Singular algebraic curves and infinite symplectic staircases

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Abstract

We show that the infinite staircases which arise in the ellipsoid embedding functions of rigid del Pezzo surfaces can be entirely explained in terms of rational sesquicuspidal symplectic curves. Moreover, we show that these curves can all be realized algebraically, giving various new families of algebraic curves with one cusp singularity. Our main techniques are (i) a generalized Orevkov twist, and (ii) the interplay between algebraic $\mathbb{Q}$-Gorenstein smoothings and symplectic almost toric fibrations. Along the way we develop various methods for constructing singular algebraic (and hence symplectic) curves which may be of independent interest.

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1 Introduction

1.1 Brief summary

One starting point for this paper is the observation that the numerics of the following two mathematical objects coincide:

(a) the family of unicuspidal rational plane curves constructed by Orevkov in [Ore02] (see also [Kas87; Fer+06])

(b) the outer corners of the steps of the Fibonacci staircase for the symplectic ellipsoid embedding function $c_{\mathbb{CP}^2}(x)$ of the complex projective plane (see e.g. [McSch12]).

A priori these belong to rather distinct subfields: the former pertains to the classical problem of characterizing algebraic plane curves of given degree and genus with prescribed singularities (see e.g. [GLS18]), while the latter belongs to the burgeoning area of quantitative symplectic embeddings (see e.g. [Sch18]). In [McS23] we showed that symplectic unicuspidal curves give (stable) symplectic embedding obstructions, and in particular that the family (a) recovers the Fibonacci staircase outer corners (b). In this paper:

- We show that the infinite staircases for rigid del Pezzo surfaces found in [Cri+20] can be *entirely understood* in terms of genus zero sesquicuspidal symplectic curves. Here the obstructions at outer corners come from index zero curves, while the embeddings at inner corners come from higher index curves (via a version of symplectic inflation in §3). As a byproduct, all of these staircases stabilize.
- We show that all of these curves can be realized *algebraically*. In particular, this gives new families of unicuspidal algebraic curves whose existence is suggested by (and has applications to) quantitative symplectic geometry. As an application, we give a new classification theorem for algebraic unicuspidal rational curves in the first Hirzebruch surface.

The core of this paper develops new techniques for constructing algebraic (and hence symplectic) unicuspidal curves. First, in §2 we give a generalization of Orevkov's twist from [Ore02] which holds in any rigid del Pezzo surface. We apply this to construct algebraic curves for each of the relevant outer corners, and later in §7.1 to produce a new sequence of algebraic plane curves responsible for the stabilized ghost stairs from [CHM18].

Then, in §4 we give a perspective on $\mathbb{Q}$-Gorenstein smoothings of singular toric surfaces which closely parallels the theory of symplectic almost toric fibrations. Using this we establish general constructions of algebraic unicuspidal rational curves in §5 and §6. These take as input tropical curves in a base polygon Q and reflect a rich combinatorial theory of polygon mutations. This approach naturally produces algebraic curves for both the inner and outer corners of the rigid del Pezzo infinite staircases, as well as more general curve families.

The remainder of this extended introduction is structured as follows. In §1.2 we first provide some context and motivation for the study of unicuspidal algebraic curves, as well as symplectic ellipsoid embeddings and infinite staircases. Then in §1.3 we give precise formulations of our main results.

1.2 Context and motivation

1.2a Singular plane curves

To set the stage, let us first recall a few basics about singular algebraic curves. In this paper all algebraic curves will be defined over the complex numbers. By "plane curve" we mean a complex algebraic curve in $\mathbb{CP}^2$, which concretely is of the form $V(F) := \{F(x,y,z) = 0\}$ for some homogeneous polynomial $F(x,y,z)$. A point $p_0 = [x_0 : y_0 : z_0] \in V(F)$ is singular if and only if we have $\partial_x F(p_0) = \partial_y F(p_0) = \partial_z F(p_0) = 0$. The following local models, written in affine coordinates with the singular point at the origin, will be relevant for us:

- $\{x^2 = y^2\}$ is the **ordinary double point** (a.k.a. the A_1 singularity)
- $\{x^3 = y^2\}$ is the **ordinary cusp**
- more generally, $\{x^p = y^q\}$ for $p, q \in \mathbb{Z}_{\geqslant 1}$ coprime is the **(p,q) cusp**.

Note that topologically the (p,q) cusp is the cone over the (p,q) torus knot, and if $p = 1$ or $q = 1$ this is just a smooth point.

Example 1.2.1. The plane curve $C = \{X^p = Y^q Z^{p-q}\} \subset \mathbb{CP}^2$ has two singularities: a (p,q) cusp at $[0:0:1]$ and a $(p, p-q)$ cusp at $[0:1:0]$. Moreover, it is rational since it admits a parametrization $\mathbb{CP}^1 \to C$, $[s:t] \mapsto [s^q t^{p-q} : s^p : t^p]$. $\Diamond$

A (reduced and irreducible) algebraic curve[1] is called **(p,q)-unicuspidal** if it has a single (p,q) cusp (with $\gcd(p,q) = 1$) and no other singularities. More generally, it is called **(p,q)-sesquicuspidal** if in addition it has some ordinary double points.

To anchor the discussion, let us recall the following classification result. We denote the Fibonacci numbers by $\mathrm{Fib}_1 = 1, \mathrm{Fib}_2 = 1, \mathrm{Fib}_3 = 2$ and so on.

Theorem 1.2.2 ([Fer+06])**.** *There exists a (p,q)-unicuspidal rational plane curve of degree d if and only if (d,p,q) is one of the following:*

(a) $(p,q) = (d, d-1)$ *for* $d \in \mathbb{Z}_{\geqslant 1}$

(b) $(p,q) = (2d-1, d/2)$ *for* $d \in 2\mathbb{Z}_{\geqslant 1}$

(c) $(p,q) = (\mathrm{Fib}_k^2, \mathrm{Fib}_{k-2}^2)$ *and* $d = \mathrm{Fib}_{k-2}\mathrm{Fib}_k$ *for* $k \in 2\mathbb{Z}_{\geqslant 2} + 3$

(d) $(p,q) = (\mathrm{Fib}_{k+2}, \mathrm{Fib}_{k-2})$ *for* $d = \mathrm{Fib}_k$ *for* $k \in 2\mathbb{Z}_{\geqslant 2} + 3$

(e) $(p,q) = (22, 3)$ *and* $d = 8$

(f) $(p,q) = (43, 6)$ *for* $d = 16$.

Family (a) is the specialization of Example 1.2.1 with $(p,q) = (d, d-1)$, while family (b) is given by $\{(zy - x^2)^{d/2} = xy^{d-1}\} \subset \mathbb{CP}^2$. See Remark 2.1.3 below for constructions of (e) and (f). The curves in family (c) are more complicated but are described by explicit equations in [Fer+06, §5], following [Kas87]. Family (d) corresponds to the aforementioned Orevkov curves [Ore02].

To make further sense of Theorem 1.2.2, it will be helpful to introduce the following:

Definition 1.2.3. *The (real)* ***index*** *of a (p,q)-sesquicuspidal rational curve C in a complex surface or symplectic four-manifold manifold M is*

$$\mathrm{ind}_{\mathbb{R}}(C) := 2c_1(A) - 2p - 2q, \tag{1.2.1}$$

where $A \in H_2(M)$ denotes the homology class of C and $c_1(A)$ is its first Chern number.

[1] The curves considered in this paper will be rational, that is parametrizable by $\mathbb{CP}^1$ (and in particular irreducible), unless explicit mention is made to the contrary.

As explained in detail in [McS23], the index corresponds to the expected (real) dimension of the space of rational curves in homology class A with a (p,q) cusp and satisfying a maximal order tangency constraint at the cusp. For instance, for an ordinary $(3,2)$ cusp there is a well-defined complex tangent line at the singular point, and the constraint corresponds to specifying both the location of the singularity and its tangent line at that point. For a general (p,q) cusp the constraint also involves higher jet constraints. Equivalently, the index is the (real) Fredholm index of the normal crossing resolution (c.f. [McS23, §4.1] or §3.1 below).

In particular, for a (p,q)-unicuspidal rational plane curve C of degree d we have $\mathrm{ind}_{\mathbb{R}}(C) = 6d - 2p - 2q$, and the indices for the curves in Theorem 1.2.2 are as follows:

	(a)	(b)	(c)	(d)	(f)	(g)
index	$2d+2$	$d+2$	2	0	-2	-2

. (1.2.2)

For index zero curves such as those in family (d) one expects to get a finite count, and indeed these are encoded by the Gromov–Witten-type invariants $N_{\mathbb{CP}^2,d[L]}\langle\!\langle\mathcal{C}^{(p,q)}pt\rangle\!\rangle$ defined in [McS23, §3]. The curves in family (c) cannot quite be counted (they occur in complex 1-parameter families), but they naturally degenerate to those in family (d) (c.f. [Fer+06, §5], based on [Kas87; MiSu81]). Meanwhile, the sporadic cases (e) and (f) have negative index, so they should disappear for a generic almost complex structure. In this article, families (c) and (d) (and their generalizations) will be the most significant, as they precisely correspond to the inner and outer corners respectively of the Fibonacci staircase. Incidentally, in §6.5 we exploit this connection with symplectic geometry to give an alternative proof that the list in (d) above is complete, and we extend to the first Hirzebruch surface (for which the corresponding list is much more complicated).

1.2b Symplectic embeddings and infinite staircases

Let us now briefly recall some notions surrounding symplectic ellipsoid embeddings and infinite staircases. Given a four-dimensional symplectic manifold X, its ellipsoid embedding function is defined by

$$c_X(x) := \inf\{\lambda \in \mathbb{R}_{>0} \mid E(\tfrac{1}{\lambda}, \tfrac{x}{\lambda}) \overset{s}{\hookrightarrow} X\}. \tag{1.2.3}$$

Here the infimum is over all $\lambda \in \mathbb{R}_{>0}$ for which there exists a symplectic embedding of the ellipsoid

$$E(\tfrac{1}{\lambda}, \tfrac{x}{\lambda}) := \{(z_1,z_2) \mid \pi|z_1|^2\lambda + \pi|z_2|^2\lambda/x \leqslant 1\} \subset \mathbb{C}^2$$

(endowed with the restriction of the standard symplectic form) into X. In [McSch12], the ellipsoid embedding function for the four-ball $B^4(1) = E(1,1)$ was explicitly worked out. In particular, the portion for $1 \leqslant x \leqslant \tau^4 := \frac{3\sqrt{5}+7}{2}$ is a piecewise linear function whose graph is a zigzag, that alternately slopes up and is horizontal, with infinitely many nonsmooth points that accumulate at τ^4 and have coordinates given by ratios of odd index Fibonacci numbers – see [McSch12, Fig 1.1]. Subsequently, similar infinite staircases were discovered for other target spaces such as $B^2(1) \times B^2(1)$ [FM15], $E(1,3/2)$

[CK20], and more (see e.g. [Ush19]). More recently, the authors in [Cri+20] gave a unified description of infinite staircases for the six rigid del Pezzo surfaces with their monotone symplectic forms, namely $\mathbb{CP}^2(3)\#^{\times k}\overline{\mathbb{CP}}^2(1)$ for $k = 0, 1, 2, 3, 4$ and $\mathbb{CP}^1(2) \times \mathbb{CP}^1(2)$.[2]

Theorem 1.2.4 ([Cri+20]). *For each rigid del Pezzo surface M, the ellipsoid embedding function $c_M(x)$ has an infinite staircase with explicitly described accumulation point and step coordinates.*

By elementary scaling and monotonicity considerations, establishing these infinite staircases boils down to (a) obstructing symplectic embeddings at the outer corners and (b) constructing symplectic embeddings at the inner corners. Embeddings corresponding to the inner corners were constructed in [Cri+20; CV22] using almost toric fibrations and their mutations (see e.g. [Sym; Eva23] or §4.2 below), and hence are also related to generalized Markov equations and exotic Lagrangian tori as in [Via17]. Meanwhile, obstructions corresponding to outer corners were established in [Cri+20] using embedded contact homology (ECH) capacities (see e.g. [Hut14]). In this paper our approach to the inner corners and one of our approaches to the outer corners are also based on almost toric fibrations, but used in a quite distinctive way through the lens of sesquicuspidal curves.

Remark 1.2.5. By definition a del Pezzo surface is a smooth complex projective surface with ample anticanonical bundle. Up to diffeomorphism these are $\mathbb{CP}^2\#^{\times k}\overline{\mathbb{CP}}^2$ for $k = 0, \ldots, 8$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. Up to biholomorphism there is a unique del Pezzo surface having smooth type $\mathbb{CP}^2\#^{\times k}\overline{\mathbb{CP}}^2$ for $k = 0, \ldots, 4$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ (these are the rigid ones), while the remaining cases appear in nontrivial moduli spaces.

Each del Pezzo surface admits a unique monotone symplectic form up to symplectomorphism and scaling (see e.g. [Sal13]), and unless explicit mention is made to the contrary we work with the monotone symplectic structure normalized to have monotonicity constant 1, i.e. $c_1(M) = [\omega_M] \in H^2(M; \mathbb{R})$ (e.g. $\mathbb{CP}^2(3)$). One should keep in mind that the moduli spaces of complex and symplectic structures on these smooth manifolds are quite distinct,[3] but it should be clear from the context whether we view M in the complex, symplectic, or smooth category. $\Diamond$

Remark 1.2.6. The treatment in [Cri+20] emphasizes the 12 convex toric domains $X_1, \ldots, X_{12}$ pictured in Figure 7 below, which includes $B^4(1), B^2(1) \times B^2(1)$, and $E(1, 3/2)$ as special cases. It is shown in [Cri+20] that the ellipsoid embedding function for each X_i is equivalent to the ellipsoid embedding function for one of the monotone rigid del Pezzo surfaces, namely the one with the same negative weight expansion (for instance we have $c_{B^4(a)}(x) = c_{\mathbb{CP}^2(a)}(x)$). Thus for simplicity of exposition we will mostly restrict our discussion to the closed target spaces (except when discussing the stable folding curve in §7.2). $\Diamond$

[2] Here $\mathbb{CP}^2(a)$ is endowed with the Fubini–Study form normalized so that a line has area a. The qualifier "rigid" is a slight misnomer since it refers to the complex rather than symplectic structure – see Remark 1.2.5 below.

[3] Roughly speaking, in the complex category the locations of blowup points matter, while in the symplectic category the sizes of blowups matters.

1.3 Main results

1.3a Singular curves and symplectic embeddings

We first explain how singular symplectic curves both obstruct and construct symplectic ellipsoid embeddings. For $p, q \in \mathbb{Z}_{\geqslant 1}$ coprime, a **(p, q)-sesquicuspidal symplectic curve** in a symplectic four-manifold M^4 is a subset $C \subset M$ which has one point $x_0 \in C$ locally modeled on a (p, q) cusp point of an algebraic curve in $\mathbb{C}^2$, and such that C is otherwise an immersed symplectic submanifold with only positive double points (see [McS23, Def. 3.5.1]).

