GLOBAL CHERN CURRENTS OF COHERENT SHEAVES AND BAUM BOTT CURRENTS

LUCAS KAUFMANN & RICHARD LÄRKÄNG & ELIZABETH WULCAN

ABSTRACT. We provide global extensions of previous results about representations of characteristic classes of coherent analytic sheaves and of Baum-Bott residues of holomorphic foliations. We show in the first case that they can be represented by currents with support on the support of the given coherent analytic sheaf, and in the second case, by currents with support on the singular set of the foliation. In previous works, we have constructed such representatives provided global resolutions of the appropriate sheaves existed. In this article, we show that the definition of Chern classes of Green and the associated techniques, which work on arbitrary complex manifolds without any assumption on the existence of global resolutions, may be combined with our previous constructions to yield the desired representatives.

We also prove a transgression formula for such representatives, which is new even in the case when global resolutions exist. More precisely, the representatives depend on local resolutions of the sheaf, and on choices of metrics and connections on these bundles, i.e., the currents for two different choices differ by a current of the form dN, where N is an explicit current, which in the first case above has support on the support of the given coherent analytic sheaf, and in the second case above has support on the singular set of the foliation.

1. INTRODUCTION

Let $\mathscr{G}$ be a coherent analytic sheaf on a complex manifold M of dimension n. In case M has the so-called resolution property, i.e., if any coherent analytic sheaf on M admits a resolution by a complex (E,φ) of holomorphic vector bundles of finite length, say N, then there is a well-defined notion of Chern forms associated to $\mathscr{G}$, which may be defined as the alternating product of the Chern and Segre forms associated to connections D_k of the involved vector bundles E_k, $k=0,\dots,N$. This provides an extension of the definition of Chern forms from vector bundles to coherent sheaves, since in case $\mathscr{G}$ is a vector bundle, and the resolution E is taken to consist of simply $\mathscr{G}$ in degree 0, the two notions of Chern forms coincide. Furthermore, at the level of de Rham cohomology classes, the corresponding Chern class so defined is the unique extension of the definition of Chern class from vector bundles to coherent sheaves which is multiplicative on short exact sequences of vector bundles.

We recall that for example projective manifolds satisfy the resolution property, but also that, as shown by Voisin, [Voi02], compact Kähler manifolds do not necessarily satisfy the resolution property. Furthermore, if M is non-compact, the resolution property always fails, as one may for any discrete sequence $(x_k)\in M$ construct a sheaf $\mathscr{G}$ for which one must have, say, that $\operatorname{rank} E_0^{x_k}\geq k$ for any resolution (E^{x_k},φ^{x_k}) of $\mathscr{G}_{x_k}$.

However, for any complex manifold M one may find an open cover $\mathcal{U}=(U_\alpha)_{\alpha\in I}$ of M such that $\mathscr{G}|_{U_\alpha}$ admits a resolution $(E^\alpha,\varphi^\alpha)$ of holomorphic vector bundles of finite length $N\leq n$. If one equips each complex of vector bundles E^α with a connection D^α (i.e., a tuple of connections $(D_N^\alpha,\dots,D_0^\alpha)$, where D_k^α is a connection on E_k^α), and we

Date: April 24, 2024.

fix a partition of unity (ψ_α) subordinate to $\mathcal{U}$, Green showed in [Gre80] that one may associate to this data a Chern form $c_\ell((D^\alpha)_{\alpha\in I})$, which is a d-closed smooth form on M. The corresponding de Rham cohomology class only depends on $\mathscr{G}$, and we may denote it as $c_\ell(\mathscr{G}) := [c_\ell((D^\alpha)_{\alpha\in I})] \in H^{2\ell}_{dR}(M)$. Here $0 \le \ell \le n$. In case M has the resolution property, and one takes a single global resolution (E, φ) equipped with a connection D and lets $(E^\alpha, \varphi^\alpha) = (E|_{U_\alpha}, \varphi|_{U_\alpha})$ and $D^\alpha = D|_{U_\alpha}$, then $c_\ell((D^\alpha)_{\alpha\in I})$ coincides with the Chern form mentioned in the first paragraph.

More generally, for any homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1, \dots, z_n]$, there is an associated characteristic form $\Phi((D^\alpha)_{\alpha\in I})$, which is a smooth closed form, whose de Rham cohomology class only depends on $\mathcal{G}$. Recall that the Chern forms are the forms associated to the elementary symmetric polynomials e_ℓ, $\ell = 0, 1, \dots, n$.

A coherent sheaf $\mathscr{G}$ will generically be a vector bundle. Let Z denote the analytic subset of M where $\mathscr{G}$ is not a vector bundle, that is, the smallest proper analytic subset Z of M such that $\mathscr{G}|_{M\setminus Z}$ is locally free. In [LW22] and [KLW23], we showed that associated to certain coherent sheaves $\mathscr{G}$, *if $\mathscr{G}$ admits a global resolution* (E, φ), then one may construct a family of connections $(\widehat{D}^\epsilon)_{\epsilon>0}$ on E such that associated to homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ of appropriate degree, the sequence of smooth forms $\Phi(\widehat{D^\epsilon})$ admits a limit

$$R^\Phi := \lim_{\epsilon\to 0} \Phi(\widehat{D^\epsilon}), \tag{1.1}$$

where R^Φ is a closed current on M whose de Rham cohomology class represents $\Phi(\mathscr{G})$. The key property of the current R^Φ is that it "localizes" $\Phi(\mathscr{G})$ at the singularities of $\mathscr{G}$, meaning that R^Φ has support on Z. In [LW22], this is done for sheaves $\mathscr{G}$ whose support Z has codimension ≥ 1, and Φ is of any degree ≥ 1. Explicit descriptions are also obtained for R^Φ when $\deg \Phi \le \operatorname{codim} Z$. In [KLW23], this is done for $\mathscr{G} = N\mathscr{F}$ being the normal sheaf of a (singular) holomorphic foliation $\mathscr{F}$ of M, provided the degree of Φ is larger than the corank of $\mathscr{F}$. For the compact connected components Z' of Z, this also yields representations of the corresponding so-called Baum-Bott residue of $\mathscr{F}$ along Z', as introduced in [BB72].

In this article, we show that the constructions from [LW22] and [KLW23] may be generalized to arbitrary complex manifolds M, i.e., that we may drop the assumption of $\mathscr{G}$ having a global resolution (E, φ).

Recall that the *(fundamental) cycle* of $\mathscr{G}$ is the cycle

$$[\mathscr{G}] = \sum_k m_k [Z_k] \tag{1.2}$$

(considered as an integration current), where Z_k are the irreducible components of $\operatorname{supp} \mathscr{G}$, and m_k is the *geometric multiplicity* of Z_k in $\mathscr{G}$, see *e.g.* [Ful98, Chapter 1.5].

Our global generalization of the results in [LW22] is the following.

Theorem 1.1. *Let M be a complex manifold of dimension n, let $\mathscr{G}$ be a coherent analytic sheaf on M, such that $Z = \operatorname{supp} \mathscr{G}$ has codimension $p \ge 1$ and let Φ be a symmetric polynomial $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ of degree ℓ with $1 \le \ell \le n$. Let $(U_\alpha)_{\alpha\in I}$ be a Stein open cover of M, and let $N \in \mathbb{N}$ be such that each $\mathscr{G}|_{U_\alpha}$ admits a finite resolution $(E^\alpha, \varphi^\alpha)$ of holomorphic vector bundles of length $\le N$. Assume that each E^α is equipped with Hermitian metrics and a connection D^α of type $(1,0)$. Then, for $\epsilon > 0$, there are explicit connections $\widehat{D}^{\alpha,\epsilon}$ on E^α such that the limit*

$$R^\Phi := \lim_{\epsilon\to 0} \Phi((\widehat{D}^{\alpha,\epsilon})_{\alpha\in I})$$

exists as a current, which represents $\Phi(\mathcal{G})$, and which has support on Z. In addition, if Z has pure codimension p, then

$$R^{e_p} = (-1)^{p-1}(p-1)![\mathscr{G}]. \tag{1.3}$$

If $\ell < p$, then

$$R^{e_\ell} = 0, \tag{1.4}$$

and if $\ell_1 + \cdots + \ell_m \le p$, where $m \ge 2$, then

$$R^{e_{\ell_1} \cdots e_{\ell_m}} = 0. \tag{1.5}$$

The connections $\widehat{D}^{\alpha,\epsilon}$ depend on the choice of connections D^α, as well as Hermitian metrics on E^α (where the connection and the metrics do not need to be related to each other) and a partition of unity of the cover. In general, the resulting current R^Φ will also depend on these choices. However, as follows by (1.3),(1.4) and (1.5), R^Φ is independent of all these choices if $\deg \Phi \le \operatorname{codim} \operatorname{supp} \mathscr{G}$. In general, we prove a transgression formula which says that they are independent of these choices up to a current of the form dN^Φ, where N^Φ is a current with support on Z, see Theorem 8.2. This last part is new, also in the case when $\mathscr{G}$ admits a global resolution.

In particular, at the level of cohomology,

$$c_p(\mathscr{G}) = (-1)^{p-1}(p-1)![\mathscr{G}], \tag{1.6}$$

where now the right hand side should be interpreted as a de Rham class. If for example $\mathscr{G}$ is the pushforward of a vector bundle from a submanifold, the fact that (1.6) holds is a well-known consequence of the Grothendieck-Riemann-Roch theorem, [OTT85], *cf.* [Ful98, Examples 15.2.16 and 15.1.2].

Just as in [LW22], in case $\mathscr{G}$ has codimension p, but not necessarily *pure* codimension p, then Theorem 1.1 still holds if we replace the first equation by

$$R^{e_p} = (-1)^{p-1}(p-1)![\mathscr{G}]_p, \tag{1.7}$$

where $[\mathscr{G}]_p$ denotes the part of $[\mathscr{G}]$ of codimension p, *i.e.* in (1.2), one only sums over the components Z_i of codimension p.

Let now $\mathscr{F}$ be a holomorphic foliation of rank κ on M and denote by $N\mathscr{F}$ its normal sheaf, see Section 2.1 for the definitions. If Z' is a compact connected component of the singular set of $\mathscr{F}$, sing $\mathscr{F}$, and $\Phi \in \mathbb{C}[z_1, \ldots, z_n]$ is a homogeneous symmetric polynomial of degree ℓ with $n - \kappa < \ell \le n$, then Baum-Bott, [BB72], defined a class $\operatorname{res}^\Phi(\mathscr{F}; Z')$, an object in $H_{2n-2\ell}(Z', \mathbb{C})$, which may also be represented as a de Rham cohomology class on M with compact support, whose representatives have support in arbitrarily small neighborhoods of Z'. Provided sing $N\mathscr{F}$ is compact, these classes together represent $\Phi(N\mathscr{F})$, i.e.,

$$\sum_{Z' \subset \text{sing}\, \mathscr{F}} \operatorname{res}^\Phi(\mathscr{F}; Z') = \Phi(N\mathscr{F}) \quad \text{in} \quad H^{2\ell}(M, \mathbb{C}),$$

where the sum is over all the connected components of Z'. This should be seen as a localization formula for $\Phi(N\mathscr{F})$ around the singularities of $\mathscr{F}$.

We may assume that M has a Stein open cover (U_α) such that for each α, $N\mathscr{F}|_{U_\alpha}$ admits a resolution

$$0 \to E_N^\alpha \xrightarrow{\varphi_N^\alpha} E_{N-1}^\alpha \xrightarrow{\varphi_{N-1}^\alpha} \ldots \xrightarrow{\varphi_2} E_1^\alpha \xrightarrow{\varphi_1^\alpha} E_0^\alpha = TM \xrightarrow{\varphi_0^\alpha} N\mathscr{F} \to 0. \tag{1.8}$$

By the syzygy theorem, one may assume that $N \le n$.

The global version of the results in [KLW23] that we obtain is the following.

Theorem 1.2. *Let M be a complex manifold of dimension n, let $\mathscr{F}$ be a holomorphic foliation of rank κ on M, and let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial of degree ℓ with $n - \kappa < \ell \leq n$. Let (U_α) be a Stein open cover of M, and let $N \in \mathbb{N}$ be such that each $N\mathscr{F}|_{U_\alpha}$ admits resolutions of the form* (1.8). *Assume that each TM is equipped with a torsion free connection D^{TM} of type $(1,0)$, and that each E_k^α is equipped with a connection D_k^α for $k = 1, \dots, N$. Then, for $\epsilon > 0$ there are explicit connections $\widehat{D}^{\alpha,\epsilon}$ on E^α such that the limit*

$$R^\Phi := \lim_{\epsilon \to 0} \Phi((\widehat{D}^{\alpha,\epsilon})_{\alpha \in I})$$

exists as a current, which represents $\Phi(N\mathscr{F})$, and which has support on $\operatorname{sing} \mathscr{F}$.

Write $R^\Phi = \sum R^\Phi_{Z'}$, where the sum runs over all connected components of $\operatorname{sing} N\mathscr{F}$*, and $R^\Phi_{Z'}$ has support on Z'. If Z' is a compact connected component of* $\operatorname{sing} \mathscr{F}$*, then $R^\Phi_{Z'}$ represents the Baum-Bott residue $\operatorname{res}^\Phi(\mathscr{F}; Z') \in H_{2n-2\ell}(Z', \mathbb{C})$. Furthermore, if $\ell = \operatorname{codim} Z'$, then $R^\Phi_{Z'}$ only depends on $\mathscr{F}$, and if $\ell < \operatorname{codim} Z'$, then $R^\Phi_{Z'} = 0$.*

As in the previous theorem, the connections and resulting currents R^Φ and $R^\Phi_{Z'}$ in general depend on the choice of connections D^α, as well as Hermitian metrics on E^α and a partition of unity of the cover. However, we prove a transgression formula which says that they are also independent of these choices up to a current of the form dN^Φ, where N^Φ is a current with support on $\operatorname{sing} \mathscr{F}$, see Theorem 8.2. Also in this case, this last extends the previous results when $\mathscr{G}$ admits a global resolution. In [KLW23], we only proved such a result when the resolution was fixed, but possibly equipped with different sets of connections and metrics.

Chern classes of coherent sheaves, without the assumption of the existence of a global locally free resolution, has been studied in various recent papers, including [Gri10, Qia16, BSW23, Wu23], in finer cohomology theories than de Rham cohomology, more precisely in (rational or complex) Bott-Chern and Deligne cohomology. In the present paper, our focus has been to find explicit representatives of Chern classes of a coherent sheaf with support on the support of the sheaf (or in its singular support in the foliation case), a type of result which as far as we can tell, none of the above mentioned works seems to consider. Our methods do not seem to yield representatives in the finer cohomology theories mentioned above, as for example our construction is based on Chern forms of connections that are not Chern connections of a hermitian metric.

