ALEXANDROV SPACES ARE CS SETS

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ABSTRACT. We prove that the extremal stratification of an Alexandrov space by Perelman–Petrunin is a CS stratification in the sense of Siebenmann. We also show that every space of directions of an Alexandrov space without proper extremal subsets is homeomorphic to a sphere. In the appendix we give an example of a primitive extremal subset of codimension 2 that is not an Alexandrov space with respect to the induced intrinsic metric.

1. Introduction

The main purpose of this paper is to prove the following.

Theorem 1.1. The extremal stratification of an Alexandrov space by Perelman–Petrunin is a CS stratification in the sense of Siebenmann.

An Alexandrov space is a metric space with a lower curvature bound introduced by Burago–Gromov–Perelman [6]. They naturally arise as Gromov–Hausdorff limits of Riemannian manifolds with sectional curvature bounded below or quotient spaces of Riemannian manifolds by isometric group actions. In general, a stratification is a well-behaved decomposition of a topological space into topological manifolds of different dimensions.

In fact, from the geometrical point of view, our proof is just a combination of known results and provides nothing new. Nevertheless, the author believes that it is worth publishing because it connects two concepts in different fields, yet even a statement cannot be found in the literature. Another motivation comes from the collapsing theory of Riemannian manifolds, where the extremal stratification of limit Alexandrov spaces plays an essential role ([28], [13], [14]; especially in the sheaf-theoretical conjectures by Alesker [1]).

For this reason, we begin by reviewing what is known about the stratification of Alexandrov spaces. There are four types of stratification appearing in this paper — extremal, MCS, CS, and WCS stratifications with the following relationships:

extremal
$$\xrightarrow{(2)}$$
 MCS
$$(1) \downarrow \qquad \qquad \downarrow (4)$$
CS $\xrightarrow{(3)}$ WCS.

The precise meaning of each arrow will be explained below. Here we focus only on the histories of the four notions and defer the definitions to Section 2.

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The notion of a stratification of an Alexandrov space was first introduced by Perelman [25], [26], as a result (actually a part) of his Morse theory for distance functions on Alexandrov spaces. He proved that any Alexandrov space is an MCS space (= multiple conical singularity), and in particular that it is stratified into topological manifolds. Note that the MCS stratification is purely topological.

At the same time as [26], Perelman–Petrunin [29] introduced the notion of an extremal subset, which is a singular set of an Alexandrov space defined by a purely geometrical condition. By extending the Morse theory to extremal subsets, they proved that any Alexandrov space is, in a certain sense, uniquely stratified into its own extremal subsets. We refer to it as the extremal stratification in this paper. The extremal stratification is finer than the MCS stratification (the arrow (2)): the former reflects not only the topological singularity but also the geometrical singularity of an Alexandrov space. For example, the extremal stratification of a squire consists of the four vertices, the interiors of the four edges, and the interior of the squire itself, whereas the MCS stratification consists only of the boundary and the interior of the squire.

The above-mentioned Morse theory relies heavily on the work of Siebenmann [34] on the deformation of homeomorphisms on stratified sets. In [34], Siebenmann introduced two notions of stratified sets: the one is a CS set (= cone-like stratified) and the other is a WCS set (= weakly cone-like stratified). As the names suggest, CS sets are WCS sets (the arrow (3)). From our viewpoint, the main difference of these two is whether the local normal trivialization preserves the stratification or not (see Remark 2.3). Still, the fundamental deformation theorem of Siebenmann works in the weaker setting of WCS sets.

In [25], [26], Perelman only stated that the MCS stratification of an Alexandrov space is a WCS stratification (the arrow (4)), which was enough for him to develop the Morse theory by using the deformation theorem of Siebenmann for WCS sets. However, there was no mention of the CS structure in [25], [26], even in [29], and as already remarked, since then there have been no papers about it (except for a closely related result of Kapovitch [20, §9] mentioned below). This is why we consider this problem and the answer is, as stated in Theorem 1.1, given by the extremal stratification of Perelman–Petrunin (the arrow (1)).

Actually the proof of Theorem 1.1 is almost the same as the original proof of the existence of the extremal stratification by Perelman–Petrunin [29, 3.8, 1.2]. The only difference is that we will make use of the relative fibration theorem with respect to extremal subsets developed later by Kapovitch [20, §9], instead of the absolute one given in [29, §2].

The extremal stratification theory in particular implies that if an Alexandrov space contains no proper extremal subsets, then it must be a topological manifold. However, even if an Alexandrov space is a topological manifold, it does not necessarily mean that each link (i.e., the space of directions) is a topological sphere. See Example 2.22 based on the double suspension theorem of Cannon [7] and Edwards [9], [10]. The infinitesimal characterization of Alexandrov topological manifolds by Wu [35] only tells us that the link is a homology manifold with the homotopy type of a sphere (cf. [23]). Regarding the structure of the link, for now we will show the following simplest case (cf. [32, Ch. 3 Problem 32], [20]).