The following explicit link between sesquicuspidal curves and symplectic embedding obstructions was established in [McS23]:

Theorem 1.3.1 (Theorems A(b), D, and E in [McS23])**.** *Let (M^4, ω_M) be a four-dimensional closed symplectic manifold, and suppose there exists an index zero (p, q)-sesquicuspidal rational symplectic curve in M in homology class $A \in H_2(M)$. Then any symplectic embedding $E(cq, cp) \overset{s}{\hookrightarrow} M$ must satisfy $c \leqslant \frac{[\omega_M]\cdot A}{pq}$. Moreover, the same is true for any symplectic embedding $E(cq, cp) \times \mathbb{C}^N \overset{s}{\hookrightarrow} M \times \mathbb{C}^N$ for $N \in \mathbb{Z}_{\geqslant 1}$, provided that $M \times \mathbb{C}^N$ is semipositive.*[4]

In other words, the existence of C implies $c_M(p/q) \geqslant \frac{p}{[\omega_M]\cdot[C]}$.[5] Since any unicuspidal algebraic curve in a complex projective surface is in particular a unicupisdal symplectic curve, applying Theorem 1.3.1 to family (d) in Theorem 1.2.2 (with $N = 0$) immediately gives the obstructive part (i.e. outer corners) of the Fibonacci staircase in $c_{\mathbb{CP}^2}(x)$. Moreover, the case $N \geqslant 1$ shows that these obstructions stabilize, i.e. we have $c_{\mathbb{CP}^2 \times \mathbb{C}^N}(x) = c_{\mathbb{CP}^2}(x)$ for all $1 \leqslant x \leqslant \tau^4$ (this is the main result of [CH18], originally proved using embedded contact homology), where we put

$$c_{X \times \mathbb{C}^N}(a) := \inf\{\lambda \in \mathbb{R}_{>0} \mid E(\tfrac{1}{\lambda}, \tfrac{a}{\lambda}) \times \mathbb{C}^N \overset{s}{\hookrightarrow} X \times \mathbb{C}^N\} \tag{1.3.1}$$

for any symplectic four-manifold X^4 and $N \in \mathbb{Z}_{\geqslant 1}$.

As for constructing symplectic embeddings, the following theorem is proved in §3 below via the method of symplectic inflation. Recall that any local branch of a holomorphic curve near a singularity is homeomorphic to the cone over an iterated torus knot (see [EN85]). The cabling parameters can be read off from the Puiseux pairs $(n_1, d_1), \ldots, (n_g, d_g)$, which can in turn be read off from a Puiseux series parametrization $x(t) = t^m, y(t) = \sum\limits_{k=m}^{\infty} a_k t^k$

[4]Here semipositivity is a technical condition which allows one to rule out sphere bubbling using only classical perturbations. Note that $M \times \mathbb{C}^N$ is automatically semipositive if $N = 1$ or if M is monotone. Using e.g. [McS23, Cor. 2.7.2], we can also quantify the above stable symplectic embedding obstructions by replacing the domain $E(1, a) \times \mathbb{C}^N$ with $E(1, a, b_1, \ldots, b_N)$ for suitable finite $b_1, \ldots, b_N \in \mathbb{R}_{>0}$. A similar remark applies to all other stable obstructions which follow, although for simplicity we will formulate results without this quantification.

[5]The basic reason for the existence of this obstruction is that one can construct an SFT-type curve in the complement of (a slight perturbation of) the ellipsoid that must have positive symplectic area. Equivalently, the exceptional divisor given by the normal crossing resolution of the singular curve must have positive area.

– see §3.3 for more details. In particular, a (p,q) cusp corresponds to a single Puiseux pair $(n_1,d_1)=(p,q)$, and our other main examples will be cusps with two Puiseux pairs $(n_1,d_1)=(p,q)$, $(n_2,d_2)=(kp+1,k)$ for some $k\in\mathbb{Z}_{\geqslant 1}$.

Theorem A. *Let (M^4,ω_M) be a four-dimensional closed symplectic manifold.*

(i) Let C be a (p,q)-sesquicuspidal symplectic curve in M whose homology class satisfies $[C]=c\,\mathrm{PD}[\omega_M]\in H^2(M;\mathbb{R})$ for some $c\in\mathbb{R}_{>0}$ and $[C]\cdot[C]\geqslant pq$. Then there exists a symplectic embedding $E(\frac{q}{c'},\frac{p}{c'})\overset{s}{\hookrightarrow} M$ for any $c'>c$. In particular, if $[C]\cdot[C]=pq$ then this is a full filling, i.e the domain and target have arbitrarily close volume.[6]

(ii) More generally, let C be a sesquicuspidal symplectic curve in M with Puiseux pairs $(p,q),(p_2,q_2),\dots,(p_g,q_g)$, whose homology class satisfies $[C]=c\mathrm{PD}[\omega_M]\in H^2(M;\mathbb{R})$ for some $c\in\mathbb{R}_{>0}$ and $[C]\cdot[C]\geqslant k^2pq$ with $k=q_2\cdots q_g$. Then there exists a symplectic embedding $E(\frac{kq}{c'},\frac{kp}{c'})\overset{s}{\hookrightarrow} M$ for any $c'>c$.

In particular the existence of C in (i) implies $c_M(p/q)\leqslant\frac{c}{q}$ (assuming $p>q$). Note that the last sentence of (i) follows since we have

$$\mathrm{vol}(M,\omega_M)=\frac{1}{2}\int_M\omega_M\wedge\omega_M=\frac{1}{2}\mathrm{PD}[\omega_M]\cdot\mathrm{PD}[\omega_M]=\frac{1}{2c^2}[C]\cdot[C].$$

The conditions on $[C]\cdot[C]$ imply that the index of the (partial) resolution of C along which we inflate is positive. Indeed, the expression $[C]\cdot[C]-pq$ in (i) corresponds to the self-intersection number of the normal crossing resolution of C, while, when $g=2$, the expression $[C]\cdot[C]-k^2pq$ in (ii) corresponds to the self-intersection number of the minimal resolution of C (see §3).

Applying Theorem A to family (c) from Theorem 1.2.2 recovers the constructive part (i.e. inner corners) for $c_{\mathbb{CP}^2}(x)$. We will see below that a similar picture holds for all of the monotone rigid del Pezzo surfaces.

1.3b Outer and inner corner curves

We first construct singular algebraic curves responsible for the obstructions at outer corners, generalizing family (d) from Theorem 1.2.2.

Theorem B. *In each rigid del Pezzo surface M there is a countable family of rational index zero unicuspidal algebraic*[7] *curves which correspond precisely to the outer corners of the steps of the infinite staircase in $c_M(x)$. More specifically, if (x,y) is an outer corner point on the graph of c_M, then the corresponding (p,q)-unicuspidal curve C in M satisfies $p/q=x$ and $\frac{p}{[\omega_M]\cdot[C]}=y$.*

[6] Note that if M is a symplectic blowup of $\mathbb{CP}^2$ (and more generally) this actually implies the existence of a symplectic embedding of the open ellipsoid $\mathring{E}(\frac{q}{c},\frac{p}{c})\overset{s}{\hookrightarrow} M$ that fills the entire volume of M (c.f. [Cri19, Proof of Prop. 1.5]).

[7] Note that by Chow's theorem we may speak interchangeably about "algebraic" and "holomorphic" curves, although in the body of the paper we work mostly in the holomorphic category.

Corollary C. *Each of the rigid del Pezzo infinite staircases stabilizes, i.e. for each monotone rigid del Pezzo surface M we have $c_{M\times\mathbb{C}^N}(x) = c_M(x)$ for all $1 \leqslant x \leqslant a_{\mathrm{acc}}(M)$, where $a_{\mathrm{acc}}(M)$ denotes the accumulation point of the infinite staircase in c_M.*

Our first proof of Theorem B in §2 is based on a generalization of Orevkov's birational transformation $\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ to a birational transformation $\Phi_M : M \dashrightarrow M$ for each rigid del Pezzo surface M. In brief, we start with two or three "seed curves" in M, and then we iteratively apply Φ_M to produce the rest of the family. The key point is that for seed curves which are "well-placed" (see Definition 2.2.4), successive applications of Φ_M lead to curves with increasingly singular cusps.

Remark 1.3.2. The number of "strands" of the infinite staircase is determined by the number of initial seed curves, which is three for $\mathbb{CP}^2(3)\#\overline{\mathbb{CP}}^2(1)$ and $\mathbb{CP}^2(3)\#^{\times 2}\overline{\mathbb{CP}}^2(1)$ and two in the remaining cases (this corresponds to J in Table 1). This number can also be seen in terms of the almost toric structures supported by the symplectic manifold M (i.e. triangles or quadrilaterals, see Figure 4). $\Diamond$

We also give a different construction of these outer corner curves in §6 based on almost toric fibrations and $\mathbb{Q}$-Gorenstein deformations.

Theorem D. *Let $\pi : \mathbb{A} \to Q$ be an almost toric fibration, where $Q \subset \mathbb{R}^2$ is a polygon*[8] *and $\mathbb{A}$ is a (not necessarily monotone) closed symplectic four-manifold which is diffeomorphic to a rigid del Pezzo surface M. Suppose that Q contains consecutive edges pointing in the directions $(-mr^2, mra-1), (0,-1), (1,0)$ for some $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r,a) = 1$.*[9] *Then M contains an index zero (r,a)-unicuspidal rational algebraic curve.*

The rough idea is first to construct unicuspidal symplectic curves in $\mathbb{A}$ which are "visible" in Q (and lie over a line segment connecting a vertex to a base-node as in Figure 5 right), and then to compare $\mathbb{A}$ with a $\mathbb{Q}$-Gorenstein smoothing of the corresponding singular toric surface V_Q in order to upgrade these to algebraic curves. Theorem D is a corollary of Theorem 6.1.7, which holds beyond rigid del Pezzo surfaces and which we state using the purely combinatorial language introduced in §4. A useful consequence of Theorem D is that we can directly observe "visible" obstructions for ellipsoid embeddings into $\mathbb{A}$ in terms of triangles in the polygon Q (see the shading in Figure 5 right and see §6.1 for more details). Another noteworthy feature of Theorem D is that $\mathbb{A}$ need not be monotone (or equivalently the polygon Q not need be dual Fano in the sense of §4.1b). This potentially allows us to construct a much larger class of unicuspidal curves than we could just by looking at monotone ATFs, as we illustrate with Theorem F below.

Remark 1.3.3. One reason for giving two different proofs of Theorem B is that the underlying techniques naturally extend in different directions. For instance, the generalized Orevkov twist is used in §7.1 to construct sesquicuspidal algebraic curves in $\mathbb{CP}^2$ (the "ghost stairs" curves) which we do not currently know how to see using almost toric fibrations. $\Diamond$

[8] All polygons in this paper are assumed to be convex.

[9] Equivalently, Q has a vertex $\mathfrak{v}$ with edge directions $(1,0), (0,1)$ and a vertex on the edge in direction $(0,1)$ with eigenray in the direction $(r,-a)$ – see Figure 5 or §4 for more details.

We now discuss curves responsible for constructing symplectic embeddings, generalizing family (c) in Theorem 1.2.2. We consider the two-stranded and three-stranded cases separately, as they behave somewhat differently, with the latter requiring more complicated cusp singularities.

Theorem E.

(a) For M each of the rigid del Pezzo surfaces $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$, and $\mathbb{CP}^2 \#^{\times j} \overline{\mathbb{CP}}^2$, $j = 3, 4$, there is a countable family of index two unicuspidal rational algebraic curves in M which correspond precisely to the inner corners of the infinite staircase in $c_M(x)$. More specifically, if (x, y) is an inner corner point on the graph of c_M, then the corresponding (p, q)-unicuspidal curve C in M satisfies $[C] \cdot [C] = pq$ and $[C] = c\,\mathrm{PD}([\omega_M])$ for $c = \frac{pq}{p+q+1}$, with $p/q = x$ and $c/q = y$.

(b) For M each of the rigid del Pezzo surfaces $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \#^{\times 2} \overline{\mathbb{CP}}^2$, there is a countable family of weakly[10] sesquicuspidal rational algebraic curves in M which correspond precisely to the inner corners of the infinite staircase in $c_M(x)$. More specifically, if $(x = p/q, y)$ is an inner corner point on the graph of c_M, then the corresponding curve C in M satisfies $[C] = kqy\,\mathrm{PD}([\omega_M])$ and $[C] \cdot [C] \geqslant k^2 pq$ for some $k \in \mathbb{Z}_{\geqslant 1}$ and has a cusp with one Puiseux pair (p, q) if $k = 1$ and two Puiseux pairs $(p, q), (kp + 1, k)$ if $k \in \mathbb{Z}_{\geqslant 2}$.

Our proof of Theorem E is based on an analogue of Theorem D, namely Theorem 5.3.1, which says roughly that every "visible ellipsoid embedding" (in the sense of §5.1b) corresponds to an algebraic (weakly) sesquicuspidal rational curve. The proof of Theorem 5.3.1 is similarly based on almost toric fibrations and $\mathbb{Q}$-Gorenstein smoothings, except that the relevant curves no longer correspond to straight line segments in the polygon Q but rather to tropical curves which intersect every edge.

Remark 1.3.4. In the above we have observed that we can recover embeddings corresponding to inner corners of infinite staircases by inflating along suitable singular symplectic curves, and moreover these curves can be explicitly constructed with the aid of almost toric fibrations. However, it should be emphasized that the very same almost toric fibrations can be used much more directly to construct symplectic ellipsoid embeddings (c.f. Proposition 5.1.2), which is the approach taken in [Cri+20; CV22]. The main novelty of Theorem E is the connection with singular algebraic (and in particular symplectic) curves. $\Diamond$

1.3c Unicuspidal curves in the first Hirzebruch surface

Let $F_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ denote the first Hirzebruch surface, and let $\ell, e \in H_2(F_1)$ denote the line and exceptional classes. We showed in [McS23, Thm. G] that there exists an index

[10]The word "weakly" indicates that these curves might have other singularities in addition to the distinguished cusp and some ordinary double points. In the symplectic category we can always perturb such a curve to a genuine sesquicuspidal one, but this is not guaranteed in the algebraic category. It seems plausible that this qualifier here could be removed by a more careful analysis of the curves in the proof of Proposition 5.2.1.

zero (p,q)-unicuspidal rational symplectic curve in F_1 in homology class $A = d\ell - me$ if and only if $A \in H_2(F_1)$ is a (p,q)-perfect exceptional class (see [McS23, Def. 4.4.2]).[11] The set

$$\mathrm{Perf}(F_1) := \{(p,q,d,m) \mid A = d\ell - me \text{ is a } (p,q)\text{-perfect exceptional class}\}$$

is quite complicated but was recently worked out explicitly in [MM24; MMW22] in the course of classifying infinite staircases for all (not necessarily monotone) symplectic forms on F_1 (see also §6.5 below for a detailed overview). In §6.5 we associate a polygon as in Theorem D to each $(p,q,d,m) \in \mathrm{Perf}(F_1)$, and thereby construct an *algebraic* (p,q)-unicuspidal curve:

Theorem F. *For any coprime positive integers $p > q$ and homology class $A \in H_2(F_1)$, the following are equivalent:*

- *there exists an index zero (p,q)-unicuspidal rational* symplectic *curve C in F_1 with $[C] = A$*
- *there exists an index zero (p,q)-unicuspidal rational* algebraic *curve C in F_1 with $[C] = A$.*

Remark 1.3.5. It follows by Theorem F and the results in [MM24; MMW22] that every rational unicuspidal algebraic curve with one Puiseux pair in F_1 corresponds to the outer corner of a staircase in $c_{\mathcal{H}_b}(x)$ for some $b \in [0,1)$, where $\mathcal{H}_b := \mathbb{CP}^2(1)\#\overline{\mathbb{CP}}^2(b)$ denotes F_1 with the symplectic form such that a line has area 1 and the exceptional divisors has area b (this is unique up to symplectomorphism). Note that $\mathcal{H}_b$ is monotone only if $b = 1/3$. ◊

We prove Theorem F by showing that such (p,q) is realized by an almost toric fibration which satisfies the hypotheses of Theorem D. The analogous statement holds for $\mathbb{CP}^2$ (see Lemma 6.5.1) and is expected for $\mathbb{CP}^1 \times \mathbb{CP}^1$ (c.f. [Ush19; Far+22] and Remark 6.5.4 below). Thus it is natural to ask whether something analogous holds also for the remaining rigid del Pezzo surfaces:

Conjecture G. *Let M be a rigid del Pezzo surface. The following are equivalent:*

(a) There exists an index zero (p,q)-unicuspidal rational symplectic curve in M

(b) There exists an almost toric fibration $\pi : \mathbb{A} \to Q$ as in Theorem D, where $\mathbb{A}$ is diffeomorphic to M and Q has consecutive edges pointing in the directions $(-mq^2, mpq-1), (0,-1), (1,0)$ for some $m \in \mathbb{Z}_{\geqslant 1}$. In particular, there exists an index zero (p,q)-unicuspidal rational algebraic curve in M.