At the final stages of the preparation of the present article appeared a preprint by Han, [Han24], with similar results as our Theorem 1.1, i.e., the main result states that there exists representations of characteristic classes of $\mathscr{F}$ as pseudomeromorphic currents with support on $Z = \operatorname{supp} \mathscr{F}$. Both in Han's work and in the present article, the approach is based on the local construction from [LW22]. In [Han24] however, the globalization builds on a theory of global resolutions of coherent sheaves by cohesive modules due to Block, [Blo10], and the associated characteristic forms as introduced by Qiang, [Qia16], cf., also [BSW23], while here we adopt Green's approach.

2. Coherent sheaves, holomorphic foliations, vector bundle complexes and characteristic classes

Throughout the paper M will be a complex manifold of dimension n.

2.1. **Holomorphic foliations.** A *holomorphic foliation* $\mathscr{F}$ on M is the data of a coherent analytic subsheaf $T\mathscr{F}$ of TM, called the *tangent sheaf* of $\mathscr{F}$, such that $T\mathscr{F}$ is *involutive*, that is, for any pair of local sections u, v of $T\mathscr{F}$, the Lie bracket $[u, v]$ belongs to $T\mathscr{F}$. The *normal sheaf* of $\mathscr{F}$ is $N\mathscr{F} := TM/T\mathscr{F}$.

The generic rank of $T\mathscr{F}$ is called the *rank of* $\mathscr{F}$. Note that $N\mathscr{F}$ is a coherent analytic sheaf. The *singular set* of $\mathscr{F}$ is, by definition, the smallest subset $\operatorname{sing}\mathscr{F} \subset M$ outside of which $N\mathscr{F}$ is locally free. We say that $\mathscr{F}$ is *regular* if $\operatorname{sing}\mathscr{F}$ is empty. By definition, the restriction of $N\mathscr{F}$ to $M \setminus \operatorname{sing}\mathscr{F}$ defines a regular foliation whose normal sheaf is a holomorphic vector bundle of rank $n-\kappa$, where κ is the rank of $\mathscr{F}$. Moreover, by Frobenius Theorem, over $M \setminus \operatorname{sing}\mathscr{F}$, the bundle $T\mathscr{F}$ is locally given by vectors tangent to the fibers of a (local) holomorphic submersion.

It is standard in the literature to also assume that a foliation $\mathscr{F}$ is *saturated* or *full*, that is, $N\mathscr{F} := TM/T\mathscr{F}$ is assumed to be torsion free. In that case, one avoids certain “artificial” singularities which for example could be caused by having a generator of $T\mathscr{F}$ where all the entries share a common factor. We do not make use of this condition in our proofs, so in this article, we do not add this assumption.

2.2. **Vector bundle complexes, connections, and superstructure.** Consider a vector bundle complex

$$0 \to E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} E_0 \tag{2.1}$$

over M. Following [AW07], we equip $E := \bigoplus_{k=0}^N E_k$ with a superstructure by letting $E^+ = \bigoplus_{2k} E_k$ (resp. $E^- = \bigoplus_{2k+1} E_k$) be the even (resp. odd) parts of $E = E^+ \oplus E^-$. This is a $\mathbb{Z}/2\mathbb{Z}$-grading that simplifies some of the formulas and computations. An endomorphism $\varphi \in \operatorname{End}(E)$ is even (resp. odd) if it preserves (resp. switches) the $\pm$-components.

The superstructure affects how form-valued endomorphisms act. Assume that $\alpha = \omega \otimes \gamma$ is a form-valued section of $\operatorname{End}(E)$, where ω is a smooth form of degree m and γ is a section of $\operatorname{Hom}(E_\ell, E_k)$. We let $\deg_f \alpha = m$ and $\deg_e \alpha = k - \ell$ denote the *form* and *endomorphism degrees*, respectively, of α. The *total degree* is $\deg \alpha = \deg_f \alpha + \deg_e \alpha$. If β is a form-valued section of E, i.e., $\beta = \eta \otimes \xi$, where η is a smooth form, and ξ is a section of E, both homogeneous in degree, then the we define the action of α on β by

$$\alpha(\beta) := (-1)^{(\deg_e \alpha)(\deg_f \beta)} \omega \wedge \eta \otimes \gamma(\xi). \tag{2.2}$$

If furthermore, $\alpha' = \omega' \otimes \gamma'$, where γ' is a holomorphic section of $\operatorname{End}(E)$, and ω' is a smooth form, both homogeneous in degree, then we define

$$\alpha\alpha' := (-1)^{(\deg_e \alpha)(\deg_f \alpha')} \omega \wedge \omega' \otimes \gamma \circ \gamma'. \tag{2.3}$$

Assume that each E_k is equipped with a connection D_k. Then there is an induced connection D_E on E, that in turn induces a connection D_{End} on $\operatorname{End}(E)$, that takes the superstructure into account, which is defined by

$$D_{\mathrm{End}}\alpha = D_E \circ \alpha - (-1)^{\deg \alpha} \alpha \circ D_E.$$

We will often denote D_{End} simply by D.

Similarly to [BB72] we say that the collection of connections $(D_N, \dots, D_0)$ is *compatible* with (2.1) if $D\varphi_k = 0$ for $k = 1, \dots, N$. If $E_{-1} := \operatorname{coker} \varphi_1$ is a vector bundle, one may consider the augmented complex

$$0 \to E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} E_{-1} \to 0. \tag{2.4}$$

Then, one may say that the collection of connections $(D_N, \dots, D_{-1})$ is compatible with (2.4) if $D\varphi_k = 0$ for $k = 0, \dots, N$, cf., i.e., [KLW23, Section 2.2] and [BB72, Definition 4.16]. Note that if (2.1) is equipped with connections $(D_N, \dots, D_0)$ compatible with (2.1), then there is an induced connection D_{-1} on E_{-1} which makes $(D_N, \dots, D_{-1})$ compatible with (2.4), and conversely, such a set of connections makes $(D_N, \dots, D_0)$ compatible with (2.1), and D_{-1} to be the induced connection on E_{-1}.

2.3. **Characteristic classes and forms.** We begin by briefly reviewing the definition of characteristic classes and forms of coherent sheaves which admit global locally free resolutions. In Section 6, we will discuss how the definition may be extended to the case when global resolutions do not exist. Most of the material in this section can be found in [BB72, Sections 1 and 4].

Let E be a smooth complex vector bundle of rank r over M equipped with a connection D. We define $2j$-forms $e_j(D)$ associated to (E, D) through

$$\det\left(I + \frac{i}{2\pi}\Theta(D)\right) = 1 + e_1(D) + \cdots + e_n(D). \tag{2.5}$$

Note that $e_j(D)$ is just the jth Chern form of (E, D).

Consider now a complex (E, φ) of smooth vector bundles

$$0 \to E_N \to \cdots \to E_0, \tag{2.6}$$

and assume that the vector bundles in (2.6) are equipped with connections $D_0, \dots, D_N$, and let $\Theta(D_k)$, $k = 0, \dots, N$, be the corresponding curvature forms. We will sometimes use short-hand notation, $D = (D_N, \dots, D_0)$ and say that E is equipped with a connection D. Generalizing (2.5) we let $e_j(D)$ be the $2j$-form defined by

$$\prod_{k=0}^{N}\left(\det\left[I + \frac{i}{2\pi}\Theta(D_k)\right]\right)^{(-1)^k} = e(D) = 1 + e_1(D) + \cdots + e_n(D). \tag{2.7}$$

Let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial of degree $\ell \leq n$. In case $\Phi = e_j$, we have defined $e_j(D)$ above. In general, there is a unique polynomial $\widehat{\Phi}$, such that

$$\Phi(z_1, \dots, z_n) = \widehat{\Phi}\big(e_1(z), \dots, e_\ell(z)\big),$$

and $\Phi(D)$ is then be defined as

$$\Phi(D) = \widehat{\Phi}\big(e_1(D), \dots, e_\ell(D)\big). \tag{2.8}$$

One motivation for the definition (2.8) at the level of forms is the following result:

Lemma 2.1 ([BB72] - Lemma 4.22). *Assume that the complex* (2.1) *is pointwise exact and that $E_1 = \operatorname{coker} \varphi_1$ is a vector bundle over some open set $U \subset M$. If $D = (D_N, \dots, D_0)$ is compatible with (E, φ), and D_{-1} is the induced connection on E_{-1}, then*

$$\Phi(D) = \Phi(D_{-1}) \quad \textit{on} \quad U.$$

Next, let $\mathscr{G}$ be a coherent analytic sheaf over M and assume that $\mathscr{G}$ admits a resolution of the form (2.6), i.e.,

$$0 \to E_N \to \cdots \to E_0 \to \mathscr{G} \to 0. \tag{2.9}$$

is exact, and equip E with some connection D. The jth Chern class of $\mathscr{G}$ is defined as $c_j(\mathscr{G}) = [e_j(D)] \in H^{2j}_{dR}(M)$, where $H^{\bullet}_{dR}(M)$ denotes de Rham cohomology (of smooth forms or currents).

In case M has the *resolution property*, i.e., if any coherent sheaf $\mathscr{G}$ admits a resolution (2.9), then it is classical that $[e_j(D)]$ is a well-defined class, i.e., it is independent of the choice of resolution and connection, and it is the unique extension of the definition of Chern classes from vector bundles to coherent sheaves which is multiplicative on short exact sequences, i.e., if $0 \to \mathscr{G}'' \to \mathscr{G} \to \mathscr{G}' \to 0$ is an exact sequence of coherent analytic sheaves, then $c(\mathscr{G}) = c(\mathscr{G}'')c(\mathscr{G}')$.

It follows more generally by the construction of Green of Chern classes of coherent sheaves that we discuss later, that $c_j(\mathscr{G}) = [e_j(D)]$ is well-defined even if M does not have the resolution property, provided that a resolution (2.9) exists for $\mathscr{G}$.

If $\Phi \in \mathbb{C}[z_1, \dots, z_n]$, $\mathscr{G}$, E and D are as above, then $\Phi(\mathscr{G}) \in H^{2\ell}(M, \mathbb{C})$ is defined as $\Phi(\mathscr{G}) = [\Phi(D)]$.

2.4. **Elementary sequences.** A vector bundle complex (E, φ) is said to be an *elementary sequence* in the vector bundles $E^1, \dots, E^L$ if it is a direct sum of trivial complexes

$$(0 \to E^j \stackrel{\mathrm{id}}{\to} E^j \to 0)[m], \tag{2.10}$$

where $j \in \{1, \dots, L\}$, $m \in \mathbb{Z}$, and $[m]$ denotes the complex shifted in degree m steps.

Note that the pullback by a smooth map of an elementary sequence is again an elementary sequence. Assume that each E^j is equipped with a connection D^j and let D be the induced connection on (E, φ). Note that then D is compatible with (E, φ).

Lemma 2.2. *Let (E, φ) be complex of vector bundles as in* (2.9)*, and let (F, η) be an elementary sequence in some vector bundles $F^1, \dots, F^L$. Assume that (E, φ) is equipped with a connection D^E, and that (F, ψ) is equipped with a connection D^F induced by a set of connections $D^{F^1}, \dots, D^{F^L}$ on $F^1, \dots, F^L$. Let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial. Then*

$$\Phi(D^E \oplus D^F) = \Phi(D^E). \tag{2.11}$$

Proof. It follows by (2.7) and the multiplicative property of the determinant on block diagonal matrices that

$$e(D^E \oplus D^F) = e(D^E)e(D^F).$$

Hence, (2.11) holds for Φ being any of the e_ℓ, $\ell = 1, \dots, n$ by Lemma 2.1, since (F, η) is a resolution of the sheaf 0 and is equipped with a compatible connection. In then follows that (2.11) holds for general Φ by (2.8). $\square$

2.5. **Interpolation of connections.** Consider a locally free resolution (E, φ) of $\mathcal{G}$ of length N as in (2.9). Assume that E is equipped with a set of connections $D^0, \dots, D^p$, i.e., $D^j = (D^j_N, \dots, D^j_0)$, where D^j_k is a connection on E_k for $k = 0, \dots, N$, $j = 0, \dots, p$. Let

$$\Delta_p = \big\{(t_0, \dots, t_p) \in [0,1]^p \mid t_0 + \dots + t_p = 1\big\} \tag{2.12}$$

be the standard p-simplex, with the orientation determined by the form $dt_1 \wedge \dots \wedge dt_n$, and let π denote the projection $M \otimes \Delta_p \to M$. Define a connection D on $\pi^* E$ by

$$D_k = \sum_{j=0}^{p} t_j \pi^* D^j_k.$$

Let $\Phi \in \mathbb{C}[z_1, \ldots, z_n]$ be a homogeneous symmetric polynomial of degree ℓ. We define

$$\Phi(D^0, \ldots, D^p) := \pi_* \Phi(D). \tag{2.13}$$

By [Suw98, equation (II.8.2)],

$$\sum_{k=0}^{p} (-1)^k \Phi(D^0, \ldots, \widehat{D^k}, \ldots, D^p) + (-1)^p d\Phi(D^0, \ldots, D^p) = 0, \tag{2.14}$$

where $\widehat{D^k}$ denotes that D^k is removed.

2.6. **The Čech-de Rham complex.** Given a locally finite open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of M, let $N\mathcal{U}$ denote the nerve of $\mathcal{U}$. Throughout we will tacitly assume that all covers are locally finite. For $\Delta = (\alpha_0, \ldots, \alpha_p) \in N\mathcal{U}$, let

$$U_\Delta = U_{\alpha_0, \ldots, \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$

Let

$$\check{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet) = \bigoplus_{p,q \geq 0} \check{C}^p(\mathcal{U}, \mathcal{E}^q)$$

be the (non-alternating) Čech-de Rham complex, where $\check{C}^p(\mathcal{U}, \mathcal{E}^\bullet)$ denote the q-form-valued Čech p-cochains

$$\gamma = (\gamma_{\alpha_0, \ldots, \alpha_p}) \in \prod_{(\alpha_0, \ldots, \alpha_p) \in I^{p+1}} \mathcal{E}^q(U_{\alpha_0, \ldots, \alpha_p}),$$

and $\mathcal{E}^q$ denotes the sheaf of smooth q-forms. We say that $\gamma \in \check{C}^p(\mathcal{U}, \mathcal{E}^q)$ has Čech degree p, form degree q, and total degree $p+q$. If each $\gamma_{\alpha_0, \ldots, \alpha_p}$ has bidegree (r, s) we will sometimes use the notation $\gamma \in \check{C}^p(\mathcal{U}, \mathcal{E}^{r,s})$. Let $\delta = \delta^p : \check{C}^p(\mathcal{U}, \mathcal{E}^\bullet) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{E}^\bullet)$ be the Čech differential

$$(\delta\gamma)_{\alpha_0, \ldots, \alpha_{p+1}} = \sum_{0 \leq j \leq p+1} (-1)^j \gamma_{\alpha_0, \ldots \widehat{\alpha}_j \ldots, \alpha_{p+1}}|_{U_{\alpha_0, \ldots, \alpha_{p+1}}},$$

and let $d = d^p : \check{C}^p(\mathcal{U}, \mathcal{E}^q) \to \check{C}^p(\mathcal{U}, \mathcal{E}^{q+1})$ be the exterior derivative

$$(d\gamma)_{\alpha_0, \ldots, \alpha_p} = (-1)^p d\gamma_{\alpha_0, \ldots, \alpha_p}.$$

Let

$$\nabla = d + \delta$$

be the (total) differential of the Čech-de Rham complex.

