - (\mathrm{I},\ \mathrm{O}) \; \alpha_0^2 \; (\mathrm{I},\ \mathrm{O})^T \\
&\quad + \widetilde{p}_0 - \left(\widetilde{A}(y, y_m) - 2\widetilde{\omega}_m \right) \widetilde{\alpha}_0 + \partial_{y_m} \widetilde{\alpha}_0 - \left(\partial_{y_m} \widetilde{\omega_m} + \widetilde{\omega_m} \widetilde{\omega_m} - \widetilde{A}(y, y_m) \widetilde{\omega}_m \right) \Bigg\},
\end{aligned}$$

where at $(x, 0) \in Y$,

$$(\mathrm{I},\ \mathrm{O}) \; \alpha_0^2(y, 0) \; (\mathrm{I},\ \mathrm{O})^T \;=\; \left(\widetilde{\alpha}_0(y, 0) \right)^2 \;-\; \frac{1}{|\xi|^2 + \lambda} \sum_{\alpha=1}^{m-1} \; \widetilde{\omega_\alpha \omega_\beta}(y, 0) \; \xi_\alpha \xi_\beta.$$

We next denote the homogeneous symbol of the resolvent $(\mu - Q^q_{\text{abs}}(\lambda))^{-1}$ by

$$\sigma\left((\mu - Q^q_{\text{abs}}(\lambda))^{-1}\right)(y,\xi,\lambda,\mu) \quad \sim \quad \widetilde{r}_{-1}(y,\xi,\lambda,\mu) + \widetilde{r}_{-2}(y,\xi,\lambda,\mu) + \widetilde{r}_{-3}(y,\xi,\lambda,\mu) + \cdots. \tag{3.38}$$

Then,

$$\begin{aligned} \widetilde{r}_{-1}(y,\xi,\lambda,\mu) &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \widetilde{\mathrm{Id}}, \\ \widetilde{r}_{-1-j}(y,\xi,\lambda,\mu) &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \sum_{k=0}^{j-1} \sum_{|\omega|+l+k=j} \frac{1}{\omega!} \partial_\xi^\omega \widetilde{\alpha}_{1-l} D_y^\omega \widetilde{r}_{-1-k}, \end{aligned} \tag{3.39}$$

which shows that the first three terms are given as follows.

$$\begin{aligned} \widetilde{r}_{-1} &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \widetilde{\mathrm{Id}}, \\ \widetilde{r}_{-2} &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \left\{\partial_\xi \widetilde{\alpha}_1 \cdot D_y \widetilde{r}_{-1} + \widetilde{\alpha}_0 \cdot \widetilde{r}_{-1}\right\}, \\ \widetilde{r}_{-3} &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \Big\{ \sum_{|\omega|=2} \frac{1}{\omega!} \partial_\xi^\omega \widetilde{\alpha}_1 \cdot D_y^\omega \widetilde{r}_{-1} + \partial_\xi \widetilde{\alpha}_1 \cdot D_y \widetilde{r}_{-2} + \partial_\xi \widetilde{\alpha}_0 \cdot D_y \widetilde{r}_{-1} + \widetilde{\alpha}_0 \cdot \widetilde{r}_{-2} + \widetilde{\alpha}_{-1} \cdot \widetilde{r}_{-1} \Big\}. \end{aligned} \tag{3.40}$$

4. The constant term a_0 in 2 and 3 dimensional manifolds

In this section we are going to compute a_0 in Theorem 2.4 in terms of curvature tensors on Y when $\dim Y = 1$ and 2. By Lemma 2.3 with (2.10), $a_0(y)$ is expressed by

$$a_0(y) \quad = \quad \frac{\partial}{\partial s}\bigg|_{s=0} \left(\frac{1}{(2\pi)^{m-1}} \int_{T_y^*Y} \frac{1}{2\pi i} \int_\gamma \mu^{-s} \operatorname{Tr} \widetilde{r}_{-m}\left(y, \xi, \frac{\lambda}{|\lambda|}, \mu\right) d\mu d\xi \right). \tag{4.1}$$