Note that (a) is equivalent to the existence of a (p,q)-perfect exceptional homology class $A \in H_2(M)$ (again by [McS23, Thm. G]), although the set $\mathrm{Perf}(M)$ remains to be worked out (and presumably becomes more complicated with more blowups).

[11] Note that this does not depend on the choice of symplectic form on F_1, by e.g. Theorem E and Theorem 3.3.2 in [McS23], along with the fact that all symplectic forms on F_1 are deformation equivalent.

Remark 1.3.6. We do not attempt to construct inner corner curves for staircases in nonmonotone manifolds such as $\mathcal{H}_b$ for $b \neq 1/3$. One main issue is that in all known cases such staircases exist only when the symplectic form has no rational multiple, so that one cannot construct optimal ellipsoid embeddings by simply inflating along a curve in a class Poincaré dual to a multiple of the symplectic form. $\Diamond$

1.3d Sesquicuspidal curves and obstructions beyond staircases

Another natural question in the spirit of Theorem 1.2.2 is to try to classify index zero (p,q)-sesquicuspidal rational curves in $\mathbb{CP}^2$. Note that for such a curve we have $d = (p+q)/3$, and by the adjunction formula the number of ordinary double points must be $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}(p-1)(q-1)$. This question is known to be closely related to the study of the stabilized ellipsoid embedding function $c_{\mathbb{CP}^2 \times \mathbb{C}^N}(x)$ beyond the Fibonacci staircase, i.e. for $x > \tau^4$ (see e.g. [McS23, §1] and the references therein). In §7 we apply the Orevkov twist $\Phi_{\mathbb{CP}^2}$ to more interesting seed curves to produce a new (to our knowledge) family of rational algebraic plane curves:

Theorem H. *There is an infinite sequence of index zero rational algebraic curves $C_1, C_2, C_3, \ldots$ in $\mathbb{CP}^2$ which correspond precisely to the "ghost stairs" obstructions from [McSch12]. More specifically, for $k \in \mathbb{Z}_{\geqslant 1}$, C_k has degree d_k and a (p_k, q_k) cusp, where $(p_k, q_k) = (\mathrm{Fib}_{4k+2}, \mathrm{Fib}_{4k-2})$ and $d_k = \frac{1}{3}(p_k + q_k) = \mathrm{Fib}_{4k}$.*

Combined with Theorem 1.3.1, this recovers the main result from [CHM18], namely there is an infinite sequence $x_1 > x_2 > x_3 > \cdots$ of positive real numbers with $\lim\limits_{i \to \infty} x_i = \tau^4$ and such that $c_{B^4(1) \times \mathbb{C}^N}(x_i) = \frac{3x_i}{x_i+1}$ for all $i, N \in \mathbb{Z}_{\geqslant 1}$. We recall the significance of the "folding function" $\frac{3x}{x+1}$ and discuss its (partly conjectural) analogue for other target spaces in §7.2. We also note that the same techniques allow for a vast generalization of Theorem H conditional on the existence of "higher degree seed curves", which we take up in the forthcoming work [McS] (see Remark 7.1.3).

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2 Unicuspidal curves and the generalized Orevkov twist

Our goal in this section is to introduce the generalized Orevkov twist and use it to prove Theorem B. We first formalize the twist $\mathbb{CP}^2$ in §2.1, with a view towards allowing more general seed curves (e.g. those considered in §7.2) and also extending the ambient space to rigid del Pezzo surfaces, which we take up in §2.2. After a brief interlude in §2.3 to recall some staircase numerics from [Cri+20], we complete the proof by constructing the relevant seed curves in §2.3.

2.1 The Orevkov twist in $\mathbb{CP}^2$

In this subsection we recall the definition and basic properties of the birational transformation $\Phi_{\mathbb{CP}^2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ from [Ore02], which for brevity we refer to as the "Orevkov twist". Let $\mathcal{N} \subset \mathbb{CP}^2$ be a fixed nodal cubic, which for concreteness we can take to be $\{y^2 z = x^2(x+z)\}$ (any other nodal cubic is projectively equivalent to this one). Let $\mathfrak{p} \in \mathcal{N}$ denote the double point, and let $\mathcal{B}_+, \mathcal{B}_-$ denote the two local branches near $\mathfrak{p}$.

Construction 2.1.1. *The birational transformation $\Phi_{\mathbb{CP}^2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is defined as follows. Let $\mathrm{Bl}^1\mathbb{CP}^2$ denote the blowup*[12] *of $\mathbb{CP}^2$ at the point $\mathfrak{p}$, with resulting exceptional divisor $\mathbb{F}_1^1$. Let $\mathcal{N}^1 \subset \mathrm{Bl}^1\mathbb{CP}^2$ denote the proper transform of $\mathcal{N}$, and let $\mathcal{B}_+^1 \subset \mathcal{N}^1$ denote the proper transform of the local branch $\mathcal{B}_+$. Put $\mathfrak{p}^1 := \mathcal{B}_+^1 \cap \mathbb{F}_1^1 \subset \mathcal{N}^1$.*

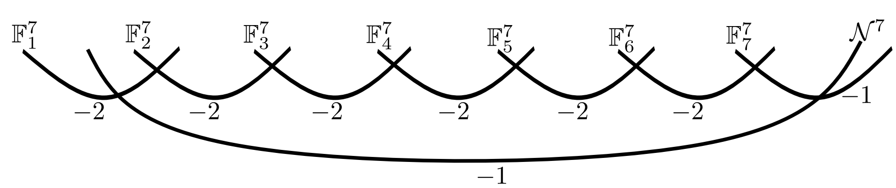
Next, let $\mathrm{Bl}^2\mathbb{CP}^2$ denote the blowup of $\mathrm{Bl}^1\mathbb{CP}^2$ at the point $\mathfrak{p}^1$, with resulting exceptional divisor $\mathbb{F}_2^2$, and with $\mathbb{F}_1^2 \subset \mathrm{Bl}^2\mathbb{CP}^2$ the proper transform of $\mathbb{F}_1^1$. Let $\mathcal{N}^2 \subset \mathrm{Bl}^2\mathbb{CP}^2$ denote the proper transform of $\mathcal{N}^1$, and let $\mathcal{B}_+^2 \subset \mathcal{N}^2$ denote the proper transform of the local branch $\mathcal{B}_+^1$. Put $\mathfrak{p}^2 := \mathcal{B}_+^2 \cap \mathbb{F}_2^2 \subset \mathcal{N}^2$.

Continuing in this manner, after a total of 7 blowups we arrive at $\mathrm{Bl}^7\mathbb{CP}^2$, which contains a chain of rational curves $\mathcal{N}^7, \mathbb{F}_1^7, \ldots, \mathbb{F}_7^7$ with intersection graph as in Figure 1. In terms of the natural identification $H_2(\mathrm{Bl}^7\mathbb{CP}^2) \cong H_2(\mathbb{CP}^2) \oplus \mathbb{Z}\langle e_1, \ldots, e_7\rangle$, we have

- $[\mathcal{N}^7] = 3\ell - 2e_1 - e_2 - \cdots - e_7$
- $[\mathbb{F}_1^7] = e_1 - e_2$
- $[\mathbb{F}_2^7] = e_2 - e_3$
- $[\mathbb{F}_3^7] = e_3 - e_4$
- $[\mathbb{F}_4^7] = e_4 - e_5$
- $[\mathbb{F}_5^7] = e_5 - e_6$
- $[\mathbb{F}_6^7] = e_6 - e_7$
- $[\mathbb{F}_7^7] = e_7$,

where $\ell \in H_2(\mathbb{CP}^2)$ denotes the line class.

Figure 1: The chain of rational curves (decorated by their self-intersection numbers) which arise in the half part of the Orevkov twist.



Since $[\mathcal{N}^7] \cdot [\mathcal{N}^7] = -1$, we can blow down $\mathrm{Bl}^7\mathbb{CP}^2$ along $\mathcal{N}^7$ to obtain $\mathrm{Bl}^{7;1}\mathbb{CP}^2$. Let $\mathbb{F}_1^{7;1}, \ldots, \mathbb{F}_7^{7;1} \subset \mathrm{Bl}^{7;1}\mathbb{CP}^2$ denote the proper transforms of $\mathbb{F}_1^7, \ldots, \mathbb{F}_7^7$ respectively. Then since $[\mathbb{F}_1^{7;1}] \cdot [\mathbb{F}_1^{7;1}] = -1$, we can blow down $\mathrm{Bl}^{7;1}\mathbb{CP}^2$ along $\mathbb{F}_1^{7;1}$ to obtain $\mathrm{Bl}^{7;2}\mathbb{CP}^2$. Let $\mathbb{F}_2^{7;2}, \ldots, \mathbb{F}_7^{7;2} \subset \mathrm{Bl}^{7;2}\mathbb{CP}^2$ denote the proper transforms of $\mathbb{F}_2^{7;1}, \ldots, \mathbb{F}_7^{7;1}$ respectively. Continuing in this manner, after a total of 7 blowdowns (along $\mathcal{N}^7, \mathbb{F}_1^{7;1}, \ldots, F_6^{7;6}$), we arrive at $\mathrm{Bl}^{7;7}\mathbb{CP}^2$, which contains the nodal rational curve $\mathbb{F}_7^{7;7}$ with $[\mathbb{F}_7^{7;7}] \cdot [\mathbb{F}_7^{7;7}] = 9$.

[12]All blowups in this section are at points and occur in the complex category. In later sections we also consider symplectic blowups, which depend on a symplectic embedding of a ball.

Finally, by composing this sequence of 7 blowups and 7 blowdowns with a biholomorphism $\mathrm{Bl}^{7;7}\mathbb{CP}^2 \cong \mathbb{CP}^2$ *sending* $\mathbb{F}_7^{7;7}$ *to* $\mathcal{N}$*, we arrive at the birational transformation* $\Phi_{\mathbb{CP}^2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Given a curve $C \subset \mathbb{CP}^2$, we denote its proper transform under the above birational transformation by $\Phi_{\mathbb{CP}^2}(C) \subset \mathbb{CP}^2$. In the following, we say that a curve C satisfies the constraint $\lessdot \mathcal{T}^{(m)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$ if C passes through $\mathfrak{p}$ and has a branch with contact order at least m (i.e. tangency order at least $m-1$) to $\mathcal{B}_-$. Note that a curve satisfying $\lessdot \mathcal{T}^{(m)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$ must have intersection multiplicity at least $m+1$ with $\mathcal{N}$, since it also intersects the branch $\mathcal{B}_+$ at $\mathfrak{p}$.

Theorem 2.1.2 ([Ore02])**.** *Let* $L \subset \mathbb{CP}^2$ *denote the unique line which satisfies* $\lessdot \mathcal{T}^{(2)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$*, and put* $C_{2k+1} := \Phi^k_{\mathbb{CP}^2}(L)$ *for* $k \in \mathbb{Z}_{\geqslant 0}$*. Similarly, let* $Q \subset \mathbb{CP}^2$ *denote the unique (irreducible) conic*[13] *which satisfies* $\lessdot \mathcal{T}^{(5)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$*, and put* $C_{2k+2} := \Phi^k_{\mathbb{CP}^2}(Q)$ *for* $k \in \mathbb{Z}_{\geqslant 0}$*. Then for* $k \in \mathbb{Z}_{\geqslant 1}$*,* C_k *is a* (p_k, q_k)*-unicuspidal rational plane curve of degree* d_k*, where* $(p_k, q_k) = (\mathrm{Fib}_{2k+1}, \mathrm{Fib}_{2k-3})$ *and* $d_k = \frac{1}{3}(p_k + q_k) = \mathrm{Fib}_{2k-1}$*.*

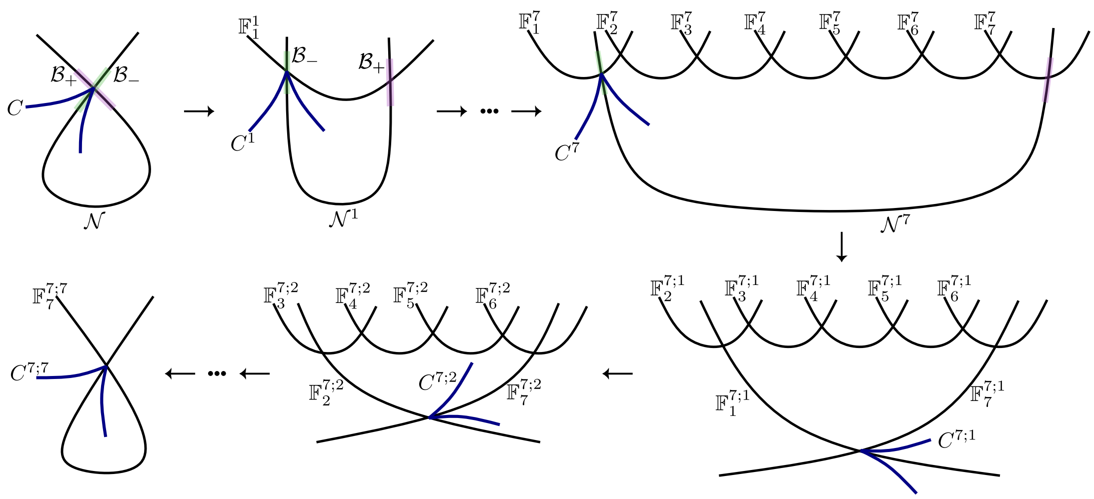
Here Fib_k is the kth Fibonacci number, i.e. $\mathrm{Fib}_1 = \mathrm{Fib}_2 = 1$ and $\mathrm{Fib}_{k+2} = \mathrm{Fib}_k + \mathrm{Fib}_{k+1}$ (it will also be convenient to put $\mathrm{Fib}_{-1} := 1$). Recall that the x-coordinate of the kth outer corner point of the Fibonacci staircase in [McSch12] is precisely the odd index Fibonacci ratio $\mathrm{Fib}_{2k+1}/\mathrm{Fib}_{2k-3}$.

