Transonic shocks for steady Euler flows with an external force in an axisymmetric perturbed cylinder

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Abstract

We concern the structural stability of transonic shocks for the steady Euler system with an external force in an axisymmetric perturbed cylinder. For a class of external forces, we first prove the existence and uniqueness of the transonic shock solution to the one-dimensional steady Euler system with an external force, which shows that the external force has a stabilization effect on the transonic shock in the flat cylinder and the shock position is uniquely determined. We then establish the existence and stability of the transonic shock solution under axisymmetric perturbations of the incoming supersonic flow, the nozzle boundary, the exit pressure and the external force. Different from the transonic shock problem in two-dimensional nozzles, there exists a singularity along the symmetric axis for axisymmetric flows. We introduce an invertible modified Lagrangian transformation to overcome this difficulty and straighten the streamline. One of the key elements in the analysis is to utilize the deformation-curl decomposition to effectively decouple the hyperbolic and elliptic modes in the steady axisymmetric Euler system with an external force. Another one is an equivalent reformulation of the Rankine-Hugoniot conditions so that the shock front is uniquely determined by an algebraic equation.

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1 Introduction and the main result

In this paper, we study the transonic shock problem for steady Euler flows of isentropic polytropic gases in an axisymmetric perturbed cylinder under the external force. Assume the flow enters the nozzle with a supersonic state and leaves it with a relatively high pressure, then it is expected that a shock front occurs in the nozzle such that the flow pressure rises to coincide with the pressure at the exit. Then catching the position of the shock front is one of the important ingredients in determining the flow field in the nozzle. This paper shows that the external force has a stabilization effect on the transonic shocks in the flat cylinder and the shock position is uniquely determined. Then we further investigate the structural stability of the transonic shock solution under axisymmetric perturbations of the incoming supersonic flow, the nozzle boundary, the exit pressure and the external force.

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The steady flow of inviscid compressible gas with an external force in $\mathbb{R}^3$ is governed by the following system:

$$\begin{cases} \partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) + \partial_{x_3}(\rho u_3) = 0, \\ \partial_{x_1}(\rho u_1^2 + P) + \partial_{x_2}(\rho u_1 u_2) + \partial_{x_3}(\rho u_1 u_3) = \rho \partial_{x_1} \Phi, \\ \partial_{x_1}(\rho u_1 u_2) + \partial_{x_2}(\rho u_2^2 + P) + \partial_{x_3}(\rho u_2 u_3) = \rho \partial_{x_2} \Phi, \\ \partial_{x_1}(\rho u_1 u_3) + \partial_{x_2}(\rho u_2 u_3) + \partial_{x_3}(\rho u_3^2 + P) = \rho \partial_{x_3} \Phi. \end{cases} \tag{1.1}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, ρ is the density, P is the pressure and Φ is the potential force, respectively. We consider the isentropic polytropic gases, therefore the equation of state is given by $P = A\rho^\gamma$, where A is a positive constant and γ is the adiabatic constant with $\gamma > 1$. For convenience, we take $A = 1$ in this paper. Denote the sound speed by $c(\rho) = \sqrt{P'(\rho)}$. It is well-known that system (1.1) is hyperbolic for supersonic flows (i.e. $|\mathbf{u}| > c(\rho)$) and hyperbolic-elliptic mixed for subsonic flows (i.e. $|\mathbf{u}| < c(\rho)$).

The stability analysis of transonic shock solutions in a flat nozzle have been studied extensively. For steady flows with shocks in finitely and infinitely long flat nozzles, there exists a class of transonic shock solutions with both upstream supersonic state and downstream subsonic state being constant and its shock position being arbitrary. The structural stability of these transonic shocks for steady potential flows in nozzles was studied in [2, 3, 4, 26, 27]. The authors in [5, 6] established the existence of transonic shocks to steady Euler flows in 2-D nozzles with slowly varying cross-sections. The existence and stability of the transonic shock for 2-D and 3-D steady Euler flows in flat or almost flat nozzles with the prescribed pressure at the exit up to a constant were studied in [7, 28] and [8, 9]. Both existence results are established under the assumption that the shock front passes through a given point. Recently, without such an artificial assumption, the authors in [10] established the stability and existence of transonic shock solutions to the two dimensional steady compressible Euler system in an almost flat finite nozzle with the exit pressure, where the shock position was uniquely determined. This was generalized to three dimensional axisymmetric case in [11].

On the other hand, there were many studies on the stability of the radially symmetric transonic shock in a divergent nozzle. The authors in [1] studied the stability of transonic shocks for multi-dimensional steady potential flows in divergent nozzles. The well-posedness of the transonic shock problem in two dimensional divergent nozzles under the perturbations of the exit pressure was first established in [17] when the opening angle of the nozzle is suitably small. This restriction was removed in [16] and the transonic shock in a 2-D straight divergent nozzle is shown in [19] to be structurally stable under the perturbations of the nozzle walls and the exit pressure. The existence and stability of three-dimensional axisymmetric transonic shock flows in a conic nozzle were studied in [15, 18, 21, 29]. In [21], the authors introduced a modified Lagrangian transformation to deal with the corner singularities near the intersection points of the shock surface and nozzle boundary and the artificial singularity near the axis simultaneously. The stability of spherically symmetric transonic shocks in a spherical shell was studied in [20] by requiring that the background transonic shock solutions satisfy some "Structure Conditions". Recently, the authors in [24] had made a substantial progress and established the existence and stability of cylindrical transonic shock solutions under three dimensional perturbations of the incoming flows and the exit pressure without any restriction on the background transonic shock solutions.

Let $L_1, L_2 (> L_1)$ be fixed positive constants. The axisymmetric cylinder is described as

$$\mathcal{N}_b := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : L_1 < x_1 < L_2, 0 \le x_2^2 + x_3^2 < 1\}.$$

We first consider the one-dimensional steady Euler system with an external force in $\mathcal{N}_b$, which is governed by

$$
\begin{cases}
(\rho_b u_b)'(x_1) = 0, \\
(\rho_b u_b u_b')(x_1) + P_b'(x_1) = (\rho_b g)(x_1), \\
\rho_b(L_1) = \bar{\rho} > 0, \quad u_b(L_1) = \bar{u} > 0, \\
P_b(L_2) = P_e,
\end{cases}
\tag{1.2}
$$

where the flow state at the entrance $x_1 = L_1$ is supersonic, i.e., $\bar{u}^2 > c^2(\bar{\rho}) = \gamma\bar{\rho}^{\gamma-1}$. By employing the monotonicity relation between the shock position and the end pressure, the following Lemma was established in [25] shows that there is a unique transonic shock solution to (1.2) when the end pressure is a suitably prescribed constant P_e and $g(x_1) > 0$ for any $x_1 \in [L_1, L_2]$. Meanwhile, it is shown that the external force has a stabilization effect on the transonic shock in the cylinder and the shock position is uniquely determined.

Lemma 1.1. *Suppose that the initial state $(\bar{\rho}, \bar{u})$ at $x_1 = L_1$ is supersonic and the external force g satisfying $g(x_1) > 0$ for any $x_1 \in [L_1, L_2]$, there exist two positive constants P_1, P_2 such that if the end pressure $P_e \in (P_1, P_2)$, there exists a unique piecewise transonic shock solution*

$$
\mathbf{\Psi}_b(\mathbf{x}) = (\mathbf{u}_b, P_b)(\mathbf{x}) =
\begin{cases}
\mathbf{\Psi}_b^-(\mathbf{x}) := (u_b^-(x_1), 0, 0, P_b^-(x_1)), & \text{if } L_1 < x_1 < L_b, \\
\mathbf{\Psi}_b^+(\mathbf{x}) := (u_b^+(x_1), 0, 0, P_b^+(x_1)), & \text{if } L_b < x_1 < L_2,
\end{cases}
\tag{1.3}
$$

with a shock front located at $x_1 = L_b \in (L_1, L_2)$. Across the shock, the following Rankine-Hugoniot conditions and entropy condition are satisfied:

$$
\begin{cases}
[\rho_b u_b](L_b) = 0, \\
[\rho_b u_b^2 + P_b](L_b) = 0, \\
[P_b](L_b) > 0.
\end{cases}
$$

Moreover, the shock position $x_1 = L_b$ increases as the exit pressure P_e decreases. In addition, the shock position $x_1 = L_b$ approaches to L_1 if P_e goes to P_2 and $x_1 = L_b$ approaches to L_2 if P_e goes to P_1.

The 1-D transonic shock solution $\mathbf{\Psi}_b$ with a shock occurring at $x_1 = L_b$ will be called the background solution in this paper. Clearly, one can extend the supersonic and subsonic parts of $\mathbf{\Psi}_b$ in a natural way, respectively. For convenience, we still call the extended subsonic and supersonic solutions $\mathbf{\Psi}_b^+$ and $\mathbf{\Psi}_b^-$. This paper is going to establish the structural stability of this transonic shock solution under axisymmetric perturbations of the incoming supersonic flows, the nozzle walls, the exit pressure and the external force.

Let (x, r, θ) be the cylindrical coordinates of $(x_1, x_2, x_3) \in \mathbb{R}^3$, that is

$$
x = x_1, \; r = \sqrt{x_2^2 + x_3^2}, \; \theta = \arctan\frac{x_3}{x_2}.
$$

Any function $v(\mathbf{x})$ can be represented as $v(\mathbf{x}) = v(x, r, \theta)$, and a vector-valued function $\mathbf{h}(\mathbf{x})$ can be represented as $\mathbf{h}(\mathbf{x}) = h_x(x, r, \theta)\mathbf{e}_x + h_r(x, r, \theta)\mathbf{e}_r + h_\theta(x, r, \theta)\mathbf{e}_\theta$, where

$$
\mathbf{e}_x = (1, 0, 0), \quad \mathbf{e}_r = (0, \cos\theta, \sin\theta), \quad \mathbf{e}_\theta = (0, -\sin\theta, \cos\theta).
$$

We say that a function $v(\mathbf{x})$ is axisymmetric if its value is independent of θ and that a vector-valued function $\mathbf{h} = (h_x, h_r, h_\theta)$ is axisymmetric if each of functions $h_x(\mathbf{x}), h_r(\mathbf{x})$ and $h_\theta(\mathbf{x})$ is axisymmetric.

Assume that

$$\rho(\mathbf{x}) = \rho(x,r), \quad P(\mathbf{x}) = P(x,r), \quad \mathbf{u}(\mathbf{x}) = u_x(x,r)\mathbf{e}_x + u_r(x,r)\mathbf{e}_r + u_\theta(x,r)\mathbf{e}_\theta.$$

Then (1.1) can be simplified as

$$\begin{cases} \partial_x(r\rho u_x) + \partial_r(r\rho u_r) = 0, \\ \rho(u_x\partial_x + u_r\partial_r)u_x + \partial_x P = \rho\partial_x\Phi, \\ \rho(u_x\partial_x + u_r\partial_r)u_r - \dfrac{\rho u_\theta^2}{r} + \partial_r P = \rho\partial_r\Phi, \\ \rho(u_x\partial_x + u_r\partial_r)(ru_\theta) = 0. \end{cases} \tag{1.4}$$

The axisymmetric perturbed cylinder is given by

$$\mathcal{N} := \{(x,r) \in \mathbb{R}^2 : L_1 < x < L_2,\ 0 \le r < 1 + \sigma f(x)\},$$

where σ is sufficiently small and $f \in C^{2,\alpha}([L_1, L_2])$ satisfies

$$f(L_1) = f'(L_1) = 0. \tag{1.5}$$

Let the potential force Φ and the supersonic incoming flow at the inlet $x = L_1$ be prescribed as

$$\begin{cases} \Phi(x,r) = \Phi_b(x) + \sigma\Phi_e(x,r), \\ \mathbf{\Psi}^-(L_1, r) = \mathbf{\Psi}_b^-(L_1) + \sigma(u_{en}^-, v_{en}^-, w_{en}^-, P_{en}^-)(r). \end{cases} \tag{1.6}$$

Here $\Phi_b' = g$ and $\Phi_e(x,r) \in C^{2,\alpha}(\overline{\mathcal{N}})$ and $(u_{en}^-, v_{en}^-, w_{en}^-, P_{en}^-)(r) \in (C^{2,\alpha}[0,1])^4$. On the nozzle wall, the flow satisfies the slip condition $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the outer normal of the nozzle wall. Using cylindrical coordinates, the slip boundary condition can be rewritten as

$$u_r = \sigma f'(x) u_x, \quad \text{on } \Gamma = \{(x,r) : r = 1 + \sigma f(x),\ L_1 \le x \le L_2\}. \tag{1.7}$$

On the exit of the nozzle, the end pressure is prescribed by

$$P(L_2, r) = P_e + \sigma P_{ex}(r), \tag{1.8}$$

where $P_{ex}(r) \in C^{1,\alpha}(\bar{\mathbb{R}}^+)$.

In this paper, we want to look for a piecewise smooth solution $\mathbf{\Psi}$, which jumps only at a shock front $\mathcal{S} = \{(x,r) : x = \xi(r), r \in [0, r_*]\}$. Here $(\xi(r_*), r_*)$ stand for the shock front and the intersection circle of the shock surface with the nozzle wall. More precisely, $\mathbf{\Psi}$ has the following form

$$\mathbf{\Psi} = \begin{cases} \mathbf{\Psi}^- := (u_x^-, u_r^-, u_\theta^-, P^-)(x,r), & \text{in } \mathcal{N}_- = \{L_1 < x < \xi(r),\ 0 \le r < 1 + \sigma f(x)\}, \\ \mathbf{\Psi}^+ := (u_x^+, u_r^+, u_\theta^+, P^+)(x,r), & \text{in } \mathcal{N}_+ = \{\xi(r) < x < L_2,\ 0 \le r < 1 + \sigma f(x)\}, \end{cases} \tag{1.9}$$

and satisfies the following Rankine-Hugoniot conditions on the shock surface $\mathcal{S}$:

$$\begin{cases} [\rho u_x] - \xi'(r)[\rho u_r] = 0, \\ [\rho u_x^2 + P] - \xi'(r)[\rho u_x u_r] = 0, \\ [\rho u_x u_r] - \xi'(r)[\rho u_r^2 + P] = 0, \\ [\rho u_x u_\theta] - \xi'(r)[\rho u_r u_\theta] = 0. \end{cases} \tag{1.10}$$

The existence and uniqueness of the supersonic flow to (1.4) follows from the the classical theory to the boundary value problem for quasi-linear hyperbolic systems (See [13]).