Let ∇^Y be the Levi-Civita connection on Y associated to the induced metric from g. We denote by $R_{\alpha\beta\gamma\delta}$ and $\mathrm{Ric}_{\alpha\beta}$ the Riemann curvature tensor and Ricci tensor on Y associated to ∇^Y defined by

$$R_{\alpha\beta\gamma\delta} \quad = \quad \left\langle \nabla^Y_{\partial_{x_\alpha}} \nabla^Y_{\partial_{x_\beta}} \partial_{x_\gamma} - \nabla^Y_{\partial_{x_\beta}} \nabla^Y_{\partial_{x_\alpha}} \partial_{x_\gamma} - \nabla^Y_{[\partial_{x_\alpha}, \partial_{x_\beta}]} \partial_{x_\gamma},\ \partial_{x_\delta} \right\rangle_Y, \quad \mathrm{Ric}_{\alpha\beta} = \sum_{\gamma=1}^{m-1} R_{\alpha\gamma\gamma\beta}. \tag{4.2}$$

The following lemma is shown in [25] and [30].

Lemma 4.1. *We consider the boundary normal coordinate system on an open neighborhood U_{ϵ_0} of $y_0 \in Y$ with metric tensor $g = (g_{ij})$ and $y_0 = (0, \cdots, 0)$. Then, we have the following equalities:*

(1) $g^{\alpha\beta;\alpha\beta}(y_0) = -\dfrac{1}{3} R_{\alpha\beta\beta\alpha}(y_0), \qquad g^{\alpha\alpha;\beta\beta}(y_0) = \dfrac{2}{3} R_{\alpha\beta\beta\alpha}(y_0),$

(2) $\partial_{y_\alpha}\partial_{y_\alpha} \ln |g|(y_0) = -\dfrac{2}{3} \operatorname{Ric}_{\alpha\alpha}(y_0), \qquad \tau_Y(y_0) = \displaystyle\sum_{\alpha,\beta=1}^{m-1} R_{\alpha\beta\beta\alpha}(y_0) = -\sum_{\alpha,\gamma=1}^{m-1} R_{\alpha\beta\alpha\beta}(y_0),$

(3) $g^{\alpha\beta;m}(y_0) = 2\kappa_\alpha \delta_{\alpha\beta} = -g_{\alpha\beta;m}(y_0), \quad \displaystyle\int_{\mathbb{R}^{m-1}} |\xi|^k \sum_{\alpha,\beta,\gamma,\epsilon=1}^{m-1} g^{\alpha\beta;\gamma\epsilon} \xi_\alpha \xi_\beta \xi_\gamma \xi_\epsilon d\xi \;=\; 0$ *for* $k < -3-m,$

(4) $\displaystyle\sum_{\alpha=1}^{m-1} g^{\alpha\alpha;mm}(y_0) = 8 \sum_{\alpha=1}^{m-1} \kappa_\alpha^2(y_0) - \sum_{\alpha=1}^{m-1} g_{\alpha\alpha;mm}(y_0),$

(5) $\displaystyle\sum_{\alpha=1}^{m-1} g_{\alpha\alpha;mm}(y_0) = -\left\{ \tau_M(y_0) - \tau_Y(y_0) - 2(m-1)^2 H_1^2(y_0) + 3(m-1)(m-2) H_2(y_0) \right\},$

where $\tau_M(y_0)$ and $\tau_Y(y_0)$ are scalar curvatures of M and Y at $y_0 \in Y$, respectively, and H_1 and H_2 are defined in (2.32).

The following lemma is straightforward.

Lemma 4.2. *Let $\mathbb{C}^+ = \mathbb{C} - \{r \in \mathbb{R} \mid r \leq 0\}$. For $z \in \mathbb{C}^+$ let γ be a counterclockwise contour in $\mathbb{C}^+$ with z inside γ. Then for $\operatorname{Re} s > 2$ the following integrals are all well defined and one computes:*

$$\frac{1}{2\pi i}\int_\gamma \frac{\mu^{-s}}{\mu - z} d\mu = z^{-s}, \quad \frac{1}{2\pi i}\int_\gamma \frac{\mu^{-s}}{(\mu - z)^2} d\mu = -sz^{-s-1}, \quad \frac{1}{2\pi i}\int_\gamma \frac{\mu^{-s}}{(\mu - z)^3} d\mu = \frac{1}{2}s(s+1)z^{-s-2},$$
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} (|\xi|^2+1)^{-\frac{s}{2}} d\xi \;=\; \frac{1}{2\pi}\frac{1}{s-2}, \qquad \frac{1}{4\pi^2}\int_{\mathbb{R}^2} (|\xi|^2+1)^{-\frac{s}{2}-1} d\xi \;=\; \frac{1}{2\pi}\frac{1}{s},$$
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \xi_1^2 (|\xi|^2+1)^{-\frac{s}{2}-2} d\xi \;=\; \frac{1}{2\pi}\frac{1}{s(s+2)},$$
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \xi_1^2\xi_2^2 (|\xi|^2+1)^{-\frac{s}{2}-3} d\xi \;=\; \frac{1}{2\pi}\frac{1}{s(s+2)(s+4)},$$
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \xi_1^4 (|\xi|^2+1)^{-\frac{s}{2}-3} d\xi \;=\; \frac{3}{2\pi}\frac{1}{s(s+2)(s+4)}.$$