Recall that $H^k(\check{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet)) \cong H^k_{dR}(M)$ and that one may construct an explicit morphism of complexes $\Psi : \check{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet) \to \mathcal{E}^\bullet(M)$ which induces this isomorphism, cf., i.e., [BT82, Proposition 9.5][1]. Assume that $(\psi_\alpha)_{\alpha \in I}$ is a partition of unity subordinate to $\mathcal{U}$, and define $\psi : \check{C}^p(\mathcal{U}, \mathcal{E}^\bullet) \to \check{C}^{p-1}(\mathcal{U}, \mathcal{E}^\bullet)$ for $p \geq 1$ by

$$(\psi\gamma)_{\alpha_0, \ldots, \alpha_{p-1}} := -\sum_{\alpha \in I} \psi_\alpha \gamma_{\alpha_0, \ldots, \alpha_{p-1}, \alpha}.$$

If $\gamma \in \check{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet)$, and γ_p denotes the part of γ of Čech degree p, then we let

$$\Psi'\gamma := \sum_{p \geq 0} (d\psi)^p \gamma_p + \psi(d\psi)^p (D\gamma)_{p+1}, \tag{2.15}$$

[1]In [BT82], they consider the alternating Čech-de Rham complex. However, the proof of [BT82, Proposition 9.5] works equally well for the non-alternating Čech-de Rham complex considered in this article.

where $(d\psi)^p$ denotes the composition $d \circ \psi$ repeated p times. Note that if γ is a cocycle, then

$$\Psi'\gamma = \sum_{p\geq 0} (d\psi)^p \gamma_p. \tag{2.16}$$

The cochain $\Psi'\gamma$ defined by (2.15) lies in $\check{C}^0(\mathcal{U}, \mathcal{E}^\bullet)$, but as shown in [BT82, Proposition 9.5], it is in fact δ-closed, so it induces a global form in $\mathcal{E}^\bullet(M)$. We denote this global form by $\Psi(\gamma)$, and note that it may thus either be defined by $\Psi(\gamma)|_{U_\alpha} = \Psi'(\gamma)_\alpha$, or by

$$\Psi\gamma := \sum \psi_\alpha (\Psi'\gamma)_\alpha. \tag{2.17}$$

Assume that $\mathcal{V} = (V_\beta)_{\beta\in J}$ is an open cover which refines $\mathcal{U}$, i.e., that there is a *refinement map* $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for all $\beta \in J$. Then there are induced morphisms $\rho = \rho^p : \check{C}^p(\mathcal{U}, \mathcal{E}^\bullet) \to \check{C}^p(\mathcal{V}, \mathcal{E}^\bullet)$, defined by

$$(\rho\gamma)_{\beta_0,\dots,\beta_p} = \gamma_{\rho(\beta_0),\dots,\rho(\beta_p)}|_{V_{\beta_0,\dots,\beta_p}}.$$

Assume that ρ_1 and ρ_2 are two refinement maps and let $h = h^p : \check{C}^p(\mathcal{U}, \mathcal{E}^\bullet) \to \check{C}^{p-1}(\mathcal{V}, \mathcal{E}^\bullet)$ be the homotopy operator defined by

$$(h\gamma)_{\beta_0,\dots,\beta_{p-1}} = \sum_{k=0}^{p-1} (-1)^k \gamma_{\rho_1(\beta_0),\dots,\rho_1(\beta_k),\rho_2(\beta_k),\dots,\rho_2(\beta_{p-1})}|_{V_{\beta_0,\dots,\beta_{p-1}}}. \tag{2.18}$$

A calculation yields that

$$\nabla h + h\nabla = \rho_2 - \rho_1. \tag{2.19}$$

3. Baum-Bott theory

The theory of Baum-Bott residues was developed in [BB72], extending the theory of rank one foliations in [BB70] to general foliations. The main outcome of Baum-Bott's theory is the fact that high degree characteristic classes $\Phi(N\mathscr{F})$ of $N\mathscr{F}$ localize around sing $\mathscr{F}$, cf. the introduction. This is a consequence of a vanishing theorem for the normal bundle of a regular foliation due to the existence of special connections. Recall that a connection is said to be of *type* $(1,0)$, or a $(1,0)$-*connection*, if its $(0,1)$-part equals $\bar\partial$.

Definition 3.1. ([BB72] - Definition 3.24) Let $\mathscr{F}$ be a regular foliation on M and let $\varphi_0 : TM \to N\mathscr{F}$ be the canonical surjection. A connection D on $N\mathscr{F}$ is *basic* if it is of type $(1,0)$ and

$$i(u)D(\varphi_0 v) = \varphi_0[u,v] \tag{3.1}$$

for any smooth sections u of $T\mathscr{F}$ and v of TM.

It is not hard to see that basic connections always exist, see [BB72, §3].

Theorem 3.2 (Baum-Bott's Vanishing theorem, [BB72] - Proposition 3.27)**.** *Let $\mathscr{F}$ be a regular foliation of rank κ on a complex manifold M of dimension n. If D is a basic connection on $N\mathscr{F}$, then $\Phi(D) = 0$ for every homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1,\dots,z_n]$ of degree ℓ with $n-\kappa < \ell \leq n$.*

3.1. **Baum-Bott residues.** In the presence of singularities, one cannot work directly with connections on $N\mathscr{F}$, so the use of suitable resolutions is necessary.

Let Z' be a compact connected component of $\operatorname{sing}\mathscr{F}$. Then one can find an open neighborhood U of Z' in M such that $U \cap \operatorname{sing}\mathscr{F} = Z'$ and Z' is a deformation retract of U, and a locally free resolution (E,φ) of $N\mathscr{F}$ of $\mathcal{A}$-modules on U, where $\mathcal{A}$ denotes the sheaf of germs of real analytic functions, cf. [BB72, Proposition 6.3].

Given a basic connection D_{-1} on $N\mathscr{F}|_{M\setminus \operatorname{sing}\mathscr{F}}$, and a compact neighborhood $\Sigma \subset U$ of Z', as is shown in [BB72, Lemma 5.26], one may construct a connection D on E which is compatible with (E,φ) on $U \setminus \Sigma$, and such that D_{-1} is the connection on $N\mathscr{F}|_{U\setminus\Sigma}$ induced by D. If $\Phi \in \mathbb{C}[z_1,\dots,z_n]$ is a homogeneous symmetric polynomial of degree ℓ with $n-\kappa < \ell \leq n$, then it follows by Theorem 3.2 and Lemma 2.1 that $\Phi(D)$ vanishes outside of Σ, and in particular, it has compact support in U. Since Z' is a deformation retract of U, the homology groups of U and Z' are naturally isomorphic. Composing this isomorphism with the Poincaré duality $H^{2\ell}_c(U,\mathbb{C}) \simeq H_{2n-2\ell}(U,\mathbb{C})$ yields an isomorphism $H^{2\ell}_c(U,\mathbb{C}) \simeq H_{2n-2\ell}(Z',\mathbb{C})$. Now $\operatorname{res}^\Phi(\mathscr{F};Z') \in H_{2n-2\ell}(Z',\mathbb{C})$ is defined as the class of $\Phi(D)$ in $H^{2\ell}_c(U,\mathbb{C})$ under this isomorphism. It is proved in [BB72, Sections 5,6,7] that the class of $\Phi(D)$ is independent of the choices of $U,(E,\varphi),D$, and only depends on the local behaviour of $\mathscr{F}$ around Z'.

4. Residue currents

We say that a function $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *smooth approximant of the characteristic function* $\chi_{[1,\infty)}$ of the interval $[1,\infty)$ and write

$$\chi \sim \chi_{[1,\infty)}$$

if χ is smooth, increasing and $\chi(t) \equiv 0$ for $t \ll 1$ and $\chi(t) \equiv 1$ for $t \gg 1$.

4.1. **Pseudomeromorphic currents.** Let f be a (generically nonvanishing) holomorphic function on a (connected) complex manifold M. Herrera and Lieberman, [HL71], proved that the *principal value*

$$\lim_{\epsilon\to 0}\int_{|f|^2>\epsilon}\frac{\xi}{f}$$

exists for test forms ξ and defines a current, that we with a slight abuse of notation denote by $1/f$. It follows that $\bar\partial(1/f)$ is a current with support on the zero set $Z(f)$ of f; such a current is called a *residue current*. Assume that $\chi \sim \chi_{[1,\infty)}$ and that s is a generically nonvanishing holomorphic section of a Hermitian vector bundle such that $Z(f) \subseteq \{s=0\}$. Then

$$\frac{1}{f} = \lim_{\epsilon\to 0}\frac{\chi(|s|^2/\epsilon)}{f} \quad\text{and}\quad \bar\partial\left(\frac{1}{f}\right) = \lim_{\epsilon\to 0}\frac{\bar\partial\chi(|s|^2/\epsilon)}{f},$$

see, e.g., [AW18]. In particular, the limits are independent of χ and s. Note that $\chi(|s|^2/\epsilon)$ vanishes identically in a neighborhood of $\{s=0\}$, so that $\chi(|s|^2/\epsilon)/f$ and $\bar\partial\chi(|s|^2/\epsilon)/f$ are smooth. More generally, if f is a generically non-vanishing holomorphic section of a line bundle $L \to M$ and ω is an L-valued smooth form, then the current ω/f is well-defined. Such currents are called *semi-meromorphic*, cf. [AW18, Section 4].

In [AW10] the sheaf $\mathcal{PM}_M$ of *pseudomeromorphic currents* on M was introduced in order to obtain a coherent approach to questions about residue and principal value currents; it consists of direct images under holomorphic mappings of products of test forms and currents like $1/f$ and $\bar\partial(1/f)$, and also suitable products of such currents. See, e.g., [AW18, Section 2.1] for a precise definition. The sheaf $\mathcal{PM}_M$ is closed under ∂ and $\bar\partial$ and under multiplication by smooth forms. Pseudomeromorphic currents have a geometric nature, similar to closed positive (or normal) currents. For instance, the *dimension principle* states that if the pseudomeromorphic current μ has bidegree $(*,p)$ and support on a variety of codimension strictly larger than p, then μ vanishes.

The sheaf $\mathcal{PM}_M$ admits natural restrictions to constructible subsets of M. In particular, if W is a subvariety of the open subset $U \subseteq M$, and s is a holomorphic section of a Hermitian vector bundle such that $\{s = 0\} = W$, then the restriction to $U \setminus W$ of a pseudomeromorphic current μ on U is the pseudomeromorphic current on U defined by

$$\mathbf{1}_{U\setminus W}\mu := \lim_{\epsilon\to 0} \chi(|s|^2/\epsilon)\mu|_U,$$

where $\chi \sim \chi_{[1,\infty)}$. It follows that

$$\mathbf{1}_W\mu := \mu - \mathbf{1}_{U\setminus W}\mu$$

has support on W. These definitions are independent of the choice of s and χ.

4.2. **Almost semi-meromorphic currents.** We refer to [AW18, Section 4] for details of the results mentioned in this section.

We say that a current a is *almost semi-meromorphic* in M, $a \in ASM(M)$, if there exists a modification $\pi : M' \to M$ and a semi-meromorphic current ω/f on M' such that $a = \pi_*\big(\omega/f\big)$. More generally, if E is a vector bundle over M, an E-valued current a is *almost semi-meromorphic* on M if $a = \pi_*\big(\omega/f\big)$, where π is as above, ω is a smooth form with values in $L \otimes \pi^*E$ and f is a holomorphic section of a line bundle $L \to M'$.

Clearly almost semi-meromorphic currents are pseudomeromorphic. In particular, if $a \in ASM(M)$, then ∂a and $\bar\partial a$ are pseudomeromorphic currents on M.

Lemma 4.1 (Proposition 4.16 in [AW18]). *Assume that $a \in ASM(M)$ is smooth in $M \setminus W$, where W is subvariety of M. $\partial a \in ASM(M)$ and $\mathbf{1}_{M\setminus W}\bar\partial a \in ASM(M)$.*

Given $a \in ASM(M)$, let $ZSS(a)$ (the *Zariski-singular support*) denote the smallest Zariski-closed set $V \subset M$ such that a is smooth outside V. The pseudomeromorphic current $r(a) := \mathbf{1}_{ZSS(a)}\bar\partial a$ is called the *residue* of a.

Almost semi-meromorphic currents have the so-called *standard extension property (SEP)* meaning that $\mathbf{1}_W a = 0$ in U for each subvariety $W \subset U$ of positive codimension, where U is any open set in M. In particular, if $a \in ASM(M)$, $\chi \sim \chi_{[1,\infty)}$, and s is any generically non-vanishing holomorphic section of a Hermitian vector bundle over M, then

$$\lim_{\epsilon\to 0} \chi(|s|^2/\epsilon)a = a. \tag{4.1}$$

It follows in view of Lemma 4.1 that, if $\{s = 0\} \supset ZSS(a)$, then

$$r(a) = \lim_{\epsilon\to 0} \bar\partial\chi(|s|^2/\epsilon) \wedge a = \lim_{\epsilon\to 0} d\chi(|s|^2/\epsilon) \wedge a. \tag{4.2}$$

Lemma 4.2. *Let χ be a cutoff function, $Z \subseteq M$ an analytic subset of positive codimension, and let $s_1, \dots, s_k$ be sections of vector bundles $F_1, \dots, F_k$ such that $s_j \not\equiv 0$ and $\{s_k = 0\} \supseteq Z$, and let β be an almost semi-meromorphic current on M. Then*

$$\lim_{\epsilon\to 0} \chi(|s_1|^2/\epsilon)\cdots\chi(|s_k|^2/\epsilon)\beta = \beta.$$

Proof. We prove more generally that

$$\chi(|s_1|^2/\epsilon_1)\cdots\chi(|s_k|^2/\epsilon_k)\beta \tag{4.3}$$

is continuous as a function of $(\epsilon_1, \dots, \epsilon_k) \in [0,\infty)^k$, and the value at 0 equals β. The desired equality then holds since β has the SEP, and thus the iterated limit of (4.3) when $\epsilon_k \to 0$ one at a time equals β.