Theorem 2.1.2 follows from the identity $\mathrm{Fib}_{2k+5} = 7\mathrm{Fib}_{2k+1} - \mathrm{Fib}_{2k-3}$, together with the fact that for a curve $C \subset \mathbb{CP}^2$ with a well-placed (p,q) cusp (see Definition 2.2.4 below), its twist $\Phi_{\mathbb{CP}^2}(C)$ has a well-placed $(7p-q, p)$ cusp. Indeed, let us analyze the construction of $\Phi_{\mathbb{CP}^2}(C)$ in more detail as follows (see Figure 2). We assume $p > 2q$, and let $C \subset \mathbb{CP}^2$ be a curve which has a (p,q) cusp maximally tangent to the branch $\mathcal{B}_-$ of the nodal cubic $\mathcal{N}$ at its double point $\mathfrak{p}$. The proper transform $C^1 \subset \mathrm{Bl}^1\mathbb{CP}^2$ of C then has a $(p-q,q)$ cusp maximally tangent to the branch $\mathcal{B}^1_-$ of $\mathcal{N}^1$. After 6 further blowups (which do not affect C^1 since it is disjoint from the blowup points), we arrive at the curve $C^7 \subset \mathrm{Bl}^7\mathbb{CP}^2$, which has a $(p-q,q)$ cusp maximally tangent to the branch $\mathcal{B}^7_-$ of $\mathcal{N}^7$. We then blow down to obtain the curve $C^{7;1} \subset \mathrm{Bl}^{7;1}\mathbb{CP}^2$, which has a $(p, p-q)$ cusp maximally tangent to $\mathbb{F}_1^{7;1}$. In the next blowdown we obtain $C^{7;2} \subset \mathrm{Bl}^{7;2}\mathbb{CP}^2$, which has a $(2p-q, p)$ cusp maximally tangent to $\mathbb{F}_7^{7;2}$. Finally, after 5 further blowdowns we arrive at $\Phi_{\mathbb{CP}^2}(C) = C^{7;7} \subset \mathrm{Bl}^{7;7}\mathbb{CP}^2$, which has a $(7p-q,p)$ cusp maximally tangent to a branch of the nodal curve $\mathbb{F}_7^{7;7}$ at its double point.

Remark 2.1.3. Suppose that C is a degree d plane curve which intersects the nodal cubic $\mathcal{N}$ at some point (necessarily distinct from the double point $\mathfrak{p}$) with contact order $3d$. Then one can check that the twist $\Phi_{\mathbb{CP}^2}(C)$ has a $(21d+1, 3d)$ cusp. In particular, in the case $d = 1$, C is a flex line, and $\Phi_{\mathbb{CP}^2}(C)$ has degree 8 by the adjunction formula, so $\Phi_{\mathbb{CP}^2}(L)$ is a sporadic unicuspidal curve as in Theorem 1.2.2(e). Similarly, in the case

[13] As an alternative to the conic Q we could take the unique line in $\mathbb{CP}^2$ which satisfies $\lessdot \mathcal{T}^{(2)}_{\mathcal{B}_+}\mathfrak{p} \gtrdot$, as this transforms into Q under one application of the Orevkov twist $\Phi_{\mathbb{CP}^2}$ (this is actually the approach taken in [Ore02]).

Figure 2: The Orevkov twist in $\mathbb{CP}^2$. Notice that $C^{7;1}$ is tangent to $F_1^{7;1}$ because its blowup C^7 intersects F_1^7. However, because the curve $C^{7;2}$ intersects $F_1^{7;1}$ at its intersection with $F_7^{7;1}$ rather than $F_2^{7;1}$, when $F_1^{7;1}$ is blown down $C^{7;2}$ is tangent to $F_7^{7;2}$ rather than to $F_2^{7;2}$, and its blowdowns remain tangent to $F_7^{7;k}$ for $k > 2$.



$d = 2$, $\Phi_{\mathbb{CP}^2}(C)$ has a $(43, 6)$ cusp and degree 16, and hence represents Theorem 1.2.2(f). $\Diamond$

2.2 The generalized Orevkov twist

We now generalize the Orevkov twist to any rigid complex del Pezzo surface M, i.e. either $\mathrm{Bl}^k\mathbb{CP}^2$ for $k = 0, 1, 2, 3, 4$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. Note that $\mathrm{PD}(c_1(M)) \in H_2(M)$ is given by $3\ell - e_1 - \cdots - e_k$ if $M = \mathrm{Bl}^k\mathbb{CP}^2$, or by $2\ell_1 + 2\ell_2$ in the case $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ (here we put $\ell_1 := [\mathbb{CP}^1 \times \{pt\}]$ and $\ell_2 := [\{pt\} \times \mathbb{CP}^1]$).

In the following we consider $\mathcal{N} \subset M$ be a rational nodal curve which is anticanonical (i.e. $[\mathcal{N}] = \mathrm{PD}(c_1(M))$) and has a unique double point (note that this is consistent with the adjunction formula). Concretely, in the case $M = \mathrm{Bl}^k\mathbb{CP}^2$ for $k = 0, \ldots, 4$ we can assume up to biholomorphism that M is given by blowing up $\mathbb{CP}^2$ at k points $\mathfrak{n}_1, \ldots, \mathfrak{n}_k$ on the standard nodal cubic $\mathcal{N}_0 := \{y^2z = x^2(x + z)\}$, and we could take $\mathcal{N}$ to be the proper transform of $\mathcal{N}_0$ in M (note that any tuple of 4 points in $\mathbb{CP}^2$, no 3 of which are collinear, is projectively equivalent to any other such tuple). In the case $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, we could take $\mathcal{N}$ to be $\left(\mathbb{CP}^1 \times \{q_1, q_2\}\right) \cup \left(\{p_1, p_2\} \times \mathbb{CP}^1\right)$ for $q_1, q_2, p_1, p_2 \in \mathbb{CP}^1$ with $q_1 \neq q_2$ and $p_1 \neq p_2$, after smoothing the nodes at $(p_1, q_2), (p_2, q_1), (p_2, q_2)$.

The degree of the del Pezzo surface M is by definition $[\mathcal{N}] \cdot [\mathcal{N}]$, and it will also be convenient to put $K := [\mathcal{N}] \cdot [\mathcal{N}] - 2$ (see Table 1).

Construction 2.2.1. *For M a rigid del Pezzo surface as above, the birational transformation $\Phi_M : M \dashrightarrow M$ is defined as follows. Let $\mathcal{N} \subset M$ be a rational nodal anticanonical* *curve, with local branches $\mathcal{B}_+, \mathcal{B}_-$ near the unique double point $\mathfrak{p}$. Let $\mathrm{Bl}^1 M$ denote the blowup of M at the point $\mathfrak{p}$, with resulting exceptional divisor $\mathbb{F}_1^1$. Let $\mathcal{N}^1 \subset \mathrm{Bl}^1 M$ denote the proper transform of $\mathcal{N}$, and let $\mathcal{B}_+^1 \subset \mathcal{N}^1$ denote the proper transform of the local branch $\mathcal{B}_+$. Put $\mathfrak{p}^1 := \mathcal{B}_+^1 \cap \mathbb{F}_1^1 \subset \mathcal{N}^1$.*

Next, let $\mathrm{Bl}^2 M$ denote the blowup of $\mathrm{Bl}^1 M$ at the point $\mathfrak{p}^1$, with resulting exceptional divisor $\mathbb{F}_2^2$, and with $\mathbb{F}_1^2 \subset \mathrm{Bl}^2 M$ the proper transform of $\mathbb{F}_1^1$. Let $\mathcal{N}^2 \subset \mathrm{Bl}^2 M$ denote the proper transform of $\mathcal{N}^1$, and let $\mathcal{B}_+^2 \subset \mathcal{N}^2$ denote the proper transform of the local branch $\mathcal{B}_+^1$. Put $\mathfrak{p}^2 := \mathcal{B}_+^2 \cap \mathbb{F}_2^2 \subset \mathcal{N}^2$.

Continuing in this manner, after a total of K blowups we arrive at $\mathrm{Bl}^K M$, which contains a chain of rational curves $\mathcal{N}^K, \mathbb{F}_1^K, \ldots, \mathbb{F}_K^K$. Since $[\mathcal{N}^K] \cdot [\mathcal{N}^K] = -1$, we can blow down $\mathrm{Bl}^K M$ along $\mathcal{N}^K$ to obtain $\mathrm{Bl}^{K;1} M$. Let $\mathbb{F}_1^{K;1}, \ldots, \mathbb{F}_K^{K;1} \subset \mathrm{Bl}^{K;1} M$ denote the proper transforms of $\mathbb{F}_1^K, \ldots, \mathbb{F}_K^K$ respectively. Then since $[\mathbb{F}_1^{K;1}] \cdot [\mathbb{F}_1^{K;1}] = -1$, we can blow down $\mathrm{Bl}^{K;1} M$ along $\mathbb{F}_1^{K;1}$ to obtain $\mathrm{Bl}^{K;2} M$. Let $\mathbb{F}_2^{K;2}, \ldots, \mathbb{F}_K^{K;2} \subset \mathrm{Bl}^{K;2} M$ denote the proper transforms of $\mathbb{F}_2^{K;1}, \ldots, \mathbb{F}_K^{K;1}$ respectively. Continuing in this manner, after a total of K blowdowns (along $\mathcal{N}^K, \mathbb{F}_1^{K;1}, \ldots, F_{K-1}^{K;K-1}$), we arrive at $\mathrm{Bl}^{K;K} M$, which contains the rational nodal curve $\mathbb{F}_K^{K;K}$ with $[\mathbb{F}_K^{K;K}] \cdot [\mathbb{F}_K^{K;K}] = K + 2$.

Finally, by composing this sequence of K blowups and K blowdowns with a biholomorphism $\Psi : \mathrm{Bl}^{K;K} M \cong M$ (which exists by Lemma 2.2.3 below), we arrive at the birational transformation $\Phi_M := M \dashrightarrow M$, which contains the rational nodal curve $\mathcal{N}' := \Psi(\mathbb{F}_K^{K;K})$.

We will sometimes denote generalized Orevkov twist Φ_M by $\Phi_{M;\mathcal{N}}$ when we wish to emphasize the role of the anticanonical curve $\mathcal{N}$.

We begin with some lemmas to justify Construction 2.2.1.

Lemma 2.2.2. *In the setting of Construction 2.2.1, $\mathcal{N}'$ is also an anticanonical curve in M, i.e. we have $[\mathcal{N}'] = \mathrm{PD}(c_1(M)) \in H_2(M)$.*

Proof. It suffices to check that $\mathbb{F}_K^{K;K}$ is an anticanonical curve in $\mathrm{Bl}^{K;K} M$. To see this, observe that if C is a (reduced but not necessarily irreducible) anticanonical curve in a smooth complex surface X, and if X' denotes the blowup of X at an ordinary double point of C, then the total transform of C in X' is again anticanonical. Conversely, if C is an anticanonical curve in a smooth complex surface Y' with an exceptional component $C_0 \subset C$ such that C_0 intersects $\overline{C \setminus C_0}$ transversely in two points, and if Y denotes the blowdown of Y' along C_0, then the image of C under the blowdown map $Y' \to Y$ is again anticanonical. Since $\mathbb{F}_K^{K;K} \subset \mathrm{Bl}^{K;K} M$ is obtained from the anticanonical curve $\mathcal{N} \subset M$ by a sequence of blowups and blowdowns of these forms, it follows that $\mathbb{F}_K^{K;K}$ is again anticanonical. □

Lemma 2.2.3. *In the setting of Construction 2.2.1, there is a biholomorphism $\mathrm{Bl}^{K;K} M \cong M$.*

Proof. We first claim that $\mathrm{Bl}^{K;K} M$ is Fano. By Lemma 2.2.2, $\mathbb{F}_K^{K;K} \subset \mathrm{Bl}^{K;K} M$ is anticanonical, so it suffices to check that this is ample, and this follows easily by the

Nakai–Moishezon criterion, since any curve in $\mathrm{Bl}^{K;K}M$ disjoint from $\mathbb{F}_K^{K;K}$ would imply a curve in M disjoint from $\mathcal{N}$.

Since $\mathrm{Bl}^{K;K}M$ has the same integral homology as M, by the classification of del Pezzo surfaces the only possible ambiguity lies in distinguishing $\mathbb{CP}^1 \times \mathbb{CP}^1$ from the first Hirzebruch surface $\mathrm{Bl}^1\mathbb{CP}^2$. In the case $M = \mathrm{Bl}^1\mathbb{CP}^2$, let $\mathbb{E} \subset M$ be the exceptional curve, and let $\mathbb{E}^{6;6}$ be its proper transform in $\mathrm{Bl}^{6;6}M$. One can check (e.g. using a similar analysis to the proof of Lemma 2.2.2) that we have $c_1([\mathbb{E}^{6;6}]) = 7$, and hence $\mathrm{Bl}^{6;6}M \cong \mathrm{Bl}^1\mathbb{CP}^2$, as all homology classes in $\mathbb{CP}^1 \times \mathbb{CP}^1$ have even Chern number.

A similar but slightly more complicated argument shows directly that if $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ then we again must have $\mathrm{Bl}^{6;6}M \cong M$. However, one can also argue using symmetry: the inverse of the Orevkov twist is given by exactly the same number of blowups and blowdowns but with the two branches of $\mathcal{N}$ reversed. The above argument concerning $\mathrm{Bl}^1\mathbb{CP}^2$ shows that any twist on $\mathrm{Bl}^1\mathbb{CP}^2$ gives $\mathrm{Bl}^1\mathbb{CP}^2$. Therefore, if the twist done to $\mathbb{CP}^1 \times \mathbb{CP}^1$ did give $\mathrm{Bl}^1\mathbb{CP}^2$, then the inverse operation done on the other branch would have also to give $\mathrm{Bl}^1\mathbb{CP}^2$, which is impossible. □

The following language will be useful for understanding how cusps transform under successive applications of the generalized Orevkov twist Φ_M.

Definition 2.2.4. *A curve C in M is* ***(p, q)-well-placed*** *with respect to $\mathcal{N}$ if we have $C \cap \mathcal{N} = \{\mathfrak{p}\}$, C is locally irreducible near $\mathfrak{p}$, and we have $(C \cdot \mathcal{B}_-)_{\mathfrak{p}} = p$ and $(C \cdot \mathcal{B}_+)_{\mathfrak{p}} = q$.*

Here $\mathcal{N}$ is any rational nodal anticanonical curve as in Construction 2.2.1, and we will sometimes simply say that C is "well-placed" if $\mathcal{N}$ is clear from the context. Note that this implies that C has a (p, q) cusp at $\mathfrak{p}$ which is maximally tangent to the branch $\mathcal{B}_-$ (in the sense of [McS23, §3.5]). We also allow the case $q = 1$, i.e. C is $(p, 1)$-well-placed if it has a single branch passing through $\mathfrak{p}$ which is smooth and strictly satisfies the tangency condition $\lessdot\mathcal{T}_{\mathcal{B}_-}^{(p)}\mathfrak{p}\gtrdot$, and C is otherwise disjoint from $\mathcal{N}$. Note that the line L (resp. conic Q) in Theorem 2.1.2 is $(2, 1)$-well-placed (resp. $(5, 1)$-well-placed) with respect to $\mathcal{N}$.