Now we proceed the computation as in [17]. For two integrable functions $f(\xi)$ and $g(\xi)$ on $\mathbb{R}^{m-1}$, we define an equivalence relation " $\approx$ " as follows:

$$f \quad \approx \quad g \qquad \text{if and only if} \qquad \int_{\mathbb{R}^{m-1}} f(\xi)\, d\xi \;=\; \int_{\mathbb{R}^{m-1}} g(\xi)\, d\xi. \tag{4.3}$$

We first suppose that Y is a 1-dimensional manifold, *i.e.* $m = 2$. Using (3.1), we have, at $(y, 0) \in Y$,

$$\begin{aligned}
\widetilde{r}_{-2} &= \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-1} \left\{ \partial_\xi \widetilde{\alpha}_1 \cdot D_y \widetilde{r}_{-1} + \widetilde{\alpha}_0 \cdot \widetilde{r}_{-1} \right\} \approx \left(\mu - \sqrt{|\xi|^2+\lambda}\right)^{-2} \cdot \widetilde{\alpha}_0 \qquad (4.4) \\
&= \frac{1}{(\mu - \sqrt{|\xi|^2+\lambda})^2} \left\{ \frac{-\partial_\xi \widetilde{\alpha}_1 \cdot D_y \widetilde{\alpha}_1 + \widetilde{p}_1}{2\sqrt{|\xi|^2+\lambda}} + (\widetilde{\omega}_m - \frac{1}{2}\widetilde{A}(y,0)) + \frac{\partial_{y_m}\widetilde{\alpha}_1}{2\sqrt{|\xi|^2+\lambda}} \right\} \\
&\approx \frac{1}{(\mu - \sqrt{|\xi|^2+\lambda})^2} \left\{ (\widetilde{\omega}_m - \frac{1}{2}\widetilde{A}(y,0) + \frac{\partial_{y_m}\widetilde{\alpha}_1}{2\sqrt{|\xi|^2+\lambda}} \right\} \\
&\approx \frac{1}{(\mu - \sqrt{|\xi|^2+\lambda})^2} \left\{ \widetilde{\omega}_m - \frac{\kappa}{2} \cdot \frac{\lambda}{|\xi|^2+\lambda} \, \widetilde{\mathrm{Id}} \right\},
\end{aligned}$$

where $\kappa(y)$ is the principal curvature on $y \in Y$. Hence,

$$\begin{aligned}
&\frac{1}{2\pi} \int_{T_y^* Y} \frac{1}{2\pi i} \int_\gamma \mu^{-s} \operatorname{Tr} \widetilde{r}_{-2} \left(y, \xi, \frac{\lambda}{|\lambda|}, \mu \right) d\mu d\xi \qquad (4.5) \\
= \; &\frac{1}{2\pi} \int_{-\infty}^{\infty} (-s) \operatorname{Tr} \widetilde{\omega}_m \frac{1}{\sqrt{\xi^2+1}^{s+1}} d\xi + s \cdot \frac{\kappa}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\xi^2+1}^{s+3}} d\xi \\
= \; &-s \cdot \operatorname{Tr} \widetilde{\omega}_m \cdot \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\xi^2+1}^{s+1}} d\xi \; + \; s \cdot \left(\frac{\kappa}{2\pi} + O(s) \right).
\end{aligned}$$

Setting $\xi^2 = t$ and using the identity $\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ [2, 21], we obtain

$$\frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{\xi^2+1}^{s+1}} d\xi \;=\; \frac{1}{2\pi} \int_0^\infty \frac{t^{-\frac{1}{2}}}{(t+1)^{\frac{s+1}{2}}} dt \;=\; \frac{1}{s \cdot \pi} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s+1}{2}\right)},$$

which leads to

$$\begin{aligned}
&\frac{1}{2\pi} \int_{T_y^* Y} \frac{1}{2\pi i} \int_\gamma \mu^{-s} \operatorname{Tr} \widetilde{r}_{-2} \left(y, \xi, \frac{\lambda}{|\lambda|}, \mu \right) d\mu d\xi \qquad (4.6) \\
= \; &-\frac{1}{\pi} \cdot \operatorname{Tr} \widetilde{\omega}_m \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s+1}{2}\right)} \; + \; s \cdot \left(\frac{\kappa}{2\pi} + O(s) \right).
\end{aligned}$$