Theorem 1.2. Every space of directions of an Alexandrov space without proper extremal subsets is homeomorphic to a sphere.

Organization. In Section 2, we recall background material including the definitions of the four stratifications mentioned above. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively. In Appendix A, we give an example of a primitive extremal subset of codimension 2 that is not an Alexandrov space with respect to the induced intrinsic metric, which also cannot be found in the literature.

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2. Preliminaries

Here we begin with the definition of a CS set and then recall some notions and results from Alexandrov geometry which will be used to prove the main theorem 1.1. For the latter, we will focus on stating their properties rather than giving the formal definitions. Actually, once those properties are assumed, the proof requires almost no geometrical knowledge.

2.1. **CS sets.** First we recall the definition of a CS set by Siebenmann [34, §1]. For simplicity, we restrict our attention to the finite-dimensional case.

A metrizable space X is a stratified set if there exists a filtration by closed subsets

$$\emptyset = X^{-1} \subset X^0 \subset \dots \subset X^n = X$$

such that the components of $X_k := X^k \setminus X^{k-1}$ are open in X_k . We call X^k and X_k the k-dimensional skeleta and strata, respectively. In addition, if the components of X_k are k-dimensional topological manifolds (without boundary), then we call X a TOP stratified set.

An isomorphism between (not necessarily TOP) stratified sets X and Y is a homeomorphism between X and Y preserving their strata of the same dimension, which will be denoted by $X \cong Y$.

For stratified sets X and Y, the direct product $X \times Y$ has a natural stratification. The open cone $C(X) := X \times [0,1)/X \times \{0\}$ has a natural stratification as well, which is induced from the product structure of $X \times (0,1)$ together with the zero-dimensional stratum consisting only of the vertex of the cone. An open subset U of X also has a natural stratification by restriction.

Definition 2.1. A TOP stratified set X is called a CS set if for any $x \in X_k$ there exist an open neighborhood U of x in X and a compact stratified set L such that there is an isomorphism

$$h: U \xrightarrow{\cong} B^k \times C(L),$$

where B^k denotes an open Euclidean ball of dimension k. Note that both sides have natural stratifications defined above, where the stratification of B^k is trivial. We call L a (normal) link and h a local normal trivialization at x.

Remark 2.2. In general, L need not be either a CS or TOP stratified set. Moreover, its homeomorphism type is not necessarily unique (see Example 2.22).

Remark 2.3. There is a weaker notion of a *WCS set* introduced by Siebenmann [34, §5]. Roughly speaking, there are two relaxations of the above conditions:

- (1) C(L) may be a *mock cone*, not necessarily a real cone;
- (2) h may be a homeomorphism preserving only the k-dimensional stratum, not necessarily an isomorphism preserving the higher-dimensional strata.

Any Alexandrov space is known to be a WCS set, but not known to be a CS set in the literature. As we will see below, the problem lies in (2) rather than (1) (i.e., C(L) is a real cone but h might not be an isomorphism). In particular, we do not need the definition of the mock cone here.

- 2.2. **Alexandrov spaces.** An *Alexandrov space* is a complete finite-dimensional geodesic metric space with a lower curvature bound in terms of triangle comparison, introduced by Burago–Gromov–Perelman [6]. More precisely, any geodesic triangle is not thinner than the comparison triangle with the same sidelengths in the model plane of constant curvature. Typical examples are
 - Riemannian manifolds with sectional curvature bounded below;
 - their convex subsets and convex hypersurfaces;
 - their Gromov–Hausdorff limits;
 - their quotient spaces by isometric compact group actions.

See also [4], [3] for the formal definition and properties. Note that the *dimension* of an Alexandrov space means its Hausdorff dimension, which coincides with the topological dimension.

2.3. MCS spaces. The following definition is due to Perelman [25], [26].

Definition 2.4. A metrizable space X is called an MCS space of dimension n if every point of X has a neighborhood pointed homeomorphic to an open cone over a compact MCS space of dimension n-1. Here we assume that the empty set is the unique compact (-1)-dimensional MCS space, the cone over which is a point.

Remark 2.5. The conical neighborhood is unique up to pointed homeomorphism, as shown by Kwun [21] (though the link may be different).

Every MCS space has a natural stratification. The k-dimensional stratum consists of those points whose conical neighborhood topologically splits into $B^k \times C(L)$ but not $B^{k+1} \times C(L')$, where L and L' are compact MCS spaces. It easily follows from its inductive definition that the MCS stratification is a WCS stratification mentioned in Remark 2.3. The only reason we cannot say that it is a CS stratification is the lack of the isomorphism preserving the stratification in the normal directions as in Remark 2.3(2).

Theorem 2.6 ([25], [26]). Every n-dimensional Alexandrov space is an MCS space of dimension n. In particular, it is a WCS set.