Lemma 1.2. *Assume that the potential force and the supersonic incoming data given in (1.6) satisfying the following compatibility conditions*

$$\begin{cases} \partial_r\Phi_e(x,0) = 0, \\ v_{en}^-(0) = w_{en}^-(0) = (v_{en}^-)''(0) = (w_{en}^-)'(0) = (P_{en}^-)'(0) = 0, \\ v_{en}^-(1) = 0,\ (P_{en}^-)'(1) = \rho_{en}^-(1)((w_{en}^-)^2(1) + \sigma\partial_r\Phi_e(L_1,1)). \end{cases} \tag{1.11}$$

Then there exists a constant $\sigma_0 > 0$ depending only on the background solution and the boundary data, such that for any $0 < \sigma < \sigma_0$, there exists a unique axisymmetric solution $\mathbf{\Psi}^- = (u_x^-, u_r^-, u_\theta^-, P^-)(x,r) \in C^{2,\alpha}(\overline{\mathcal{N}})$ to (1.4) with (1.6) and (1.7), which satisfies

$$\|(u_x^-, u_r^-, u_\theta^-, P^-) - (u_b^-, 0, 0, P_b^-)\|_{C^{2,\alpha}(\overline{\mathcal{N}})} \le C_0\sigma \tag{1.12}$$

and

$$(u_r^-, \partial_r^2 u_r^-)(x,0) = (u_\theta^-, \partial_r u_\theta^-)(x,0) = \partial_r(u_x^-, P^-)(x,0) = 0, \quad \forall x \in [L_1, L_2]. \tag{1.13}$$

Therefore, the problem is reduced to solve a free boundary value problem for the steady axisymmetric Euler system with an external force in which the shock front and the downstream subsonic flows are unknown. Then to state the main result, we first introduce some weighted Hölder spaces and their norms. For any bounded domain $\mathcal{P}$, let $\mathcal{H}$ be a closed portion of $\mathcal{P}$. For $\mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{P}$, define

$$\delta_{\mathbf{x}} := \mathrm{dist}(\mathbf{x}, \mathcal{H}) \quad \text{and} \quad \delta_{\mathbf{x},\tilde{\mathbf{x}}} := \min(\delta_{\mathbf{x}}, \delta_{\tilde{\mathbf{x}}}).$$

For any positive integer m, $\alpha \in (0,1)$ and $\kappa \in \mathbb{R}$, we define

$$\begin{aligned} [u]_{k,0;\mathcal{P}}^{(\kappa;\mathcal{H})} &:= \sum_{|\beta|=k} \sup_{\mathbf{x}\in\mathcal{P}} \delta_{\mathbf{x}}^{\max(|\beta|+\kappa,0)} |D^\beta u(\mathbf{x})|,\ k = 0,1,\cdots,m; \\ [u]_{m,\alpha;\mathcal{P}}^{(\kappa;\mathcal{H})} &:= \sum_{|\beta|=k} \sup_{\mathbf{x},\tilde{\mathbf{x}}\in\mathcal{P},\mathbf{x}\neq\tilde{\mathbf{x}}} \delta_{\mathbf{x},\tilde{\mathbf{x}}}^{\max(m+\alpha+\kappa,0)} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\tilde{\mathbf{x}})|}{|\mathbf{x}-\tilde{\mathbf{x}}|^\alpha}; \\ \|u\|_{m,\alpha;\mathcal{P}}^{(\kappa;\mathcal{H})} &:= \sum_{k=0}^m [u]_{k,0;\mathcal{P}}^{\kappa;\mathcal{H}} + [u]_{m,\alpha;\mathcal{P}}^{\kappa;\mathcal{H}} \end{aligned}$$

with the corresponding function space defined as

$$C_{m,\alpha}^{(\kappa;\mathcal{H})}(\mathcal{P}) = \{u : \|u\|_{m,\alpha;\mathcal{P}}^{(\kappa,\mathcal{H})} < \infty\}.$$

The main result in this paper is stated as follows.

Theorem 1.3. *Assume that the compatibility conditions (1.5) and (1.11) hold. There exist suitable positive constants σ_0 and C_* depending only on the background solution $\mathbf{\Psi}_b$ defined in (1.3) and the boundary data $\mathbf{\Psi}^-(L_1,\cdot)$, f, P_{ex} ,Φ_e such that if $0 < \sigma \le \sigma_0$, the problem (1.4) with (1.6), (1.7), (1.8) and (1.10) has a unique axisymmetric solution $\mathbf{\Psi}^+ = (u_x^+, u_r^+, u_\theta^+, P^+)(x,r)$ with the shock front $\mathcal{S}$ satisfying the following properties.*

(1) *The function $\xi(r) \in C_{3,\alpha}^{(-1-\alpha;\{r_*\})}([0,r_*))$ satisfies*

$$\|\xi(r) - L_b\|_{3,\alpha;[0,r_*)}^{(-1-\alpha;\{r_*\})} \le C_*\sigma \tag{1.14}$$

and

$$\xi'(0) = \xi^{(3)}(0) = 0. \tag{1.15}$$

(2) *The solution* $\mathbf{\Psi}^+ = (u_x^+, u_r^+, u_\theta^+, P^+)(x,r) \in C_{2,\alpha}^{(-\alpha;\Gamma_{p,s})}(\mathcal{N}_+)$ *satisfies the entropy condition*

$$P^+(\xi(r), r) > P^-(\xi(r), r), \ \forall r \in [0, r_*] \tag{1.16}$$

and the estimate

$$\|\mathbf{\Psi}^+ - \mathbf{\Psi}_b^+\|_{2,\alpha;\mathcal{N}_+}^{(-\alpha;\Gamma_{p,s})} \leq C_*\sigma \tag{1.17}$$

with the compatibility conditions

$$(u_r^+, \partial_r^2 u_r^+)(x,0) = (u_\theta^+, \partial_r u_\theta^+)(x,0) = \partial_r(u_x^+, P^+)(x,0) = 0, \quad \forall x \in [\xi(r), L_2], \tag{1.18}$$

where

$$\Gamma_{p,s} = \{(x,r) : \xi(r) \leq x \leq L_2, r = 1 + \sigma f(x)\}.$$

Remark 1.4. Compared with the two-dimensional case in [25], one of the major difficulties for axisymmetric flows is the possible singularity near the symmetric axis. Inspired by [21], the singular term r in the density equation is of order $O(r)$ near the axis $r = 0$, hence we can find a simple modified Lagrangian transformation such that it is invertible near the axis and also straightens the streamline.

Remark 1.5. The previous work [11, 15, 21] reduced the steady axisymmetric Euler system in the subsonic region into a elliptic system of the flow angle and pressure. One of main ingredients of our analysis here is quite different from those in [11, 15, 21], we utilize the deformation-curl decomposition introduced in [22, 23] to effectively decouple the hyperbolic and elliptic modes in the steady axisymmetric Euler system with an external force. This decomposition is based on a simple observation that one can rewrite the density equation as a Frobenius inner product of a symmetric matrix and the deformation matrix by using the Bernoulli's law and representing the density as functions of the Bernoulli's quantity and the velocity field. The vorticity is resolved by an algebraic equation of the Bernoulli's quantity.

The rest of this article is organized as follows. In Section 2, we introduce the modified Lagrangian transformation and decompose the hyperbolic and elliptic modes for the steady axisymmetric Euler system with an external force in the subsonic region in terms of the deformation and curl, and the corresponding reformulation of the Rankine-Hugoniot conditions. We also introduce a coordinate transformation such that the free boundary becomes fixed. In Section 3, we design an iteration scheme to prove Theorem 1.3.

2 The reformulation of the transonic shock problem

In this section, we first introduce the modified Lagrangian transformation, then the deformation-curl reformulation developed in [22, 23] is employed to rewrite the steady axisymmetric Euler system with an external force. Finally, we reformulate the Rankine-Hugoniot conditions and boundary conditions and introduce another coordinates transformation to reduce the transonic shock problem into a fixed boundary value problem.

2.1 The modified Lagrangian transformation

For generic perturbations of the cylinder wall, one can only expect the C^α boundary regularity for the solution in the subsonic region (see [28, Remark 3.2]). In order to avoid the difficulty in uniquely determining the trajectory, we introduce a Lagrangian transformation to straighten the streamline.

However, in the three-dimensional axisymmetric setting, there is a singular term r in the density equation. Inspired by [21], we can find a simple modified Lagrangian transformation to overcome this difficulty and apply this modified Lagrangian transformation to rewrite (1.4) and (1.10).

Define $(\tilde{y}_1, \tilde{y}_2) = (x, \tilde{y}_2(x, r))$ such that

$$\begin{cases} \frac{\partial \tilde{y}_2}{\partial x} = -r\rho^- u_r^-, \ \frac{\partial \tilde{y}_2}{\partial r} = r\rho^- u_x^-, & \text{if } (x, r) \in \overline{\mathcal{N}_-}, \\ \frac{\partial \tilde{y}_2}{\partial x} = -r\rho^+ u_r^+, \ \frac{\partial \tilde{y}_2}{\partial r} = r\rho^+ u_x^+, & \text{if } (x, r) \in \overline{\mathcal{N}_+}, \\ \tilde{y}_2(L_1, 0) = 0, \quad \tilde{y}_2(L_2, 0) = 0. \end{cases} \tag{2.1}$$

On the axis $r = 0$ and the nozzle wall Γ, one derives that

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{y}_2(x, 0) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tilde{y}_2(x, 1 + \sigma f(x)) = 0.$$

Thus for any $x \in [L_1, L_2]$, we can assume $\tilde{y}_2(x, 0) = 0$. Then one has

$$\begin{cases} \tilde{y}_2(x, 1 + \sigma f(x)) = \mathcal{M}^2, & \forall x \in [L_1, L_*], \\ \tilde{y}_2(x, 1 + \sigma f(x)) = \mathcal{M}_1^2, & \forall x \in [L_*, L_2], \end{cases}$$

where $\mathcal{M}$ and $\mathcal{M}_1$ are two constants to be determined, and $(L_*, 1 + \sigma f(L_*))$ is the intersecting point of the shock front $\mathcal{S}$ with the nozzle wall Γ. Next, we need to verify that $\tilde{y}_2(x, r)$ is well-defined and belongs to $\mathrm{Lip}(\overline{\mathcal{N}})$. Indeed, by using the first equation in (1.4), one obtains

$$\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}r}(\xi(r) + 0, r) = \frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}r}(\xi(r) - 0, r).$$

This yields $\mathcal{M}_1 = \mathcal{M}$, which can be computed as follows

$$\mathcal{M}^2 = \int_0^1 s\rho^- u_x^-(L_1, s)\mathrm{d}s > 0.$$

Define the modified Lagrangian transformation as

$$\begin{cases} y_1 = x, \\ y_2 = \tilde{y}_2^{\frac{1}{2}}(x, r). \end{cases} \tag{2.2}$$

If $(\rho^\pm, u_x^\pm, u_r^\pm, u_\theta^\pm)$ are close to the background solution $(\rho_b^\pm, u_b^\pm, 0, 0)$, there exist two positive constants C_1 and C_2, depending on the background solution, such that

$$C_1 r^2 \leq \tilde{y}_2(x, r) = \int_0^r s\rho^- u_x^-(L_1, s)\mathrm{d}s \leq C_2 r^2.$$

Then one gets $\sqrt{C_1} r \leq y_2(x, r) \leq \sqrt{C_2} r$. Therefore, the Jacobian of the modified Lagrangian transformation satisfies

$$\frac{\partial(y_1, y_2)}{\partial(x, r)} = \begin{vmatrix} 1 & 0 \\ -\frac{r\rho u_r}{2y_2} & \frac{r\rho u_x}{2y_2} \end{vmatrix} = \frac{r\rho u_x}{2y_2} \geq C_3 > 0. \tag{2.3}$$

Hence the modified Lagrangian transformation is invertible.

Under the modified Lagrangian transformation, the shock front $\mathcal{S}$ and the flows before and behind $\mathcal{S}$ are denoted by $y_1 = \psi(y_2)$ and $(u_x^\pm, u_r^\pm, u_\theta^\pm, P^\pm)(y_1, y_2)$ respectively. Then the domains $\mathcal{N}_-$ and $\mathcal{N}_+$ are changed into

$$\begin{cases} \mathcal{D}_- = \{(y_1, y_2) : L_1 < y_1 < \psi(y_2), y_2 \in [0, \mathcal{M})\}, \\ \mathcal{D}_+ = \{(y_1, y_2) : \psi(y_2) < y_1 < L_2, y_2 \in [0, \mathcal{M})\}. \end{cases}$$

The nozzle wall $\Gamma_{p,s}$ is straightened to be

$$\Gamma_{p,y} = \{(y_1, y_2) : \psi(\mathcal{M}) \le y_1 \le L_2,\ y_2 = \mathcal{M}\}. \tag{2.4}$$

On the other hand, under this transformation, r as a function of (y_1, y_2) becomes nonlinear and nonlocal. In fact, it follows from the inverse transformation that

$$\frac{\partial r}{\partial y_1} = \frac{u_r}{u_x}, \quad \frac{\partial r}{\partial y_2} = \frac{2y_2}{r\rho u_x}, \quad r(y_1, 0) = 0.$$

Thus one obtains

$$r(y_1, y_2) = 2\left(\int_0^{y_2} \frac{s}{\rho u_x(y_1, s)} \mathrm{d}s\right)^{\frac{1}{2}}. \tag{2.5}$$

In particular, for the background solution $(\rho_b^+, u_b^+, 0, 0)$, one has

$$r_b(y_2) = \kappa_b y_2, \tag{2.6}$$

where $\kappa_b = \left(\frac{2}{(\rho_b^+ u_b^+)(y_1)}\right)^{\frac{1}{2}}$ is a positive constant for any $y_1 \in [L_b, L_2]$.