Taking the derivative with respect to s gives

$$a_0(y) \;=\; -\frac{1}{\pi} \operatorname{Tr}(\widetilde{\omega}_m) \ln 2 + \frac{\kappa(y)}{2\pi}. \qquad (4.7)$$

Lemma 3.2 shows that if $q = 0$ then $\widetilde{\omega}_m = 0$ and if $q = 1$ then $\widetilde{\omega}_m = \kappa(y)$. This leads to the following result.

Theorem 4.3. *When* $\dim Y = 1$*, the constant* a_0 *in Theorem 2.4 is given as follow.*

$$a_0 \;=\; \begin{cases} \frac{1}{2\pi} \int_Y \kappa(y) \; dy & \text{for} \quad q = 0 \\ \frac{1}{2\pi}(1 - 2\ln 2) \int_Y \kappa(y) \; dy & \text{for} \quad q = 1. \end{cases}$$

Remark : It follows from (2.10) that

$$q_1 \quad = \quad \begin{cases} 0 & \text{for} \quad q = 0 \\ -\frac{1}{2\pi}\int_Y \kappa(y)\; dy & \text{for} \quad q = 1, \end{cases} \tag{4.8}$$

which agrees with $\mathfrak{a}_2 - \mathfrak{b}_2$ in Corollary 2.9 and Corollary 2.10.

We next consider the case that Y is a 2-dimensional compact Riemannian manifold, *i.e.* $m = 3$. We refer to [17] for details. Before computing $a_0(y)$, we first consider $a_1(y)$, which is by Lemma 2.3

$$a_1(y) \quad = \quad (\mathfrak{a}_1(y) - \mathfrak{b}_1(y)) - \pi_0(y). \tag{4.9}$$

Simple computation shows that for $\mathfrak{r}_0 = \binom{2}{q}$,

$$\begin{aligned} \pi_0(y) \quad &= \quad -\frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\,\mathrm{Tr}\,\widetilde{r}_{-1}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right)\; d\mu d\xi\right) \\ &= \quad -\mathfrak{r}_0\,\frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \frac{\mu^{-s}}{\mu - \sqrt{|\xi|^2+1}}\, d\mu d\xi\right) \\ &= \quad \frac{\mathfrak{r}_0}{8\pi}. \end{aligned} \tag{4.10}$$

It is well known (for example, Theorem 3.4.1 and Theorem 3.6.1 in [11] or Section 4.2 and 4.5 in [15]) that

$$\mathfrak{a}_1(y) - \mathfrak{b}_1(y) \quad = \quad \frac{\mathfrak{r}_0}{8\pi}, \tag{4.11}$$

which yields the following result.

Lemma 4.4. *When* $\dim Y = 2$*, the constant* a_1 *in Lemma 2.2 is zero.*

We next compute $a_0(x)$. We recall that

$$\begin{aligned} a_0(y) \quad &= \quad \frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\,\mathrm{Tr}\,\widetilde{r}_{-3}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right) d\mu d\xi\right) \\ &= \quad \frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\,\mathrm{Tr}\left\{(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V})\right\} d\mu d\xi\right), \end{aligned} \tag{4.12}$$

where

$$(\mathrm{I}) \;=\; \frac{\sum_{|\omega|=2}\frac{1}{\omega!}\partial_\xi^\omega\widetilde{\alpha}_1\cdot D_y^\omega\widetilde{r}_{-1}}{\mu - \sqrt{|\xi|^2+\lambda}}, \qquad (\mathrm{II}) \;=\; \frac{\partial_\xi\widetilde{\alpha}_1\cdot D_y\widetilde{r}_{-2}}{\mu - \sqrt{|\xi|^2+\lambda}}, \qquad (\mathrm{III}) \;=\; \frac{\partial_\xi\widetilde{\alpha}_0\cdot D_y\widetilde{r}_{-1}}{\mu - \sqrt{|\xi|^2+\lambda}},$$

$$(\mathrm{IV}) \;=\; \frac{\widetilde{\alpha}_0\cdot\widetilde{r}_{-2}}{\mu - \sqrt{|\xi|^2+\lambda}}, \qquad (\mathrm{V}) \;=\; \frac{\widetilde{\alpha}_{-1}\cdot\widetilde{r}_{-1}}{\mu - \sqrt{|\xi|^2+\lambda}}.$$