Remark 2.7. The proof actually shows that every Alexandrov space is a *non-branching* MCS space, which is defined in the same inductive way as in Definition 2.4, but allowing only 1-dimensional manifolds with boundary to be 1-dimensional nonbranching MCS spaces.

In order to prove the above theorem, Perelman [25], [26] developed the Morse theory for distance maps on Alexandrov spaces, which we will review in the next section (see also [27, §2], [20]).

2.4. Canonical neighborhoods. The following canonical neighborhood theorem is proved by reverse induction on k, using the deformation theorem of Siebenmann for WCS sets [34, 5.4, 6.9, 6.14]. Let X be an Alexandrov space of dimension n.

Theorem 2.8 ([26, 1.3]). For any $p \in X$, there exist an open neighborhood K of p in X and a compact MCS space L such that there is a homeomorphism

$$h: K \to B^k \times C(L),$$

where $0 \le k \le n$. More precisely, there exists a map $(f,g): K \to B^k \times [0,1)$ with the following properties:

- (1) g(p) = 0 (possibly $g \equiv 0$ when k = n);
- (2) f is regular on K and (f,g) is regular on $K \setminus g^{-1}(0)$;
- (3) f restricted to $g^{-1}(0)$ is a homeomorphism to B^k and (f,g) restricted to $K \setminus g^{-1}(0)$ is a trivial fiber bundle over $B^k \times (0,1)$ with fiber L.

In particular, the homeomorphism h is constructed from (3): the first B^k -coordinate is given by f and the radial function of C(L) is given by q.

We will not state the definition of a regular map here (see [26], [20]). It is more meaningful to give a basic example: for k+1 (!) points $a_i \in X$ ($0 \le i \le k$), the distance map

$$f = (d(a_1, \cdot), \dots, d(a_k, \cdot)) : X \to \mathbb{R}^k,$$

is regular at p if the angles $\angle a_i p a_j$ are greater than $\pi/2$ for all $0 \le i \ne j \le k$. However, the general definition is more complicated to include the function g above.

Remark 2.9. Any regular map is open. Moreover, the second half of (3) is a consequence of the regularity of (f,g) and the *fibration theorem* proved simultaneously in [26], asserting that any proper regular map is a locally trivial fibration.

Remark 2.10. Given a regular map f_0 at p to \mathbb{R}^{k_0} $(k_0 \leq k)$, the map f can be chosen to respect f_0 , that is, the first k_0 -coordinates of f coincide with f_0 .

Definition 2.11. The above K is called a *canonical neighborhood* of p associated with the map (f,g). We also denote it by K(f,g) to indicate (f,g). If f respects f_0 in the above sense, we say that K respects f_0 .

- 2.5. Extremal subsets. An extremal subset, introduced by Perelman–Petrunin [29], is (informally) a closed subset of an Alexandrov space such that the set of normal directions at each point has diameter $\leq \pi/2$. Typical examples are
 - a point at which all angles $\leq \pi/2$, called an *extremal point*;
 - the closures of the MCS strata of an Alexandrov space; especially the *bound-ary*, i.e., the closure of the codimension 1 stratum;
 - the projection of the fixed point set into the quotient space of a Riemannian manifold by an isometric compact group action; more generally, the projections of the fixed point sets of closed subgroups.

See also [27, §3], [31, §4], [11], [12] for the formal definition and properties.

We first recall how to define the extremal stratification of an Alexandrov space. The following basic properties were proved in [29].

- The union and intersection of two extremal subsets are extremal. Moreover, the closure of the difference of the two is extremal.
- The collection of extremal subsets is locally finite.

These facts lead to the following definition.

Definition 2.12. An extremal subset is called *primitive* if it cannot be represented as a union of two proper extremal subsets. For a primitive extremal subset E, its main part \mathring{E} is the relative complement of all proper extremal subsets in E.

Remark 2.13. In other words, E is a minimal extremal subset containing some point x. The main part of E is the set of those x for which E is the minimal extremal subset containing it.

Clearly, the main parts of all primitive extremal subsets define a disjoint covering of an Alexandrov space. Furthermore, Perelman–Petrunin [29, §2] developed the Morse theory on extremal subsets by extending the canonical neighborhood theorem 2.8 to the restriction (!) of a regular map to extremal subsets. In particular, they proved in [29, 3.8, 1.2] that

- any primitive extremal subset is an MCS space;
- the closures of its MCS strata are extremal subsets.

These imply that the main part of a primitive extremal subset is a (connected) topological manifold contained in some MCS stratum of the ambient Alexandrov space. Let us summarize the above as follows.

Definition 2.14. The *extremal stratification* of an Alexandrov space is defined in such a way that the k-dimensional stratum consists of the k-dimensional main parts of primitive extremal subsets.

Theorem 2.15 ([29, 3.8]). The extremal stratification of an Alexandrov space is a TOP stratification that is a refinement of the MCS stratification.