Since β is a locally finite sum of push-forwards under modifications of currents of the form $\pi_*(\beta_0/f)$, where β_0 is a smooth form with compact support and f is a holomorphic function, we may in fact assume that $\beta = \beta_0/f$. The result then follows by [BS10, Theorem 1.1] and the paragraph after the proof of Theorem 1.1 on [BS10, p. 52]. □

Proposition 4.3. *Let (U_α) be a cover of M, let (ψ_α) be a partition of unity subordinate to (U_α), and let β be an almost semi-meromorphic current on M, which is smooth outside of the analytic subset Z. Let for each α, $s_\alpha \not\equiv 0$ be a section of some Hermitian vector bundle over U_α such that $\{s_\alpha = 0\} \supseteq Z$, let $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a smooth cut-off function, and define $\chi_\epsilon := \sum \psi_\alpha \chi(|s_\alpha|^2/\epsilon)$. Then for $k \geq 1$,*

$$\lim_{\epsilon \to 0} \chi_\epsilon^k \beta = \beta \tag{4.4}$$

$$\lim_{\epsilon \to 0} d\chi_\epsilon^k \wedge \beta = r(\beta), \tag{4.5}$$

so in particular, both limits are pseudomeromorphic and depend only on β.

Proof. The equality (4.4) follows immediately from Lemma 4.2. To prove (4.5), we have that

$$\lim_{\epsilon \to 0} d\chi_\epsilon^k \wedge \beta = \lim_{\epsilon \to 0} d(\chi_\epsilon^k \beta) - \chi_\epsilon^k(\partial\beta + \mathbf{1}_{M\setminus Z}\bar\partial\beta) = d\beta - \partial\beta - \mathbf{1}_{M\setminus Z}\beta = r(\beta),$$

where the first equality holds since $\mathbf{1}_{M\setminus Z}\bar\partial\beta = \bar\partial\beta$ on $\operatorname{supp} \chi_\epsilon^k$, and the second equality holds by (4.4) since since $\partial\beta$ and $\mathbf{1}_{M\setminus Z}\bar\partial\beta$ are almost semi-meromorphic by Lemma 4.1. □

If $a_1, a_2 \in ASM(M)$, then $a_1 + a_2 \in ASM(M)$, and moreover there is a well-defined product $a_1 \wedge a_2 \in ASM(M)$, so that $ASM(M)$ is an algebra over smooth forms, see [AW18, Section 4.1]. Note that if $\chi \sim \chi_{[1,\infty)}$ and s is a generically nonvanishing holomorphic section of a Hermitian vector bundle such that $\{s = 0\}$ contains the Zariski-singular supports of a_1 and a_2, then $a_1 \wedge a_2$ is the limit of the smooth form $\chi(|s|^2/\epsilon) a_1 \wedge a_2$.

5. Residue currents as localization of characteristic classes

We briefly recall the relevant parts of the constructions from [LW22] and [KLW23] of residue currents representing characteristic classes of coherent sheaves.

Assume that $\mathscr{G}$ is a coherent analytic sheaf on M, and that $\mathscr{G}$ admits a locally free resolution

$$0 \to E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} E_0 \to \mathscr{G} \to 0, \tag{5.1}$$

and assume that E is equipped with Hermitian metrics and a connection D. The construction of the residue currents in both these articles is done through a procedure as follows: If Z denotes the analytic subset where $\mathscr{G}$ is not a vector bundle, then under suitable hypothesis, a connection $\tilde{D}$ on $E|_{M\setminus Z}$ is constructed,and $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ is a homogeneous symmetric polynomial of appropriate degree, then $\Phi(\tilde{D}) = 0$.

Let s be a holomorphic section of some holomorphic vector bundle over M such that $\{s = 0\} = Z$. For example, we may take $s = \bigwedge^\rho \varphi_1$, where ρ is the (generic) rank of φ_1. Let $\chi \sim \chi_{[1,\infty)}$ and define $\chi_\epsilon = \chi(|s|^2/\epsilon)$. Note that $\chi_\epsilon \equiv 0$ in a neighborhood of Z, and is such that if that the set Σ_ϵ where χ_ϵ is not identically 1 shrinks to Z when ϵ tends to 0. One may then define a new family of connections

$$\widehat{D}^\epsilon = (1-\chi_\epsilon)\tilde{D} + \chi_\epsilon D. \tag{5.2}$$

Then, $\Phi(\widehat{D}^\epsilon)$ will be a family of characteristic forms, which since $\widehat{D}^\epsilon = \tilde{D}$ outside of Σ_ϵ, this family will satisfy that the support of $\Phi(\widehat{D}^\epsilon)$ shrinks to Z as ϵ tends to 0. The main point of the construction is then to choose the connections $\tilde{D}$ appropriately such that the limit $R^\Phi = \lim_{\epsilon\to 0}\Phi(\widehat{D}^\epsilon)$ exists as a current, which will then have support on Z, and represent $\Phi(\mathscr{G})$. We describe below how this is done in the two cases.

In both cases, the connections are constructed such that if one for $k = 0,\dots,N$, writes on $M\setminus Z$

$$\tilde{D}_k = D_k + a_k, \tag{5.3}$$

where a_k is a smooth $\mathrm{End}(E_k)$-valued 1-form on $M\setminus Z$, then a_k admits an extension as an almost semi-meromorphic current to Z. It will then follow from the theory of almost semi-meromorphic currents that the limit $\Phi(\widehat{D}^\epsilon)$ exists. Note that when $\tilde{D}_k$ is of this form, the regularization (5.2) may be written as

$$\widehat{D}^\epsilon = \chi_\epsilon a + D. \tag{5.4}$$

If $D = (D_N,\dots,D_0)$, we say that D is of type $(1,0)$, or that D is a $(1,0)$-connection, if D_k is of type $(1,0)$ for each k.

Remark 5.1. Note that if D and $\tilde{D}$ are of type $(1,0)$, then so is $\tilde{D}^\epsilon$.

In both cases, the morphisms a_k are constructed in terms of the following morphisms. For $k = 1,\dots,N$, we let $\sigma_k : E_{k-1}\to E_k$ be the *minimal inverse* of φ_k. These are smooth vector bundle morphisms defined outside the analytic set $Z_k\subset Z$ where φ_k does not have optimal rank and are determined by the following properties:

$$\varphi_k\sigma_k\varphi_k = \varphi_k,\quad \operatorname{im}\sigma_k\perp\operatorname{im}\varphi_{k+1}\quad\text{and}\quad\sigma_{k+1}\sigma_k = 0.$$

The morphisms σ_k may be continued as almost semi-meromorphic sections of $\mathrm{End}(E)$, see, e.g., the proof of Lemma 2.1 in [LW22].

Lemma 5.2. *Let $\mathscr{G}, Z, (E,\varphi), D$ be as above, and assume that $Z = \operatorname{supp}\mathscr{G}$ has codimension ≥ 1. Define a connection $\tilde{D}$ on E through* (5.3) *with $a_k = -\sigma_k D\varphi_k$ for $k = 1,\dots,N$. Then $\tilde{D}$ is a connection on $M\setminus Z$ which is compatible with (E,φ) and a_k admits an extension as an almost semi-meromorphic current to M. If D is of type $(1,0)$, then so is $\tilde{D}$.*

Proof. This definition of $\tilde{D}_k$ coincides with the definition in [LW22, Remark 4.5]. It also follows by that remark that $\tilde{D}$ is compatible with (E,φ) since (E,φ) is pointwise exact on $M\setminus Z$. Since σ_k extends as an almost semi-meromorphic current to M, and almost semi-meromorphic currents are closed under multiplication with smooth forms, it follows that a_k extends as an almost semi-meromorphic current to all of M.

If D_k and D_{k-1} are of type $(1,0)$, then since φ_k is holomorphic, $D\varphi_k$ will be of bidegree $(1,0)$, and hence $\tilde{D}_k = D_k + a_k$ will be of type $(1,0)$ as well. □

We now consider the case of foliations from [KLW23], i.e., we assume that $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation on M. We furthermore assume that $N\mathscr{F}$ admits a resolution of the form

$$0\to E_N \xrightarrow{\varphi_N}\cdots\xrightarrow{\varphi_1} TM\to N\mathscr{F}\to 0, \tag{5.5}$$

i.e., at level 0, the bundle equals TM, and that the connection D_0 on TM is of type $(1,0)$ and torsion free. In [KLW23], we then defined a certain smooth morphism $\mathcal{D}\varphi_1 : E_0 \otimes TM \to E_1$, which is obtained from $D\varphi_1$ by the identification of 1-forms with $\mathrm{Hom}(TM, \mathcal{O})$, see [KLW23, equation (2.11)] for a precise definition. Using this morphism, one may produce a new morphism

$$b = \mathcal{D}\varphi_1\sigma_1\left(dz \cdot \frac{\partial}{\partial z}\right), \tag{5.6}$$

which takes values in $\mathrm{End}(E_0)$ and is smooth on $M\ Z$, see [KLW23, equation (5.7)], and where $dz \cdot \frac{\partial}{\partial z}$ denotes the 1-form valued vector in $E_0 = TM$ which invariantly may be described as the vector induced by Id_{TM} through identifying elements of $\mathrm{End}(TM)$ with elements of $\mathrm{Hom}(TM, \mathcal{O}) \otimes TM$, i.e., 1-form valued sections of TM.

Lemma 5.3. *Let $\mathscr{G}, Z, (E, \varphi), D$ be as above, and assume that $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation on M. Define a connection $\tilde{D}$ on E through* (5.3) *with*

$$a_0 = b(\mathrm{Id}_{E_0} - \varphi_1\sigma_1) - D\varphi_1\sigma_1 \quad \text{and} \quad a_k = -D\varphi_{k+1}\sigma_{k+1} \quad \text{for} \quad k = 1, \dots, N,$$

where b is defined by [KLW23, equation (5.7)]. *Then $\tilde{D}$ is a connection on $M \setminus Z$ which is compatible with (E, φ), a_k admits an extension as an almost semi-meromorphic current to M, and if $\tilde{D}_{-1}$ denotes the connection on $E_{-1} := \operatorname{coker} \varphi_1|_{M\setminus Z} \cong N\mathscr{F}|_{M\setminus Z}$ induced by $\tilde{D}_0$, then $\tilde{D}_{-1}$ is a basic connection on E_{-1}. Furthermore, the form b is locally defined in terms of (E, φ) and its Hermitian metrics and connection D. If $D_1, \dots, D_N$ are of type $(1,0)$, then so is $\tilde{D}$.*

Proof. It follows by [KLW23, equation (5.7), (5.16), (5.17) and (5.18)] that $\tilde{D}$ defined here coincides with the connections $\tilde{D}$ in [KLW23] (while we remark that the smooth reference connection on E_0 that we here denote by D_0 is denoted D^{TM} in [KLW23], and the D_0 that appears in [KLW23, Section 5.1] has a different meaning). It follows similarly as above that a_k is smooth on $M \setminus Z$, and by additionally using that almost semi-meromorphic currents form an algebra, it follows that a_k admits an extension as an almost semi-meromorphic current on M, see also [KLW23, Lemma 5.5] (where we remark that the form a_0 defined here is different, but related to the form a_0 in [KLW23]). By [KLW23, Proposition 5.4], $(\tilde{D}_N, \dots, \tilde{D}_0, D_{basic})$ is compatible with the augmented complex (2.4), so by the discussion in Section 2.2,$\tilde{D} = (\tilde{D}_N, \dots, \tilde{D}_0)$ is compatible with (E, φ), and D_{basic} is the connection on E_{-1} induced by $\tilde{D}_0$. By the definition of b, it follows that on an open set $U \subseteq M$, it only depends (E, φ) and its Hermitian metrics and connections (at levels 0 and 1) restricted to U. $\square$

6. Global Chern forms

In this section we briefly recall the necessary parts of the construction of simplicial resolutions and characteristic classes due to Green, [Gre80]. Further details can be found in [Gre80, TT86, Hos23a, Hos23b].

6.1. **Simplicial resolutions in the sense of Green.**

Assume that $\mathscr{G}$ is a coherent analytic sheaf on M, and that $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a (locally finite) Stein open cover of M. Moreover, assume that for some $N \geq 0$ and for each $\alpha \in I$, there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$,

$$0 \to E^\alpha_N \xrightarrow{\varphi^\alpha_N} \cdots \xrightarrow{\varphi^\alpha_1} E^\alpha_0 \to \mathscr{G}|_{U_\alpha} \to 0, \tag{6.1}$$

of $\mathcal{O}$-modules over U_α. By Hilbert's syzygy theorem, locally we can always find locally free resolutions of length at most n, so after possibly refining $\mathcal{U}$ we may assume that we have resolutions (6.1) for each vertex α of $N\mathcal{U}$.

Starting from the locally free resolutions $(E^\alpha, \varphi^\alpha)_{\alpha\in I}$ Green, [Gre80], see also [TT86], constructed a "simplicial resolution" of $\mathscr{G}$. The simplicial resolution consists of, for each $\Delta = (\alpha_0, \dots, \alpha_p) \in N\mathcal{U}$, a locally free resolution $(E^\Delta, \varphi^\Delta)$ of $\mathscr{G}|_{U_\Delta}$,

$$0 \to E_N^\Delta \xrightarrow{\varphi_N^\Delta} \cdots \xrightarrow{\varphi_1^\Delta} E_0^\Delta \to \mathscr{G}|_{U_\Delta} \to 0, \tag{6.2}$$

and the different resolutions fit together in a certain way. In fact, from the $(E^\alpha, \varphi^\alpha)$ one can construct a so-called twisted resolution of $\mathscr{G}$ in the sense of Toledo and Tong, [TT78], and given this Green constructed his resolution. How the resolutions fit together is summarised in [Gre80, Proposition 1.4], see also [TT86, p. 264]. In particular, for each subsimplex τ of $\Delta = (\alpha_0, \dots, \alpha_p)$, there is an isomorphism (in the category of $\mathcal{O}_{U_\Delta}$-modules)

$$E^\Delta \cong E^\tau \oplus E_\tau^\Delta, \tag{6.3}$$

where E_τ^Δ is an elementary sequence in the $E^{\alpha_j}, j = 0, \dots, p$. Furthermore, if ω is in turn a subsimplex of τ, then there is an isomorphism (in the category of $\mathcal{O}_{U_\Delta}$-modules)

$$E_\omega^\Delta \cong E_\omega^\tau \oplus E_\tau^\Delta. \tag{6.4}$$

We will say that $(E^\Delta, \varphi^\Delta)$ is a *(holomorphic) simplicial resolution of $\mathscr{G}$ with respect to $\mathcal{U}$.*

Remark 6.1. The construction of Green applies also for sheaves of $\mathcal{E}_M$-modules (smooth functions) or $\mathcal{A}_M$-modules (real analytic functions), to produce smooth or real analytic simplicial resolutions out of a family of locally free smooth or real analytic resolutions. By flatness, a holomorphic simplicial resolution is also a real analytic simplicial resolution, and a real analytic simplicial resolution is also a smooth simplicial resolution. All the results in this section hold also for such resolutions, except for the results concerning $(1,0)$-connections.