Moreover, we denote

$$(\mathrm{V}) \;=\; \frac{\widetilde{\alpha}_{-1}\cdot\widetilde{r}_{-1}}{\mu-\sqrt{|\xi|^2+\lambda}} \;=\; \frac{\widetilde{\alpha}_{-1}}{(\mu-\sqrt{|\xi|^2+\lambda})^2} \;=\; (\mathrm{V}_1)+(\mathrm{V}_2)+(\mathrm{V}_3)+(\mathrm{V}_4)+(\mathrm{V}_5)+(\mathrm{V}_6)+(\mathrm{V}_7)+(\mathrm{V}_8),$$

where

$$\begin{aligned}
(\mathrm{V}_1) &= \frac{\sum_{|\omega|=2}\frac{1}{\omega!}\partial_\xi^\omega\widetilde{\alpha}_1\cdot\partial_y^\omega\widetilde{\alpha}_1}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}}, &\quad (\mathrm{V}_2) &= \frac{i(\partial_\xi\widetilde{\alpha}_0)\cdot(\partial_y\widetilde{\alpha}_1)}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}},\\
(\mathrm{V}_3) &= \frac{i(\partial_\xi\widetilde{\alpha}_1)\cdot(\partial_y\widetilde{\alpha}_0)}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}}, &\quad (\mathrm{V}_4) &= \frac{-(\mathrm{I},\ \mathrm{O})\ \alpha_0^2\ (\mathrm{I},\ \mathrm{O})^T}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}},\\
(\mathrm{V}_5) &= \frac{\widetilde{p}_0}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}}, &\quad (\mathrm{V}_6) &= \frac{(2\widetilde{\omega}_m-\widetilde{A}(y,0))\widetilde{\alpha}_0}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}},\\
(\mathrm{V}_7) &= \frac{\partial_{y_m}\widetilde{\alpha}_0}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}}, &\quad (\mathrm{V}_8) &= \frac{-(\partial_{y_m}\widetilde{\omega}_m+\widetilde{\omega}_m\widetilde{\omega}_m-\widetilde{A}(y,0)\widetilde{\omega}_m)}{2(\mu-\sqrt{|\xi|^2+\lambda})^2\sqrt{|\xi|^2+\lambda}}.
\end{aligned}$$

Direct and tedious computations show the followings (cf. [17]). Here, as before, we denote $\mathfrak{r}_0 = \binom{2}{q}$ so that $\mathfrak{r}_0 = 1$ for $q = 0,\ 2$ and $\mathfrak{r}_0 = 2$ for $q = 1$.