Remark 2.2.5. For any (rational) curve $C \subset M$ which is (p, q)-well-placed with respect to $\mathcal{N}$ we have $c_1([C]) = [C] \cdot [\mathcal{N}] = p + q$, i.e. C must have index zero. Note also that the singularities of $C \setminus \mathfrak{p}$ are in bijective correspondence with the singularities of $\Phi_M(C) \setminus \mathfrak{p}$. In particular, if C is unicuspidal (with $p, q > 1$) then so is $\Phi_M(C)$. ◊

The singularity analysis given at the end of §2.1 (and depicted in Figure 2) immediately extends to the generalized Orevkov twist as follows.

Proposition 2.2.6. *If a curve $C \subset M$ is (p, q)-well-placed with respect to $\mathcal{N}$, then $\Phi_M(C)$ is $(Kp - q, p)$-well-placed with respect to $\mathcal{N}'$.*

Crucially, since $\mathcal{N}'$ is itself a rational nodal anticanonical curve by Lemma 2.2.2, we can subsequently apply the generalized Orevkov twist using $\mathcal{N}'$ to obtain a curve $\Phi_{M;\mathcal{N}'}(\Phi_{M;\mathcal{N}}(C))$ which is $(K[Kp - q] - p, Kp - q)$-well-placed with respect to $\mathcal{N}''$, and so on.

Remark 2.2.7. In the case of $\Phi_{\mathbb{CP}^2}$ there is a biholomorphism taking $\mathcal{N}'$ to $\mathcal{N}$, but this is not a priori clear (or needed) in the general case. $\Diamond$

2.3 Staircase numerics and seed curves

Before completing the proof of Theorem B, we briefly recall some numerical aspects of the rational infinite staircases. Our discussion is largely informed by [Cri+20; CV22], and we refer the reader to these references for more details.

Recall that associated to each rigid del Pezzo surface M is a sequence of nonnegative integers $g_1, g_2, g_3, \ldots$ which determines the locations of the outer and inner corners of the corresponding infinite staircase. These sequences are explicated in [Cri+20, Table 1.18], which is reproduced in Table 1. Here J denotes the number of "strands", $K+2$ is the degree of the corresponding del Pezzo surface, and a_{acc} is the accumulation point, i.e. the limiting x-value. More explicitly, the sequence $g_1, g_2, g_3, \ldots$ determines the locations of the outer and inner corner points in the graph of $c_M(x)$ as follows:

- kth outer corner: x-coordinate $\frac{g_{k+J}}{g_k}$, y-coordinate $\frac{g_{k+J}}{g_k+g_{k+J}}$
- kth inner corner: x-coordinate $\frac{g_{k+J}(g_{k+1}+g_{k+1+J})}{g_{k+1}(g_k+g_{k+J})}$, y-coordinate $\frac{g_{k+J}}{g_k+g_{k+J}}$.

In particular, if p/q is the x-coordinate of an outer corner, then $(Kp-q)/p$ is the x-coordinate of the outer corner J steps away. This means that the full set of outer corners is obtained by iteratively applying the recursion $p/q \mapsto (Kp-q)/p$ to the seeds $\frac{g_{1+J}}{g_1}, \ldots, \frac{g_{2J}}{g_J}$. Note that the generalized Orevkov twist achieves precisely the recursion $(p,q) \mapsto (Kp-q, p)$ by Proposition 2.2.6.

Example 2.3.1. In the case $M = \mathbb{CP}^2$, the sequence $g_1, g_2, g_3, \ldots$ corresponds to the odd index Fibonacci numbers. In the case $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, the even index entries of $g_1, g_2, g_3, \ldots$ correspond to the odd index Pell numbers, while the odd index entries of $g_1, g_2, g_3, \ldots$ correspond to the even index half-companion Pell numbers. $\Diamond$

Remark 2.3.2. Note that if M is endowed with its monotone symplectic form ω_M, and if C is a (p,q)-sesquicuspidal rational symplectic curve in M of index zero, then by Theorem 1.3.1 we have $c_M(p/q) \geqslant \frac{p}{[\omega_M]\cdot[C]} = \frac{p}{c_1([C])} = \frac{p}{p+q}$. Meanwhile, the outer corners described above are all of the form $(x,y) = (\frac{p}{q}, \frac{p}{p+q})$ for $p,q \in \mathbb{Z}_{\geqslant 1}$. $\Diamond$

By the discussion in the previous subsection, in order to complete the proof of Theorem B it remains to construct seed curves. Namely, for M a rigid del Pezzo surface with corresponding integer sequence $g_0, g_1, g_2, \ldots$, we must construct a well-placed (g_{k+J}, g_k)-unicuspidal rational algebraic curve in M for $k = 0, \ldots, J-1$. More explicitly, inspecting Table 1, it suffices to find a well-placed (p,q)-unicuspidal rational algebraic curve C with (p,q) ranging as follows:

- $M = \mathbb{CP}^2$: $(p,q) = (1,2), (2,1)$
- $M = \mathbb{CP}^1 \times \mathbb{CP}^1$: $(p,q) = (1,1), (3,1)$
- $M = \mathrm{Bl}^1\mathbb{CP}^2$: $(p,q) = (1,1), (2,1), (4,1)$

Table 1: The sequences controlling the rational infinite staircases, reproduced from [Cri+20, Table 1.18].

rigid del Pezzo surface	negative weight expansion	K	J	recursion $g_{k+2J} = Kg_{k+J} - g_k$	seeds $g_0, \dots, g_{2J-1}$	acc. pt. a_{acc}
$\mathbb{CP}^2(3)$	(3)	7	2	$g_{k+4} = 7g_{k+2} - g_k$	$2, 1, 1, 2$	$\frac{7+3\sqrt{5}}{2}$
$\mathbb{CP}^1(2) \times \mathbb{CP}^1(2)$	$(4; 2, 2)$	6	2	$g_{k+4} = 6g_{k+2} - g_k$	$1, 1, 1, 3$	$3 + 2\sqrt{2}$
$\mathbb{CP}^2(3)\#\overline{\mathbb{CP}}^2(1)$	$(3; 1)$	6	3	$g_{k+6} = 6g_{k+3} - g_k$	$1, 1, 1, 1, 2, 4$	$3 + 2\sqrt{2}$
$\mathbb{CP}^2(3)\#^{\times 2}\overline{\mathbb{CP}}^2(1)$	$(3; 1, 1)$	5	3	$g_{k+6} = 5g_{k+3} - g_k$	$1, 1, 1, 1, 2, 3$	$\frac{5+\sqrt{21}}{2}$
$\mathbb{CP}^2(3)\#^{\times 3}\overline{\mathbb{CP}}^2(1)$	$(3; 1, 1, 1)$	4	2	$g_{k+4} = 4g_{k+2} - g_k$	$1, 1, 1, 2$	$2 + \sqrt{3}$
$\mathbb{CP}^2(3)\#^{\times 4}\overline{\mathbb{CP}}^2(1)$	$(3; 1, 1, 1, 1)$	3	2	$g_{k+4} = 3g_{k+2} - g_k$	$1, 2, 1, 3$	$\frac{3+\sqrt{5}}{2}$

- $M = \mathrm{Bl}^2\mathbb{CP}^2$: $(p, q) = (1, 1), (2, 1), (3, 1)$
- $M = \mathrm{Bl}^3\mathbb{CP}^2$: $(p, q) = (1, 1), (2, 1)$
- $M = \mathrm{Bl}^4\mathbb{CP}^2$: $(p, q) = (1, 1), (3, 2)$.

As above, for $k = 1, 2, 3, 4$ we take $\mathrm{Bl}^k\mathbb{CP}^2$ to be the blowup of $\mathbb{CP}^2$ at k points $\mathfrak{n}_1, \dots, \mathfrak{n}_k$ on the standard nodal cubic $\mathcal{N}_0 = \{Y^2Z = X^2(X + Z)\}$, and we take $\mathcal{N}$ to be the proper transform of $\mathcal{N}$. Meanwhile in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ we take $\mathcal{N}$ to be the smoothing of $(\mathbb{CP}^1 \times \{q_1, q_2\}) \cup (\{p_1, p_2\} \times \mathbb{CP}^1)$ at the nodes $(p_1, q_2), (p_2, q_1), (p_2, q_2)$.

<u>Case $M = \mathbb{CP}^2$</u>: For $(p, q) = (1, 2)$ and $(p, q) = (5, 1)$ (i.e. $(7 \cdot 1 - 2, 1)$) we take the line L and conic Q respectively mentioned in Theorem 2.1.2.

<u>Case $M = \mathrm{Bl}^1\mathbb{CP}^2$</u>:

- For $(p, q) = (1, 1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which passes through $\mathfrak{n}_1$ and the double point $\mathfrak{p}$ of $\mathcal{N}_0$. Note that we have $[C] = \ell - e \in H_2(\mathrm{Bl}^1\mathbb{CP}^2)$.
- For $(p, q) = (2, 1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which is tangent to $\mathcal{B}_-$ at $\mathfrak{p}$. Note that C is necessarily disjoint from $\mathfrak{n}_1$ so we have $[C] = \ell \in H_2(\mathrm{Bl}^1\mathbb{CP}^2)$.
- For $(p, q) = (4, 1)$, we take C to be the unique conic in $\mathbb{CP}^2$ which satisfies $\lessdot \mathcal{T}^{(4)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$ and also passes through $\mathfrak{n}_1$ (this is easily constructed using a linear system, or by a deformation argument similar to the ones given below).

<u>Case $M = \mathbb{CP}^1 \times \mathbb{CP}^1$</u>:

- For $(p, q) = (1, 1)$, we take C to be the unique line in class ℓ_1 (or alternatively ℓ_2) which passes through the double point $\mathfrak{p}$ of $\mathcal{N}$.
- For $(p, q) = (3, 1)$, we take C to be the unique rational curve of bidegree $(1, 1)$ which has contact order 3 to a branch of $\mathcal{N}$ at $\mathfrak{p}$. To construct such a curve, we can start with a bidegree $(1, 1)$ curve D passing through $\mathfrak{p}$ and two other nearby points $x_1, x_2 \in \mathcal{B}_-$ (e.g. D can be realized as the graph of a holomorphic map

$\mathbb{CP}^1 \to \mathbb{CP}^1$). As we move the points x_1, x_2 into $\mathfrak{p}$ along $\mathcal{B}_-$, D correspondingly deforms into a curve of the desired kind.

Case $M = \mathrm{Bl}^2\mathbb{CP}^2$:

- For $(p,q) = (1,1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which passes through $\mathfrak{p}$ and $\mathfrak{n}_1$ (or alternatively $\mathfrak{n}_2$)
- For $(p,q) = (2,1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which is tangent to $\mathcal{B}_-$ at $\mathfrak{p}$.
- For $(p,q) = (3,1)$, we take C to be the proper transform of the unique conic in $\mathbb{CP}^2$ which satisfies $\lessdot \mathcal{T}^{(3)}_{\mathcal{B}_-}\mathfrak{p} \gtrdot$ and passes through $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

Case $M = \mathrm{Bl}^3\mathbb{CP}^2$:

- For $(p,q) = (1,1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which passes through $\mathfrak{p}$ and $\mathfrak{n}_1$.
- For $(p,q) = (2,1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which is tangent to $\mathcal{B}_-$ at $\mathfrak{p}$.

Case $M = \mathrm{Bl}^4\mathbb{CP}^2$:

- For $(p,q) = (1,1)$, we take C to be the proper transform of the unique line in $\mathbb{CP}^2$ which passes through $\mathfrak{p}$ and $\mathfrak{n}_1$.
- For $(p,q) = (3,2)$, we take C to be the proper transform of a rational cubic in $\mathbb{CP}^2$ which has a $(3,2)$ cusp with contact order 3 to $\mathcal{B}_-$ at $\mathfrak{p}$ and which passes through points $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4$ on the standard nodal cubic $\mathcal{N}_0$, as guaranteed by Lemma 2.3.3 below.

We end this subsection by constructing the above $(3,2)$ seed curve, which then completes the proof of Theorem B. Similar to [McS23], we will denote by $\lessdot \mathcal{C}^{(p,q)}\mathfrak{p} \gtrdot$ the constraint that a curve has a holomorphic parametrization $u : \mathbb{CP}^1 \to M$ such that $u([0:0:1]) = \mathfrak{p}$, and u has contact order at least p with $\mathcal{B}_-$ at $[0:0:1]$ and contact order at least q with $\mathcal{B}_+$ at $[0:0:1]$.

Lemma 2.3.3. *There exists a rational cubic algebraic curve in $\mathbb{CP}^2$ which satisfies the cuspidal constraint $\lessdot \mathcal{C}^{(3,2)}\mathfrak{p} \gtrdot$ as well as the point constraints $\lessdot \mathfrak{n}_1, \ldots, \mathfrak{n}_4 \gtrdot$.*

Proof. Let D be a cuspidal cubic in $\mathbb{CP}^2$ whose cusp has contact order 3 with $\mathcal{B}_-$ at $\mathbf{p}$ (this is a codimension 4 constraint), and let $x_1, \ldots, x_4$ be distinct points in $D \smallsetminus \{\mathfrak{p}\}$. Let $x_1^t, \ldots, x_4^t$, $t \in [0,1]$, be an isotopy in $\mathbb{CP}^2 \smallsetminus \{\mathfrak{p}\}$ such that $x_1^1, \ldots, x_4^1 \in \mathcal{N}_0$ and $x_1^t, \ldots, x_4^t, \mathfrak{p}$ are in general position for each $t \in [0,1]$. We consider the parametrized moduli space

$$\{(t,u) \mid t \in [0,1], u \in \mathcal{M}_{\mathbb{CP}^2, 3\ell} \lessdot \mathcal{C}^{(3,2)}\mathfrak{p}, x_1^t, \ldots, x_4^t \gtrdot\},$$

where, for $t \in [0,1]$, $\mathcal{M}_{\mathbb{CP}^2,3\ell}\lessdot \mathcal{C}^{(3,2)}\mathfrak{p}, x_1^t, \ldots, x_4^t \gtrdot$ denotes the moduli space of holomorphic maps[14] $u : \mathbb{CP}^1 \to \mathbb{CP}^2$ (modulo biholomorphic reparametrization) which satisfy

[14] Here we formulate the argument in terms of holomorphic maps in order to match the setup of [McS23], but one could also formulate the argument using only the language of subvarieties.

the cuspidal constraint $\langle\!\langle \mathcal{C}^{(3,2)}\mathfrak{p}\rangle\!\rangle$ and the point constraints $\langle\!\langle x_1^t,\dots,x_4^t\rangle\!\rangle$. We seek to show that $\mathcal{M}_{\mathbb{CP}^2,3\ell}\langle\!\langle \mathcal{C}^{(3,2)}\mathfrak{p},\mathfrak{n}_1,\dots,\mathfrak{n}_4\rangle\!\rangle$ is nonempty, and for this it suffices to show that a sequence of curves in the parametrized moduli space cannot degenerate into a configuration D_δ having more than one component.