Remark 2.16. The dimension of the main part of a primitive extremal subset E coincides with the Hausdorff and topological dimensions of E, which we simply call the *dimension* of E ([12]). It also coincides with the maximal integer k such that there exists a regular map to \mathbb{R}^k at some point of E ([29, §2], [20, p.133]).

Next we discuss the relationship between extremal subsets and canonical neighborhoods. The following characterization of extremal subsets is due to Perelman [28, 2.3] (see also [13, 5.1]). Let X be an Alexandrov space.

Lemma 2.17. Suppose a subset S of X satisfies the following property:

(*) if S intersects a canonical neighborhood K(f,g), then S contains $g^{-1}(0)$. Then the closure of S is an extremal subset.

Remark 2.18. The converse is true, that is, if an extremal subset E intersects K(f,g), then E contains $g^{-1}(0)$. This follows from the openness of a regular map restricted to an extremal subset, [29, §2].

The next relative version of the canonical neighborhood theorem 2.8 with respect to extremal subsets was proved by Kapovitch [20, §9].

Complement 2.19 (to Theorem 2.8). The homeomorphism h of Theorem 2.8 can be chosen to respect (all) extremal subsets. More precisely, for any extremal subset E intersecting K, the restriction of h gives a homeomorphism to the subcone:

$$K \cap E \to B^k \times C(L \cap E)$$
.

As in Remark 2.9, this also follows from the relative fibration theorem proved simultaneously in [20, §9], asserting that any proper regular map is a locally trivial fibration respecting extremal subsets. Indeed, applying it to the regular map (f, g) on $K \setminus g^{-1}(0)$ and gluing the homeomorphism f on $g^{-1}(0) \subset E$ yield the desired homeomorphism h. The proof relies on the relative version of the deformation theorem of Siebenmann [34, 5.10, 6.10, 6.16].

Remark 2.20. In the above statement, it is possible that $L \cap E = \emptyset$; in this case $K \cap E = g^{-1}(0)$ corresponds to $B^k \times \{o\}$, where o is the vertex of the cone.

Remark 2.21. Kapovitch only discussed the case of one extremal subset, but in view of the relative deformation theorem of Siebenmann cited above, it is easy to generalize it to the multiple case.

The key difference between the absolute version of the Morse theory on extremal subsets by Perelman–Petrunin [29, §2] and the relative version by Kapovitch [20, §9] is whether it includes information on the normal directions or not. The latter is essential for proving the main theorem 1.1.

We conclude this section with a well-known example from geometric topology. From our point of view, this gives an example for which the extremal stratification does not coincide with the MCS stratification.

Example 2.22. Let Σ be the Poincaré homology 3-sphere of constant curvature 1. Then the spherical suspension $S(\Sigma)$ is not a topological manifold, but the double spherical suspension $S^2(\Sigma)$ is a topological sphere by the theorem of Cannon [7] and Edwards [9], [10]. Thus the MCS stratification of $S^2(\Sigma)$ is trivial. On the other hand, since diam $\Sigma \leq \pi/2$ (cf. [16]), the extremal stratification consists of the subsuspension $S(\{\xi,\eta\})$ and its complement, where ξ,η are the poles of $S(\Sigma)$.

Remark 2.23. Theorem 1.2 tells us that the above proper extremal subset must exist. See Theorem 4.1 and Proposition 4.4 for further discussions.

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. Indeed, almost all the background knowledge needed for the proof is covered in the previous section.

Proof of Theorem 1.1. Let X be an Alexandrov space and E a primitive extremal subset. We will show that there exists a compact stratified set L (depending only on E) such that for any point x in the main part \mathring{E} of E there is a neighborhood U of x in X admitting an isomorphism as stratified sets

$$U \cong B^m \times C(L),$$

where $m = \dim E$ and the stratifications of the left- and right-hand sides are the natural ones induced by those of X and L, respectively.

Step 1. We first prove the claim for any regular point of E. A regular point of E is a point where the tangent cone of E is isometric to Euclidean space of dimension $m = \dim E$ (the tangent cone means the Gromov–Haussdorff blowup of E at p). Note that regular points are contained in \mathring{E} . See [12] for more information; but actually all we need here is the second half of Remark 2.16.

Let $p \in E$ be a regular point. Then there exists a regular map f to \mathbb{R}^m at p (consider the distance map from m points in E close to p that make obtuse but almost right angles at p). Since dim E = m, there is no function f_1 such that (f, f_1) is regular at p (Remark 2.16). By Theorem 2.8 and Remark 2.10, one can construct a function g and the associated canonical neighborhood K = K(f, g) of p with a homeomorphism

$$h: K \to B^m \times C(L_p),$$

where L_p is a fiber of (f,g) in $K \setminus g^{-1}(0)$.

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It remains to show that h preserves the stratifications of both sides. Note that not only K, but also L_p has a natural stratification as a subset of X, by restricting the extremal stratification. Since L_p is not open, we have to show that this is a stratified set in the sense of Section 2.1, i.e., the components of the strata are open. However, we defer it to the next step and first observe that h preserves these stratifications.