$$\begin{aligned}
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{I})d\mu d\xi &= -\mathfrak{r}_0\cdot\frac{\tau_Y}{24\pi}\cdot\frac{s+1}{s+2},\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{II})d\mu d\xi &= \mathfrak{r}_0\cdot\frac{\tau_Y}{12\pi}\cdot\frac{s+1}{s+2} \;-\; \frac{1}{4\pi}\,\mathrm{Tr}\left(\partial_{y_\alpha}\widetilde{\omega}_\alpha\right)\cdot\frac{s+1}{s+2},\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{III})d\mu d\xi &= 0,\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{IV})d\mu d\xi &= \mathfrak{r}_0\cdot\frac{H_1^2}{4\pi}\cdot\frac{(s+1)^2(s+3)}{(s+2)(s+4)} \;-\; \mathfrak{r}_0\cdot\frac{H_2}{4\pi}\cdot\frac{s+1}{(s+2)(s+4)}\\
&\quad -\frac{H_1}{2\pi}\,\mathrm{Tr}(\widetilde{\omega}_m)\frac{(s+1)^2}{s+2}+\frac{1}{4\pi}\,\mathrm{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m)\cdot(s+1),\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_1)d\mu d\xi &= -\mathfrak{r}_0\cdot\frac{\tau_Y}{24\pi}\cdot\frac{1}{s+2},\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_2)d\mu d\xi &= 0,\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_3)d\mu d\xi &= \mathfrak{r}_0\cdot\frac{\tau_Y}{12\pi}\cdot\frac{1}{s+2} \;-\; \frac{1}{4\pi}\,\mathrm{Tr}\left(\partial_{y_\alpha}\widetilde{\omega}_\alpha\right)\cdot\frac{1}{s+2},\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_4)d\mu d\xi &= \mathfrak{r}_0\cdot\frac{H_1^2}{4\pi}\cdot\frac{(s+1)(s+3)}{(s+2)(s+4)}-\mathfrak{r}_0\cdot\frac{H_2}{4\pi}\cdot\frac{1}{(s+2)(s+4)}\\
&\quad -\frac{1}{4\pi}\,\mathrm{Tr}(\widetilde{\omega_\alpha\omega_\alpha})\cdot\frac{1}{s+2}-\frac{H_1}{2\pi}\,\mathrm{Tr}(\widetilde{\omega}_m)\cdot\frac{s+1}{s+2}+\frac{1}{4\pi}\,\mathrm{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m),\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_5)d\mu d\xi &= \frac{1}{4\pi}\,\mathrm{Tr}\left(\partial_{y_\alpha}\widetilde{\omega}_\alpha\right) \;+\; \frac{1}{4\pi}\,\mathrm{Tr}\left(\widetilde{\omega_\alpha\omega_\alpha}\right)+\frac{1}{4\pi}\,\mathrm{Tr}\left(\widetilde{E}_q\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_6)d\mu d\xi &= -\mathfrak{r}_0\cdot\frac{H_1^2}{2\pi}\cdot\frac{s+1}{s+2}+\frac{H_1}{2\pi}\operatorname{Tr}(\widetilde{\omega}_m)\cdot\frac{2s+3}{s+2}-\frac{1}{2\pi}\operatorname{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m),\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_7)d\mu d\xi &= \mathfrak{r}_0\cdot\frac{1}{16\pi}(\tau_M-\tau_Y)\cdot\frac{s+1}{s+2}+\mathfrak{r}_0\cdot\frac{H_1^2}{2\pi}\cdot\frac{s+1}{s+4}\\
&\quad-\mathfrak{r}_0\cdot\frac{H_2}{8\pi}\cdot\frac{s^2+s-4}{(s+2)(s+4)}-\frac{1}{4\pi}\operatorname{Tr}(\partial_{y_m}\widetilde{\omega}_m),\\
\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}(\mathrm{V}_8)d\mu d\xi &= -\frac{H_1}{2\pi}\operatorname{Tr}(\widetilde{\omega}_m)+\frac{1}{4\pi}\operatorname{Tr}(\partial_{y_m}\widetilde{\omega}_m)+\frac{1}{4\pi}\operatorname{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m).
\end{aligned}
$$

Adding up the above terms, we obtain

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{T_y^*Y}\frac{1}{2\pi i}\int_\gamma \mu^{-s}\operatorname{Tr}\widetilde{r}_{-3}\left(y,\xi,\frac{\lambda}{|\lambda|},\mu\right)d\mu d\xi \qquad (4.13)\\
=\;& \mathfrak{r}_0\cdot\left\{\frac{\tau_M}{16\pi}\cdot\frac{s+1}{s+2}-\frac{\tau_Y}{48\pi}\cdot\frac{s-1}{s+2}+\frac{H_1^2}{4\pi}\cdot\frac{s^3+6s^2+7s+2}{(s+2)(s+4)}-\frac{H_2}{8\pi}\cdot\frac{s(s+3)}{(s+2)(s+4)}\right\}+\frac{1}{4\pi}\operatorname{Tr}(\widetilde{E}_q)\\
&+\frac{1}{4\pi}\operatorname{Tr}(\widetilde{\omega_\alpha\omega_\alpha})\cdot\frac{s+1}{s+2}-\frac{H_1}{2\pi}\operatorname{Tr}(\widetilde{\omega}_m)\cdot\frac{s^2+2s+1}{s+2}+\frac{1}{4\pi}\operatorname{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m)\cdot(s+1),
\end{aligned}
$$

which shows that

$$
\begin{aligned}
a_0(y) &= \mathfrak{r}_0\cdot\left(\frac{\tau_M}{64\pi}-\frac{\tau_Y}{64\pi}+\frac{11}{64\pi}H_1^2-\frac{3}{64\pi}H_2\right) \qquad (4.14)\\
&\quad+\frac{1}{16\pi}\sum_{\alpha=1}^{2}\operatorname{Tr}(\widetilde{\omega_\alpha\omega_\alpha})-\frac{3H_1}{8\pi}\operatorname{Tr}(\widetilde{\omega}_m)+\frac{1}{4\pi}\operatorname{Tr}(\widetilde{\omega}_m\widetilde{\omega}_m).
\end{aligned}
$$