For simplicity of the notations, we neglect the superscript"+"for the solution in the subsonic region. Then under the transformation (2.2), the system (1.4) becomes

$$\begin{cases} \partial_{y_1}\left(\frac{2y_2}{r\rho u_x}\right) - \partial_{y_2}\left(\frac{u_r}{u_x}\right) = 0, \\ \partial_{y_1}\left(u_x + \frac{P}{\rho u_x}\right) - \frac{r}{2y_2}\partial_{y_2}\left(\frac{P u_r}{u_x}\right) - \frac{P u_r}{r\rho u_x^2} = \partial_{y_1}\Phi - \frac{r\rho u_r}{2y_2 u_x}\partial_{y_2}\Phi, \\ \partial_{y_1} u_r + \frac{r}{2y_2}\partial_{y_2} P - \frac{u_\theta^2}{r u_x} = \frac{r\rho}{2y_2}\partial_{y_2}\Phi, \\ \partial_{y_1}(r u_\theta) = 0. \end{cases} \tag{2.7}$$

The Rankine-Hugoniot conditions (1.13) across the shock front $\mathcal{S}$ become

$$\begin{cases} \frac{2y_2}{r}\left[\frac{1}{\rho u_x}\right] + \psi'(y_2)\left[\frac{u_r}{u_x}\right] = 0, \\ \left[u_x + \frac{P}{\rho u_x}\right] + \psi'(y_2)\frac{r}{2y_2}\left[\frac{P u_r}{u_x}\right] = 0, \\ [u_r] - \psi'(y_2)\frac{r}{2y_2}[P] = 0, \\ [u_\theta] = 0. \end{cases} \tag{2.8}$$

2.2 The deformation-curl decomposition for the steady axisymmetric Euler system with an external force

The steady axisymmetric Euler system with an external force is hyperbolic-elliptic coupled in the subsonic region. An effective decomposition of the elliptic and hyperbolic modes is crucial for the solvability of the nonlinear free boundary problem. Here we will use the deformation-curl decomposition introduced in [22, 23] to derive an equivalent system (2.13), where the hyperbolic quantities and elliptic quantities are effectively decoupled.

Define the Bernoulli's function B by

$$B = \frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma P}{(\gamma-1)\rho} - \Phi. \tag{2.9}$$

Then one has

$$\rho\mathbf{u}\cdot B = 0. \tag{2.10}$$

By using the Bernoulli's quantity, the density ρ can be represented as

$$\rho = H(B,\Phi,|\mathbf{u}|^2) = \left(\frac{\gamma-1}{\gamma}(B+\Phi-\frac{1}{2}|\mathbf{u}|^2)\right)^{\frac{1}{\gamma-1}}. \tag{2.11}$$

Define the vorticity $\omega = \operatorname{curl}\mathbf{u} = \omega_x\mathbf{e}_x + \omega_r\mathbf{e}_r + \omega_\theta\mathbf{e}_\theta$, where

$$\omega_x = \frac{1}{r}\partial_r(ru_\theta),\ \omega_r = -\partial_x u_\theta,\ \omega_\theta = \partial_x u_r - \partial_r u_x.$$

From the third equation in (1.4) and the Bernoulli's law, one derives that

$$\omega_\theta = \frac{u_\theta\omega_x}{u_x} - \frac{\partial_r B}{u_x}. \tag{2.12}$$

Substituting (2.11) into the density equation and combining (2.10) and (2.12), the system (1.4) is equivalent to the following system

$$\begin{cases}(c^2(\rho)-u_x^2)\partial_x u_x + (c^2(\rho)-u_r^2)\partial_r u_r - u_x u_r(\partial_x u_r + \partial_r u_x) + u_r\dfrac{c^2(\rho)+u_\theta^2}{r}\\ \quad + (u_x\partial_x\Phi + u_r\partial_r\Phi) = 0,\\ u_x(\partial_x u_r - \partial_r u_x) = u_\theta\partial_r u_\theta + \dfrac{u_\theta^2}{r} - \partial_r B,\\ (u_x\partial_x + u_r\partial_r)(ru_\theta) = 0,\\ (u_x\partial_x + u_r\partial_r)B = 0.\end{cases} \tag{2.13}$$

Under the modified Lagrangian transformation, the system (2.13) can be rewritten as

$$\begin{cases}(c^2(\rho)-u_x^2)\left(\partial_{y_1}u_x - \dfrac{r\rho u_r}{2y_2}\partial_{y_2}u_x\right) + (c^2(\rho)-u_r^2)\left(\dfrac{r\rho u_x}{2y_2}\partial_{y_2}u_r\right) + \dfrac{c^2(\rho)+u_\theta^2}{r}u_r\\ = -u_x\partial_{y_1}\Phi + u_x u_r\left(\partial_{y_1}u_r - \dfrac{r\rho u_r}{2y_2}\partial_{y_2}u_r + \dfrac{r\rho u_x}{2y_2}\partial_{y_2}u_x\right),\\ u_x\left(\partial_{y_1}u_r - \dfrac{r\rho u_r}{2y_2}\partial_{y_2}u_r - \dfrac{r\rho u_x}{2y_2}\partial_{y_2}u_x\right) = \dfrac{r\rho u_x}{2y_2}u_\theta\partial_{y_2}u_\theta + \dfrac{u_\theta^2}{r} - \dfrac{r\rho u_x}{2y_2}\partial_{y_2}B,\\ \partial_{y_1}(ru_\theta) = 0,\\ \partial_{y_1}B = 0.\end{cases} \tag{2.14}$$

2.3 The reformulation of the Rankine-Hugoniot conditions and boundary conditions

Due to the mixed elliptic-hyperbolic structure of the steady axisymmetric Euler system with an external force in the subsonic region, it is important to formulate proper boundary conditions and their compatibility.

Define

$$
\begin{aligned}
&v_1(y_1,y_2)=u_x(y_1,y_2)-u_b^+(y_1,y_2), \quad v_2(y_1,y_2)=u_r(y_1,y_2), \quad v_3(y_1,y_2)=u_\theta(y_1,y_2),\\
&v_4(y_1,y_2)=B(y_1,y_2)-B_b^+(y_1,y_2), \quad \mathbf{v}=(v_1,v_2,v_3,v_4), \quad v_5(y_2)=\psi(y_2)-L_b.
\end{aligned}
$$

Then the density and the pressure can be expressed as

$$
\begin{aligned}
&\rho(y_1,y_2)=\rho(\mathbf{v})=\Big(\frac{\gamma-1}{\gamma}\Big)^{\frac{1}{\gamma-1}}\Big(B_b^++v_4+\Phi_b+\sigma\Phi_e-\frac{1}{2}(u_b^++v_1)^2-\sum_{j=2}^{3}|v_j|^2\Big)^{\frac{1}{\gamma-1}},\\
&P(y_1,y_2)=P(\mathbf{v})=\Big(\frac{\gamma-1}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}}\Big(B_b^++v_4+\Phi_b+\sigma\Phi_e-\frac{1}{2}(u_b^++v_1)^2-\sum_{j=2}^{3}|v_j|^2\Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{2.15}
$$

By the third equation in (2.8), one derives

$$
\psi'(y_2)=\frac{2y_2[u_r]}{r[P]}=a_1v_2(\psi(y_2),y_2)+h_1(\mathbf{\Psi}^-(L_b+v_5,y_2)-\mathbf{\Psi}_b^-(L_b+v_5),\mathbf{v}(\psi,y_2),v_5). \tag{2.16}
$$

where $a_1=\frac{1}{\kappa_b[P_b(L_b)]}>0$ and

$$
\begin{aligned}
&h_1(\mathbf{\Psi}^-(L_b+h_5,y_2)-\mathbf{\Psi}_b^-(L_b+h_5),\mathbf{v}(\psi,y_2),h_5)\\
&=\frac{2y_2[u_r]}{r[P]}-a_1v_2(\psi(y_2),y_2)=v_2\left(\frac{2y_2}{r[P]}-a_1\right)-\frac{2y_2u_r^-(\psi(y_2),y_2)}{r[P]}.
\end{aligned}
$$

The functions h_1 is regarded as the error term which can be bounded by

$$
|h_1|\le C\Big(|\mathbf{\Psi}^-(r_b+v_5,y_2)-\mathbf{\Psi}_b^-(r_b+v_5)|+|\mathbf{v}(\psi,y_2)|^2+|v_5|^2\Big). \tag{2.17}
$$

Using the equation (2.16), we can eliminate the quantity ψ' in the first two equations of (2.8) to obtain

$$
\begin{cases}
[\frac{1}{\rho u_x}]+\frac{[u_r]}{[P]}\left[\frac{u_r}{u_x}\right]=0,\\
[u_x+\frac{P}{\rho u_x}]+\frac{[u_r]}{[P]}\left[\frac{Pu_r}{u_x}\right]=0.
\end{cases}
\tag{2.18}
$$

Next, a simple calculation gives

$$
\begin{cases}
[\rho u_x]=\rho u_x\rho^-u_x^-\frac{[u_r]}{[P]}\frac{[u_r]}{[u_x]},\\
[\rho u_x^2+P]=-\rho^-u_x^-\frac{[u_r]}{[P]}\frac{[Pu_r]}{[u_x]}+(\rho u_x^2+P)\rho^-u_x^-\frac{[u_r]}{[P]}\frac{[u_r]}{[u_x]}.
\end{cases}
\tag{2.19}
$$

Denote $\dot\rho(y_1,y_2)=\rho(y_1,y_2)-\rho_b^+(y_1)$. Then the first equation in (2.19) implies that

$$
\begin{aligned}
&\rho_b^+(L_b)v_1(\psi,y_2)+u_b^+(L_b)\dot\rho(\psi,y_2)\\
&=-[\rho_bu_b](\psi)+\rho u_x\rho^-u_x^-\frac{[u_r]}{[P]}\frac{[u_r]}{[u_x]}+(\rho^-u_x^-)(\psi,y_2)-(\rho_b^-u_b^-)(\psi)\\
&\quad-(u_x+u_b^+(L_b+v_5)-u_b^+(L_b))\dot\rho(\psi,y_2)-(\rho_b^+(L_b+v_5)-\rho_b^+(L_b))v_1(\psi,y_2)\\
&:=R_{11}(\mathbf{\Psi}^-(L_b+v_5,y_2)-\mathbf{\Psi}_b^-(L_b+v_5),\mathbf{v}(\psi,y_2),v_5).
\end{aligned}
$$

Similarly, one can follow from the second equation in (2.19) that at $(\psi(y_2), y_2)$, there holds

$$\begin{cases} \rho_b^+(L_b)v_1(\psi, y_2) + u_b^+(L_b)\dot{\rho}(\psi, y_2) = R_{11}(\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5), \mathbf{v}(\psi, y_2), v_5), \\ 2(\rho_b^+u_b^+)(L_b)v_1(\psi, y_2) + \Big((u_b^+(L_b))^2 + c^2(\rho_b^+(L_b))\Big)\dot{\rho}(\psi, y_2) \\ = -((\rho_b^+ - \rho_b^-)g)(L_b)v_5 + R_{12}(\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5), \mathbf{v}(\psi, y_2), v_5), \end{cases} \tag{2.20}$$

where

$$\begin{aligned} R_{12} = &-\Big\{[\rho_b u_b^2 + P_b](L_b+v_5) - ((\rho_b^+ - \rho_b^-)g)(L_b)v_5\Big\} + (\rho^-(u_x^-)^2 + P^-)(\psi, y_2) - (\rho_b^-(u_b^-)^2 + P_b^-)(\psi) \\ &- \Big\{(\rho u_x^2 + P)(\psi, y_2) - (\rho_b^+(u_b^+)^2 + P_b^+)(\psi) - 2(\rho_b^+u_b^+)(L_b)v_1 \\ &- \{(u_b^+(L_b))^2 + c^2(\rho_b^+(L_b))\}\dot{\rho}\Big\} - \rho^-u_x^-\frac{[u_r]}{[P]}\frac{[Pu_r]}{[u_x]} + (\rho u_x^2 + P)\rho^-u_x^-\frac{[u_r]}{[P]}\frac{[u_r]}{[u_x]}. \end{aligned}$$

Note that

$$\frac{d}{dx}(\rho_b u_b)(x) = 0, \quad \frac{d}{dx}(\rho_b u_b^2 + P_b)(x) = (\rho_b g)(x).$$

Then

$$[\rho_b u_b](L_b+v_5) = O(v_5^2), \quad [\rho_b u_b^2 + P_b](L_b+v_5) - ((\rho_b^+ - \rho_b^-)g)(L_b)v_5 = O(v_5^2).$$

Therefore, there exists a constant $C > 0$ depending only on the background solution such that

$$|R_{1i}| \leq C\Big(|\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5)| + |\mathbf{v}(\psi, y_2)|^2 + |v_5|^2\Big). \tag{2.21}$$

By solving the algebraic equations (2.20), one derives

$$\begin{cases} \dot{\rho}(\psi, y_2) = b_1v_5(y_2) + R_1(\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5), \mathbf{v}(\psi, y_2), v_5), \\ v_1(\psi, y_2) = b_2v_5(y_2) + R_2(\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5), \mathbf{v}(\psi, y_2), v_5), \end{cases} \tag{2.22}$$

where

$$b_1 = -\frac{(\rho_b^+(L_b) - \rho_b^-(L_b))g(L_b)}{c^2(\rho_b^+(L_b)) - (u_b^+(L_b))^2} < 0, \quad b_2 = \frac{u_b^+(L_b)(\rho_b^+(L_b) - \rho_b^-(L_b))g_b(L_b)}{\rho_b^+(L_b)(c^2(\rho_b^+(L_b)) - (u_b^+(L_b))^2)} > 0,$$

and

$$\begin{aligned} R_1 =& \frac{-2u_b^+(L_b)R_{11} + R_{12}}{(c^2(\rho_b^+(L_b)) - (u_b^+(L_b))^2)}, \\ R_2 =& \frac{\Big((u_b^+(L_b))^2 + c^2(\rho_b^+(L_b))\Big)R_{11} - u_b^+(L_b)R_{12}}{\rho_b^+(L_b)(c^2(\rho_b^+(L_b)) - (u_b^+(L_b))^2)}. \end{aligned}$$

In the following, it follow from the Bernoulli's quantity and (2.22) that

$$v_4(\psi, y_2) = b_3v_5(y_2) + R_3(\mathbf{\Psi}^-(L_b+v_5, y_2) - \mathbf{\Psi}_b^-(L_b+v_5), \mathbf{v}(\psi, y_2), v_5), \tag{2.23}$$

where

$$b_3 = \frac{(\rho_b^-(L_b) - \rho_b^+(L_b))g(L_b)}{\rho_b^+(L_b)} < 0, \quad R_3 = \frac{-u_b^+(L_b)R_{11} + R_{12}}{\rho_b^+(L_b)} + R_{13},$$

$$\begin{aligned} R_{13} = &(u_b^+(L_b+v_5) - u_b^+(L_b))v_1(\psi, y_2) - \frac{c^2(\rho_b^+(L_b))}{\rho_b^+(L_b)}\dot{\rho}(\psi, y_2) \\ &+ \frac{1}{2}\sum_{i=1}^{3} v_j^2(\psi, y_2) + \frac{\gamma}{\gamma-1}(\rho^+(\psi, y_2)^{\gamma-1} - (\rho_b^+(\psi))^{\gamma-1}) - \sigma\Phi_e(\psi, y_2). \end{aligned}$$