By Complement 2.19, h can be chosen to preserve extremal subsets. Thus the only thing we should check here is that there is a bottom stratum in the left-hand side corresponding to $B^m \times \{o\}$ in the right-hand side. This is nothing but E itself by the same reasoning dim E = m as above (Theorem 2.8(2), Remarks 2.16).

Step 2. As mentioned above, we have to prove that the components of the strata of L_p are open. It is sufficient to show that the strata are locally path-connected. Let E_1 be a primitive extremal subset intersecting L_p (thus containing E) and let $x \in L_p \cap \mathring{E}_1$. By Theorem 2.8 and Remark 2.10, there exists a canonical neighborhood K_1 of x respecting (f,g) (see Definition 2.11). By Complement 2.19, it follows that $K_1 \cap L_p \cap \mathring{E}_1$ is homeomorphic to a product of a ball and a cone, which is locally path-connected at x.

Step 3. Finally we prove the claim for any point of the main part \mathring{E} with fixed L. Fix a regular point $p \in E$ and set $L := L_p$. Define a subset F of E by

 $F := \{x \in E \mid \text{no open neighborhood in } X \text{ is isomorphic to } B^m \times C(L)\},$

where the isomorphism is as stratified sets. Note that F is closed since the bottom stratum of $B^m \times C(L)$ has dimension m, while \mathring{E} is the unique m-dimensional stratum contained in E. However, we will not need it in view of Lemma 2.17 used below (but see also Remark 3.1, where it is needed).

We prove that F is an extremal subset of X. It suffices to verify the condition (*) of Lemma 2.17 for F. Suppose $x \in E$ has a (small) neighborhood isomorphic to $B^m \times C(L)$. For a canonical neighborhood $K(f_1, g_1)$, we show that if $x \in g_1^{-1}(0)$, then any other $y \in g_1^{-1}(0)$ is contained in E and has a neighborhood isomorphic to $B^m \times C(L)$. This follows from Complement 2.19 (clearly $y \in E$ by Remark 2.18). Indeed, since $x, y \in g_1^{-1}(0)$, there is a homeomorphism between neighborhoods of x and y via a translation with respect to the f_1 -coordinate of $K(f_1, g_1)$. Since the homeomorphism of Complement 2.19 respects extremal subsets, so does the translation. This implies that x and y have isomorphic neighborhoods. Thus the condition (*) holds and hence F is an extremal subset of X.

Moreover, F is not equal to E since $p \notin F$. By the primitiveness, F must be contained in $E \setminus \mathring{E}$. Therefore, any point of \mathring{E} has a neighborhood isomorphic to $B^m \times C(L)$. This complete the proof.

Remark 3.1. Actually in Step 3, we do not need Lemma 2.17 to prove that F is an extremal subset. One can just repeat the same argument as in [29, 3.8, 1.2], which showed that \mathring{E} is a topological manifold, replacing the absolute fibration theorem on extremal subsets in [29, §2] by the relative one in [20, §9]. The reason we did not take this approach is because our exposition puts emphasis on canonical neighborhoods (in fact, we have not even given the precise definition of extremal subsets). Another reason is that our argument can be viewed as a fixed space version of [13, 5.6, 5.11], which showed that the homotopy type of a regular fiber in a collapsing space is invariant over a component of a stratum of the limit space.

By the theorem of Handel [18, 2.4], we obtain the following corollary.

Corollary 3.2. The intrinsic stratification of an Alexandrov space is CS.

Here the intrinsic stratification (or minimal stratification) is defined as follows. The k-dimensional stratum consists of those points whose conical neighborhood topologically splits into $B^k \times C(L)$ but not $B^{k+1} \times C(L')$, where L and L' are compact topological spaces. Compare with the MCS stratification in Section 2.3, where L and L' are MCS spaces. Although the MCS stratification might be less important now, the following question still remains.

Question 3.3. Is the MCS stratification of an Alexandrov space different from the intrinsic stratification? Even if so, is it CS?

Another question remains regarding the link, which leads to Theorem 1.2.

Question 3.4. Is the stratification of the link L in the above proof TOP, or more strongly CS? If it is CS, what about iterated links? (i.e., a link of a link of ...)

Remark 3.5. Note that L is an MCS space (Theorem 2.8). Moreover, in view of the stability theorem [25, 4.3], L will be homeomorphic to a space Σ of curvature ≥ 1 , which is the subspace of directions at p normal to the tangent cone of E. Therefore L admits a CS structure by the extremal stratification of Σ . However, at present it is unclear to the author whether the above stratification of L induced from the extremal stratification of the ambient space L has such nice properties. In view of the relative stability theorem [20, 9.2], this stratification will exist in between the first MCS and the second CS stratifications mentioned above.