Assume by contradiction that such a D_δ exists, say at $t = t_\delta \in [0,1]$, and suppose first that some nonconstant component D_δ^1 of D_δ carries the cuspidal constraint $\langle\!\langle \mathcal{C}^{(3,2)}\mathfrak{p}\rangle\!\rangle$. Then D_δ^1 must be a conic (necessarily a double cover of a line), and the remaining nonconstant component D_δ^2 of D_δ must be a line. Since $D_\delta^1 \cdot \mathcal{N}_0 = 6$, D_δ^1 carries at most 1 of the point constraints $\langle\!\langle x_1^{t_\delta},\dots,x_4^{t_\delta}\rangle\!\rangle$, since the cuspidal constraint contributes a local intersection multiplicity of at least 5. Then the line component D_δ^2 must carry at least three of the point constraints, which is a contradiction since we assumed they are in general position.

Now suppose that the cuspidal constraint $\langle\!\langle \mathcal{C}^{(3,2)}\mathfrak{p}\rangle\!\rangle$ is carried by a ghost component, so that the constraint itself decomposes into several constraints $\langle\!\langle \mathcal{C}^{(p_1,q_1)}\mathfrak{p}\rangle\!\rangle,\dots,\langle\!\langle \mathcal{C}^{(p_k,q_k)}\mathfrak{p}\rangle\!\rangle$ carried by nearby nonconstant components of D_δ. By [McS23, Prop. 3.2.4] (or [McS23, Ex. 3.2.7]), we must have $\sum_{i=1}^k (p_i+q_i) \geqslant 3+2+1 = 6$. Taking into account the point constraints $\langle\!\langle x_1^{t_\delta},\dots,x_4^{t_\delta}\rangle\!\rangle$, this gives $D_\delta \cdot \mathcal{N} \geqslant 6+4 > 9$, which is a contradiction since D_δ is degree 3. □

3 Inflating along sesquicuspidal curves

The main goal of this section is to prove Theorem A, using the following basic outline:

1) construct a (partial) resolution $\widetilde{C}$ of C in a suitable iterated blowup $\widetilde{M}$ of M

2) apply the technique of symplectic inflation to $\widetilde{C}$ to modify the symplectic form on $\widetilde{M}$

3) blow down again to obtain a symplectic manifold M' which is symplectomorphic to M and by construction contains a large symplectic ellipsoid.

The main technicality is that we need to perform the blowdowns in families in the symplectic category, where blowups and proper transforms are more delicate than in the complex category. Indeed, recall that whereas complex blowups are performed at a point, symplectic blowups require the data of a symplectic ball embedding $\iota : B^{2n}(R) \overset{s}{\hookrightarrow} M$ for some $R \in \mathbb{R}_{>0}$.[15] The symplectic blowup $\mathrm{Bl}_\iota M$ is then defined roughly by removing the interior of $\iota(B^{2n}(R))$ and collapsing the boundary along the fibers of the Hopf fibration. Some precise relations between complex and symplectic blowups are detailed in [McSa17, §7.1].

In §3.1, we first discuss a model for the resolution $\widetilde{M} \to M$ in the case of a (p,q) cusp singularity using toric moment maps, and we use this to prove Theorem A(i) in §3.2. In §3.3 we extend the discussion to cusps with multiple Puiseux pairs, and finally we prove

[15] Strictly speaking the construction requires choosing an extension of this embedding to $B^{2n}(R+\varepsilon)$ for some $\varepsilon > 0$, but we will suppress this from the discussion.

Theorem A(ii) in §3.3. Along the way we also discuss some generalities on resolutions of cusp singularities which will be needed elsewhere in the paper.

3.1 Toric resolution of a (p,q) cusp

Recall that any cusp singularity of a holomorphic curve $C \subset \mathbb{C}^2$ can be resolved by a finite sequence of point blowups (see e.g. [Wal04, §3.3]). We will denote the exceptional divisor resulting from the ith blowup by $\mathbb{F}_i^i$, and for $j > i$ we denote its proper transform in the jth blowup $\mathrm{Bl}^j\mathbb{C}^2$ by $\mathbb{F}_i^j$. We arrive at the minimal resolution C^K after some number $K \in \mathbb{Z}_{\geqslant 1}$ of blowups, and after $L-K$ further blowups for some $L \in \mathbb{Z}_{\geqslant 1}$ we arrive at the normal crossing resolution C^L, in which the total transform $C^L \cup \mathbb{F}_1^L \cup \cdots \cup \mathbb{F}_L^L \subset \mathrm{Bl}^L\mathbb{C}^2$ of C is a normal crossing divisor. We have $[\mathbb{F}_L^L]\cdot[\mathbb{F}_L^L] = -1$ and $[\mathbb{F}_i^L]\cdot[\mathbb{F}_i^L] \leqslant -2$ for $i = 1,\dots,L-1$, and the spheres $\mathbb{F}_1^L,\dots,\mathbb{F}_{L-1}^L$ are disjoint from C^L, while $\mathbb{F}_L^L$ intersects C^L transversely in one point. In the case of a (p,q) cusp, the combinatorics of the normal crossing resolution are related to the continued fraction expansion of p/q and are neatly encoded in the so-called box diagram for (p,q) (see [McS23, §4.1]).

Let $\mu_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}$, $\mu(z_1,z_2) = (\pi|z_1|^2, \pi|z_2|^2)$, denote the moment map for the standard torus action on $\mathbb{C}^2$. Given $p > q$ coprime positive integers, let $\Delta_{(q,p)} \subset \mathbb{R}^2_{\geqslant 0}$ denote the triangle with vertices $(0,0),(q,0),(0,p)$, let $\Omega_{(q,p)} \subset \mathbb{R}^2_{\geqslant 0}$ denote the closure of its complement, and let $X_{(q,p)}$ denote the corresponding toric symplectic orbifold with moment map $\mu_{X_{(q,p)}} : X_{(q,p)} \to \Omega_{(q,p)}$ (this can be viewed as a weighted blowup of $\mathbb{C}^2$). Note that $X_{(q,p)}$ has two cyclic quotient singularities of types $\frac{1}{p}(1,p-q)$ and $\frac{1}{q}(1,q-r)$, where r is the remainder when p is divided by q,[16] and these can each be resolved by finitely many toric blowups (c.f. [McS23, Rmk. 4.3.4]). On the level of moment polygons, a toric blowup at a corner adjacent to edges having primitive inward normals $(1,0),(a,b) \in \mathbb{Z}^2$ with $a < b$ amounts to chopping off the corner so as to introduce a new edge with inward normal $(1,1)$ (the general case reduces to this one by an integral affine transformation). We denote by $\widetilde{\Omega}_{(q,p)} \subset \mathbb{R}^2_{\geqslant 0}$ any (noncompact) polygon obtained from $\Omega_{(q,p)}$ after resolving both of the singularities by successive toric blowups. The corresponding (noncompact) toric symplectic manifold $\mu_{\widetilde{X}_{(q,p)}} : \widetilde{X}_{(q,p)} \to \widetilde{\Omega}_{(q,p)}$ is also obtained from $\mathbb{C}^2$ by a sequence of L toric blowups.

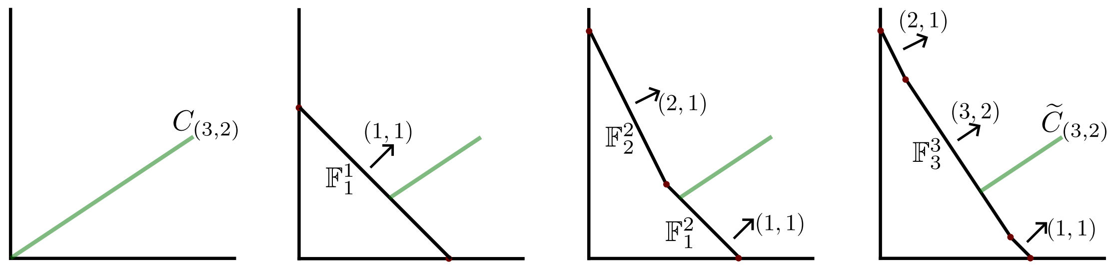
Example 3.1.1. Figure 3 illustrates the construction of $\widetilde{X}_{(2,3)}$ from $\mathbb{C}^2$ by 3 toric blowups, with corresponding inward normal vectors $(1,1),(2,1),(3,2)$. $\Diamond$

Let $\widetilde{\mathbf{D}}_{(q,p)} := \mathbb{F}_1^L \cup \cdots \cup \mathbb{F}_L^L$ denote the compact components of the toric boundary divisor in $\widetilde{X}_{(q,p)}$. Note that, for $1 \leqslant i < j \leqslant L$, $\mathbb{F}_i^L$ and $\mathbb{F}_j^L$ are either disjoint or intersect symplectically orthogonally in one point. For $i = 1,\dots,L$, the symplectic area of $\mathbb{F}_i^L$ is given by the affine length of the corresponding edge $\mu_{\widetilde{X}_{(q,p)}}(\mathbb{F}_i^L) \subset \partial\widetilde{\Omega}_{(q,p)}$, and evidently in the construction of $\widetilde{X}_{(q,p)}$ we can independently choose arbitrary values $\lambda_1,\dots,\lambda_L \in \mathbb{R}_{>0}$ for these affine lengths.

Given a neighborhood U of $\mu_{\widetilde{X}_{(q,p)}}(\widetilde{\mathbf{D}}_{(q,p)})$ in $\widetilde{\Omega}_{(q,p)}$, there is a corresponding neighbor-

[16] For more information, see §4.1a.

Figure 3: The sequence of toric blowups starting at $\mathbb{C}^2$ and ending at $\widetilde{X}_{(2,3)}$. The green lines represent the visible $(3,2)$-cuspidal curve $C_{(3,2)} \subset \mathbb{C}^2$ and its resolution $\widetilde{C}_{(3,2)} \subset \widetilde{X}_{(2,3)}$.



hood $V = U \cup \widetilde{\Delta}_{(q,p)}$ of the origin in $\mathbb{R}^2_{\geqslant 0}$, where $\widetilde{\Delta}_{(q,p)}$ denotes the closure of $\mathbb{R}^2_{\geqslant 0} \setminus \widetilde{\Omega}_{(q,p)}$, so that $\partial U \cap \mathbb{R}^2_{>0} = \partial V \cap \mathbb{R}^2_{>0}$. Then the corresponding toric domains $X_V := \mu^{-1}_{\mathbb{C}^2}(V)$ and $\widetilde{X}_U := \mu^{-1}_{\widetilde{X}_{(q,p)}}(U)$ coincide away from compact subsets. Our model for the symplectic resolution $\widetilde{M} \to M$ of a (p,q) cusp singularity will roughly amount to excising X_V and gluing in $\widetilde{X}_U$. Note that the ellipsoid $E(q,p)$ naturally sits in X_V.

Now suppose that $C_1, \ldots, C_L$ is any configuration of symplectically embedded two-spheres in a symplectic four-manifold W which have the same respective areas as $\mathbb{F}_1, \ldots, \mathbb{F}_L$ and the same intersection graph with symplectically orthogonal intersections. Then by a version of the symplectic neighborhood theorem (see e.g. [Sym98, Prop. 3.5]), there is a neighborhood $\mathcal{W}$ of $C_1 \cup \cdots \cup C_L$ in W which is symplectomorphic to a neighborhood of $\mathbb{F}_1 \cup \cdots \cup \mathbb{F}_L$ in $\widetilde{X}_{(q,p)}$ of the form $\widetilde{X}_U$ for some $U \subset \widetilde{\Omega}_{(q,p)}$ containing $\mu_{\widetilde{X}_{(q,p)}}(\widetilde{\mathbf{D}}_{(q,p)})$. This means that there is a symplectic surgery of W which excises $\mathcal{W}$ and glues in X_V, with $V = U \cup \widetilde{\Delta}_{(q,p)}$ as above. This gives an explicit model for the symplectic blowdown of W along $C_L, \ldots, C_1$, which by construction contains the ellipsoid $E(q,p)$.

Observe that is a (p,q)-unicuspidal symplectic curve $C_{(p,q)}$ in $\mathbb{C}^2$ whose image under $\mu_{\mathbb{C}^2}$ is the ray $\mathbb{R}_{\geqslant 0} \cdot (p,q)$, given explicitly by

$$C_{(p,q)} = \{(r\sqrt{p}e^{2\pi ipt}, r\sqrt{q}e^{2\pi iqt}) \in \mathbb{C}^2 \mid r \in \mathbb{R}_{\geqslant 0}, t \in [0,1]\} \tag{3.1.1}$$

(this is "visible" in the sense of §5.1a below and can be viewed as the hyperKähler twist of the Schoen–Wolfson Lagrangian discussed in [Eva23, Ex. 5.11]). Although $C_{(p,q)}$ is not equal on the nose to the model (p,q) cusp $\{x^p + y^q = 0\}$, these two curves have the same links as transverse torus knots, namely the (p,q) torus knot of maximal self-linking number, and hence they are essentially interchangeable for the purposes of this section. Similarly, there is a nonsingular visible symplectic curve $\widetilde{C}_{(p,q)}$ in $\widetilde{X}_{(q,p)}$ whose image under $\mu_{\widetilde{X}_{(q,p)}}$ is the intersection of the ray $\mathbb{R}_{\geqslant 0} \cdot (p,q)$ with $\widetilde{\Omega}_{(q,p)}$. In §3.2 we will take $\widetilde{C}_{(p,q)}$ as a model for the symplectic proper transform of $C_{(p,q)}$ in $\widetilde{X}_{(q,p)}$, noting that $\widetilde{C}_{(p,q)}$ intersects $\mathbb{F}^L_L$ positively in one point and is disjoint from $\mathbb{F}^L_i$ for $i = 1, \ldots, L-1$.

The moment map images of $C_{(p,q)}$ and $\widetilde{C}_{(p,q)}$ are illustrated in Figure 3 for the case $(p,q) = (3,2)$

3.2 Inflating along a curve with a (p,q) cusp

We first prove part (i) of Theorem A.

Proof of Theorem A(i). Let C be a (p,q)-sesquicuspidal symplectic curve in M which satisfies $[C] = c\mathrm{PD}[\omega_M]$ and $[C] \cdot [C] \geqslant pq$. After resolving any double points, we will assume that C is nonsingular away from the (p,q) cusp (but possibly of higher genus). After further modifying C near the cusp point, we can further assume that

- there is a neighborhood $\mathcal{D} \subset M$ of the cusp which is symplectomorphic to $\varepsilon \cdot X_V$, where $X_V = \mu_{\mathbb{C}^2}^{-1}(V) \subset \mathbb{C}^2$ as in §3.1 with $\widetilde{\Delta}_{(q,p)} \subset V \subset \mathbb{R}^2_{\geqslant 0}$, and $\varepsilon \cdot X_V$ is the result after scaling the symplectic form by some $\varepsilon > 0$ sufficiently small
- $C \cap \mathcal{D}$ is sent to $C_{(p,q)} \cap X_V$, with $C_{(p,q)}$ the visible symplectic curve defined in (3.1.1).