Next, the superscript "+" in $\rho_b^+, u_b^+, P_b^+, B_b^+$ will be ignored to simplify the notations. To derive the boundary condition at the exit, it follows from the definition of the Bernoulli's quantity and (1.8) that

$$v_1(L_2, y_2) = \frac{v_4(L_2, y_2)}{u_b(L_2)} - \frac{\sigma P_{ex}(r(L_2, y_2))}{(\rho_b u_b)(L_2)} - \frac{1}{2u_b(L_2)}\sum_{j=1}^{3} v_j^2(L_2, y_2) - \frac{E(\mathbf{v}(L_2, y_2))}{u_b(L_2)}, \tag{2.24}$$

where

$$E(\mathbf{v}(y_1, y_2)) = \frac{\gamma}{\gamma-1}(P(\mathbf{v}))^{\frac{\gamma-1}{\gamma}} - \frac{\gamma}{\gamma-1}P_b^{\frac{\gamma-1}{\gamma}} - \frac{1}{\rho_b(y_1)}(P(\mathbf{v}) - P_b) - \sigma\Phi_e.$$

The boundary condition on the nozzle wall is

$$v_2(y_1, \mathcal{M}) = \sigma f'(y_1)(u_b(y_1) + v_1(y_1, \mathcal{M})), \quad \text{on } \Gamma_{p,y}. \tag{2.25}$$

Finally, we derive the equations for v_j $(j = 1, \cdots, 4)$. Note that

$$\begin{cases} (c^2(B_b, u_b, \Phi_b) - u_b^2)u_b' = -u_b g, \\ c^2(B, |\mathbf{u}|, \Phi) - u^2 - c^2(B_b, u_b, \Phi_b) + u_b^2 = (\gamma-1)(v_4 + \sigma\Phi_e) - \frac{\gamma+1}{2}v_1^2 - (\gamma+1)u_b v_1 - \frac{\gamma-1}{2}\sum_{j=2}^{3} v_j^2. \end{cases}$$

Then it follows from (2.14) that

$$\begin{cases} d_1(y_1)\partial_{y_1}v_1 + d_2(y_1)\Big(\partial_{y_2}v_2 + \dfrac{v_2}{y_2}\Big) + d_3(y_1)v_1 + d_4(y_1)v_4 = F_1(\mathbf{v}), \\ \partial_{y_1}v_2 - d_2(y_1)\partial_{y_2}v_1 + d_5(y_1)\partial_{y_2}v_4 = F_2(\mathbf{v}), \\ \partial_{y_1}(rv_3) = 0, \\ \partial_{y_1}v_4 = 0, \end{cases} \tag{2.26}$$

where

$$d_1(y_1) = 1 - M_b^2(y_1) > 0, \quad M_b^2(y_1) = \frac{u_b^2(y_1)}{c^2(\rho_b(y_1))}, \quad d_2(y_1) = \frac{1}{\kappa_b(y_1)} > 0, \quad d_4(y_1) = \frac{(\gamma-1)u_b'(y_1)}{c^2(\rho_b(y_1))},$$

$$d_3(y_1) = \frac{g(y_1) - (\gamma+1)(u_b u_b')(y_1)}{c^2(\rho_b(y_1))} = \frac{(1+\gamma M_b^2)g(y_1)}{c^2(\rho_b(y_1)) - u_b^2(y_1)} > 0, \quad d_5(y_1) = \frac{1}{(\kappa_b u_b)(y_1)} > 0,$$

$$\begin{aligned} F_1(\mathbf{v}) = \frac{1}{c^2(\rho_b(z_1))}\Big(& -(c^2(\rho) - (u_b+v_1)^2 - c^2(\rho_b) + u_b^2)\partial_{y_1}v_1 + \Big(\frac{\gamma+1}{2}v_1^2 + \frac{\gamma-1}{2}(v_2^2+v_3^2)\Big)u_b' \\ & + (c^2(\rho) - (u_b+v_1)^2)\frac{r\rho v_2}{2y_2}\partial_{y_2}v_1 - c^2(\rho)\frac{r\rho(u_b+v_1)}{2y_2}\partial_{y_2}v_2 + c^2(\rho_b)\frac{r_b\rho_b u_b}{2y_2}\partial_{y_2}v_2 \\ & + c^2(\rho)v_2^2\frac{r\rho(u_b+v_1)}{2y_2}\partial_{y_2}v_2 - \frac{c^2(\rho)}{r}v_2 + \frac{c^2(\rho_b)}{r_b}v_2 - \frac{c^2(\rho)v_3^2}{r}v_2 - \sigma(u_b+v_1)\partial_{y_1}\Phi_e \\ & + (u_b+v_1)v_2\Big(\partial_{y_1}v_2 - \frac{r\rho v_2}{2y_2}\partial_{y_2}v_2 + \frac{r\rho(u_b+v_1)}{2y_2}\partial_{y_2}v_1\Big)\Big), \end{aligned}$$

$$\begin{aligned} F_2(\mathbf{v}) = & \frac{r\rho v_2}{2y_2}\partial_{y_2}v_2 + \frac{r\rho(u_b+v_1)}{2y_2}\partial_{y_2}v_1 - \frac{r_b\rho_b u_b}{2y_2}\partial_{y_2}v_1 + \frac{1}{u_b+v_1}\Big(\frac{r\rho(u_b+v_1)}{2y_2}v_3\partial_{y_2}v_3 + \frac{v_3^2}{r} \\ & - \frac{r\rho(u_b+v_1)}{2y_2}\partial_{y_2}v_4 + \frac{r_b\rho_b u_b}{2y_2}\partial_{y_2}v_4\Big). \end{aligned}$$

Here $F_1(\mathbf{v})$ and $F_2(\mathbf{v})$ are quadratic and high order terms.

Then to solve the problem (1.4) with (1.6), (1.7), (1.8) and (1.10) is equivalent to find a function v_5 defined on $[0, \mathcal{M})$ and vector functions $(v_1, \cdots, v_4)$ defined on the $\mathcal{D}_+ := \{(y_1, y_2) : L_b + v_5(y_2) < y_1 < L_2, 0 \le y_2 < \mathcal{M}\}$, which solves the system (2.26) with boundary conditions (2.16), (2.22)-(2.25).

2.4 The coordinates transformation

In order to deal with the free boundary value problem, it is convenient to reduce it into a fixed boundary value problem by setting

$$z_1 = \frac{y_1 - \psi(y_2)}{L_2 - \psi(y_2)}(L_2 - L_b) + L_b = \frac{y_1 - L_b - w_5}{L_2 - L_b - w_5}(L_2 - L_b) + L_b, \quad z_2 = y_2, \tag{2.27}$$

where

$$w_5(y_2) = \psi(y_2) - L_b.$$

Then

$$\begin{cases} y_1 = z_1 + \dfrac{L_2 - z_1}{L_2 - L_b} w_5 =: D_0^{w_5}, \\ \partial_{y_1} = \dfrac{L_2 - L_b}{L_2 - L_b - w_5(z_2)} \partial_{z_1} =: D_1^{w_5}, \\ \partial_{y_2} = \partial_{z_2} + \dfrac{(y_1 - L_2)\partial_{z_2} w_5}{L_2 - L_b - w_5} \partial_{z_1} =: D_2^{w_5}, \end{cases}$$

and the domain $\mathcal{D}_+$ becomes

$$\Omega = \{(z_1, z_2) : L_b < z_1 < L_2, 0 \le z_2 < \mathcal{M}\}.$$

Set

$$w_j(z_1, z_2) = v_j\left(z_1 + \frac{L_2 - z_1}{L_2 - L_b} w_5, z_2\right), \ j = 1, \cdots, 4, \ \mathbf{w} = (w_1, \cdots, w_4).$$

Then the functions $\rho(y_1, y_2)$ and $P(y_1, y_2)$ in (2.15) can be rewritten as

$$\begin{aligned} \tilde{\rho}(z_1, z_2) &= \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma-1}} \left(B_b(D_0^{w_5}) + w_4 + \Phi_b(D_0^{w_5}) + \sigma\Phi_e - \frac{1}{2}(u_b(D_0^{w_5}) + w_1)^2 - \sum_{j=2}^{3} |w_j|^2\right)^{\frac{1}{\gamma-1}}, \\ \tilde{P}(z_1, z_2) &= \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} \left(B_b(D_0^{w_5}) + w_4 + \Phi_b(D_0^{w_5}) + \sigma\Phi_e - \frac{1}{2}(u_b(D_0^{w_5}) + w_1)^2 - \sum_{j=2}^{3} |w_j|^2\right)^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{2.28}$$

Furthermore, after the coordinate transformation, (2.16) is changed to be

$$w_5'(z_2) = a_1 w_2(L_b, z_2) + h_1(\mathbf{\Psi}^-(L_b + w_5, z_2) - \mathbf{\Psi}_b^-(L_b + w_5), \mathbf{w}(L_b, z_2), w_5). \tag{2.29}$$

In the z-coordinates, the transonic shock problem can be reformulated as follows. By the second equation in (2.22), the shock front will be determined as follows

$$w_5(z_2) = \frac{1}{b_2} w_1(L_b, z_2) - \frac{1}{b_2} R_1(\mathbf{\Psi}^-(L_b + w_5, z_2) - \mathbf{\Psi}_b^-(L_b + w_5), \mathbf{w}(L_b, z_2), w_5). \tag{2.30}$$

The function w_3 will be determined by the following equation

$$\begin{cases} \partial_{z_1}(\tilde{r} w_3) = 0, \\ w_3(L_b, z_2) = u_\theta^-(L_b + w_5(z_2), z_2), \end{cases} \tag{2.31}$$

where

$$\tilde{r}(z_1, z_2) = 2\left(\int_0^{z_2} \frac{s}{u_b(D_0^{w_5})\tilde{\rho}(z_1, s) + (\tilde{\rho} w_1)(z_1, s)} \mathrm{d}s\right)^{\frac{1}{2}}.$$

The Bernoulli's quantity w_4 will be determined by (2.23). That is

$$\begin{cases}\partial_{z_1} w_4 = 0,\\ w_4(L_b, z_2) = b_3 w_5(z_2) + R_3(\mathbf{\Psi}^-(L_b + w_5, z_2) - \mathbf{\Psi}_b^-(L_b + w_5), \mathbf{w}(L_b, z_2), w_5).\end{cases} \tag{2.32}$$

Next, the first and second equations in (2.26) can be rewritten as

$$\begin{cases} d_1(z_1)\partial_{z_1} w_1 + d_2(z_1)\Big(\partial_{z_2} w_2 + \dfrac{w_2}{z_2}\Big) + d_3(z_1) w_1 + d_4(z_1) w_4 = F_3(\mathbf{w}, w_5),\\ \partial_{z_1} w_2 - d_2(z_1)\partial_{z_2} w_1 + d_5(z_1)\partial_{z_2} w_4 = F_4(\mathbf{w}, w_5),\end{cases} \tag{2.33}$$

where

$$\begin{aligned}
F_3(\mathbf{w}, w_5) &= F_1(\mathbf{w}, w_5) - (d_1(D_0^{w_5})D_1^{w_5} w_1 - d_1(z_1)\partial_{z_1} w_1) - (d_3(D_0^{w_5}) - d_3(z_1))w_1 - (d_4(D_0^{w_5}) - d_4(z_1))w_1\\
&\quad - \Big(d_2(D_0^{w_5})\Big(D_2^{w_5} w_2 + \frac{w_2}{D_0^{w_5}}\Big) - d_2(z_2)\Big(\partial_{z_2} w_2 + \frac{w_2}{z_2}\Big)\Big),\\
F_4(\mathbf{w}, w_5) &= F_2(\mathbf{w}, w_5) - (D_1^{w_5} w_2 - \partial_{z_1} w_2) + (d_2(D_0^{w_5})D_2^{w_5} w_2 - d_2(z_1)\partial_{z_2} w_1)\\
&\quad - (d_5(D_0^{w_5})D_2^{w_5} w_4 - d_5(z_1)\partial_{z_2} w_4),\\
F_1(\mathbf{w}, w_5) &= \frac{1}{c^2(\rho_b(D_0^{w_5}))}\Big(- (c^2(\tilde{\rho}) - (u_b + w_1)^2 - c^2(\rho_b) + u_b^2)D_1^{w_5} w_1 + \Big(\frac{\gamma+1}{2} w_1^2 + \frac{\gamma-1}{2}(w_2^2 + w_3^2)\Big)u_b'\\
&\quad + (c^2(\tilde{\rho}) - (u_b + w_1)^2)\frac{\tilde{r}\tilde{\rho} w_2}{2y_2} D_2^{w_5} w_1 - c^2(\tilde{\rho})\frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} D_2^{w_5} w_2 + c^2(\rho_b)\frac{r_b\rho_b u_b}{2y_2} D_2^{w_5} w_2\\
&\quad + c^2(\tilde{\rho}) w_2^2 \frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} D_2^{w_5} w_2 - \frac{c^2(\tilde{\rho})}{\tilde{r}} w_2 + \frac{c^2(\rho_b)}{r_b} w_2 - \frac{c^2(\tilde{\rho}) w_3^2}{\tilde{r}} w_2 - \sigma(u_b + w_1)D_1^{w_5}\Phi_e\\
&\quad + (u_b + w_1) w_2\Big(D_1^{w_5} w_2 - \frac{\tilde{r}\tilde{\rho} w_2}{2y_2} D_2^{w_5} w_2 + \frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} D_2^{w_5} w_1\Big)\Big),\\
F_2(\mathbf{w}, w_5) &= \frac{\tilde{r}\tilde{\rho} w_2}{2y_2} D_2^{w_5} w_2 + \frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} D_2^{w_5} w_1 - \frac{r_b\rho_b u_b}{2y_2} D_2^{w_5} w_1 + \frac{1}{u_b + w_1}\Big(\frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} w_3 D_2^{w_5} w_3 + \frac{w_3^2}{\tilde{r}}\\
&\quad - \frac{\tilde{r}\tilde{\rho}(u_b + w_1)}{2y_2} D_2^{w_5} w_4 + \frac{r_b\rho_b u_b}{2y_2} D_2^{w_5} w_4\Big).
\end{aligned}$$

Finally, the boundary condition (2.24) at the exit becomes

$$w_1(L_2, z_2) = \frac{w_4(L_2, z_2)}{u_b(L_2)} - \frac{\sigma P_{ex}(\tilde{r}(L_2, z_2))}{(\rho_b u_b)(L_2)} - \frac{1}{2u_b(L_2)}\sum_{j=1}^{3} w_j^2(L_2, z_2) - \frac{E(\mathbf{w}(L_2, z_2))}{u_b(L_2)}, \tag{2.34}$$

where

$$E(\mathbf{w}(z_1, z_2)) = \frac{\gamma}{\gamma - 1}(\tilde{P}(\mathbf{w}))^{\frac{\gamma-1}{\gamma}} - \frac{\gamma}{\gamma - 1} P_b^{\frac{\gamma-1}{\gamma}} - \frac{1}{\rho_b(z_1)}(\tilde{P}(\mathbf{w}) - P_b) - \sigma\Phi_e.$$

The boundary condition on the nozzle wall is

$$w_2(z_1, \mathcal{M}) = \sigma f'(D_0^{w_5})(u_b(D_0^{w_5}) + w_1(z_1, \mathcal{M})), \text{ on } \Gamma_{p,z} = \{(z_1, z_2) : L_b \le z_1 \le L_2,\ z_2 = \mathcal{M}\}. \tag{2.35}$$

Therefore, after the coordinates transformation (2.27), the problem (2.26) with boundary conditions (2.16), (2.22)-(2.25) is equivalent to solve the following problem.