6.2. **Connections and characteristic classes.** For each $\alpha \in I$, assume that E^α is equipped with a connection D^α.

Let D_α^Δ be the induced connection on the elementary sequence E_α^Δ, cf., Section 2.4, and let $\widetilde{D}^{\Delta,\alpha}$ be the connection on E^Δ induced by $D^\alpha \oplus D_\alpha^\Delta$. Note that if α is a vertex of τ, which in turn is a subsimplex of Δ, then it follows by (6.4) that

$$\widetilde{D}^{\Delta,\alpha} = \widetilde{D}^{\tau,\alpha} \oplus D_\tau^\Delta, \tag{6.5}$$

where D_τ^Δ is the connection induced by the elementary sequence E_τ^Δ.

Given $\Delta = (\alpha_0, \dots, \alpha_p) \in N\mathcal{U}$, let π denote the projection $U_\Delta \times \Delta_p \to U_\Delta$, where Δ_p is the standard simplex (2.12), and let

$$D^\Delta = \sum_{j=0}^p t_j \pi^* \widetilde{D}^{\Delta,\alpha_j}. \tag{6.6}$$

Then D^Δ is a connection on $\pi^* E^\Delta$.

Let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial of degree ℓ. On $U_\Delta \times \Delta_p$ we have the characteristic form $\Phi(D^\Delta)$ defined as in (2.8). We can associate with $D = (D^\alpha)_{\alpha\in I}$ the Čech-de Rham cochain $\check{\Phi}(D) \in \check{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet)$, defined by

$$\check{\Phi}(D)_\Delta = \pi_* \Phi(D^\Delta) = \int_{\Delta_p} \Phi(D^\Delta). \tag{6.7}$$

Note that $\check{\Phi}(D)_\Delta \in \mathcal{E}^{2\ell-p}(U_\Delta)$, so that $\check{\Phi}(D)$ has total (Čech-de Rham) degree 2ℓ.

Proposition 6.2. *Assume that $\mathscr{G}$, $\mathcal{U}$, D and Φ are as above. Then $\nabla\check{\Phi}(D) = 0$.*

When $\Phi = e_\ell$, the above proposition is [Gre80, Lemma 2.2].

Proof. Note that

$$\Phi(D^\Delta)_\Delta = \Phi(\tilde{D}^{\Delta,\alpha_0}, \dots, \tilde{D}^{\Delta,\alpha_p}),$$

where the right-hand side is defined by (2.13). It then follows by (2.14), Lemma 2.2, (6.3) and (6.4) that $\nabla\check{\Phi}(D) = 0$. □

Remark 6.3. One could alternatively have defined first Čech-de Rham cochains $\Phi(D)$ just for $\Phi = e_\ell$ being the elementary symmetric polynomials, and then defined $\check{\Phi}(D)$ in terms of $\check{e}_\ell(D)$ through a formula corresponding to (2.8). This would yield a different definition of $\check{\Phi}(D)$, which would be cohomologous to our definition by an explicit cochain, cf., [Suw00, Lemma 1.5 and Proposition 1.6]. The results in this article would also hold for this alternative definition, but the one we have chosen seems to be the most natural and convenient for our results.

We say that a form on $U_\Delta \times \Delta_p$ has degree (r, s, q) if it (in local coordinates) can be written as a sum of forms of the form

$$f dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s} \wedge dt_{k_1} \wedge \wedge dt_{k_q},$$

where f is a smooth function. We say that a connection on $\pi^* E^\Delta$ is a $(1,0)$-connection if its connection matrices in frames induced by local holomorphic frames of E^Δ are of degree $(1, 0, 0)$. Note that when $p = 0$, it coincides with the usual notion of $(1, 0)$-connections.

Note that if the D^{α_j} are $(1, 0)$-connections then so is D^Δ. Moreover, in view of Section 2.4 note that if D^{α_j} is compatible with $(E^{\alpha_j}, \varphi^{\alpha_j})$ for $j = 0, \dots, p$, then D^Δ is compatible with $(\pi^* E^\Delta, \pi^*\varphi^\Delta)$.

Lemma 6.4. *Assume that $\mathscr{G}$, $\mathcal{U}$, D and Φ are as above. Assume furthermore that for each $\alpha \in I$, D^α is a $(1, 0)$-connection. Then the elements of $\check{\Phi}(D)$ are sums of forms of bidegree $(\ell + r, s)$, where $r, s \geq 0$ and $r + s + p = \ell$.*

Proof. Let us fix $\Delta = (\alpha_0, \dots, \alpha_p) \in N\mathcal{U}$. Recall that since the D^{α_j} are $(1, 0)$-connections, then so are the connections $\widetilde{D}^{\Delta,\alpha_j}$. Since the statement is local we may work in a local trivialization. Let $\theta_k^{\alpha_j}$ be the connection matrix of $\widetilde{D}_k^{\Delta,\alpha_j}$; with a slight abuse of notation we will let it denote also the connection matrix of $\pi^* \widetilde{D}_k^{\Delta,\alpha_j}$. Then the connection matrix of D_k^Δ equals

$$\theta_k^\Delta = \sum_{j=0}^{p} t_j \theta_k^{\alpha_j},$$

cf. (6.6). Now the curvature matrix of D_k^Δ equals

$$\Theta(D_k^\Delta) = \theta_k^\Delta \wedge \theta_k^\Delta + \sum_{j=0}^{p} (dt_j \wedge \theta_k^{\alpha_j} + t_j \partial\theta_k^{\alpha_j} + t_j \bar{\partial}\theta_k^{\alpha_j}).$$

Since θ_k^Δ is a $(1, 0)$-form it follows that $\Theta(D_k^\Delta)$ is a sum of terms of degree $(2, 0, 0)$, $(1, 0, 1)$, and $(1, 1, 0)$. Since $e_i(D^\Delta)$ is a polynomial of degree j in the entries of $\Theta(D_k^\Delta)$, $k = 0, \dots, N$ it follows that it is a sum of forms of degree $(j + r, s, q), r + s + q = j$ (since each factor has at least degree 1 in dz_j). Since Φ is homogeneous of degree ℓ, $\Phi(D^\Delta)$ is a sum of forms of degree $(\ell+r,s,q)$, $r+s+q=\ell$, where $r,s,q\geq 0$. It follows that $\check{\Phi}(D)_\Delta=\int_{\Delta_p}\Phi(D^\Delta)$ is a sum of forms of bidegree $(\ell+r,s)$, where $r,s\geq 0$ and $r+s+p=\ell$, since integration of a form of degree $(*,*,q)$ over Δ_p vanishes unless $q=p$. $\square$

The next result shows that $[\check{\phi}(D)]$ only depends on $\mathcal{G}$ in a suitable sense. To formulate the result, we consider two Stein open covers $\mathcal{U}_1=(U_\alpha)_{\alpha\in I_1}$ and $\mathcal{U}_2=(U_\beta)_{\beta\in I_2}$ of M. We define a new cover

$$\mathcal{U}_{12}=\mathcal{U}_1\cap\mathcal{U}_2=(U_\alpha\cap U_\beta)_{(\alpha,\beta)\in I_1\times I_2}.$$

Let $r_1: I_1\times I_2\to I_1$ be the projection $(\alpha,\beta)\mapsto\alpha$ and define r_2 analogously; then for $j=1,2$, $\mathcal{U}_{12}$ refines $\mathcal{U}_j$ by the refinement map r_j.

Proposition 6.5. *Assume that $\mathscr{G}$, $\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_{12},r_1,r_2$ and Φ are as above. Assume there is some $N\geq 0$ such that for $j=1,2$, and for each $\alpha\in I^j$, $\mathscr{G}|_{U_\alpha}$ admits a locally free resolution $(E^\alpha,\varphi^\alpha)$*

$$0\to E_N^\alpha\xrightarrow{\varphi_N^\alpha}E_{N-1}^\alpha\xrightarrow{\varphi_{N-1}^\alpha}\dots\xrightarrow{\varphi_2}E_1^\alpha\xrightarrow{\varphi_1^\alpha}E_0^\alpha\xrightarrow{\varphi_0^\alpha}\mathcal{G}|_{U_\alpha}\to 0, \tag{6.8}$$

and assume that E^α is equipped with a connection D^α. For $j=1,2$, let $D^j=(D^\alpha)_{\alpha\in I^j}$.

Then there is a cochain $\check{\eta}^\Phi\in\oplus_{p+q=2\ell-1}\check{C}^p(\mathcal{U}_{12},\mathcal{E}^q)$ such that

$$\nabla\check{\eta}^\Phi=r_2\check{\Phi}(D^2)-r_1\check{\Phi}(D^1). \tag{6.9}$$

The elements of $\check{\eta}^\Phi$ are polynomials in the entries of θ^α and $d\theta^\alpha$, where $\alpha\in I_1\coprod I_2$, and θ^α is the connection matrix of D^α in some local frame.

If each D^α is a $(1,0)$-connection, then

$$\check{\eta}^\Phi\in\bigoplus_{r\geq 0,\ r+s+p=\ell-1}\check{C}^p(\mathcal{U}_{12},\mathcal{E}^{\ell+r,s}). \tag{6.10}$$

Proof. We let

$$\mathcal{V}=\mathcal{U}_1\coprod\mathcal{U}_2=(U_\alpha)_{\alpha\in I_1\coprod I_2}.$$

For $j=1,2$ let $\rho_j: I_j\to I_1\coprod I_2$ be the inclusion $\alpha\mapsto\alpha$; then $\mathcal{U}_j$ refines $\mathcal{V}$ by the refinement map ρ_j. Let $\varrho_j=\rho_j\circ r_j: I_1\times I_2\to I_1\coprod I_2$ be the corresponding refinement maps of the refinement of $\mathcal{V}$ in $\mathcal{U}$, and let $h=h^p:\check{C}^p(\mathcal{V},\mathcal{E}^\bullet)\to\check{C}^{p-1}(\mathcal{U}_{12},\mathcal{E}^\bullet)$ be the associated homotopy (2.18).

From $(E^\alpha,\varphi^\alpha)_{\alpha\in I_1\coprod I_2}$ one can construct a simplicial resolution $(E^\Delta,\varphi^\Delta)_{\Delta\in N\mathcal{V}}$ of $\mathcal{G}$ with respect to $\mathcal{V}$ such that on "pure" subsets $U_\Delta,\Delta\in N\mathcal{U}_j$, it coincides with the original simplicial resolutions $(E^\Delta,\varphi^\Delta)_{\Delta\in N\mathcal{U}_j}$ with respect to $\mathcal{U}_j$, see, e.g., the proof of Theorem 2.4 in [Gre80].

As above we equip $(E^\Delta,\varphi^\Delta)_{\Delta\in N\mathcal{V}}$ with the connections $(D_N^\Delta,\dots,D_0^\Delta)$ constructed as above, and let $D=(D^\alpha)_{\alpha\in I_1\coprod I_2}$. We denote the Čech-de Rham cocycle $\check{\Phi}(D)$ by $\check{\Phi}_\mathcal{V}=\check{\Phi}_\mathcal{V}(D)$ to emphasize the dependence on $\mathcal{V}$. Then, since $(E^\Delta,\varphi^\Delta)_{\Delta\in N\mathcal{V}}$ coincides with $(E^\Delta,\varphi^\Delta)_{\Delta\in N\mathcal{U}_j}$ for $\Delta\in N\mathcal{U}_j$, it follows that for $\Delta\in N\mathcal{U}_j$, $\check{\Phi}_\mathcal{V}(D)_\Delta=\check{\Phi}_{\mathcal{U}_j}(D^j)_\Delta$, where $\check{\Phi}_{\mathcal{U}_j}=\check{\Phi}_{\mathcal{U}_j}(D)$ denotes the Čech-de Rham cocycle associated with the simplicial resolutions with respect to $\mathcal{U}_j$. In other words

$$\rho_j\check{\Phi}_\mathcal{V}=\check{\Phi}_{\mathcal{U}_j}. \tag{6.11}$$

Let

$$\check{\eta}^\Phi=h\check{\Phi}_\mathcal{V}(D)\in\bigoplus_{p+q=2\ell-1}\check{C}^p(\mathcal{U}_{12},\mathcal{E}^q).$$

Then

$$\nabla\check{\eta}^{\Phi} = (\nabla h + h\nabla)\check{\Phi}_{\mathcal{V}} = \varrho_2\check{\Phi}_{\mathcal{V}} - \varrho_1\check{\Phi}_{\mathcal{V}} = r_2\check{\Phi}_{\mathcal{U}_2} - r_1\check{\Phi}_{\mathcal{U}_1}, \tag{6.12}$$

where we have used that $\check{\Phi}_{\mathcal{V}}$ is a cocycle for the first equality, (2.19) for the second equality, and (6.11) for the last equality.

By the definition of $\Phi_{\mathcal{V}}$ and h, (2.18), it follows that the elements of $\check{\eta}^{\Phi}$ are polynomials in the entries of θ^{α} and $d\theta^{\alpha}$, where $\alpha \in I_1 \coprod I_2$, and θ^{α} is the connection matrix of D^{α} in some local frame.

Assume now that for each $\alpha \in I_1 \coprod I_2$, D^{α} is a $(1,0)$-connection. By Lemma 6.4,

$$\check{\Phi}_{\mathcal{V}} \in \bigoplus_{r\geq 0,\ r+s+p=\ell} \check{C}^p(\mathcal{V}, \mathcal{E}^{\ell+r,s}).$$

Moreover, note that h maps $\check{C}^p(\mathcal{V}, \mathcal{E}^{\ell+r,s})$ to $\check{C}^{p-1}(\mathcal{U}_{12}, \mathcal{E}^{\ell+r,s})$, and thus

$$\check{\eta}^{\Phi} \in \bigoplus_{r\geq 0,\ r+s+p=\ell-1} \check{C}^p(\mathcal{U}_{12}, \mathcal{E}^{\ell+r,s}).$$

It then follows that (6.10) holds. □

If $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$ in the above proposition, then one may alternatively find $\check{\eta}^{\Phi} \in \oplus_{p+q=2\ell}\check{C}^p(\mathcal{U}, \mathcal{E}^q)$ such that

$$\nabla\check{\eta}^{\Phi} = \check{\Phi}(D^2) - \check{\Phi}(D^1). \tag{6.13}$$

This follows from making minor adjustments to the above proof, i.e., letting $\mathcal{U}_{12} = \mathcal{U}$, and letting r_1 and r_2 be the identity maps.