4. Proof of Theorem 1.2

Next we prove Theorem 1.2. Here we will assume some more familiarity with Alexandrov geometry. The following are basic properties of Alexandrov topological manifolds and homology manifolds (with integer coefficients). We refer to [35], [23], and [33, 7.2]: the second one is concerned with geodesically complete spaces with curvature bounded above, but the methods there can also be applied to Alexandrov spaces with curvature bounded below (if correctly translated).

- Any space of directions of an Alexandrov homology manifold is a homology manifold with the homology of a sphere.
- Any nonmanifold point of an Alexandrov homology manifold is isolated; in particular, it is an extremal point.
- A point in an Alexandrov homology manifold is a manifold point if and only if its space of directions is simply-connected, or equivalently homotopy equivalent to a sphere (by the Whitehead theorem).

Moreover, in dimension ≤ 4 , any space of directions of an Alexandrov topological manifold is homeomorphic to a sphere. Therefore we may restrict our attention to dimension ≥ 5 . We will prove the following slightly stronger statement.

Theorem 4.1. Let X be an Alexandrov topological manifold. Let E be the set of points in X where the space of directions is not homeomorphic to a sphere. If E is nonempty, then it is a 1-dimensional extremal subset.

Since an Alexandrov space without proper extremal subsets is a topological manifold, this immediately implies Theorem 1.2.

Proof of Theorem 4.1. We denote by Σ_p the space of directions at $p \in X$. Since X is a topological manifold, Σ_p is a homology manifold with the homotopy type of a sphere as stated above. In particular, Σ_p is homeomorphic to a sphere if and only if it is a topological manifold (by the generalized Poincaré conjecture). Together with the above-mentioned properties, we see that

- Σ_p is a topological sphere iff its all spaces of directions are simply-connected;
- the nonmanifold points of Σ_p , or equivalently, points having nonsimply-connected spaces of directions, are isolated extremal points.

These facts will be used implicitly and frequently below.

If $p \in E$, we also denote by $\Sigma_p E$ the space of directions of E at p, that is, the subset of Σ_p consisting of limits of the directions of shortest paths from p to $p_i \in E \setminus \{p\}$ that converges to p. To prove that E is an extremal subset, it is sufficient to show that E is closed and $\Sigma_p E$ is an extremal subset of Σ_p for any $p \in E$ ([29, 1.4]). For the latter, we will prove that $\Sigma_p E$ coincides with the set of nonmanifold points in Σ_p , which is a discrete set of extremal points as stated above (but see also Step 1 below).

Step 1. We first show that if Σ_p contains only one nonmanifold point ξ , then Σ_p is contained in the closed $\pi/2$ -neighborhood of ξ . This is an additional condition imposed on a one-point extremal subset in the space of directions so that the above infinitesimal characterization [29, 1.4] holds (see [29, 1.1], [31, §4.1 Property (2)]). Actually we prove that diam $\Sigma_p \leq \pi/2$.

Suppose diam $\Sigma_p > \pi/2$. Then the suspension theorem of Perelman [25, 4.5] implies that Σ_p is homeomorphic to a suspension over a space of curvature ≥ 1 . Moreover, its proof shows that the distance function to each pole of the suspension is regular except at the poles. In particular, any point other than the poles is not an extremal point, thus a manifold point. Therefore ξ must be one of the poles of the suspension, and the other pole, which also has a nonsimply-connected link, is a nonmanifold point. This is a contradiction.

Step 2. We next prove the following

Claim 4.2. Let $p \in X$ and $\xi \in \Sigma_p$. Then ξ is a nonmanifold point if and only if there exists a sequence $p_i \in E \setminus \{p\}$ converging to p such that the directions to p_i converges to ξ .

Proof of Claim 4.2. We denote by B(p,r) and S(p,r) the open r-ball and r-sphere around p, respectively. First suppose r is small enough and let $x \in S(p,r)$. Then the point p is a 1-strainer at x, that is, there exists a direction at x making an angle almost π with the directions to p, which we call an almost antipode. Next choose $\varepsilon > 0$ small enough compared to r. Then the pair of points (p,x) is a 2-strainer at any point of $S(p,r) \cap B(x,\varepsilon) \setminus \{x\}$, i.e., the directions to p and x and their almost antipodes are almost orthogonal. By the stability theorem and the suspension theorem [25, 4.3, 4.5], we see that Σ_x is homeomorphic to the suspension over $\Sigma := S(p,r) \cap S(x,\varepsilon)$ (more precisely, the direction(s) at x to p may not be a pole of the suspension, but they are very close to the pole, so the homeomorphism type of the equator of the suspension turns out to be Σ). Moreover, as mentioned in Step 1, any point other than the poles of the suspension is not an extremal point, thus a manifold point. Therefore, Σ_x is a topological manifold if and only if Σ is simply-connected.