Problem 2.1. Find a function w_5 defined on $[0, \mathcal{M})$ and vector functions $(w_1, \cdots, w_4)$ defined on the Ω, which solve the transport equations (2.31)-(2.32) and (2.33) with boundary conditions (2.29), (2.30), (2.34) and (2.35).

Theorem 1.3 then follows directly from the following result.

Theorem 2.2. *Assume that the compatibility conditions (1.5) and (1.11) hold. There exist suitable positive constants σ_0 and C_* depending only on the background solution $\mathbf{\Psi}_b$ defined in (1.3) and the boundary data $\mathbf{\Psi}^-(L_1, \cdot)$, f, P_{ex} ,Φ_e such that if $0 < \sigma \leq \sigma_0$, the problem (2.31)-(2.33) with boundary conditions (2.29), (2.30), (2.34) and (2.35) has a unique solution $(w_1, w_2, w_3, w_4)(z_1, z_2)$ with the shock front $\mathcal{S} : z_1 = w_5(z_2)$ satisfying the following properties.*

(1) *The function $w_5(z_2) \in C_{3,\alpha}^{(-1-\alpha;\{\mathcal{M}\})}([0, \mathcal{M}))$ satisfies*

$$\|w_5\|_{3,\alpha;[0,\mathcal{M})}^{(-1-\alpha;\{\mathcal{M}\})} \leq C_*\sigma \tag{2.36}$$

and

$$w_5'(0) = w_5^{(3)}(0) = 0. \tag{2.37}$$

(2) *The solution $(w_1, w_2, w_3, w_4)(z_1, z_2) \in C_{2,\alpha}^{(-\alpha;\Gamma_{p,z})}(\Omega)$ satisfies the estimate*

$$\sum_{i=1}^{4} \|w_i\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} \leq C_*\sigma \tag{2.38}$$

with the compatibility conditions

$$(w_2, \partial_{z_2}^2 w_2)(z_1, 0) = (w_3, \partial_{z_2} w_3)(z_1, 0) = \partial_{z_2}(w_1, w_4)(z_1, 0) = 0, \quad \forall z_1 \in [L_b, L_2]. \tag{2.39}$$

3 Proof of Theorem 2.2

In this section, we first construct a suitable iteration scheme to linearize the problem (2.31)-(2.33) with boundary conditions (2.29), (2.30), (2.34) and (2.35) . Especially, a linear first order elliptic system with a nonlocal term can be derived. Then one can introduce a potential function to reduce the first order elliptic system into a second order elliptic equation with a nonlocal term involving only the trace of the potential function on the shock front and a free parameter. We solve this second order nonlocal elliptic equation with a free parameter and establish some prior estimates and then complete the proof of Theorem 2.2.

3.1 An iteration scheme

In this subsection, we develop an iteration scheme to linearize the problem (2.31)-(2.33) with boundary conditions (2.29), (2.30), (2.34) and (2.35). The solution class $\mathcal{J}$ consists of the vector functions $(w_1, \cdots, w_4, w_5) \in \left(C_{2,\alpha}^{(-\alpha;\Gamma_{p,z})}(\Omega))^4 \times C_{3,\alpha}^{(-1-\alpha;\{\mathcal{M}\})}([0, \mathcal{M})\right)$ satisfying the estimate

$$\|(\mathbf{w}, w_5)\|_{\mathcal{J}} = \sum_{j=1}^{4} \|w_j\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|w_5\|_{3,\alpha;[0,\mathcal{M})}^{(-1-\alpha;\{\mathcal{M}\})} \leq \delta \tag{3.1}$$

and the following compatibility conditions

$$w_5'(0) = w_5^{(3)}(0) = (w_2, \partial_{z_2}^2 w_2)(z_1, 0) = (w_3, \partial_{z_2} w_3)(z_1, 0) = \partial_{z_2}(w_1, w_4)(z_1, 0) = 0, \tag{3.2}$$

where $\delta > 0$ to be determined later. Given $(\hat{\mathbf{w}}, \hat{w}_5) \in \mathcal{J}$, we will construct an iterative procedure that generates a new $(\mathbf{w}, w_5) \in \mathcal{J}$, and thus one can define a mapping from $\mathcal{J}$ to itself by choosing a suitably small positive constant δ.

Step 1. The iteration scheme for w_5.

The shock front w_5 is uniquely determined by

$$w_5(z_2) = \frac{1}{b_2} w_1(L_b, z_2) - \frac{1}{b_2} R_1(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5), \tag{3.3}$$

provided that $w_1(L_b, z_2)$ is obtained.

Step 2. The iteration scheme for w_3 and w_4.

We solve the transport equations for the swirl velocity and the Bernoulli's quantity . The swirl velocity w_3 will be determined by

$$\begin{cases} \partial_{z_1}(\hat{r} w_3) = 0, \\ w_3(L_b, z_2) = u_\theta^-(L_b + \hat{w}_5(z_2), z_2), \end{cases} \tag{3.4}$$

where

$$\hat{r}(z_1, z_2) = 2\left(\int_0^{z_2} \frac{s}{u_b(D_0^{\hat{w}_5})\hat{\rho}(z_1, s) + (\hat{\rho}\hat{w}_1)(z_1, s)} \mathrm{d}s\right)^{\frac{1}{2}}.$$

Then w_3 can be solved as follows

$$w_3(z_1, z_2) = \frac{\hat{r}(L_b, z_2) u_\theta^-(L_b + \hat{w}_5(z_2), z_2)}{\hat{r}(z_1, z_2)}. \tag{3.5}$$

The Bernoulli's quantity w_4 satisfies

$$\begin{cases} \partial_{z_1} w_4 = 0, \\ w_4(L_b, z_2) = b_3 w_5(z_2) + R_3(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5). \end{cases} \tag{3.6}$$

This, together with (3.3), yields that

$$\begin{aligned} w_4(z_1, z_2) &= b_3 w_5(z_2) + R_3(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5) \\ &= \frac{b_3}{b_2} w_1(L_b, y_2) + R_4(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &R_4(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5) \\ &= R_3(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5) \\ &\quad - \frac{b_3}{b_2} R_1(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5). \end{aligned}$$

Step 3. The iteration scheme for w_1 and w_2.

We derive the equations for w_1 and w_2. Firstly, it follows from (2.29) that

$$w_5'(z_2) = a_1 w_2(L_b, z_2) + h_1(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5). \tag{3.8}$$

Substituting (3.7) into (2.33) and combining (3.3), (3.8), (2.34) and (2.35), one gets

$$
\begin{cases}
d_1(z_1)\partial_{z_1} w_1 + d_2(z_1)\Big(\partial_{z_2} w_2 + \dfrac{w_2}{z_2}\Big) + d_3(z_1) w_1 + d_4(z_1)\dfrac{b_3}{b_2} w_1(L_b, z_2) = G_1(\hat{\mathbf{w}}, \hat{w}_5),\\
\partial_{z_1} w_2 - d_2(z_1)\partial_{z_2} w_1 + d_5(z_1)\dfrac{b_3}{b_2}\partial_{z_2} w_1(L_b, z_2) = G_2(\hat{\mathbf{w}}, \hat{w}_5),\\
\partial_{z_2} w_1(L_b, z_2) = a_1 b_2 w_2(L_b, z_2) + h_2(z_2),\\
w_1(L_2, z_2) = \dfrac{b_3}{b_2 u_b(L_2)} w_1(L_b, z_2) + h_3(z_2),\\
w_2(z_1, 0) = 0,\\
w_2(z_1, \mathcal{M}) = h_4(z_1),
\end{cases}
\tag{3.9}
$$

where

$$
\begin{aligned}
G_1(\hat{\mathbf{w}}, \hat{w}_5) &= F_3(\hat{\mathbf{w}}, \hat{w}_5) - d_4(z_1) R_4(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5),\\
G_2(\hat{\mathbf{w}}, \hat{w}_5) &= F_4(\hat{\mathbf{w}}, \hat{w}_5) - d_5(z_1)\partial_{z_2} R_4(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5),\\
h_2(z_2) &= h_1(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5)\\
&\quad + \partial_{z_2} R_1(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5),\\
h_3(z_2) &= \frac{R_4(\mathbf{\Psi}^-(L_b + \hat{w}_5, z_2) - \mathbf{\Psi}_b^-(L_b + \hat{w}_5), \hat{\mathbf{w}}(L_b, z_2), \hat{w}_5)}{u_b(L_2)} - \frac{\sigma P_{ex}(\hat{r}(L_2, z_2))}{(\rho_b u_b)(L_2)}\\
&\quad - \frac{1}{2u_b(L_2)}\sum_{j=1}^{3} \hat{w}_j^2(L_2, z_2) - \frac{E(\hat{\mathbf{w}}(L_2, z_2))}{u_b(L_2)},\\
h_4(z_1) &= \sigma f'(D_0^{\hat{w}_5})(u_b(D_0^{\hat{w}_5}) + \hat{w}_1(z_1, \mathcal{M})).
\end{aligned}
$$

Then it follows from the expressions of F_i and R_i together with direct computations that

$$
\sum_{j=1}^{2} \|G_i(\hat{\mathbf{w}}, \hat{w}_5)\|_{1,\alpha;\Omega}^{(1-\alpha;\Gamma_{p,z})} + \|h_2\|_{1,\alpha;[0,\mathcal{M})}^{(1-\alpha;\{\mathcal{M}\})} + \|h_3\|_{1,\alpha;[0,\mathcal{M})}^{(-\alpha;\{\mathcal{M}\})} + \|h_4\|_{0,\alpha;[L_b,L_2]} \le C\left(\sigma + \|(\hat{\mathbf{w}}, \hat{w}_5)\|_{\mathcal{J}}^2\right). \tag{3.10}
$$

Next, the second eqation in (3.9) can be rewritten as

$$
\partial_{z_1} w_2 - \partial_{z_2}\Big(d_2(z_1) w_1 - d_5(z_1)\frac{b_3}{b_2} w_1(L_b, z_2) - \int_{z_2}^{\mathcal{M}} G_2(\hat{\mathbf{w}}, \hat{w}_5)(z_1, s)\mathrm{d}s\Big) = 0,
$$

which implies that there exists a potential function ϕ such that

$$
\partial_{z_2}\phi = w_2,\ \ \partial_{z_1}\phi = d_2(z_1) w_1 - d_5(z_1)\frac{b_3}{b_2} w_1(L_b, z_2) - \int_{z_2}^{\mathcal{M}} G_2(\hat{\mathbf{w}}, \hat{w}_5)(z_1, s)\mathrm{d}s,\ \ \phi(L_b, \mathcal{M}) = 0. \tag{3.11}
$$

Therefore, one obtians

$$
\begin{aligned}
&w_1(L_b, z_2) = b_4\Big(\partial_{z_1}\phi(L_b, z_2) + \int_{z_2}^{\mathcal{M}} G_2(\hat{\mathbf{w}}, \hat{w}_5)(L_b, s)\mathrm{d}s\Big),\ \ b_4 = d_2(L_b) - d_5(L_b)\frac{b_3}{b_2} = \frac{c^2(\rho_b(L_b))}{\kappa_b(L_b)} > 0,\\
&w_1(z_1, z_2) = \frac{1}{d_2(z_1)}\Big(\partial_{z_1}\phi + d_5(z_1)\frac{b_3 b_4}{b_2}\partial_{z_1}\phi(L_b, z_2) + \int_{z_2}^{\mathcal{M}} (G_2(\hat{\mathbf{w}}, \hat{w}_5)(z_1, s) + b_4 d_5(z_1) G_2(\hat{\mathbf{w}}, \hat{w}_5)(L_b, s))\mathrm{d}s\Big)
\end{aligned}
$$

Then the problem (3.9) is reduced to

$$\begin{cases}\partial_{z_1}(\lambda_1(z_1)\partial_{z_1}\phi)+\lambda_2(z_1)\Big(\partial_{z_2}^2\phi+\dfrac{\partial_{z_2}\phi}{z_2}\Big)-\lambda_3(z_1)\partial_{z_1}\phi(L_b,z_2)=\partial_{z_1}\mathcal{G}_1+\partial_{z_2}\mathcal{G}_2,\\ \partial_{z_2}(\partial_{z_1}\phi(L_b,z_2)-b_5\phi(L_b,z_2))=q_2(z_2),\\ \partial_{z_1}\phi(L_2,z_2)=q_3(z_2),\\ \partial_{z_2}\phi(z_1,0)=0,\\ \partial_{z_2}\phi(z_1,\mathcal{M})=h_4(z_1),\end{cases}\tag{3.12}$$

where

$$\begin{aligned}&\lambda_1(z_1)=\frac{\lambda_0(z_1)}{d_2(z_1)}>0,\quad \lambda_2(z_1)=\lambda_0(z_1)\frac{d_2}{d_1}(z_1)>0,\quad \lambda_0(z_1)=\exp\Big(\int_{L_b}^{z_1}\frac{d_3}{d_1}(s)\mathrm{d}s\Big),\\ &\lambda_3(z_1)=-\frac{b_3b_4}{b_2}\lambda_0(z_1)\Big(\frac{d_2d_3}{d_1d_5}(z_1)+\Big(\frac{d_5}{d_2}\Big)'(z_1)+\frac{d_4}{d_1}(z_1)\Big)\\ &\qquad\quad=-\frac{b_3b_4}{b_2}\lambda_0(z_1)\frac{(u_bg)(z_1)}{c^2(\rho_b(z_1))-u_b^2(z_1)}\frac{2+(\gamma-1)M_b^4(z_1)}{1-M_b^2(z_1)}>0,\\ &\mathcal{G}_1(z_1,z_2)=-\lambda_1(z_1)\int_{z_2}^{\mathcal{M}}(G_2(\hat{\mathbf{w}},\hat{w}_5)(z_1,s)+b_4d_5(z_1)G_2(\hat{\mathbf{w}},\hat{w}_5)(L_b,s))\mathrm{d}s,\\ &\mathcal{G}_2(z_1,z_2)=\frac{\lambda_0(z_1)}{d_1(z_1)}\int_0^{z_2}G_1(\hat{\mathbf{w}},\hat{w}_5)(z_1,s)\mathrm{d}s,\quad b_5=\frac{a_1b_2}{b_4}>0,\\ &q_2(z_2)=G_2(\hat{\mathbf{w}},\hat{w}_5)(L_b,z_2)+\frac{h_2(z_2)}{b_4},\\ &q_3(z_2)=-\int_{z_2}^{\mathcal{M}}G_2(\hat{\mathbf{w}},\hat{w}_5)(L_2,s)\mathrm{d}s+h_3(z_2).\end{aligned}$$