If $p=0$, then $\mathfrak{r}_0=1$ and $\widetilde{\omega}_\alpha=\widetilde{\omega}_m=0$. Eq.(3.31) and (3.32) show that if $p=1$, then $\mathfrak{r}_0=2$ and

$$
\widetilde{\omega}_m = \begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix},\quad \widetilde{\omega}_m\widetilde{\omega}_m = \begin{pmatrix}\kappa_1^2 & 0\\ 0 & \kappa_2^2\end{pmatrix},\quad \widetilde{\omega_1\omega_1} = -\kappa_1^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \widetilde{\omega_2\omega_2} = -\kappa_2^2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.
$$

If $p=2$, then $\mathfrak{r}_0=1$ and

$$
\widetilde{\omega}_m = \kappa_1+\kappa_2=2H_1,\quad \widetilde{\omega}_m\widetilde{\omega}_m = (\kappa_1+\kappa_2)^2=4H_1^2,\quad \widetilde{\omega_1\omega_1} = -\kappa_1^2,\quad \widetilde{\omega_2\omega_2} = -\kappa_2^2.
$$

These facts lead to the following result.

Theorem 4.5. *When* $\dim Y=2$*, the constant* a_0 *and* a_1 *in Theorem 2.4 and Lemma 2.2 are given as follows.*

$$
\begin{aligned}
a_1 &= 0,\\
a_0 &= \begin{cases}\frac{1}{64\pi}\int_Y\left(\tau_M-\tau_Y+11H_1^2-3H_2\right)dy & \text{for}\quad q=0\\ \frac{1}{32\pi}\int_Y\left(\tau_M-\tau_Y+11H_1^2-15H_2\right)dy & \text{for}\quad q=1\\ \frac{1}{64\pi}\int_Y\left(\tau_M-\tau_Y+11H_1^2+5H_2\right)dy & \text{for}\quad q=2.\end{cases}
\end{aligned}
$$

Remark : When $\dim Y = 2$, we get from (2.10), (3.35), (4.13) and Lemma 2.3

$$\begin{aligned} q_2 &= \int_Y q_2(y)\, dy \;=\; \frac{1}{2}\int_Y \Big\{ \mathfrak{r}_0 \cdot \left(\frac{\tau_M}{32\pi} + \frac{\tau_Y}{96\pi} + \frac{H_1^2}{16\pi} \right) + \frac{1}{4\pi}\operatorname{Tr}(\widetilde{E}_q) \\ &\quad + \frac{1}{8\pi}\sum_{\alpha=1}^{2} \operatorname{Tr}(\widetilde{\omega_\alpha \omega_\alpha}) - \frac{H_1}{4\pi}\operatorname{Tr}(\widetilde{\omega}_m) + \frac{1}{4\pi}\operatorname{Tr}(\widetilde{\omega}_m \widetilde{\omega}_m) \Big\}\, dy \\ &= \begin{cases} \frac{1}{8\pi}\int_Y \left(\frac{1}{8}\tau_M + \frac{1}{24}\tau_Y + \frac{1}{4}H_1^2 \right) dy & \text{for} \quad q = 0 \\ \frac{1}{8\pi}\int_Y \left(-\frac{1}{4}\tau_M - \frac{5}{12}\tau_Y + \frac{1}{2}H_1^2 \right) dy & \text{for} \quad q = 1 \\ \frac{1}{8\pi}\int_Y \left(-\frac{3}{8}\tau_M + \frac{13}{24}\tau_Y + \frac{1}{4}H_1^2 \right) dy & \text{for} \quad q = 2 \end{cases} \\ &= \mathfrak{b}_3 - \mathfrak{c}_3 \;=\; \frac{1}{2}\left(\zeta_{Q^q_{\mathrm{abs}}(0)}(0) + \ell_q \right), \end{aligned} \tag{4.15}$$

which agrees with Corollary 2.10.