Remark 6.6. Assume that $\Delta \in N\mathcal{U}$ is a vertex, i.e., $\Delta = (\alpha)$. Then note that $\Delta_p = \{1\}$. In this case we identify $U_\Delta \times \Delta_p$ with U_Δ and then π is just the identity. Then $D^{\Delta} = D^{\alpha}$ and hence $\check{\Phi}(D)_\Delta = \Phi(D^{\alpha})$.

6.3. **Characteristic forms.** Assume that $(\psi_\alpha)_{\alpha\in I}$ is a partition of unity subordinate to $\mathcal{U}$ and let Ψ be the corresponding morphism (2.17). Then we define the characteristic form $\Phi(D)$ as

$$\Phi(D) = \Psi(\check{\Phi}(D)). \tag{6.14}$$

Since $\check{\Phi}(D)$ is a Čech-de Rham cocycle of total degree 2ℓ and Ψ is an isomorphism on cohomology, it follows that $\Phi(D)$ is a closed form of degree 2ℓ. The de Rham cohomology class that $\Phi(D)$ defines indeed only depends on $\mathcal{G}$, cf., the proof of Lemma 7.3 below.

Example 6.7. Assume that

$$0 \to E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} E_0 \to \mathscr{G} \to 0 \tag{6.15}$$

is a locally free resolution of $\mathscr{G}$ in M, and assume that $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is a Stein cover of M. Then $(E^{\Delta}, \varphi^{\Delta})_{\Delta\in N\mathcal{U}}$, where $(E^{\Delta}, \varphi^{\Delta}) = (E, \varphi)$, is a simplicial resolution of $\mathscr{G}$ with respect to $\mathcal{U}$. In this case each E^{Δ}_{α} is the zero complex, cf. (6.3).

Assume that (E, φ) is equipped with a connection D, and for $\alpha \in I$, equip $(E^{\alpha}, \varphi^{\alpha})$ with the connection $D^{\alpha} := D$. Take $\Delta = (\alpha_0, \dots, \alpha_p)$, $p > 0$, and let D^{Δ} be the connection on π^*E^{Δ} as defined in Section 6.2. Using the notation in that section, note that $\widetilde{D}^{\Delta,\alpha} = D^{\alpha} = D$ and thus, in view of (6.6) $D^{\Delta} = \pi^*D$. If follows that $\Phi(D^{\Delta})$ is independent of t and thus $\check{\Phi}(D)_\Delta = 0$. If $\Delta = (\alpha)$, then $D^{\Delta} = D$ and thus $\check{\Phi}(D)_{(\alpha)} = \Phi(D)|_{U_\alpha}$, where $\Phi(D)$ within this example denotes the characteristic form (2.8), cf. Remark 6.6 defined from (6.15).

Assume that $(\psi_\alpha)_{\alpha\in I}$ is a partition of unity subordinate to $\mathcal{U}$, and let Ψ be the corresponding morphism (2.17). Then the associated characteristic form (6.14), which we within this example denote by $\Phi_{Green}(D)$, is given by

$$\Phi_{Green}(D) = \Psi(\check{\Phi}(D)) = \sum_{\alpha\in I} \psi_\alpha \check{\Phi}(D)_{(\alpha)} = \sum_{\alpha\in I} \psi_\alpha \Phi(D) = \Phi(D).$$

Lemma 6.8. *Assume that for each $\alpha \in I$, D^α is a $(1,0)$-connection. Then $\Phi(D)$ is a sum of forms of bidegree $(\ell+j, \ell-j)$, $0 \le j \le \ell$.*

Proof. Note that Ψ maps an element in $\check{C}^p(\mathcal{U}, \mathcal{E}^{q_1,q_2})$ to a sum of forms of bidegree $(q_1 + p_1, q_2 + p_2)$, where $p_1, p_2 \ge 0$ and $p_1 + p_2 = p$. It then follows by Lemma 6.4 that $\Phi(D) = \Psi(\check{\Phi}(D))$ is a sum of forms of bidegree $(\ell + r + p_1, s + p_2)$, where $r, p_1, s, p_2 \ge 0$ and $\ell + r + p_1 + s + p_2 = 2\ell$, i.e., $\Phi(D)$ is of the desired form. $\square$

7. Characteristic forms of coherent sheaves

Assume that $\mathscr{G}$ is a coherent sheaf on M, and that Z denotes the set where $\mathscr{G}$ is not a vector bundle, and that either $\operatorname{supp}\mathscr{G}$ has codimension ≥ 1 and then let $\kappa = n$, or that $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation of rank κ on M. Assume also that $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is a Stein open cover and $N \in \mathbb{N}$ is such that in each U_α there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$ of length $\le N$. Let $(E^\Delta, \varphi^\Delta)_{\Delta\in N\mathcal{U}}$ be the corresponding simplicial resolution. Assume also that Φ is a symmetric polynomial in $\mathbb{C}[z_1, \dots, z_n]$ of degree ℓ, where $n - \kappa < \ell \le n$.

In this section, all the results, except for the last part of Lemma 7.1 which concerns connections of type $(1,0)$, hold when the resolutions $(E^\alpha, \varphi^\alpha)$ are complexes of smooth or real analytic vector bundles, cf., Remark 6.1.

Lemma 7.1. *Assume that M, $\mathscr{G}$, Z, Φ, κ, $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, $(E^\alpha, \varphi^\alpha)$ are as above. Moreover, assume that there is a closed neighborhood $\Sigma \subset M$ of Z, and for each $\alpha \in I$, assume that E^α is equipped with a connection D^α which is compatible with $(E^\alpha, \varphi^\alpha)$ over $U_\alpha \setminus \Sigma$, and in case $\mathscr{G} = N\mathscr{F}$, assume that the connection D^α_{-1} on $N\mathscr{F}|_{U_\alpha}$ induced by D^α_0 on E_0 is a basic connection on $U_\alpha \setminus \Sigma$. Let $\Phi(D)$ be the corresponding characteristic form* (6.7). *Then $\check{\Phi}(D)$ and $\Phi(D)$ have support in Σ.*

For the proof in the case when $\mathscr{G} = N\mathscr{F}$, we need the following slight generalization of Baum-Bott's vanishing theorem, Theorem 3.2.

Lemma 7.2. *Assume that $\mathscr{F}$ is a regular foliation of rank κ on a complex manifold M of dimension n, that Φ is a homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ of degree ℓ with $n - \kappa < \ell \le n$, and that $D^0, \dots, D^p$ are basic connections on $N\mathscr{F}$. Let Δ_p be the simplex* (2.12)*, let π denote the projection $M \times \Delta_p \to M$, and let D be the connection*

$$D = \sum_{j=0}^{p} t_j \pi^* D^j$$

on $M \times \Delta_p$; here $\pi^* D^j$ denotes the pullback connection. Then

$$\Phi(D) = 0 \quad \text{on} \quad M \times \Delta_p.$$

Proof. This follows essentially by the arguments in the proof of [BB72, Proposition 3.27]. As in that proof, choose local coordinates $z_1, \dots, z_n$ on M such that $T\mathscr{F}$ is generated by $\partial/\partial z_1, \dots, \partial/\partial z_\kappa$. Then for each $j = 0, \dots, p$, by the arguments in that proof each entry of the connection matrix θ_j of D^j is in the ideal generated by $dz_{\kappa+1},\dots,dz_n$. It follows that each entry the connection matrix $\theta = \sum t_j\pi^*\theta_j$ of D is in the ideal $\widetilde{I}$ generated by $\pi^*dz_{\kappa+1},\dots,\pi^*dz_n$, and thus each entry in the curvature matrix $\Theta(D)$ is in $\widetilde{I}$. As in that proof it follows that $\Phi(D)=0$ since $\ell > n-\kappa$. □

Proof of Lemma 7.1. Take $\Delta = (\alpha_0,\dots,\alpha_p)$, $p\geq 0$, and let $\tilde{D}^\Delta$ be the connection on π^*E^Δ associated to $(\tilde{D}^\alpha)$, as defined in Section 6.2. Since $(D^\Delta_N,\dots,D^\Delta_0)$ is compatible with $(\pi^*E^\Delta,\pi^*\varphi^\Delta)$ in $(U_\Delta\times\Delta_p)\setminus\pi^{-1}\Sigma$, it follows by Lemma 2.1 that

$$\Phi(D^\Delta)=\Phi(D^\Delta_{-1})$$

in $(U_\Delta\times\Delta_p)\setminus\pi^{-1}\Sigma$, where D^Δ_{-1} is the connection on $\pi^*\mathscr{G}|_{U_\Delta\setminus\Sigma}$ induced by D^Δ_0. In case $\operatorname{supp}\mathscr{G}$ has codimension ≥ 1, then $\mathscr{G}|_{U_\Delta\setminus\Sigma}=0$, so $\Phi(D^\Delta_{-1})=0$. Otherwise, $\mathscr{G}=N\mathscr{F}$, and then by Lemma 7.2 $\Phi(D^\Delta_{-1})=0$ in $(U_\Delta\times\Delta_p)\setminus\pi^{-1}\Sigma$. In any case, $\Phi(D^\Delta)=0$ in $(U_\Delta\times\Delta_p)\setminus\pi^{-1}\Sigma$. It follows that $\check{\Phi}(D)_\Delta$ has support in $U_\Delta\cap\Sigma$ and consequently $\Phi(D)$ has support in Σ as well, cf. (2.17). □

Lemma 7.3. *Assume that M, $\mathscr{G}$, Z, Σ, κ and Φ are as above. Assume that there is some $N\in\mathbb{N}$ such that for $j=1,2$, $\mathcal{U}_j=(U_\alpha)_{\alpha\in I_j}$ is a Stein open cover of M such that for each $\alpha\in I_j$, there is a locally free resolution $(E^\alpha,\varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$ of length $\leq N$. Assume that for any $\alpha\in I_j$, $j=1,2$, E^α is equipped with a connection D^α which is compatible with $(E^\alpha,\varphi^\alpha)$ in $U_\alpha\setminus\Sigma$, and in case $\mathscr{G}=N\mathscr{F}$, then the connection D^α_{-1} on $N\mathscr{F}|_{U_\alpha}$ induced by D^α_0 on E_0 is a basic connection on $U_\alpha\setminus\Sigma$. Finally, for $j=1,2$, assume that $(\psi_\alpha)_{\alpha\in I_j}$ is a partition of unity subordinate to $\mathcal{U}_j$.*

Let $\Phi(D_j)$ denote the form (6.14) *associated to D_j and $(\psi_\alpha)_{\alpha_j\in I_j}$. Then there is a form η^Φ with support in Σ such that*

$$d\eta^\Phi=\Phi(D_1)-\Phi_1(D_2). \tag{7.1}$$

The form η^Φ is locally given as a polynomial in the entries of θ^α, $d\theta^\alpha$ and ψ_α, where $\alpha\in I_1\coprod I_2$, and θ^α is the connection matrix of D^α in some local frame.

Furthermore, if each D^α is of type $(1,0)$, then η^Φ is a sum of forms of bidegree $(\ell+i,\ell-1-i)$ for $0\leq i\leq \ell-1$.

Proof. Let $\check{\eta}^\Phi\in\oplus_{p+q=2\ell-1}\check{C}^p(\mathcal{U}_1\cap\mathcal{U}_2,\mathcal{E}^q)$ in Proposition 6.5, which satisfies (6.9).

For $j=1,2$, let $\Psi_{\mathcal{U}_j}:\check{C}^\bullet(\mathcal{U}_j,\mathcal{E}^\bullet)\to\mathcal{E}^\bullet(M)$ be the morphism (2.17) associated with the partition of unity $(\psi_\alpha)_{\alpha\in I_j}$ subordinate to $\mathcal{U}_j$. Consider the refinement maps $r_j:\mathcal{U}_{12}\to\mathcal{U}_j$, $j=1,2$, from Proposition 6.5, where $\mathcal{U}_{12}=\mathcal{U}_1\cap\mathcal{U}_2$, and the partition of unity

$$(\psi_{(\alpha,\beta)})_{(\alpha,\beta)\in I_1\times I_2},\ \psi_{(\alpha,\beta)}=\psi_\alpha\psi_\beta$$

subordinate to $\mathcal{U}_{12}$. Let $\Psi_{\mathcal{U}_{12}}:\check{C}^\bullet(\mathcal{U}_{12},\mathcal{E}^\bullet)\to\mathcal{E}^\bullet(M)$ be the associated morphism (2.17). Observe that (by Fubini)

$$\Psi_{\mathcal{U}_{12}}\circ r_j=\Psi_{\mathcal{U}_j}. \tag{7.2}$$

Now let

$$\eta^\Phi=\Psi_{\mathcal{U}_{12}}(\check{\eta}^\Phi).$$

Then

$$d\eta^\Phi=\Psi_{\mathcal{U}_{12}}(r_2\check{\Phi}_{\mathcal{U}_2})-\Psi_{\mathcal{U}_{12}}(r_1\check{\Phi}_{\mathcal{U}_1})=\Psi_{\mathcal{U}_2}(\check{\Phi}_{\mathcal{U}_2})-\Psi_{\mathcal{U}_1}(\check{\Phi}_{\mathcal{U}_1})=\Phi(D_2)-\Phi(D_1); \tag{7.3}$$

here we have used that $\Psi_{\mathcal{U}_{12}}$ is a morphism of complexes and (6.12) for the first equality, (7.2) for the second equality, and (6.14) for the last equality. This proves (7.1).

Recall that $\check{\Phi}_{\mathcal{V}}$ is defined by

$$(\check{\Phi}_{\mathcal{V}})_\Delta = \int_{\Delta_p} \Phi_{\mathcal{V}}(\widehat{D}^{\Delta,\epsilon}) \tag{7.4}$$

if $\Delta = (\alpha_0, \dots, \alpha_p)$, see (6.7). Recall from Lemma 7.1 that, for $\Delta \in N\mathcal{V}$, $\Phi_{\mathcal{V}}(D^\Delta)$ has support in $(U_\Delta \cap \Sigma) \times \Delta_p$. It follows that $(\check{\Phi}_{\mathcal{V}})_\Delta$ has support in $U_\Delta \cap \Sigma$. Hence, for each $\Delta \in N\mathcal{U}_{12}$, $(h\check{\Phi}_{\mathcal{U}_V})_\Delta$ has support in $U_\Delta \cap \Sigma$. Therefore η^Φ has support in Σ.

By Proposition 6.5, the elements of $\check{\eta}^\Phi$ are polynomials in the entries θ^α and $d\theta^\alpha$, and thus, η^Φ is locally given as a polynomial in such entries and the ψ_α.