The second equation in (3.12) implies that

$$\partial_{z_1}\phi(L_b,z_2)-b_5(\phi(L_b,z_2)+\Lambda)=\tilde{q}_2(z_2),\tag{3.13}$$

where

$$\Lambda=-\frac{w_5(\mathcal{M})}{a_1},\ \tilde{q}_2(z_2)=-\int_{z_2}^{\mathcal{M}}q_2(s)\mathrm{d}s+\frac{1}{b_4}R_1(\mathbf{\Psi}^-(L_b+\hat{w}_5(\mathcal{M}),\mathcal{M})-\mathbf{\Psi}_b^-(L_b+\hat{w}_5(\mathcal{M})),\hat{\mathbf{w}}(L_b,\mathcal{M}),\hat{w}_5(\mathcal{M})).$$

Substituting (3.13) into the first equation in (3.12) yields

$$\begin{cases}\partial_{z_1}(\lambda_1(z_1)\partial_{z_1}\phi)+\lambda_2(z_1)\Big(\partial_{z_2}^2\phi+\dfrac{\partial_{z_2}\phi}{z_2}\Big)-\lambda_3(z_1)b_5(\phi(L_b,z_2)+\Lambda)=\partial_{z_1}\mathcal{G}_1+\partial_{z_2}\mathcal{G}_2+\mathcal{G}_3,\\ \partial_{z_1}\phi(L_b,z_2)-b_5(\phi(L_b,z_2)+\Lambda)=\tilde{q}_2(z_2),\\ \partial_{z_1}\phi(L_2,z_2)=\tilde{q}_3(z_2),\\ \partial_{z_2}\phi(z_1,0)=0,\\ \partial_{z_2}\phi(z_1,\mathcal{M})=h_4(z_1),\end{cases}\tag{3.14}$$

where

$$\mathcal{G}_3(z_1,z_2)=\lambda_3(z_1)\tilde{q}_2(z_2),\quad \tilde{q}_3(z_2)=q_3(z_2).$$

Furthermore, it follows from (3.10) that the following estimate holds:

$$\sum_{j=1}^{3}\|\mathcal{G}_j\|_{1,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})}+\sum_{j=2}^{3}\|\tilde{q}_j\|_{1,\alpha;[0,\mathcal{M})}^{(-\alpha;\{\mathcal{M}\})}+\|h_4\|_{0,\alpha;[L_b,L_2]}\le C\left(\sigma+\|(\hat{\mathbf{w}},\hat{w}_5)\|_{\mathcal{J}}^2\right).\tag{3.15}$$

3.2 Solving a second order elliptic equation with a nonlocal term

Let $\phi_*(z_1, z_2) = \phi(z_1, z_2) + \Lambda$. Then (3.14) is equivalent to the following problem

$$
\begin{cases}
\partial_{z_1}(\lambda_1(z_1)\partial_{z_1}\phi_*) + \lambda_2(z_1)\Big(\partial_{z_2}^2\phi_* + \dfrac{\partial_{z_2}\phi_*}{z_2}\Big) - \lambda_3(z_1)b_5\phi_*(L_b, z_2) = \partial_{z_1}\mathcal{G}_1 + \partial_{z_2}\mathcal{G}_2 + \mathcal{G}_3, \\
\partial_{z_1}\phi_*(L_b, z_2) - b_5\phi_*(L_b, z_2) = \tilde{q}_2(z_2), \\
\partial_{z_1}\phi_*(L_2, z_2) = \tilde{q}_3(z_2), \\
\partial_{z_2}\phi_*(z_1, 0) = 0, \\
\partial_{z_2}\phi_*(z_1, \mathcal{M}) = h_4(z_1).
\end{cases} \tag{3.16}
$$

In order to deal with the singularity near $z_2 = 0$, we rewrite the problem (3.16) by using the cylindrical coordinates transformation again. Define

$$
\eta_1 = z_1,\ \eta_2 = z_2\cos\tau,\ \eta_3 = z_2\sin\tau,\ \tau \in [0, 2\pi],
$$

and

$$
\begin{aligned}
&\Omega_1 = \{(\eta_1, \eta_2, \eta_3) : L_b < \eta_1 < L_2, \eta_2^2 + \eta_3^2 \le \mathcal{M}^2\}, \quad \Omega_2 = \{(\eta_2, \eta_3) : \eta_2^2 + \eta_3^2 \le \mathcal{M}^2\}, \\
&\Gamma'_\eta = \{\eta' = (\eta_2, \eta_3) : \eta_2^2 + \eta_3^2 = \mathcal{M}^2\}, \quad \Gamma_{p,\eta} = [L_b, L_2] \times \Gamma'_\eta, \\
&\Psi(\boldsymbol{\eta}) = \phi_*(\eta_1, \sqrt{\eta_2^2 + \eta_3^2}) = \phi_*(\eta_1, |\eta'|).
\end{aligned}
$$

Then $\Psi(\boldsymbol{\eta})$ satisfies

$$
\begin{cases}
\partial_{\eta_1}(\lambda_1(\eta_1)\partial_{\eta_1}\Psi) + \lambda_2(\eta_1)\displaystyle\sum_{j=2}^{3}\partial_{\eta_j}^2\Psi - \lambda_3(\eta_1)b_5\Psi(L_b, \eta') \\
\quad = \partial_{\eta_1}\mathcal{G}_1(\eta_1, |\eta'|) + \displaystyle\sum_{j=2}^{3}\partial_{\eta_j}\left(\frac{\eta_j\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|}\right) - \frac{\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|} + \mathcal{G}_3(\eta_1, |\eta'|), \\
\partial_{\eta_1}\Psi(L_b, \eta') - b_5\Psi(L_b, \eta') = \tilde{q}_2(|\eta'|), \\
\partial_{\eta_1}\Psi(L_2, \eta') = \tilde{q}_3(|\eta'|), \\
(\eta_2\partial_{\eta_2} + \eta_3\partial_{\eta_3})\Psi(\eta_1, \eta') = \mathcal{M}h_4(\eta_1).
\end{cases} \tag{3.17}
$$

Firstly, the weak solution to (3.17) can be obtained as follows. $\Psi \in H^1(\Omega_1)$ is said to be a weak solution to (3.17), if for any $\varphi \in H^1(\Omega_1)$, the following equality holds:

$$
\mathcal{L}(\Psi, \varphi) = \mathcal{F}(\varphi), \quad \forall\varphi \in H^1(\Omega_1), \tag{3.18}
$$

where

$$
\begin{aligned}
\mathcal{L}(\Psi, \varphi) = &\iiint_{\Omega_1} \lambda_1(\eta_1)\partial_\eta\Psi\partial_{\eta_1}\varphi + \lambda_2(\eta_1)(\partial_{\eta_2}\Psi\partial_{\eta_2}\varphi + \partial_{\eta_3}\Psi\partial_{\eta_3}\varphi) + \lambda_3(\eta_1)b_5\Psi(L_b, \eta')\varphi(\eta_1, \eta')\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3 \\
&+ \iint_{\Omega_2} \lambda_1(L_b)b_5\Psi(L_b, \eta')\varphi(L_b, \eta')\mathrm{d}\eta_2\mathrm{d}\eta_3, \\
\mathcal{F}(\varphi) = &\iiint_\Omega \mathcal{G}_1\partial_{\eta_1}\varphi + \sum_{j=2}^{3}\frac{\eta_j\mathcal{G}_2}{|\eta'|}\partial_{\eta_i}\varphi - \left(\mathcal{G}_3 - \frac{\mathcal{G}_2}{|\eta'|}\right)\varphi\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3 - \int_{L_b}^{L_2}\mathcal{G}_2(\eta_1, \mathcal{M})\varphi(\eta_1, \mathcal{M})\mathrm{d}\eta_1 \\
&- \iint_{\Omega_2}\mathcal{G}_1(L_2, |\eta'|))\varphi(L_2, \eta') - \mathcal{G}_1(L_b, |\eta'|))\varphi(L_b, \eta')\mathrm{d}\eta_2\mathrm{d}\eta_3 + \int_{L_b}^{L_2}\mathcal{M}h_4(\eta_1)\varphi(\eta_1, \mathcal{M})\mathrm{d}\eta_1 \\
&+ \iint_{\Omega_2}\lambda_1(L_2)\tilde{q}_3(|\eta'|))\varphi(L_2, \eta') - \lambda_1(L_b)\tilde{q}_2(|\eta'|))\varphi(L_b, \eta')\mathrm{d}\eta_2\mathrm{d}\eta_3.
\end{aligned}
$$

Lemma 3.1. *There exists a positive constant K depending only on the background solution such that the following problem has a unique weak solution in $H^1(\Omega_1)$*

$$\begin{cases}\partial_{\eta_1}(\lambda_1(\eta_1)\partial_{\eta_1}\Psi)+\lambda_2(\eta_1)\sum_{j=2}^{3}\partial_{\eta_j}^2\Psi-\lambda_3(\eta_1)b_5\Psi(L_b,\eta')+K\Psi\\ =\partial_{\eta_1}\mathcal{G}_1(\eta_1,|\eta'|)+\sum_{j=2}^{3}\partial_{\eta_j}\left(\frac{\eta_j\mathcal{G}_2(\eta_1,|\eta'|)}{|\eta'|}\right)-\frac{\mathcal{G}_2(\eta_1,|\eta'|)}{|\eta'|}+\mathcal{G}_3(\eta_1,|\eta'|),\\ \partial_{\eta_1}\Psi(L_b,\eta')-b_5\Psi(L_b,\eta')=\tilde{q}_2(|\eta'|),\\ \partial_{\eta_1}\Psi(L_2,\eta')=\tilde{q}_3(|\eta'|),\\ (\eta_2\partial_{\eta_2}+\eta_3\partial_{\eta_3})\Psi(\eta_1,|\eta'|)=\mathcal{M}h_4(\eta_1).\end{cases}\tag{3.19}$$

Proof. The system (3.19) has the following bilinear form on $H^1(\Omega_1)\times H^1(\Omega_1)$:

$$\mathcal{L}_K(\Psi,\varphi)=\mathcal{L}(\Psi,\varphi)+\iiint_{\Omega_1}\Psi\varphi d\eta_1 d\eta_2 d\eta_3=\mathcal{F}(\varphi),\quad\forall\varphi\in H^1(\Omega_1).\tag{3.20}$$

For any $\epsilon>0$, one can use Cauchy's inequality to get

$$\begin{aligned}&\iiint_{\Omega_1}\Psi(L_b,\eta')\Psi(\eta_1,\eta')d\eta_1 d\eta_2 d\eta_3\\ &\leq\frac{C_1}{\epsilon}\iiint_{\Omega_1}\Psi^2(\eta_1,\eta'))d\eta_1 d\eta_2 d\eta_3+\epsilon\iiint_{\Omega_1}(\partial_{\eta_1}\Psi)^2(\eta_1,\eta'))d\eta_1 d\eta_2 d\eta_3.\end{aligned}$$

Note that $\mathcal{G}_2(\eta_1,0)=0$. Thus the boundedness and coercivity of $\mathcal{L}_K$ can be verified as follows

$$\begin{aligned}&|\mathcal{L}_K(\Psi,\varphi)|\leq C\|\Psi\|_{H^1(\Omega_1)}\|\varphi\|_{H^1(\Omega)},\\ &|\mathcal{F}(\varphi)|\leq C\left(\sum_{j=1}^{3}\|\mathcal{G}_j\|_{C^{0,\alpha}(\overline{\Omega_1})}+\sum_{j=2}^{3}\|\tilde{q}_j\|_{C^{0,\alpha}(\overline{\Omega_2})}+\|h_4\|_{C^{0,\alpha}[L_b,L_2]}\right)\|\varphi\|_{H^1(\Omega_1)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_K(\Psi,\Psi)=&\iiint_{\Omega_1}\lambda_1(\eta_1)(\partial_{\eta_1}\Psi)^2+\lambda_2(\eta_1)\sum_{i=2}^{3}(\partial_{\eta_i}\Psi)^2+\lambda_3(\eta_1)b_5\Psi(L_b,\eta')\Psi(\eta,\eta')+K\Psi^2 d\eta_1 d\eta_2 d\eta_3\\ &+\iint_{\Omega_2}\lambda_1(L_b)b_5(\Psi(L_b,\eta'))^2 d\eta_2 d\eta_3,\\ &\geq C_*\Big(\|\nabla\Psi\|^2_{L^2(\Omega_1)}+\|\Psi(L_b,\cdot)\|^2_{L^2(\Omega_2)}\Big)+K\|\Psi\|^2_{L^2(\Omega_1)}-\frac{C_*}{4}\|\partial_{\eta_1}\Psi\|^2_{L^2(\Omega_1)}\\ &-\frac{C_*}{4}\|\Psi(L_b,\cdot)\|^2_{L^2(\Omega_2)}-\tilde{C}_*\|\Psi\|^2_{L^2(\Omega_1)}\\ &\geq\frac{C_*}{2}\Big(\|\nabla\Psi\|^2_{L^2(\Omega_1)}+\|\Psi(L_b,\cdot)\|^2_{L^2(\Omega_2)}\Big)+\frac{K}{2}\|\Psi\|^2_{L^2(\Omega_1)}\end{aligned}$$

provided that K is sufficiently large. Then the existence and uniqueness of $H^1(\Omega_1)$ solution Ψ to (3.17) can be obtained by using the Lax-Milgram theorem, which completes the proof of Lemma 3.1. □

Now we are going to solve the problem (3.17).