Since r is small enough, it follows from the stability theorem that S(p,r) is homeomorphic to Σ_p ; in particular, it is a homology manifold (of dimension ≥ 4) with isolated nonmanifold points. Since ε is small enough, it also follows from the fibration theorem (with a gluing argument) that $S(p,r) \cap B(x,\varepsilon)$ is homeomorphic to the cone over Σ . Thus Σ is simply-connected if and only if x is a manifold point of S(p,r) (the "only if" part follows from [8], [5]; see also [23, 6.2]). Since the rescaled sphere $r^{-1}S(p,r)$ converges to Σ_p as $r \to 0$, and the stability homeomorphism is a Gromov–Hausdorff approximation, the claim follows.

Now the "if" part of Claim 4.2 implies that E is closed and $\Sigma_p E$ is contained in the set of nonmanifold points of Σ_p . On the other hand, the "only if" part implies that the nonmanifold points of Σ_p are contained in $\Sigma_p E$. This complete the proof of Theorem 4.1.

Remark 4.3. We give a more geometrical description of the proof of Claim 4.2 without relying on [8], [5] (at least explicitly). Let $\xi \in \Sigma_p$ and $r_i \to 0$. Consider the pointed Gromov–Hausdorff convergence $(r_i^{-1}X,p) \to (T_p,o)$, where (T_p,o) is the tangent cone at p based at its vertex. Then there is a canonical way of lifting ξ to $p_i \in r_i^{-1}S(p,r_i)$ so that the distance function $d(p_i,\cdot)$ is uniformly regular on a punctured ball around p_i of fixed radius (after rescaling) independent of i (the uniform regularity here means that the norm of the gradient of $d(p_i,\cdot)$ is uniformly bounded below). See [24, 1.2], [36, 3.2] (actually a similar argument is included in the proof of the stability theorem used above). This, together with the fibration and stability theorems, implies that Σ_{p_i} is homeomorphic to the space of direction of T_p at ξ (cf. [20, 5.1]). The latter is nothing but the spherical suspension over Σ_{ξ} , the space of directions of Σ_p at ξ . Therefore, ξ is a manifold point of Σ_p if and only if Σ_{p_i} is a topological sphere.

This canonical lifting argument obviously shows the "only if" part of Claim 4.2. The "if" part follows from Proposition 4.4(1) below and the following observation. Suppose another sequence $q_i \in S(p,r_i) \setminus \{p_i\}$ converges to ξ . Then there exist three directions at q_i making obtuse angles. This follows from the facts that $d(p_i,\cdot)$ is uniformly regular at q_i and that the two directions at q_i to p_i and p are almost orthogonal, where the latter has an almost antipode. Then Proposition 4.4(1) shows that Σ_{q_i} must be a topological sphere. Therefore, p_i is the only candidate having a nonspherical space of directions among all sequences converging to ξ . This shows the "if" part of Claim 4.2.

The following is a sphere theorem for an Alexandrov homology manifold of positive curvature and its link. The first half generalizes the double suspension theorem for an Alexandrov homology sphere in [35, p.750].

Proposition 4.4. Let Σ be an Alexandrov homology manifold with curvature ≥ 1 .

- (1) If Σ contains a triple of points with pairwise distances $> \pi/2$, then it is homeomorphic to a sphere.
- (2) If Σ contains a quadruple of points with pairwise distances $> \pi/2$, then every space of directions is homeomorphic to a sphere.

Proof. We will follow the same strategy as in [17, Theorem C], which generalized Perelman's diameter suspension theorem [25, 4.5] to the multiple suspension version in terms of the *packing radius*. In their terminology, our first/second assumption means that Σ has the third/fourth packing radius $> \pi/4$, respectively.

- (1) Let (ξ_1, ξ_2, ξ_3) be the triple of points in Σ with pairwise distances $> \pi/2$. Then a standard triangle comparison argument shows that
 - (i) there exists a unique point η_1 farthest from ξ_1 ;
 - (ii) ξ_2, ξ_3 are contained in the $\pi/2$ -neighborhood of η_1 ; in particular, they make an obtuse angle at η_1 ;
 - (iii) the distance function from η_1 has no critical points on $\Sigma \setminus \{\eta_1, \zeta_1\}$, where ζ_1 is the unique point farthest from η_1 .

For the proofs of (ii) and (iii), see [17, 2.1, 2.8]. In particular, (iii) together with Perelman's fibration theorem implies that Σ is homeomorphic to a suspension with poles η_1, ζ_1 . Moreover, (ii) shows that the pole η_1 is a manifold point as it is not an extremal point. This shows that Σ is a topological sphere.

(2) Suppose there is a quadruple of points in Σ with pairwise distances $> \pi/2$ but also there is a point of Σ at which the space of directions is not homeomorphic to a sphere. By Theorem 4.1, the set E of such singular points with nonspherical links is a 1-dimensional extremal subset of Σ . Note that the proof of Theorem 4.1 shows that E has no isolated points.