Let $U := V \setminus \operatorname{Int} \widetilde{\Delta}_{(q,p)}$ denote the corresponding neighborhood of the finite edges in $\widetilde{\Omega}_{(q,p)}$, with associated domain $\widetilde{X}_U = \mu^{-1}_{\widetilde{X}_{(q,p)}}(U) \subset \widetilde{X}_{(q,p)}$. Let $(\widetilde{M}, \omega_{\widetilde{M}})$ denote the result after excising $\mathcal{D}$ from M and gluing in $\varepsilon \cdot \widetilde{X}_U$ under the natural symplectic identification $\mathcal{O}p(\partial\mathcal{D}) \cong \mathcal{O}p(\partial(\varepsilon \cdot \widetilde{X}_U))$. Let $\widetilde{C} \subset \widetilde{M}$ be the unique symplectic curve which agrees with C outside of $\mathcal{D}$ and agrees with $\widetilde{C}_{(p,q)}$ in $\varepsilon \cdot \widetilde{X}_U$. In other words, $\widetilde{M}$ is a model for the L-fold symplectic blowup of M, and $\widetilde{C}$ is a model for the symplectic resolution of C at its cusp point.

We now symplectically inflate along $\widetilde{C}$ as follows. Note that $\widetilde{C}$ is smoothly embedded, and by assumption we have $[\widetilde{C}] \cdot [\widetilde{C}] = [C] \cdot [C] - pq \geqslant 0$. Therefore, using e.g. [McD94, Lem. 3.7], there exists a closed two-form η on $\widetilde{M}$ such that:

- $[\eta] = \mathrm{PD}([\widetilde{C}]) \in H^2(\widetilde{M};\mathbb{R})$
- η has support in a small neighborhood of $\widetilde{C}$ which is disjoint from $\mathbb{F}_1^L, \dots, \mathbb{F}_{L-1}^L$
- $\widetilde{\omega}_s := \omega_{\widetilde{M}} + s\eta$ is a symplectic form for all $s \in \mathbb{R}_{>0}$
- $\mathbb{F}_L^L$ is a symplectic submanifold of $(\widetilde{M}, \widetilde{\omega}_s)$ for all $s \in \mathbb{R}_{>0}$.

Note that $(\widetilde{M}, \widetilde{\omega}_s)$ contains the configuration of symplectic spheres $\mathbb{F}_1^L, \dots, \mathbb{F}_L^L$ which still intersect symplectically orthogonally with the same intersection pattern for all $s \in \mathbb{R}_{\geqslant 0}$, and we have

$$\int_{\mathbb{F}_i^L} \widetilde{\omega}_s = \begin{cases} \int_{\mathbb{F}_i^L} \omega_{\widetilde{M}} & i = 1, \dots, L-1 \\ \int_{\mathbb{F}_L^L} \omega_{\widetilde{M}} + s & i = L. \end{cases}$$

Now let (M_s, ω_s) denote the result after performing the toric model for the symplectic blowdown along $\mathbb{F}_L^L, \dots, \mathbb{F}_1^L$ as described in §3.1. By choosing the relevant symplectic neighborhoods smoothly with s and identifying M_s smoothly with M, we view $\{\omega_s\}_{s\geqslant 0}$ as a smooth family of symplectic forms on M, such that $[\omega_s] = [\omega_M] + s\mathrm{PD}([C]) = (1+sc)[\omega_M]$, and such that there is a symplectic embedding of $(\varepsilon + s)\cdot E(q,p)$ into (M,ω_s). By the Moser's stability theorem, the rescaled symplectic form $\frac{1}{1+sc}\cdot\omega_s$ is symplectomorphic to ω_M, and it admits a symplectic embedding of $\frac{\varepsilon+s}{1+sc}\cdot E(q,p)$. Since $\frac{\varepsilon+s}{1+sc}\to\frac{1}{c}$ as $s\to\infty$, the result now follows by taking s sufficiently large. □

3.3 Cusps with multiple Puiseux pairs

In this subsection, we first recall some more generalities about cusp singularities and their resolutions, in order to relate the blowup sequence for a (p,q) cusp with that of a cusp with Puiseux pairs $(p,q),(p_2,q_2),\dots,(p_k,q_k)$. In particular, we recall the definition of the Puiseux characteristic, which is a useful alternative to Puiseux pairs when discussing discussing blowups. We then state a technical lemma relating symplectic and complex blowups which will be used in the proof of Theorem A(ii) in the next subsection.

According to [Wal04, §2], for C any germ of a holomorphic curve near the origin in $\mathbb{C}^2$ which is not tangent to $\{x=0\}$ we can find a local parametrization of the form

$$x = t^m, \quad y = a_{m_1}t^{m_1} + a_{m_2}t^{m_2} + a_{m_3}t^{m_3} + \cdots,$$

with $m \leqslant m_1 < m_2 < \cdots$ and $a_{m_1}, a_{m_2}, a_{m_3}, \dots \in \mathbb{C}^*$, and such that $\gcd(m, m_1, m_2, \dots) = 1$. Here m is the multiplicity of C at the origin. We define β_1 to be the smallest m_i which is not a multiple of m, we put $e_1 := \gcd(m,\beta_1)$, and we put inductively

$$\beta_{k+1} = \min\{m_i \mid e_k \nmid m_i\}, \qquad e_{k+1} = \gcd(e_k, \beta_{k+1}).$$

We necessarily arrive at $e_g = 1$ for some $g \in \mathbb{Z}_{\geqslant 1}$, and the Puiseux characteristic of C is by definition $(m;\beta_1,\dots,\beta_g)$. One can show that this is independent of the coordinate representation of C and is obtained from the Puiseux pairs $(n_1,d_1),\dots,(n_g,d_g)$ via

$$m = d_1\cdots d_g, \qquad \beta_i = n_i d_{i+1}\cdots d_g.$$

In the reverse direction, given $(m;\beta_1,\dots,\beta_g)$ we can recover $n_1,\dots,n_g$ and $d_1,\dots,d_g$ via $\frac{\beta_i}{m} = \frac{n_i}{d_1\cdots d_i}$. In particular, note that if the first Puiseux pair is $(n_1,d_1) = (p,q)$ then the Puiseux characteristic takes the form $(kq;kp,\beta_2,\dots,\beta_g)$ with $k = d_2\cdots d_g$. One can also show that the Puiseux characteristic determines the multiplicity sequence and vice versa (see [Wal04, Thm. 3.5.6]).

Remark 3.3.1. Recall that, according to [EN85], any local branch of a singular holomorphic curve in $\mathbb{C}^2$ is homeomorphic to the cone over an iterated torus knot, where the cabling parameters (d_k,s_k) can be read off from the Puiseux pairs with $d_k \neq 1$ via $s_1 = n_1$ and $s_k = n_k - n_{k-1}d_k + d_{k-1}d_k s_{k-1}$ for $k \geqslant 2$ (here we follow the conventions of [Neu17]). ◊

The following example is of primary relevance for Theorem E(b):

Example 3.3.2. The Puiseux pairs $(p,q),(kp+1,k)$ correspond to the Puiseux characteristic $(kq;kp,kp+1)$ (and vice versa). $\Diamond$

If C has Puiseux characteristic $(m;\beta_1,\dots,\beta_g)$, then its proper transform after blowing up has Puiseux characteristic $(m',\beta_1',\dots,\beta_{g'}')$ given as follows (see [Wal04, Thm. 3.5.5]):

$$(m';\beta_1',\dots,\beta_{g'}') = \begin{cases} (m;\beta_1-m,\dots,\beta_g-m) & \beta_1 > 2m \\ (\beta_1-m;m,\beta_2-\beta_1+m,\dots,\beta_g-\beta_1+m) & \beta < 2m \text{ and } (\beta_1-m)\nmid m \\ (\beta_1-m;\beta_2-\beta_1+m,\dots,\beta_g-\beta_1+m) & (\beta_1-m)|m. \end{cases} \tag{3.3.1}$$

Using (3.3.1), the following is readily checked:

Lemma 3.3.3. *Suppose that the normal crossing resolution of a (p,q) cusp requires L blowups and results in negative self-intersection spheres $\mathbb{F}_1^L,\dots,\mathbb{F}_L^L$ as in §3.1. Let C be any cusp singularity with Puiseux pairs $(n_1,d_1),(n_1,d_2),\dots,(n_g,d_g)$ with $(n_1,d_1)=(p,q)$. Then the first L blowups of the resolution sequence for C produce spheres $\mathbb{G}_1^L,\dots,\mathbb{G}_L^L$ having the same intersection pattern (including self-intersection numbers) as $\mathbb{F}_1^L,\dots,\mathbb{F}_L^L$. The proper transform $\widetilde{C}$ of C intersects $\mathbb{G}_L^L$ in one point with contact order $k := d_2\cdots d_g$, and is disjoint from $\mathbb{G}_1^L,\dots,\mathbb{G}_{L-1}^L$.*

Note that $\widetilde{C}$ may itself have a residual cusp singularity.

Example 3.3.4 (continuation of Example 3.3.2)**.** Let C be a curve with cusp having Puiseux characteristic of the form $(kq;kp,kp+1)$ (and hence Puiseux pairs $(p,q),(kp+1,k)$), and let L be as in Lemma 3.3.3. Then the first L blowups in the resolution sequence achieve the normal crossing resolution for a (p,q) cusp, after which the proper transform $\widetilde{C}$ is nonsingular but intersects $\mathbb{G}_L^L$ in a single point of contact order k. Thus L blowups achieve the minimal resolution for C, and an additional k blowups achieve the normal crossing resolution for C.

For instance, the minimal resolution sequence for Puiseux characteristic $(3;5)$ is $(3;5)\to(2;3)\to(1;2)$, while that of $(9;15,16)$ is $(9;15,16)\to(6;9,10)\to(3;7)\to(3;4)\to(1,3)$ (this corresponds to $(q,p)=(3,5)$, $k=3$, and $L=4$). $\Diamond$

We record the following for later purposes:

Lemma 3.3.5. *Let C be a curve with a cusp having Puiseux characteristic $(kq;kp,kp+1)$, and let $\widetilde{C}$ be its minimal resolution. Then we have $[\widetilde{C}]\cdot[\widetilde{C}] = [C]\cdot[C]-k^2pq$. In particular, $[C]\cdot[C]\geqslant k^2pq$ if and only if $[\widetilde{C}]\cdot[\widetilde{C}]\geqslant 0$.*

We now are in a position to complete the proof of Theorem A. In §3.2 we assumed that C coincides with the visible symplectic curve $C_{(p,q)}\subset\mathbb{C}^2$ from (3.1.1) locally near its cusp. Since this curve has no direct analogue for a cusp with multiple Puiseux pairs, we will augment the explicit toric resolution model from §3.1 with a slightly more abstract argument.

The following technical lemma relating symplectic and complex blowups will suffice for our purposes.

Lemma 3.3.6 (see [McSa17, §7.1]). *Let (M,ω) be a symplectic manifold equipped with an ω-tame almost complex structure J which is integrable near a point $\mathfrak{p} \in M$. Let $(\mathrm{Bl}_{\mathfrak{p}} M, \widetilde{J})$ denote the complex blowup M at $\mathfrak{p}$, with exceptional divisor $\mathbb{E}_{\mathrm{Bl}_{\mathfrak{p}} M}$. Then for some $\delta > 0$ there exists a symplectic embedding $\iota : (B^{2n}(\delta), \omega_{\mathrm{std}}) \overset{s}{\hookrightarrow} (M,\omega)$ with $\iota(0) = \mathfrak{p}$ for which the corresponding symplectic blowup $(\mathrm{Bl}_\iota M, \widetilde{\omega})$ admits a diffeomorphism $\Phi : \mathrm{Bl}_\iota M \xrightarrow{\cong} \mathrm{Bl}_{\mathfrak{p}} M$ such that $\Phi^* \widetilde{J}$ is $\widetilde{\omega}$-tame and $\Phi(\mathbb{E}_{\mathrm{Bl}_\iota M}) = \mathbb{E}_{\mathrm{Bl}_{\mathfrak{p}} M}$.*

Furthermore, suppose that $\mathbf{D}_1$ and $\mathbf{D}_2$ are smooth J-holomorphic local divisors in M which intersect ω-orthogonally at p, and let $\widetilde{\mathbf{D}}_1, \widetilde{\mathbf{D}}_2 \subset \mathrm{Bl}_{\mathfrak{p}} M$ denote their $\widetilde{J}$-holomorphic proper transforms. Then we can arrange that $\Phi^{-1}(\widetilde{\mathbf{D}}_1)$ and $\Phi^{-1}\widetilde{\mathbf{D}}_2$ each intersect $\mathbb{E}_{\mathrm{Bl}_\iota M}$ $\widetilde{\omega}$-orthogonally.

Note that, in the context of Lemma 3.3.6, if C is (singular) symplectic curve in M which is preserved by J near p, then we can define its symplectic proper transform to be $\Phi^{-1}(\widetilde{C})$, where $\widetilde{C}$ is the $\widetilde{J}$-holomorphic proper transform of C in $\mathrm{Bl}_p M$.

Proof of Theorem A(ii). Let C be a sesquicuspidal symplectic curve in M with Puiseux pairs $(p,q), (p_2,q_2), \ldots, (p_g,q_g)$, whose homology class satisfies $[C] = c\mathrm{PD}[\omega_M]$ and $[C] \cdot [C] \geqslant k^2 pq$ for $k = q_2 \cdots q_g$. As before, after resolving any double points we can assume that C is embedded away from the cusp point. By assumption there is a neighborhood U of the cusp point such that $(U, C \cap U)$ is symplectomorphic to $(U', C' \cap U')$, where U' is a neighborhood of the origin in $\mathbb{C}^2$ and $C' \subset \mathbb{C}^2$ is a holomorphic curve having a cusp with Puiseux pairs $(p_1,q_1), \ldots, (p_g,q_g)$. By pulling back $J_{\mathrm{std}}|_{U'}$ to U and extending over M, we can find an ω_M-compatible almost complex structure J on M which preserves C and is integrable near the cusp.