Proposition 3.2. *Suppose that* $(\mathcal{G}_1, \mathcal{G}_2, , \mathcal{G}_3) \in (C^{(-\alpha;\Gamma_{p,\eta})}_{1,\alpha}(\Omega_1))^3, \mathcal{G}_2(\eta_1, 0) = 0$, $(\tilde{q}_2, \tilde{q}_3) \in (C^{(-\alpha;\Gamma'_\eta)}_{1,\alpha}(\Omega_2))^2$ *and* $h_4 \in (C^{0,\alpha}([L_b, L_2])$. *Then the problem* (3.17) *has a unique solution* $\Psi \in C^{(-1-\alpha;\Gamma_{p,\eta})}_{2,\alpha}(\Omega_1)$ *satisfying the estimate*

$$\|\Psi\|^{(-1-\alpha;\Gamma_{p,\eta})}_{2,\alpha;\Omega_1} \le C_\sharp \left(\sum_{j=1}^{3} \|\mathcal{G}_j\|^{(-\alpha;\Gamma_{p,\eta})}_{1,\alpha;\Omega_1} + \sum_{j=2}^{3} \|\tilde{q}_j\|^{(-\alpha;\Gamma'_\eta)}_{1,\alpha;\Omega_2} + \|h_4\|_{0,\alpha;[L_b,L_2]} \right), \tag{3.21}$$

where the constant $C_\sharp$ *depends only on the coefficients* λ_i, $i = 1, 2, 3$, b_5 *and thus depends only on the background solution.*

Proof. Firstly, we show that any $H^1(\Omega_1)$ weak solution to (3.17) has a higher regularity $C^{(-1-\alpha;\Gamma_{p,\eta})}_{2,\alpha}(\Omega_1)$. To this end, the first equation in (3.17) can be rewritten as a standard second order elliptic equation for Ψ:

$$\begin{aligned}
&\partial_{\eta_1}(\lambda_1(\eta_1)\partial_{\eta_1}\Psi) + \lambda_2(\eta_1)\sum_{j=2}^{3}\partial^2_{\eta_j}\Psi \\
&= \lambda_3(\eta_1)b_5\Psi(L_b, \eta') + \partial_{\eta_1}\mathcal{G}_1(\eta_1, |\eta'|) + \sum_{j=2}^{3}\partial_{\eta_j}\left(\frac{\eta_j\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|}\right) - \frac{\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|} + \mathcal{G}_3(\eta_1, |\eta'|).
\end{aligned}$$

Since $\Psi \in H^1(\Omega_1)$, thus the trace theorem implies that $\Psi(L_b, \eta') \in L^2(\Omega_2)$. Then one can apply [14, Theorems 5.36 and 5.45] to get

$$\begin{aligned}
\|\Psi\|_{C^{0,\alpha}(\overline{\Omega_1})} &\le C_\sharp\Bigg(\|b_5\lambda_3(z_1)\Psi(L_b, \eta')\|_{L^2(\Omega_2)} + \|\mathcal{G}_1\|_{L^4(\Omega_1)} + \sum_{j=2}^{3}\left\|\frac{\eta_j\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|}\right\|_{L^4(\Omega_1)} \\
&\qquad + \left\|\frac{\mathcal{G}_2(\eta_1, |\eta'|)}{|\eta'|}\right\|_{L^2(\Omega_1)} + \|\mathcal{G}_3\|_{L^2(\Omega_1)} + \sum_{j=2}^{3}\|\tilde{q}_j\|_{L^3(\Omega_2)} + \|h_4\|_{L^3(L_b,L_2)}\Bigg) \\
&\le C_\sharp\left(\|\Psi\|_{H^1(\Omega_1)} + \sum_{i=1}^{3}\|\mathcal{G}_i\|^{(-\alpha;\Gamma_{p,\eta})}_{1,\alpha;\Omega_1} + \sum_{j=2}^{3}\|\tilde{q}_j\|^{(-\alpha;\Gamma'_\eta)}_{1,\alpha;\Omega_2} + \|h_4\|_{0,\alpha;[L_b,L_2]}\right).
\end{aligned} \tag{3.22}$$

Hence $b_5\lambda_3(z_1)\Psi(L_b, \eta') \in C^\alpha(\overline{\Omega_1})$ and the Schauder estimate in [14, Theorem 4.6] would imply that

$$\|\Psi\|^{(-1-\alpha;\Gamma_{p,\eta})}_{2,\alpha;\Omega_1} \le C_\sharp\left(\|\Psi\|_{H^1(\Omega_1)} + \sum_{i=1}^{3}\|\mathcal{G}_i\|^{(-\alpha;\Gamma_{p,\eta})}_{1,\alpha;\Omega_1} + \sum_{j=2}^{3}\|\tilde{q}_j\|^{(-\alpha;\Gamma'_\eta)}_{1,\alpha;\Omega_2} + \|h_4\|_{0,\alpha;[L_b,L_2]}\right). \tag{3.23}$$

Next, to show the uniqueness of the $H^1(\Omega_1)$ weak solution to (3.17), we first investigate the following eigenvalue problem

$$\begin{cases}
\partial^2_{\eta_2}\beta + \partial^2_{\eta_3}\beta + \mu\beta = 0, & \text{in } \Omega_2,\\
(\eta_2\partial_{\eta_2} + \eta_3\partial_{\eta_3})\beta = 0, & \text{on } \partial\Omega_2.
\end{cases} \tag{3.24}$$

By the standard elliptic theroy in [12], for $\mu < 0$, the problem (3.24) is uniquely solvable. Note that

$$\mu\|\beta\|^2_{L^2(\Omega_2)} = \iint_{\Omega_2}(\partial_{\eta_2}\beta)^2 + (\partial_{\eta_3}\beta)^2 \mathrm{d}\eta_2\mathrm{d}\eta_3.$$

Then there exists a sequence of eigenvalues $0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots \le \mu_i \to \infty$ and the corresponding eigenfunctions $\{\beta_i(\eta_2, \eta_3)\}_{i=1}^{\infty} \in C^{\infty}(\overline{\Omega_2})$ associated with μ_i. The functions $\{\beta_i(\eta_2, \eta_3)\}_{i=1}^{\infty}$ constitute a complete orthonormal basis of $L^2(\Omega_2)$ and are also orthogonal in $H^1(\Omega_2)$.

Since $\Psi \in C^{1,\alpha}(\overline{\Omega_1}) \cap C^{2,\alpha}(\Omega_1)$, then Ψ can be represented as

$$\Psi(\boldsymbol{\eta}) = \sum_{i=0}^{\infty} X_i(\eta_1)\beta_i(\eta_2, \eta_3),$$

where $X_i(z_1) = \int_{L_b}^{L_2} \Psi(\boldsymbol{\eta})\beta_i(\eta_2, \eta_3)\mathrm{d}\eta_2\mathrm{d}\eta_3 \in C^{1,\alpha}([L_b, L_2]) \cap C^{2,\alpha}((L_b, L_2))$ solves the problem

$$\begin{cases} \lambda_1(\eta_1)X_i^{''}(\eta_1) + \lambda_1^{'}(\eta_1)X_i^{'}(\eta_1) - \lambda_2(z_1)\mu_i X_i(\eta_1) - b_5\lambda_3(\eta_1)X_i(L_b) = 0, \\ X_i^{'}(L_b) - b_5 X_i(L_b) = 0, \\ X_i^{'}(L_2) = 0. \end{cases} \tag{3.25}$$

Suppose that $X_i(L_b) = 0$, then the maximum principle and Hopf's lemma show that $X_i(\eta_1) \equiv 0$ for any $\eta_1 \in [L_b, L_2]$. Suppose that $X_i(L_b) > 0$, then

$$\begin{cases} \lambda_1(\eta_1)X_i^{''}(\eta_1) + \lambda_1^{'}(\eta_1)X_i^{'}(\eta_1) - \lambda_2(\eta_1)\mu_i X_i(\eta_1) = b_5\lambda_3(\eta_1)X_i(L_b) > 0, \\ X_i^{'}(L_b) = b_5 X_i(L_b) > 0, \\ X_i^{'}(L_2) = 0. \end{cases} \tag{3.26}$$

Assume that there exists $\tilde{\eta}_1 \in [L_b, L_2]$, such that $X_i(\tilde{\eta}_1) = \max_{\eta_1 \in [L_b, L_2]} X_i(\eta_1) > 0$. Then the second and the third equations in (3.26) imply that $\tilde{\eta}_1 \in (L_b, L_2]$. If $\tilde{\eta}_1 \in (L_b, L_2)$, then $X_i^{'}(\tilde{\eta}_1) = 0$, $X_i^{''}(\tilde{\eta}_1) \le 0$, which contradicts to the first equation in (3.26). If $\tilde{\eta}_1 = L_2$, then Hopf's lemma yields that $X_i'(\eta_2) > 0$, which also contradicts. Similarly, $X_i(L_b) < 0$ will induce a contradiction. Hence, $X_i(\eta_1) \equiv 0$ for all $\eta_1 \in [L_b, L_2]$. Therefore we get $\Psi \equiv 0$ in Ω_1. Then we complete the proof of the uniqueness of the $H^1(\Omega_1)$ weak solution to (3.17).

Next, we can use Lemma 3.1 and the Fredholm alternatives for elliptic equations and the arguments in [12, Theorem 8.6] to deduce that there exists a unique $H^1(\Omega_1)$ weak solution to (3.17). Furthermore, the uniqueness helps us to derive the estimate (3.21) from (3.23). The invariance of the equation and the boundary datum in (3.17) under the rotation transform in (η_2, η_3) plane shows that Ψ is axisymmetric. This completes the proof of the proposition. $\square$

Proposition 3.2 shows that $\phi_*(z_1, z_2)$ is uniquely determined, then $\Lambda = \phi_*(L_b, \mathcal{M})$. Hence this proposition actually implies that the following estimate for w_1 and w_2.

Proposition 3.3. *The problem (3.9) has a unique solution $(w_1, w_2) \in (C_{2,\alpha}^{(-\alpha;\Gamma_{p,z})}(\Omega))^2$ satisfying the estimate*

$$\|w_1\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|w_2\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} \le C(\delta^2 + \sigma) \tag{3.27}$$

and the compatibility conditions

$$\partial_{z_2} w_1(z_1, 0) = (w_2, \partial_{z_2}^2 w_2)(z_1, 0) = 0. \tag{3.28}$$

Proof. It follows from Proposition 3.2 and the equivalence between $\|\cdot\|^{(-\alpha;\Gamma_{p,z})}_{1,\alpha;\Omega}$ and $\|\cdot\|^{(-\alpha;\Gamma_{p,\eta})}_{1,\alpha;\Omega_1}$ that the problem (3.9) has a unique solution $(w_1, w_2) \in (C^{(-\alpha;\Gamma_{p,z})}_{1,\alpha}(\Omega))^2$ satisfying

$$\begin{aligned}
&\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{1,\alpha,\Omega} + \|w_2\|^{(-\alpha;\Gamma_{p,z})}_{1,\alpha,\Omega}\\
&\leq C\Big(\sum_{j=1}^{2}\|G_i(\hat{\mathbf{w}}, \hat{w}_5)\|^{(1-\alpha;\Gamma_{p,z})}_{1,\alpha;\Omega} + \|h_2\|^{(1-\alpha;\{\mathcal{M}\})}_{1,\alpha;[0,\mathcal{M})} + \|h_3\|^{(-\alpha;\{\mathcal{M}\})}_{1,\alpha;[0,\mathcal{M})} + \|h_4\|_{0,\alpha;[L_b,L_2]}\Big)\\
&\leq C\left(\sigma + \|(\hat{\mathbf{w}}, \hat{w}_5)\|^2_{\mathcal{J}}\right).
\end{aligned}\tag{3.29}$$

Furthermore, one can further verify that

$$w_2(z_1, 0) = \partial_{z_2} w_1(z_1, 0) = 0.$$

Next, we estimate $\|(w_1, w_2)\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha,\Omega}$. To this end, we rewrite (3.9) as

$$\begin{cases}
\partial_{z_1}(\lambda_1(z_1)w_1) + \lambda_2(z_1)\Big(\partial_{z_2} w_2 + \dfrac{w_2}{z_2}\Big) = G_3(\hat{\mathbf{w}}, \hat{w}_5),\\
\partial_{z_1} w_2 - d_2(z_1)\partial_{z_2} w_1 = G_4(\hat{\mathbf{w}}, \hat{w}_5),\\
w_1(L_b, z_2) = G_5(z_2),\\
w_1(L_2, z_2) = G_6(z_2),\\
w_2(z_1, 0) = 0,\\
w_2(z_1, \mathcal{M}) = h_4(z_1),
\end{cases}\tag{3.30}$$

where

$$\begin{aligned}
G_3(\hat{\mathbf{w}}, \hat{w}_5) &= \frac{\lambda_0(z_1)}{d_1(z_1)}\Big(G_1(\hat{\mathbf{w}}, \hat{w}_5) - d_4(z_1)\frac{b_3}{b_2} w_1(L_b, z_2)\Big),\\
G_4(\hat{\mathbf{w}}, \hat{w}_5) &= G_2(\hat{\mathbf{w}}, \hat{w}_5) - d_5(z_1)\frac{b_3}{b_2}(a_1 b_2 w_2(L_b, z_2) + h_2(z_2)),\\
G_5(z_2) &= a_1 b_2 \Lambda + b_2 R_1(L_b, \mathcal{M}) - \int_{z_2}^{\mathcal{M}}(a_1 b_2 w_2(L_b, s) + h_2(s))\mathrm{d}s,\\
G_6(z_2) &= \frac{b_3}{b_2 u_b(L_2)} w_1(L_b, z_2) + h_3(z_2).
\end{aligned}$$

Then w_1 satisfies

$$\begin{cases}
\partial_{z_1}\left(\dfrac{1}{\lambda_2(z_1)}\partial_{z_1}(\lambda_1(z_1)w_1)\right) + d_2(z_1)\left(\partial^2_{z_2} w_1 + \dfrac{1}{z_2}\partial_{z_2} w_1\right)\\
= \partial_{z_1}\left(\dfrac{G_3(\hat{\mathbf{w}}, \hat{w}_5)}{\lambda_2(z_1)}\right) + \partial_{z_2} G_4(\hat{\mathbf{w}}, \hat{w}_5) + \dfrac{G_4(\hat{\mathbf{w}}, \hat{w}_5)}{z_2},\\
w_1(L_b, z_2) = G_5(z_2), \quad w_1(L_2, z_2) = G_6(z_2),\\
\partial_{z_2} w_1(z_1, 0) = 0, \qquad w_1(z_1, \mathcal{M}) = w_1(z_1, \mathcal{M}).
\end{cases}\tag{3.31}$$