In fact any component F of E is a primitive extremal subset homeomorphic to a circle. Indeed, since F is a compact MCS space, it is topologically a finite graph. If F has a topological singularity ξ , i.e., a branching or boundary point, it must be an extremal point. However, since we have the quadruple, at least two of them are inside or outside the $\pi/2$ -neighborhood of ξ . Then triangle comparison shows that ξ is not an extremal point, a contradiction.

Similarly, for any $\xi \in F$, two of the quadruple lie inside the $\pi/2$ -neighborhood of ξ and the other two lie outside: otherwise there are three directions at ξ making obtuse angles, which contradicts the first half. On the other hand, it is known that every point of Σ is contained in the $\pi/2$ -neighborhood of F ([29, 1.4.1]). Therefore, if we go around the circle F, we will get a contradiction.

Remark 4.5. Both of the above statements are optimal in the sense that the numbers of points in the assumptions cannot be reduced. Indeed, the single and double suspensions of a homology sphere as in Example 2.22 give counterexamples to (1) and (2), respectively. On the other hand, if we assume Σ to be a topological manifold from the beginning, then two points are enough to get the conclusion of (1), as in the original sphere theorem of Grove–Shiohama [16] (cf. [23, 8.2]).

The following is an advanced problem related to Question 3.4. An *iterated space* of directions refers to a space of directions of a space of directions of ... of a space of directions.

Question 4.6. Suppose an Alexandrov space contains no proper extremal subsets. Is any iterated space of directions homeomorphic to a sphere?

It is known that the above conclusion holds for any noncollapsing limit of Riemannian manifolds with sectional curvature bounded below, [19, 1.4] (though there may be proper extremal subsets). Yet another sufficient condition that implies the above conclusion is that every space of directions has radius greater than $\pi/2$, i.e., for any direction there is another direction making an obtuse angle with it, [30, §3], [31, §5.2] (cf. [15], [29, §1]; in this case there are no proper extremal subsets). See also [23, 1.3, 1.4, 1.5] for the corresponding results in the context of geodesically complete spaces with curvature bounded above.

APPENDIX A.

In this appendix we give an example of a primitive extremal subset of codimension 2 whose induced intrinsic metric is not an Alexandrov metric with curvature bounded below. It is irrelevant to the main theorem, but like the main theorem, it cannot be found in the literature. This is why we include it here.

Let us briefly review the history. One of the long-standing problems in Alexandrov geometry is the boundary conjecture — the boundary of an Alexandrov space equipped with the induced intrinsic metric would have the same lower curvature bound as the ambient space. This is known for a convex hypersurface in a Riemannian manifold ([2]), but is still open even for a convex hypersurface in a smoothable Alexandrov space, i.e., a noncollapsing limit of Riemannian manifolds ([22]). Recall that the boundary is a codimension 1 extremal subset.

As a generalization of the boundary conjecture, Perelman–Petrunin [29, 6.1] asked whether the induced intrinsic metric of a primitive extremal subset admits a lower curvature bound or not. In [30, $\S 1$ A counterexample], Petrunin constructed an example of a primitive extremal subset of codimension 3 (and thus ≥ 3) that is not an Alexandrov space (cf. [27, 4.1]). However, other than the boundary case, the codimension 2 case remained. In [31, 9.1.2], Petrunin conjectured that a primitive extremal subset of codimension 2 might be an Alexandrov space.

The following simple example provides a negative answer to this conjecture.

Example A.1. Consider a convex subset M of \mathbb{R}^4 defined by

$$M := \{(x, y, z, w) \in \mathbb{R}^4 \mid x \ge 0, \ y \ge 0, \ z \ge 0\}.$$

Suppose \mathbb{Z} acts on M by the cyclic permutation on the (x, y, z)-coordinate and by the translation on the w-coordinate. More precisely, it is generated by

$$\sigma: (x, y, z, w) \mapsto (y, z, x, w + 1).$$

The quotient space $X := M/\mathbb{Z}$ is an Alexandrov space of nonnegative curvature.

The xw-, yw-, and zw-planes are primitive extremal subsets of M. In particular, their union is an extremal subset of M, but not primitive. Let us denote by E its projection into X, which is an extremal subset of codimension 2 ([29, 4.1]). Then E is primitive because it can no longer be divided into each plane due to the group action. However, E cannot admit any Alexandrov metric with curvature bounded below as its intrinsic geodesics branch.

An extremal subset is called *minimal* if it contains no proper extremal subset. Clearly a minimal extremal subset is primitive. Note that any minimal extremal subset must be a topological manifold because the set of topological singularities, if nonempty, is a proper extremal subset ([29, 3.8]). In particular, the above counterexample is not minimal; indeed it contains the projection of the w-axis.

The following possibility still remains.

Question A.2. Is there a minimal extremal subset of codimension 2 that is not an Alexandrov space with respect to the induced intrinsic metric?

Remark A.3. Petrunin's counterexample in [30, §1] is also not minimal: the vertex of the cone is an extremal point. However, it seems possible to slightly modify it so that the vertex becomes a nonextremal point.

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