Assume finally that for each $\alpha \in I_j$, $j = 1, 2$, that D^α is a $(1,0)$-connection. Recall from the proof of Lemma 6.8 that Ψ maps an element in $\check{C}^p(\mathcal{U}, \mathcal{E}^{q_1,q_2})$ to a sum of forms of bidegree $(q_1 + p_1, q_2 + p_2)$, where $p_1, p_2 \geq 0$ and $p_1 + p_2 = p$. It then follows from (6.10) that η^Φ is a sum of forms of bidegree $(\ell + i, \ell - 1 - i)$, where $0 \leq i \leq \ell - 1$. □

8. Global Baum-Bott currents

Assume that M, $\mathscr{G}$, Z, Φ, κ, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, $(E^\alpha, \varphi^\alpha)$ are as in the previous section. In this section, the resolutions $(E^\alpha, \varphi^\alpha)$ are assumed to be complexes of holomorphic vector bundles. Assume also that each E^α is equipped with Hermitian metrics and a connection D^α. We will apply the construction of connections $\tilde{D}_k$ from Section 5 on each $(E^\alpha, \varphi^\alpha), D^\alpha$, which yields connections defined on $U^\alpha \setminus Z$ satisfying certain properties. If $\operatorname{supp} \mathscr{G}$ has codimension ≥ 1, this is obtained from Lemma 5.2 and in case $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation of M, this is obtained from Lemma 5.3. In both cases, this yields a family of connections $\tilde{D}^\alpha$ of the form

$$\tilde{D}^\alpha_k = D^\alpha_k + a^\alpha_k, \tag{8.1}$$

where a_k is smooth on $U_\alpha \setminus Z$, and where a^α_k extends as an almost semi-meromorphic current to all of U_α.

We now define a family of regularizations of those connections. Let $\mathcal{V} = (V_\beta)_{\beta \in J}$ be an open cover of M. Choose $\chi \sim \chi_{[1,\infty)}$ and on each V_β choose a generically nonvanishing holomorphic section s^β of a Hermitian vector bundle such that $Z \cap V_\beta \subset \{s^\beta = 0\}$. Let $(\psi_\beta)_{\beta \in J}$ be a partition of unity subordinate to $\mathcal{V}$, and define a family of cut-off functions by

$$\chi_\epsilon = \sum_{\beta \in J} \psi_\beta \chi(|s^\beta|^2/\epsilon), \tag{8.2}$$

and let Σ_ϵ be the closure of $\{\chi_\epsilon < 1\}$ in M.

Given connections of the form (8.1), we define a new family of connections $\widehat{D}^{\alpha,\epsilon}$ by

$$\widehat{D}^{\alpha,\epsilon} := \chi_\epsilon \tilde{D}^\alpha + (1 - \chi_\epsilon) D^\alpha = \chi_\epsilon a^\alpha + D^\alpha. \tag{8.3}$$

Note that $\widehat{D}^{\alpha,\epsilon}$ is smooth on all of U_α.

Theorem 8.1. *Assume M is a complex manifold of dimension n, that $\mathscr{G}$ is a coherent sheaf on M, and that Z denotes the set where $\mathscr{G}$ is not a vector bundle, and that either $\operatorname{supp} \mathscr{G}$ has codimension ≥ 1 and let $\kappa = n$, or that $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation of rank κ on M. Let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial of degree ℓ with $n - \kappa < \ell \leq n$. Assume that $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a Stein open cover of M and $N \in \mathbb{N}$ be such that for each $\alpha \in I$, there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$ of length $\leq N$. In case $\mathscr{G} = N\mathscr{F}$, assume also that $E^\alpha_0 = TM|_{U_\alpha}$.*

Moreover, for each $\alpha \in I$, assume that E^α is equipped with Hermitian metrics and a connections D^α, and if $\mathscr{G} = N\mathscr{F}$, assume that D_0^α is of type $(1,0)$ and torsion free. Let χ_ϵ be a cut-off function of the form (8.2)*, and let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity subordinate to $\mathcal{U}$. Let $(\widehat{D}^{\alpha,\epsilon})$ be the family of connections defined in* (8.3)*.*

Then

$$R^\Phi := \lim_{\epsilon\to 0} \Phi((\widehat{D}^{\alpha,\epsilon})_{\alpha\in I}) \tag{8.4}$$

is a well-defined closed pseudomeromorphic current of degree 2ℓ with support on Z. Moreover, R^Φ only depends on $(E^\alpha, \varphi^\alpha)_{\alpha\in I}$, the Hermitian metrics and connections D^α, and $(\psi_\alpha)_{\alpha\in I}$ close to Z, and in particular it is independent of the choice of χ_ϵ. If we assume that each D^α is a $(1,0)$-connection, then R^Φ is a sum of currents of bidegree $(\ell+i, \ell-i)$ for $0 \le i \le \ell$.

Within the proof, we will consider connections whose entries in any local holomorphic frame over some open set U admit expansions of the form

$$A + \chi_\epsilon B, \tag{8.5}$$

where A and B are independent of χ_ϵ, A is smooth 1-form, and B is an almost semi-meromorphic current that is smooth outside of Z. If one considers products of entries C_j or dC_j, where C_j is a 1-form of the form (8.5), then we claim that one obtains an form which can be written as

$$\alpha + \sum_{k\ge 1} \chi_\epsilon^k \beta_k + \sum_{k\ge 1} \chi_\epsilon^{k-1} d\chi_\epsilon \beta_k', \tag{8.6}$$

where β_k and β_k' are almost semi-meromorphic. Indeed, this follows by using that almost semi-meromorphic currents form an algebra, that $\mathbf{1}_{U\setminus Z} dB$ is almost semi-meromorphic if B is almost semi-meromorphic by Lemma 4.1, and that $\chi_\epsilon \mathbf{1}_{U\setminus Z} dB = \chi_\epsilon dB$.

Proof. We will first show that the limit R^Φ exists as a pseudomeromorphic current. For $\alpha \in I$, let $\widehat{\theta}_k^\alpha$ be the connection matrix of $\widehat{D}_k^{\alpha,\epsilon}$. Then, in view of (8.1) and (8.3), the entries of $\widehat{\theta}_k^{\alpha,\epsilon}$ in a local frame are of the form (8.5). If $\widehat{\theta}^{\Delta,\alpha,\epsilon}$ denotes the connection matrix of the connection on E^Δ induced by $\widehat{D}^{\alpha,\epsilon}$ on E^α as defined by (6.5), it follows that the entries of $\widehat{D}^{\Delta,\alpha,\epsilon}$ are also of the same form. It follows that for each k the connection matrix of $\widehat{D}_k^{\epsilon,\Delta}$ is of the form $\sum_{j=0}^p t_j\theta_j$, where θ_j is a direct sum of forms (8.5), so the entries of the curvature matrix are polynomials in $t_1,\dots,t_p, dt_1,\dots,dt_p$ with coefficients of the form (8.6). Thus, $\Phi(\widehat{D}^{\Delta,\epsilon})$ may be expanded as a polynomial of the same form.

Since each term in (8.6) is independent of t, and linear combinations of terms of the form (8.6) are again of this form, it follows that $\check{\Phi}(\widehat{D}^\epsilon)_\Delta = \int_{\Delta_p} \Phi(\widehat{D}^{\Delta,\epsilon})$ is also of the form (8.6).

By Proposition 4.3 we conclude that the limit as $\epsilon \to 0$ of terms of the form (8.6) exist as pseudomeromorphic currents and the limits are independent of the choice of χ and the s^α. Since $\Phi(\widehat{D}^\epsilon)$ is a closed form of degree 2ℓ, R^Φ is a closed current of degree 2ℓ.

Since R^Φ is independent of the s^α we may choose s^α so that $\{s^\alpha = 0\} = U_\alpha \cap S$. Indeed, if ρ is the (generic) rank of φ_1^α, then we may take $s^\alpha = \bigwedge^\rho \varphi_1^\alpha$. Then $\bigcap_{\epsilon>0} \Sigma_\epsilon = S$. By (8.3), we have outside of Σ_ϵ that $\widehat{D}^{\alpha,\epsilon} = \widetilde{D}^\alpha$. It then follows by Lemma 5.2, Lemma 5.3 and Lemma 7.1 that $\Phi(\widehat{D}^\epsilon)$ has support in Σ_ϵ. Thus, R^Φ has support in Z.

By the definitions of $\widetilde{D}$ in Lemma 5.2 and Lemma 5.3, it follows that $\widehat{D}^{\alpha,\epsilon}$ are locally defined, in the sense that on any open set $U \subset U_\alpha$, then $\widehat{D}^{\alpha,\epsilon}$ only depend on $(E^\alpha, \varphi^\alpha)$, the

Hermitian metrics on E^α and the connections D^α on U. It follows that also $\widehat{D}^{\Delta,\epsilon}$ is locally defined, and for any open set $U \subset U_\Delta$, $\widehat{D}^{\Delta,\epsilon}$ only depends on the same data as above for any $\alpha \in \Delta$. Hence, R^Φ is locally defined in the sense that it only depends on the $(E^\alpha, \varphi^\alpha)$, the Hermitian metrics and connections D^α, and $(\psi_\alpha)_{\alpha\in I}$ close to Z.

Assume that each D^α is a $(1,0)$-connection. Then, by Remark 5.1 and Lemma 5.2 or Lemma 5.3, $\widehat{D}^{\alpha,\epsilon}$ is of type $(1,0)$. Hence, by Lemma 6.8, $\Phi(\widehat{D}^\epsilon)$ is a sum of forms of bidegree $(\ell+i, \ell-i)$, $0 \le i \le \ell$. It follows that R^Φ is a sum of currents of bidegree $(\ell+i,\ell-i)$, $0 \le i \le \ell$. □

8.1. **Independence of resolutions.** In this section we prove that the Baum-Bott currents in Theorem 8.1 only depend on Φ and $\mathcal{G}$, up to a d-exact pseudomeromorphic current with support on Z.

Theorem 8.2. *Let M be a complex manifold of dimension n, let $\mathscr{G}$ be a coherent sheaf on M, and let Z denote the set where $\mathscr{G}$ is not a vector bundle. Assume that either $\operatorname{supp}\mathscr{G}$ has codimension ≥ 1 and let $\kappa = n$, or that $\mathscr{G} = N\mathscr{F}$, where $\mathscr{F}$ is a holomorphic foliation of rank κ on M. Let $\Phi \in \mathbb{C}[z_1,\dots,z_n]$ be a homogeneous symmetric polynomial of degree ℓ with $n-\kappa < \ell \le n$.*

For $j=1,2$, assume that $\mathcal{U}_j = (U_\alpha)_{\alpha\in I_j}$ is a Stein open cover of M and $N \in \mathbb{N}$ is such that for each $\alpha \in I_1 \coprod I_2$, there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$ of length $\le N$. In case $\mathscr{G} = N\mathscr{F}$, assume also that $E_0^\alpha = TM|_{U_\alpha}$. Moreover, for each $\alpha \in I_1 \coprod I_2$, assume that E^α is equipped with Hermitian metrics and a connection D^α, and if $\mathscr{G} = N\mathscr{F}$, then assume that D_0^α is of type $(1,0)$ and torsion free.

Let for $j=1,2$, $(\psi_\alpha)_{\alpha\in I_j}$ be a partition of unity subordinate to $\mathcal{U}_j$. Let $R^\Phi_{(j)}, j=1,2$, denote the corresponding currents (8.4). *Then there exists a pseudomeromorphic current N^Φ of degree $2\ell-1$ with support on Z such that*

$$dN^\Phi = R^\Phi_{(2)} - R^\Phi_{(1)}. \tag{8.7}$$

Furthermore, if all D^α are of type $(1,0)$, then N^Φ is a sum of currents of bidegree $(\ell+i, \ell-1-i)$ for $0 \le i \le \ell-1$.

Proof. We may assume that the currents $R^\Phi_{(1)}$ and $R^\Phi_{(2)}$ are both defined with respect to the same family of cut-off functions (8.2).

We will use the notation from Proposition 6.5 and Lemma 7.3. Let N^Φ_ϵ be the form η^Φ defined in the proof of Lemma 7.3 starting from the connections $\widehat{D}^{\alpha,\epsilon}$. By the proof of Proposition 6.5, $N^\Phi_\epsilon = h\check{\Phi}_{\mathcal{V}}$, where we use the notation from the proof of Lemma 7.3 so that $\check{\Phi}_{\mathcal{V}} = \check{\Phi}_{\mathcal{V}}(\widehat{D}^{\Delta,\epsilon})$

Recall from the proofs of Theorem 8.1 and Lemma 7.1 that, for $\Delta \in N\mathcal{V}$, $(\check{\Phi}_{\mathcal{V}})_\Delta$ is of the form (8.6) and has support in $U_\Delta \cap \Sigma_\epsilon$. Hence, for each $\Delta \in N\mathcal{V}$, $(h\check{\Phi}_{\mathcal{V}})_\Delta$ is of the same form, cf. (2.18). Therefore N^Φ_ϵ is of the form (8.6) and has support in Σ_ϵ. As in the proof of Theorem 8.1 we conclude that the limit

$$N^\Phi := \lim_{\epsilon\to 0} N^\Phi_\epsilon$$

is a well-defined pseudomeromorphic current that is independent of the choice of χ_ϵ. Since the limit is independent of the choice, we may locally assume that there is a section s such that $\{s=0\} = Z$, and that $\chi_\epsilon = \chi(|s|^2/\epsilon)$. Since for each $\epsilon > 0$, N^Φ_ϵ has support in $U_\alpha \cap \Sigma_\epsilon$, it follows that N^Φ has support in $\bigcap_{\epsilon>0} \Sigma_\epsilon = Z$.

Since (7.3) holds for $\epsilon > 0$ with $\Phi_{\mathcal{U}_j} = \Phi_{\mathcal{U}_j}(\widehat{D}^\epsilon)$, it follows that

$$dN^\Phi = \lim_{\epsilon\to 0} \Phi_{\mathcal{U}_2}(\widehat{D}^\epsilon) - \lim_{\epsilon\to 0} \Phi_{\mathcal{U}_1}(\widehat{D}^\epsilon) = R^\Phi_{(2)} - R^\Phi_{(1)}.$$

Assume now that each D^α is a $(1,0)$-connection. By Lemma 7.3 N^Φ_ϵ is a sum of forms of bidegree $(\ell+i, \ell-1-i)$, where $0 \le i \le \ell-1$, and hence N^Φ is a sum of currents of the same bidegrees. □

8.2. **More precise versions of the main results.** The following theorem is a slightly more precise version of Theorem 1.1.