As an application of Corollary 2.5 and Theorem 4.3, we recover Theorem 1.1 in [13]. For this purpose let M be a 2-dimensional compact Riemann manifold with boundary Y. We consider a Laplacian Δ^0_M acting on smooth functions and the conformal variation of Corollary 2.5 as follows. For a smooth function $F : M \to \mathbb{R}$, we denote $g_{ij}(\epsilon) = e^{2\epsilon F} g_{ij}$. We also denote by $\ln \operatorname{Det} \Delta^0_{M,\mathrm{abs}}(\epsilon)$, $\ln \operatorname{Det} \Delta^0_{M,\mathrm{D}}(\epsilon)$, $\kappa(\epsilon)$, $dy(\epsilon)$, and $\ln \operatorname{Det} Q^0_{\mathrm{abs}}(0)(\epsilon)$ the corresponding objects with respect to the metric $g_{ij}(\epsilon)$, where $\Delta^0_M(\epsilon) = e^{-2\epsilon F}\Delta^0_M$ and $Q^0_{\mathrm{abs}}(0)(\epsilon) = e^{-\epsilon F} Q^0_{\mathrm{abs}}(0)$. Then, Corollary 2.5 and Theorem 4.3 for $g_{ij}(\epsilon)$ can be rewritten by

$$\begin{aligned} \ln \frac{\operatorname{Det}^* Q^0_{\mathrm{abs}}(0)(\epsilon)}{\ell(Y)(\epsilon)} &= -\frac{1}{2\pi}\int_Y \kappa(\epsilon) dx(\epsilon) - \ln V(M)(\epsilon) + \ln \operatorname{Det}^* \Delta^0_{M,\mathrm{abs}}(\epsilon) - \ln \operatorname{Det} \Delta^0_{M,\mathrm{D}}(\epsilon) \\ &= -\frac{1}{2\pi}\int_Y \kappa(\epsilon) e^{\epsilon F}\, dy - \ln \int_M e^{2\epsilon F}\, dx + \ln \operatorname{Det}^* \Delta^0_{M,\mathrm{abs}}(\epsilon) - \ln \operatorname{Det} \Delta^0_{M,\mathrm{D}}(\epsilon), \end{aligned} \tag{4.16}$$

where $dx = d\,\mathrm{vol}(M)$. For $t \to 0^+$, we put

$$\operatorname{Tr}\left(F e^{-t\Delta^0_{M,\mathrm{abs}}} \right) \sim \sum_{j=0}^{\infty} \mathfrak{a}_j(F) t^{-\frac{2-j}{2}}, \qquad \operatorname{Tr}\left(F e^{-t\Delta^0_{M,\mathrm{D}}} \right) \sim \sum_{j=0}^{\infty} \mathfrak{b}_j(F) t^{-\frac{2-j}{2}}. \tag{4.17}$$

It is well known that [4, 5]

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \ln \operatorname{Det}^* \Delta^0_{M,\mathrm{abs}}(\epsilon) &= -2\left(\mathfrak{a}_2(F) - \frac{1}{\mathrm{vol}(M)}\int_M F(x)\, dx \right), \\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} \ln \operatorname{Det}^* \Delta^0_{M,\mathrm{D}}(\epsilon) &= -2\mathfrak{b}_2(F), \qquad \frac{d}{d\epsilon}\Big|_{\epsilon=0} \kappa(\epsilon) = -F\kappa - F_{;2}, \end{aligned} \tag{4.18}$$

where $F_{;2}$ is the derivative of F with respect to the inward unit normal vector field. Moreover, it is also well known that [11, 15]

$$\mathfrak{a}_2(F) - \mathfrak{b}_2(F) = \frac{1}{4\pi}\int_Y F_{;2}\, dy. \tag{4.19}$$

Consideration of all these facts shows that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \ln \frac{\mathrm{Det}^* Q(0)(\epsilon)}{\ell(Y)(\epsilon)} \quad = \quad 0, \tag{4.20}$$

which shows that $\frac{\mathrm{Det}^* Q^0_{\mathrm{abs}}(0)}{\ell(Y)}$ is a conformal invariant, which is proved earlier in [13] (see also [8]).

Remark : A similar computation shows that $\ln \mathrm{Det}^* Q^1_{\mathrm{abs}}(0)$ depends on the conformal change of a metric, where $Q^1_{\mathrm{abs}}(0)$ is the Dirichlet-to-Neumann operator acting on 1-forms on Y with $\dim Y = 1$.

Declarations

Ethics approval and consent to participate. Not applicable.

Consent for publication. Not applicable.

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