Note that $G_4(z_1, 0) = 0$. Similar to the proof of Proposition 3.2, one has

$$\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} \leq C\left(\sum_{i=3}^{4}\|G_i\|^{(1-\alpha;\Gamma_{p,z})}_{1,\alpha;\Omega} + \sum_{i=5}^{6}\|G_i\|^{(-\alpha;\{\mathcal{M}\})}_{1,\alpha;[0,\mathcal{M})} + \|w_1\|^{(-\alpha;\Gamma_{p,z})}_{1,\alpha,\Omega}\right) \leq C(\sigma + \delta^2).\tag{3.32}$$

This, together with the second equation in (3.30), gives

$$\begin{aligned}\|\partial_{z_1}^2 w_2\|_{0,\alpha;\Omega}^{(2-\alpha;\Gamma_{p,z})} + \|\partial_{z_1z_2}^2 w_2\|_{0,\alpha;\Omega}^{(2-\alpha;\Gamma_{p,z})} &\leq C(\|w_1\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|w_2\|_{1,\alpha;\Omega_+}^{(-\alpha;\Gamma_{p,z})} + \|G_4\|_{1,\alpha;\Omega}^{(1-\alpha;\Gamma_{p,z})})\\ &\leq C(\sigma+\delta^2).\end{aligned} \tag{3.33}$$

Next, we derive that $\partial_{z_2}^2 w_2(z_1,0)=0$. It follows from the first equation in (3.30) and $w_2(z_1,0)=0$ that

$$w_2(z_1,z_2) = \frac{1}{\lambda_2(z_1)z_2}\int_0^{z_2} s(\partial_{z_1}(\lambda_1(z_1)w_1(z_1,s)) - G_3(z_1,s))\mathrm{d}s. \tag{3.34}$$

Then $w_2(z_1,z_2)$ can be rewritten as

$$w_2(z_1,z_2) = \frac{1}{z_2}\int_0^{z_2} s(R(z_1,s)-R(z_1,0))\mathrm{d}s + \frac{z_2}{2}R(z_1,0), \tag{3.35}$$

where

$$R(z_1,z_2) = \frac{1}{\lambda_2(z_1)}\partial_{z_1}(\lambda_1(z_1)w_1(z_1,z_2)) - G_3(z_1,z_2).$$

Thus one gets $\partial_{z_2}R(z_1,0)=0$. Furthermore, it follows from (3.35) that

$$\partial_{z_2}^2 w_2 = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \frac{2}{z_2^3}\int_0^{z_2} s(R(z_1,s)-R(z_1,0))\mathrm{d}s,\\ I_2 &= -\frac{1}{z_2}(R(z_1,z_2)-R(z_1,0)),\\ I_3 &= \partial_{z_2}R(z_1,z_2).\end{aligned}$$

Obviously, $I_3(z_1,0)=0$. In addition,

$$\begin{aligned} I_2 &= -\frac{1}{z_2}(R(z_1,z_2)-R(z_1,0)) = -\int_0^1 \partial_{z_2}R(z_1,sz_2)\mathrm{d}s,\\ I_1 &= \frac{2}{z_2^3}\int_0^{z_2} s(R(z_1,s)-R(z_1,0))\mathrm{d}s = \frac{2}{z_2^3}\int_0^{z_2}\left(\int_0^1 \partial_{z_2}R(z_1,ts)\mathrm{d}t\right)s^2\mathrm{d}s.\end{aligned}$$

Hence $I_1(z_1,0)=I_2(z_1,0)=0$. That is $\partial_{z_2}^2 w_2(z_1,0)=0$. The proof of Proposition 3.3 is completed. □

In the following, we are ready to estimate w_3, w_4, and w_5.

Proposition 3.4. *w_5, w_3, and w_4 are uniquely determined by (3.3), (3.5) and (3.7), which satisfy the following estimate*

$$\|w_5\|_{3,\alpha;[0,\mathcal{M})}^{(-1-\alpha;\{\mathcal{M}\})} + \|w_3\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|w_4\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} \leq C(\delta^2+\sigma) \tag{3.36}$$

and the compatibility conditions

$$w_5'(0) = w_5^{(3)}(0) = (w_3,\partial_{z_2}w_3)(z_1,0) = \partial_{z_2}w_4(z_1,0) = 0. \tag{3.37}$$

Proof. It follows from (3.3) that

$$\begin{aligned}\|w_5\|^{(-\alpha;\{\mathcal{M}\})}_{2,\alpha;[0,\mathcal{M})} &\le C\left(\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \|R_1(\mathbf{\Psi}^-(L_b+\hat{w}_5,z_2)-\mathbf{\Psi}^-_b(L_b+\hat{w}_5),\hat{\mathbf{w}}(L_b,z_2),\hat{w}_5)\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega}\right)\\ &\le C\left(\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \sigma + \|(\hat{\mathbf{w}},\hat{w}_5)\|^2_{\mathcal{J}}\right).\end{aligned} \tag{3.38}$$

Meanwhile, (3.8) shows that

$$\begin{aligned}\|w_5'\|^{(-\alpha;\{\mathcal{M}\})}_{2,\alpha;[0,\mathcal{M})} &\le C\left(\|w_2\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \|h_1(\mathbf{\Psi}^-(L_b+\hat{w}_5,z_2)-\mathbf{\Psi}^-_b(L_b+\hat{w}_5),\hat{\mathbf{w}}(L_b,z_2),\hat{w}_5)\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega}\right)\\ &\le C\left(\|w_2\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \sigma + \|(\hat{\mathbf{w}},\hat{w}_5)\|^2_{\mathcal{J}}\right).\end{aligned} \tag{3.39}$$

Furthermore, using (1.11), (3.28) and the explicit expression of h_1, one can verify that

$$w_5'(0) = w_5^{(3)}(0) = 0.$$

Next, (3.5) gives that

$$(w_3,\partial_{z_2}w_3)(z_1,0) = 0 \quad \text{and} \quad \|w_3\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} \le C\delta\sigma. \tag{3.40}$$

Fianlly, it follows from (3.7) that the following estimate and compatibility condition hold:

$$\begin{aligned}\|w_4\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} &\le C\left(\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \|R_4(\mathbf{\Psi}^-(L_b+\hat{w}_5,z_2)-\mathbf{\Psi}^-_b(L_b+\hat{w}_5),\hat{\mathbf{w}}(L_b,z_2),\hat{w}_5)\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega}\right)\\ &\le C\left(\|w_1\|^{(-\alpha;\Gamma_{p,z})}_{2,\alpha;\Omega} + \sigma + \|(\hat{\mathbf{w}},\hat{w}_5)\|^2_{\mathcal{J}}\right)\end{aligned} \tag{3.41}$$

and

$$\partial_{z_2}w_4(z_1,0) = 0.$$

Combining (3.39)-(3.41) together finishes the proof of the proposition. □

3.3 Proof of Theorem 2.2

Now, we start to prove Theorem 2.2. The proof is divided into two steps.

Step 1. The boundedness of the operator $\mathcal{T}$.

Given any $(\hat{\mathbf{w}},\hat{w}_5)\in\mathcal{J}$, we define a mapping $\mathcal{T}$ as follows

$$\mathcal{T}(\hat{\mathbf{w}},\hat{w}_5) = (\mathbf{w},w_5), \tag{3.42}$$

where $(\mathbf{w},w_5)$ is the solution obtained in Proposition 3.3 and 3.4. Combining (3.27) and (3.36), one derives that

$$\|(\mathbf{w},w_5)\|_{\mathcal{J}} \le C_*(\sigma+\delta^2). \tag{3.43}$$

Setting $\delta = 2C_*\sigma$ and choosing σ_0 small enough such that $2C_*^2\sigma_0 \le \frac{1}{2}$. Then for any $0<\sigma<\sigma_0$, $C_*(\sigma+\delta^2) = \frac{\delta}{2} + 2C_*^2\sigma\delta < \delta$, hence $\mathcal{T}$ maps $\mathcal{J}$ into itself.

Step 2. The contraction of the operator $\mathcal{T}$.

For any two elements $(\hat{\mathbf{w}}^i,\hat{w}_5^i)$, $i=1,2$, define $(\mathbf{w}^i,w_5^i) = \mathcal{T}(\hat{\mathbf{w}}^i,\hat{w}_5^i)$ for $i=1,2$. Denote

$$(\hat{\boldsymbol{k}},\hat{k}_5) = (\hat{\mathbf{w}}^{(1)},\hat{w}_5^{(1)}) - (\hat{\mathbf{w}}^{(2)},\hat{w}_5^{(2)}) \quad \text{and} \quad (\boldsymbol{k},k_5) = (\mathbf{w}^{(1)},w_5^{(1)}) - (\mathbf{w}^{(2)},w_5^{(2)}).$$

It follows from (3.9) that k_1 and k_2 satisfy

$$\begin{cases} d_1(z_1)\partial_{z_1}k_1 + d_2(z_1)\Big(\partial_{z_2}k_2 + \dfrac{k_2}{z_2}\Big) + d_3(z_1)k_1 + d_4(z_1)\dfrac{b_3}{b_2}k_1(L_b,z_2) = G_1(\mathbf{w}^{(1)},w_5^{(1)}) - G_1(\mathbf{w}^{(2)},w_5^{(2)}),\\ \partial_{z_1}k_2 - d_2(z_1)\partial_{z_2}k_1 + d_5(z_1)\dfrac{b_3}{b_2}\partial_{z_2}k_1(L_b,z_2) = G_2(\mathbf{w}^{(1)},w_5^{(1)}) - G_2(\mathbf{w}^{(2)},w_5^{(2)}),\\ \partial_{z_2}k_1(L_b,z_2) = a_1b_2k_2(L_b,z_2) + h_2^{(1)}(z_2) - h_2^{(2)}(z_2),\\ k_1(L_2,z_2) = \dfrac{b_3}{b_2u_b(L_2)}k_1(L_b,z_2) + h_3^{(1)}(z_2) - h_3^{(2)}(z_2),\\ k_2(z_1,0) = 0,\\ k_2(z_1,\mathcal{M}) = h_4^{(1)}(z_1) - h_4^{(2)}(z_1), \end{cases} \tag{3.44}$$

Then Proposition 3.3 gives that

$$\begin{aligned} \sum_{j=1}^{2}\|k_j\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} &\le C\Big(\sum_{j=1}^{2}\|G_j(\hat{\mathbf{w}}^{(1)},\hat{w}_5^{(1)}) - G_i(\hat{\mathbf{w}}^{(2)},\hat{w}_5^{(2)})\|_{1,\alpha;\Omega}^{(1-\alpha;\Gamma_{p,z})} + \|h_2^{(1)} - h_2^{(2)}\|_{1,\alpha;[0,\mathcal{M})}^{(1-\alpha;\{\mathcal{M}\})}\\ &\qquad + \|h_3^{(1)} - h_3^{(2)}\|_{1,\alpha;[0,\mathcal{M})}^{(-\alpha;\{\mathcal{M}\})} + \|h_4^{(1)} - h_4^{(2)}\|_{0,\alpha;[L_b,L_2]}\Big)\\ &\le C\sigma\|(\hat{\boldsymbol{k}},\hat{k}_5)\|_{\mathcal{J}}. \end{aligned} \tag{3.45}$$

Next, it follows (3.3) and (3.8) that

$$\begin{cases} k_5(z_2) = \dfrac{1}{b_2}k_1(L_b,z_2) - \dfrac{1}{b_2}(R_1^{(1)} - R_1^{(2)}),\\ k_5'(z_2) = a_1k_2(L_b,y_2) + h_1^{(1)} - h_1^{(2)}. \end{cases} \tag{3.46}$$

Thus one gets

$$\begin{aligned} \|k_5\|_{3,\alpha;[0,\mathcal{M})}^{(-1-\alpha;\{\mathcal{M}\})} &\le C\left(\|k_1\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|k_2\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|R_1^{(1)} - R_1^{(2)}\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|h_1^{(1)} - h_1^{(2)}\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})}\right)\\ &\le C\sigma\|(\hat{\boldsymbol{k}},\hat{k}_5)\|_{\mathcal{J}}. \end{aligned} \tag{3.47}$$

Finally, (3.5)and (3.7) yield that

$$\begin{cases} k_3(z_1,z_2) = \dfrac{\hat{r}^{(1)}(L_b,z_2)u_\theta^-(L_b + \hat{w}_5^{(1)}(z_2),z_2)}{\hat{r}^{(1)}(z_1,z_2)} - \dfrac{\hat{r}^{(2)}(L_b,z_2)u_\theta^-(L_b + \hat{w}_5^{(2)}(z_2),z_2)}{\hat{r}^{(2)}(z_1,z_2)}\\ k_4(z_1,z_2) = \dfrac{b_3}{b_2}k_1(L_b,z_2) + R_4^{(1)} - R_4^{(2)}. \end{cases} \tag{3.48}$$

Hence it holds that

$$\|k_3\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} + \|k_4\|_{2,\alpha;\Omega}^{(-\alpha;\Gamma_{p,z})} \le C\sigma\|(\hat{\boldsymbol{k}},\hat{k}_5)\|_{\mathcal{J}}. \tag{3.49}$$

Combining all the above estimates, one can conclude that

$$\|(\boldsymbol{k},k_5)\|_{\mathcal{J}} \le C_\sharp\sigma\|(\hat{\boldsymbol{k}},\hat{k}_5)\|_{\mathcal{J}}. \tag{3.50}$$

Choosing $\sigma_0 \le \min\{\frac{1}{4C_*^2},\frac{1}{2C_\sharp}\}$, then if $0 < \sigma < \sigma_0$, $\|(\boldsymbol{k},k_5)\|_{\mathcal{J}} \le \frac{1}{2}\|(\hat{\boldsymbol{k}},\hat{k}_5)\|_{\mathcal{J}}$ so that the mapping $\mathcal{T}$ is a contraction operator in the norm $\|\cdot\|_{\mathcal{J}}$. Thus there exists a unique fixed point $(\mathbf{w},w_5) \in \mathcal{J}$ such that $\mathcal{T}(\mathbf{w},w_5) = (\mathbf{w},w_5)$. It is easy to see that this fixed point is the solution for the problem (2.31)-(2.32) and (2.33) with boundary conditions (2.29), (2.30), (2.34) and (2.35).

Since the modified Lagrangian transformation is invertible, thus the soultion transformed back in (x,r)-coordinates satisfies the properties (1.14)-(1.18) in Theorem 1.3. The proof of Theorem 1.3 is completed.

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