Theorem 8.3. *Let M be a complex manifold of dimension n, let $\mathscr{G}$ be a coherent analytic sheaf on M such that $Z = \operatorname{supp}\mathscr{G}$ has pure codimension $p \ge 1$ and let $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ be a homogeneous symmetric polynomial of degree ℓ with $1 \le \ell \le n$. Assume that $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is a Stein open cover of M such that for each $\alpha \in I$, there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $\mathscr{G}|_{U_\alpha}$ of the form* (1.8). *Moreover, for each $\alpha \in I$, assume that E is equipped with Hermitian metrics and connections D^α. Let χ_ϵ be a cut-off function of the form* (8.2), *and let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity subordinate to $\mathcal{U}$. Let $\widehat{D}^{\alpha,\epsilon}$ be the family of connections defined in* (8.3).

Then

$$R^\Phi := \lim_{\epsilon\to 0} \Phi((\widehat{D}^{\alpha,\epsilon})_{\alpha\in I}) \tag{8.8}$$

is a well-defined closed pseudomeromorphic current of degree 2ℓ with support on Z. Moreover, R^Φ only depends on $(E^\alpha, \varphi^\alpha)_{\alpha\in I}$, the Hermitian metrics and connections D^α and $(\psi_\alpha)_{\alpha\in I}$ close to Z, and in particular it is independent of the choice of χ_ϵ.

If we assume that all D^α are of type $(1,0)$, then R^Φ is a sum of currents of bidegree $(\ell + j, \ell - j)$ for $0 \le j \le \ell$. In addition,

$$R^{e_p} = (-1)^{p-1}(p-1)![\mathscr{G}]. \tag{8.9}$$

If $\ell < p$, then

$$R^{e_\ell} = 0, \tag{8.10}$$

and if $\ell_1 + \cdots + \ell_m \le p$, where $m \ge 2$, then

$$R^{e_{\ell_1}\cdots e_{\ell_m}} = 0. \tag{8.11}$$

Proof of Theorem 8.3. The existence of R^Φ, and the fact that it represents $\Phi(\mathcal{G})$, and has support on Z follows immediately by Theorem 8.1.

It remains to prove the formulae (8.9), (8.10) and (8.11). Assume thus that $\Phi = e_{\ell_1}\cdots e_{\ell_m}$, where $\ell = \ell_1 + \cdots + \ell_m \le p$. Since the equalities we should prove are local statements, and R^Φ is locally defined, we may assume that all U_α are equal to M. By Theorem 8.2, given different two choices of resolutions etc., R^Φ only depends on these choices up to a current dN^Φ, where N^Φ is a sum of pseudomeromorphic currents of bidegree $\le \ell - 1 \le p - 1$. By the dimension principle, $N^\Phi = 0$, so R^Φ is independent of all those choices. We may thus assume that M is a Stein open set, equipped with a Stein open cover consisting of just $\mathcal{U} = (U_\alpha)_{\alpha\in pt} = \{M\}$, with a resolution (E, φ) and which is equipped with a connection D. If we let $\tilde{D}$ denote the connection on E obtained by (5.2) and $\widehat{D}^\epsilon$ denote the connection obtained by the construction in (5.2) associated with $D, \tilde{D}$ and χ_ϵ, and $\widehat{D}^{\alpha,\epsilon}$ the induced connection on the cover, then clearly,

$$\Phi(\widehat{D}^{\alpha,\epsilon}) = \Phi(\widehat{D}^\epsilon),$$

and by (2.8),

$$\Phi(\widehat{D}^\epsilon) = e_{\ell_1}(\widehat{D}^\epsilon) \wedge \dots e_{\ell_m}(\widehat{D}^\epsilon).$$

The limit

$$\lim_{\epsilon\to 0} e_{\ell_1}(\widehat{D}^\epsilon) \wedge \dots e_{\ell_m}(\widehat{D}^\epsilon)$$

is exactly the current that in [LW22] is denoted by $c^{Res}_{\ell_1}(E,D) \wedge \dots \wedge c^{Res}_{\ell_m}(E,D)$.

The formulae we should prove then follow by [LW22, equations (1.3)-(1.5)] □

The following theorem is a slightly more precise version of Theorem 1.2.

Theorem 8.4. *Let M be a complex manifold of dimension n, let $\mathscr{F}$ be a holomorphic foliation of rank κ on M, and let $\Phi \in \mathbb{C}[z_1,\dots,z_n]$ be a homogeneous symmetric polynomial of degree ℓ with $n-\kappa < \ell \le n$. Assume that $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is a Stein open cover of M such that for each $\alpha \in I$, there is a locally free resolution $(E^\alpha, \varphi^\alpha)$ of $N\mathscr{F}|_{U_\alpha}$ of the form* (1.8)*. Moreover, for each $\alpha \in I$, assume that E^α is equipped with Hermitian metrics and a connection D^α such that D_0^α is of type $(1,0)$ and torsion free. Let χ_ϵ be a cut-off function of the form* (8.2)*, and let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity subordinate to $\mathcal{U}$. Let $\widehat{D}^{\alpha,\epsilon}$ be the family of connections defined in* (8.3).

Then

$$R^\Phi := \lim_{\epsilon\to 0} \Phi((\widehat{D}^{\alpha,\epsilon})_{\alpha\in I}) \tag{8.12}$$

is a well-defined closed pseudomeromorphic current of degree 2ℓ with support on sing $\mathscr{F}$. *Moreover, R^Φ only depends on $(E^\alpha,\varphi^\alpha)_{\alpha\in I}$, the Hermitian metrics and connections D^α, and $(\psi_\alpha)_{\alpha\in I}$ close to* sing $\mathscr{F}$*, and in particular it is independent of the choice of χ_ϵ.*

If Z' is a compact connected component of sing $\mathscr{F}$*, then*

$$R^\Phi_{Z'} := \mathbf{1}_{Z'} R^\Phi \tag{8.13}$$

is a pseudomeromorphic current with support on Z' which represents the Baum-Bott residue $\mathrm{res}^\Phi(\mathscr{F};Z') \in H_{2n-2\ell}(Z',\mathbb{C})$.

Assume now that each D^α is of type $(1,0)$. Then R^Φ is a sum of currents of bidegree $(\ell+j,\ell-j)$ for $0 \le j \le \ell$. Furthermore, if Z' is a connected component as above, and $\ell = \operatorname{codim} Z'$, then $R^\Phi_{Z'}$ only depends on $\mathscr{F}$, and if $\ell < \operatorname{codim} Z'$, then $R^\Phi_{Z'} = 0$.

Proof. The existence of R^Φ, and the fact that it represents $\Phi(N\mathscr{F})$, and has support on sing $\mathscr{F}$ follows immediately by Theorem 8.1.

It remains to prove the properties of $R^\Phi_{Z'}$. Since $R^\Phi_{Z'}$ is locally defined, we may replace M by a neighborhood of Z', and thus assume that $Z' = \text{sing}\,\mathscr{F} = Z$. Hence, $R^\Phi_{Z'} = R^\Phi$, which is pseudomeromorphic and has support on Z.

We now prove that $R^\Phi_{Z'}$ represents $\mathrm{res}^\Phi(\mathscr{F};Z')$. After possibly shrinking M further, we may assume that Z' is a deformation retract of M, and that there is a locally free resolution (E,φ) of the form (1.8) of $\mathcal{A}$-modules of $N\mathscr{F}\otimes\mathcal{A}$, see Section 3.1. Moreover, we may assume that the E_k and $N\mathscr{F}|_{M\setminus Z}$ are equipped with connections $\widehat{D}_k$ and D_{basic} such that D_{basic} is basic, $\widehat{D} = (\widehat{D}_N,\dots,\widehat{D}_0)$ is compatible with (E,φ) in $M\setminus\Sigma$, for some compact neighborhood $Z \subset \Sigma \subset M$, and D_{basic} is the connection induced by $\widehat{D}$ on $M\setminus\Sigma$, cf., Section 3. Now the associated characteristic form $\Phi(\widehat{D})$ has support in Σ and represents $\mathrm{res}^\Phi(\mathscr{F};Z)$ by definition, see Section 3.1. By Example 6.7 we may assume that $\Phi(\widehat{D})$ is defined from a simplicial resolution with respect to a Stein open cover.

Recall that $\Phi(\widehat{D}^\epsilon)$ has support in Σ_ϵ. For $\epsilon > 0$ small enough, there is a neighborhood U of Σ such that $\Sigma_\epsilon \cap U \subset \Sigma$. By replacing M by U, we may thus assume that $\Sigma_\epsilon \subset \Sigma$ for ϵ small enough. We than obtain by Lemma 7.3 that $\Phi(\widehat{D}^\epsilon) - \Phi(D) = d\eta^\epsilon$, where η^ϵ is a form with support in Σ. It follows that $\Phi(\widehat{D}^\epsilon)$ represents $\mathrm{res}^\Phi(\mathscr{F}; Z)$ for $\epsilon > 0$ small enough, see Section 3.1. Thus so does the limit $R^\Phi = R^\Phi_Z$ by Poincaré duality.

It remains to prove the properties of independence and vanishing of $R^\Phi_{Z'}$. If $\ell < \operatorname{codim} Z'$, then since $R^\Phi_{Z'} = 0$ is a pseudomeromorphic current of bidegree $(*, \ell)$ with support on Z', it follows by the dimension principle that $R^\Phi_{Z'} = 0$ if $\ell < \operatorname{codim} Z'$.

It remains to consider the case $\ell = \operatorname{codim} Z'$. Let $R^\Phi_{Z',(j)}$, $j = 1, 2$, denote the Baum-Bott currents corresponding to two different choices of Stein open cover, resolutions, Hermitian metrics, connections, and partition of unity.

Then, by Theorem 8.2,

$$R^\Phi_{Z',(2)} - R^\Phi_{Z',(1)} = \mathbf{1}_{Z'} dN^\Phi,$$

where N^Φ is a pseudomeromorphic current with components of bidegree $(*, q)$ with $q \leq \ell - 1$, cf. (8.13).

Since N^Φ has support on Z' of codimension $\geq \ell$, it follows from the dimension principle, see Section 4.1, that N^Φ vanishes, and consequently so does dN^Φ. Furthermore, since $\mathbf{1}_{Z'} dN^\Phi$ only depends on dN^Φ, it follows that $\mathbf{1}_{Z'} dN^\Phi$ vanishes. Thus $R^\Phi_{Z',(1)} = R^\Phi_{Z',(2)}$, which proves the result. □

References

[AW07] Mats Andersson and Elizabeth Wulcan. Residue currents with prescribed annihilator ideals. *Ann. Sci. École Norm. Sup. (4)*, 40(6):985–1007, 2007.

[AW10] Mats Andersson and Elizabeth Wulcan. Decomposition of residue currents. *J. Reine Angew. Math.*, 638:103–118, 2010.

[AW18] Mats Andersson and Elizabeth Wulcan. Direct images of semi-meromorphic currents. *Ann. Inst. Fourier (Grenoble)*, 68(2):875–900, 2018.

[BB70] Paul F. Baum and Raoul Bott. On the zeros of meromorphic vector-fields. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages 29–47. Springer, New York, 1970.

[BB72] Paul Baum and Raoul Bott. Singularities of holomorphic foliations. *J. Differential Geometry*, 7:279–342, 1972.

[Blo10] Jonathan Block. Duality and equivalence of module categories in noncommutative geometry. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 311–339. Amer. Math. Soc., Providence, RI, 2010.

[BS10] Jan-Erik Björk and Håkan Samuelsson. Regularizations of residue currents. *J. Reine Angew. Math.*, 649:33–54, 2010.

[BSW23] Jean-Michel Bismut, Shu Shen, and Zhaoting Wei. *Coherent Sheaves, Superconnections, and Riemann-Roch-Grothendieck*, volume 347 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2023.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gre80] H. I. Green. *Chern classes for coherent sheaves*. PhD thesis, University of Warwick, 1 1980.

[Gri10] Julien Grivaux. Chern classes in Deligne cohomology for coherent analytic sheaves. *Math. Ann.*, 347(2):249–284, 2010.

[Han24] Zhaobo Han. Characteristic currents on cohesive modules. Preprint, arXiv:2404.09439 [math.AG], 2024.

[HL71] M. Herrera and D. Lieberman. Residues and principal values on complex spaces. *Math. Ann.*, 194:259–294, 1971.

[Hos23a] Timothy Hosgood. Simplicial Chern-Weil theory for coherent analytic sheaves, Part I. *Bull. Soc. Math. France*, 151(1):127–170, 2023.

[Hos23b] Timothy Hosgood. Simplicial Chern–Weil theory for coherent analytic sheaves, part II. *manuscripta mathematica*, 2023.

[KLW23] Lucas Kaufmann, Richard Lärkäng, and Elizabeth Wulcan. Baum-bott residue currents, 2023.

[LW22] Richard Lärkäng and Elizabeth Wulcan. Chern currents of coherent sheaves. *Épijournal Géom. Algébrique*, 6:Art. 14, 26, 2022.

[OTT85] Nigel R. O'Brian, Domingo Toledo, and Yue Lin L. Tong. A Grothendieck-Riemann-Roch formula for maps of complex manifolds. *Math. Ann.*, 271(4):493–526, 1985.

[Qia16] Hua Qiang. On the Bott-Chern characteristic classes for coherent sheaves. Preprint, arXiv:1611.04238 [math.DG], 2016.

[Suw98] Tatsuo Suwa. *Indices of vector fields and residues of singular holomorphic foliations*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1998.

[Suw00] Tatsuo Suwa. Characteristic classes of coherent sheaves on singular varieties. In *Singularities—Sapporo 1998*, volume 29 of *Adv. Stud. Pure Math.*, pages 279–297. Kinokuniya, Tokyo, 2000.

[TT78] Domingo Toledo and Yue Lin L. Tong. Duality and intersection theory in complex manifolds. I. *Math. Ann.*, 237(1):41–77, 1978.

[TT86] Domingo Toledo and Yue Lin L. Tong. Green's theory of Chern classes and the Riemann-Roch formula. In *The Lefschetz centennial conference, Part I (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 261–275. Amer. Math. Soc., Providence, RI, 1986.

[Voi02] Claire Voisin. A counterexample to the Hodge conjecture extended to Kähler varieties. *Int. Math. Res. Not.*, (20):1057–1075, 2002.

[Wu23] Xiaojun Wu. Intersection theory and Chern classes in Bott-Chern cohomology. *Ark. Mat.*, 61(1):211–265, 2023.

L. Kaufmann, Institut Denis Poisson, CNRS, Université d'Orléans, Rue de Chartres, B.P. 6759, 45067, Orléans cedex 2, FRANCE

Email address: lucas.kaufmann@univ-orleans.fr

R. Lärkäng, E. Wulcan, Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Gothenburg, SWEDEN

Email address: larkang@chalmers.se, wulcan@chalmers.se