Let $(\widetilde{M}_{\mathrm{comp}}, \widetilde{J})$ denote the L-fold complex blowup of (M,J) which achieves normal crossing resolution of a (p,q) cusp singularity as in Lemma 3.3.3, with negative self-intersection $\widetilde{J}$-holomorphic spheres $\mathbb{G}_1^L, \ldots, \mathbb{G}_L^L \subset \widetilde{M}_{\mathrm{comp}}$. Let $\widetilde{C} \subset \widetilde{M}_{\mathrm{comp}}$ be the corresponding proper transform of C (this may be smooth or have a residual cusp). Using Lemma 3.3.6, there is a corresponding L-fold symplectic blowup $(\widetilde{M}_{\mathrm{symp}}, \widetilde{\omega})$ of (M, ω_M) and a diffeomorphism $\Phi : \widetilde{M}_{\mathrm{symp}} \xrightarrow{\cong} \widetilde{M}_{\mathrm{comp}}$ such that $\Phi^*\widetilde{J}$ is $\widetilde{\omega}$-tame, and the symplectic spheres $\Phi^{-1}(\mathbb{G}_L^1), \ldots, \Phi^{-1}(\mathbb{G}_L^L)$ intersect symplectically orthogonally. After smoothing the residual cusp of $\Phi^{-1}(\widetilde{C})$ (if necessary) by replacing it with a perturbation of the corresponding Milnor fiber, we obtain a symplectically embedded curve $D \subset \widetilde{M}_{\mathrm{symp}}$ which intersects $\Phi^{-1}(\mathbb{G}_L^L)$ positively in k points and is disjoint from $\Phi^{-1}(\mathbb{G}_1^L), \ldots, \Phi^{-1}(\mathbb{G}_{L-1}^L)$. Note that we have $[\Phi^{-1}(\widetilde{C})]$ has positive self-intersection number by Lemma 3.3.5. The rest of the proof proceeds as in §3.2 by inflating along D, blowing down $\Phi^{-1}(\mathbb{G}_{L-1}^L), \ldots, \Phi^{-1}(\mathbb{G}_1^L)$ using the same toric model from §3.1, and finally rescaling the symplectic form and applying Moser's stability theorem. Note that after inflating $\Phi^{-1}(\mathbb{G}_L^L)$ has symplectic area $\varepsilon + ks$ since $[D] \cdot [\mathbb{G}_L^L] = k$, and hence the rescaled symplectic manifold (M, ω_s) admits a symplectic embedding of $\frac{\varepsilon + ks}{1+sc} E(q,p)$. □

4 $\mathbb{Q}$-Gorenstein smoothings and almost toric fibrations

In this section we collect various facts about (a) $\mathbb{Q}$-Gorenstein smoothings of singular toric algebraic surfaces (§4.1) and (b) symplectic almost toric fibrations (§4.2). Few if any of the results in this section are original, but our perspective is somewhat novel in that we emphasize the central role played by T-polygons (see §4.1c) in both algebraic and symplectic geometry. Roughly, we associate to a T polygon Q both an algebraic surface $\widetilde{V}_Q$ (defined as a $\mathbb{Q}$-Gorenstein smoothing) and a symplectic four-manifold $\mathbb{A}(Q_{\text{nodal}})$ (defined as the total space of an almost toric fibration). Proposition 4.2.6 gives a direct comparison between these two geometries, which we utilize in §5 and §6 in order to construct algebraic curves via symplectic techniques.

4.1 Toric surfaces and T-singularities

In this subsection, we begin by briefly reviewing some toric algebraic geometry and singularity theory and setting up our notation. We then recall the notion of T-singularities and their $\mathbb{Q}$-Gorenstein smoothings, and define T-polygons. We also discuss (dual) Fano polygons and their mutations, which play an important role in the mirror symmetry approach to Fano surfaces (see e.g. [GU10; Akh+16; KNP17; Coa+12]).

4.1a Cyclic quotient singularities and toric surfaces

For $\kappa \in \mathbb{Z}_{\geqslant 1}$, let

$$\mu_\kappa = \{e^{2\pi\sqrt{-1}j/\kappa} \mid j = 0, \ldots, \kappa-1\}$$

denote the group of κth roots of unity. Given $w_1, \ldots, w_n \in \mathbb{Z}_{\geqslant 0}$, we consider the action of μ_κ on $\mathbb{C}^n$ with weights $w_1, \ldots, w_n$, i.e. with $\mu \cdot (z_1, \ldots, z_n) = (\mu^{w_1} z_1, \ldots, \mu^{w_n} z_n)$ for $\mu \in \mu_\kappa$. Note that the weights $w_1, \ldots, w_n$ are only relevant modulo κ. We denote this representation of μ_κ by $\mu_\kappa^{w_1,\ldots,w_n}$.

Cyclic quotient singularities are by definition quotients of the form

$$\tfrac{1}{\kappa}(w_1, \ldots, w_n) := \mathbb{C}^n / \mu_\kappa^{w_1,\ldots,w_n}$$

for $\kappa \in \mathbb{Z}_{\geqslant 1}$ and $w_1, \ldots, w_n \in \mathbb{Z}_{\geqslant 1}$ coprime to κ. Here we have $\frac{1}{\kappa}(w_1, \ldots, w_n) = \frac{1}{\kappa}(\ell w_1, \ldots, \ell w_n)$ for any $\ell \in (\mathbb{Z}/\kappa)^\times$, so we may assume $w_1 = 1$. Note that the cyclic quotient surface singularity $\frac{1}{\kappa}(1, w)$ is the affine toric variety $U_\sigma := \operatorname{Spec}_m(\mathbb{C}[S_\sigma])$ corresponding to the cone $\sigma \subset \mathbb{R}^2$ generated by $(0, 1)$ and $(\kappa, -w)$. Here $\sigma^\vee \subset \mathbb{M}_\mathbb{R}$ is the dual cone to σ, S_σ is the semigroup of lattice points in $\sigma^\vee$, and $\mathbb{C}[S_\sigma]$ is the associated semigroup algebra (see e.g. [CLS11; Ful93; Bra04; Da 03] for more background on toric varieties).

Let $\mathbb{N}$ be lattice of rank $n \in \mathbb{Z}_{\geqslant 1}$, with dual lattice $\mathbb{M} = \operatorname{Hom}(\mathbb{N}, \mathbb{Z})$ (typically we will have $\mathbb{N} = \mathbb{Z}^n$, but this notation is still helpful in distinguishing the roles of $\mathbb{N}$ and $\mathbb{M}$). We put $\mathbb{N}_\mathbb{R} := \mathbb{N} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathbb{M}_\mathbb{R} := \mathbb{M} \otimes_\mathbb{Z} \mathbb{R}$. We will say that a polytope[17] $P \subset \mathbb{N}_\mathbb{R}$ is

[17] By **polytope** $P \subset \mathbb{N}_\mathbb{R}$ we mean the convex hull of finitely many points in $\mathbb{N}_\mathbb{R}$. We call this a **polygon** when $\mathbb{N}$ has rank two.

centered if P is n-dimensional and contains the origin in its interior. Given a centered polytope $P \subset \mathbb{N}_{\mathbb{R}}$, the **dual polytope** $P^o \subset \mathbb{M}_{\mathbb{R}}$ is by definition

$$P^o := \{u \in \mathbb{M}_{\mathbb{R}} \mid \langle u, v\rangle \geqslant -1 \; \forall \; v \in P\}.$$

Note that (unless P is reflexive) P^o is typically not a lattice polytope (i.e. having vertices in $\mathbb{M}$), even if P is. For a polytope $Q \subset \mathbb{M}_{\mathbb{R}}$, the dual polytope $Q^o \subset \mathbb{N}_{\mathbb{R}}$ is defined similarly.

We associate to P its **face fan** Σ_P in $\mathbb{N}_{\mathbb{R}}$, which has a cone σ_τ for each face τ of P, where σ_τ is generated by the vertices of τ. Equivalently, this is the **normal fan** Σ_{P^o} of P^o, which has a cone σ_η for each face η of P^o, where σ_η is generated by the inward normal vectors of those facets of P^o which contain η. We denote by V_Σ the (typically singular) toric variety associated to a fan Σ. In the case $\Sigma = \Sigma_P = \Sigma_{P^o}$ we will also denote V_Σ by V_P or V_{P^o} when we wish to emphasize the polytope P or its dual P^o.[18]

For a general polygon $Q \subset \mathbb{M}_{\mathbb{R}}$, the toric surface V_Q has cyclic quotient singularities at its toric fixed points, which correspond to the vertices of Q (or equivalently the maximal cones of the normal fan Σ_Q). Explicitly, for each vertex $v \in Q$ there is an integral affine transformation[19] of $\mathbb{M}_{\mathbb{R}}$ sending v to the origin, so that the edge directions become $(0,1)$ and $(n,-q)$ for some coprime $n, q \in \mathbb{Z}_{\geqslant 1}$, in which case the singularity has type $\frac{1}{n}(1,q)$. In the case $n = 1$, this corresponds to a smooth point of V_Q and we refer to v as a **Delzant vertex** vertex of Q. If Q has only Delzant vertices then it is **Delzant polygon**.

We end this subsection with a remark about the homology of a smooth toric surface. Let $Q \subset \mathbb{M}_{\mathbb{Q}}$ be a Delzant polygon with edges $e_1, \ldots, e_\ell$ and corresponding toric divisors $\mathbf{D}_{e_1}, \ldots, \mathbf{D}_{e_\ell} \subset V_Q$. Recall that the homology group $H_2(V_Q)$ of the associated nonsingular toric variety V_Q is generated by the toric divisors $\mathbf{D}_{e_1}, \ldots, \mathbf{D}_{e_\ell}$. More precisely, letting $\vec{n}_1, \ldots, \vec{n}_\ell \in \mathbb{N}$ be the primitive inward normal vectors to the edges, we have the short exact sequence

$$0 \to \mathbb{M} \to \mathbb{Z}\langle[\mathbf{D}_{e_1}], \ldots, [\mathbf{D}_{e_\ell}]\rangle \to H_2(V_Q) \to 0, \tag{4.1.1}$$

where the first nontrivial map sends $u \in \mathbb{M}$ to $\sum_{i=1}^{\ell} \langle \vec{n}_i, u\rangle [\mathbf{D}_{e_i}]$; see [CLS11, §5.1].

4.1b Fano and dual Fano polygons

A polytope $P \subset \mathbb{N}_{\mathbb{R}}$ is said to be **Fano** if it is centered and its vertices are primitive lattice vectors (see e.g. [Akh+12, §3]). In this case the corresponding toric surface V_P has anticanonical divisor which is $\mathbb{Q}$-Cartier and ample, i.e. V_P is a (typically singular) toric Fano variety. In particular, in the case $\dim(P) = 2$, V_P is a singular toric del Pezzo surface. We will say that a centered polygon $Q \subset \mathbb{M}_{\mathbb{R}}$ is **dual Fano** if the dual polygon $Q^o \subset \mathbb{N}_{\mathbb{R}}$ is Fano, and hence in particular a lattice polytope. Note that the dual Fano

[18] It should be clear from the context whether we are taking the face fan or normal fan since P and P^o live in different vector spaces.

[19] By **integral transformation** of $\mathbb{M}_{\mathbb{R}}$ we mean a map $\mathbb{M} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{R}$ which is a group isomorphism $\mathbb{M} \cong \mathbb{M}$ on the first factor and the identity on the second factor. By **integral affine transformation** of $\mathbb{M}_{\mathbb{R}}$ we mean the composition of an integral transformation with a translation.

condition is equivalent to each edge e of Q having height one, where we define the **height** of an edge to be the number $\mathbb{h}(e)$ such that $\langle \vec{n}, u \rangle = -\mathbb{h}(e)$ for all $u \in e$, where $\vec{n} \in \mathbb{N}$ is the primitive inward normal vector to e.

For points $v, w \in \mathbb{M}_{\mathbb{R}}$, recall that the **affine length** $\mathrm{Len}_{\mathrm{aff}}([v, w])$ of the line segment $[v, w] \subset \mathbb{M}_{\mathbb{R}}$ is $|c|$, where we put $v - w = c(v - w)_{\mathrm{prim}}$ for $c \in \mathbb{R}$ and $(v - w)_{\mathrm{prim}} \in \mathbb{M}$ a primitive lattice vector. The following gives another characterization of the dual Fano condition (c.f. Proposition 5.1.6 for a symplectic counterpart).

Lemma 4.1.1. *If a centered Delzant polygon $Q \subset \mathbb{M}_{\mathbb{R}}$ is dual Fano, then we have $c_1([\mathbf{D}_e]) = \mathrm{Len}_{\mathrm{aff}}(e)$ for each edge e. The converse also holds if Q is a lattice polygon.*

Proof. We can assume $\mathbb{M} = \mathbb{Z}^2$ and $\mathbb{M}_{\mathbb{R}} = \mathbb{R}^2$ without loss of generality. Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell$ be the vertices of Q ordered counterclockwise, and let e_i be the edge joining $\mathfrak{v}_i$ and $\mathfrak{v}_{i+1}$ for $i = 1, \ldots, \ell$ (modulo ℓ). Let $\mathbb{h}(e_i)$ denote the height of the edge e_i, and let $\vec{n}_i \in \mathbb{N}$ denote the primitive inward normal vector to the edge e_i for $i = 1, \ldots, \ell$.

After applying an integral transformation of $\mathbb{R}^2$, we can further assume that $\vec{n}_1 = (0, 1)$ and $\vec{n}_\ell = (1, 0)$, and hence

$$\mathfrak{v}_1 = (-\mathbb{h}(e_\ell), -\mathbb{h}(e_1)) \quad \text{and} \quad \mathfrak{v}_2 = \mathfrak{v}_1 + (\mathrm{Len}_{\mathrm{aff}}(e_1), 0).$$

Intersecting (4.1.1) with $[\mathbf{D}_{e_1}]$ (with $u = \mathfrak{v}_2$) gives $\sum_{i=1}^{\ell} \langle \vec{n}_i, \mathfrak{v}_2 \rangle ([\mathbf{D}_{e_i}] \cdot [\mathbf{D}_{e_1}]) = 0$. Further we have $[\mathbf{D}_{e_i}] \cdot [\mathbf{D}_{e_1}] = 0$ for $i \notin \{1, 2, \ell\}$, and $[\mathbf{D}_{e_1}] \cdot [\mathbf{D}_{e_2}] = [\mathbf{D}_{e_1}] \cdot [\mathbf{D}_{e_\ell}] = 1$ since Q is Delzant. Thus

$$\begin{aligned} 0 &= \langle \vec{n}_1, \mathfrak{v}_2 \rangle ([\mathbf{D}_{e_1}] \cdot [\mathbf{D}_{e_1}]) + \langle \vec{n}_2, \mathfrak{v}_2 \rangle + \langle \vec{n}_\ell, \mathfrak{v}_2 \rangle \\ &= -\mathbb{h}(e_1) ([\mathbf{D}_{e_1}] \cdot [\mathbf{D}_{e_1}]) - \mathbb{h}(e_2) - \mathbb{h}(e_\ell) + \mathrm{Len}_{\mathrm{aff}}(e_1), \end{aligned}$$

so that

$$[\mathbf{D}_{e_1}] \cdot [\mathbf{D}_{e_1}] = \frac{-\mathbb{h}(e_2) - \mathbb{h}(e_\ell) + \mathrm{Len}_{\mathrm{aff}}(e_1)}{\mathbb{h}(e_1)}.$$

Noting that $\mathbf{D}_{e_1}$ is an embedded two-sphere, by the adjunction formula and symmetry we have

$$c_1([\mathbf{D}_{e_i}]) = 2 + \frac{-\mathbb{h}(e_{i+1}) - \mathbb{h}(e_{i-1}) + \mathrm{Len}_{\mathrm{aff}}(e_i)}{\mathbb{h}(e_i)} \tag{4.1.2}$$

for $i = 1, \ldots, \ell$.

If Q is dual Fano, then we have $\mathbb{h}(e_i) = 1$ for $i = 1, \ldots, \ell$, so (4.1.2) becomes $c_1([\mathbf{D}_{e_i}]) = \mathrm{Len}_{\mathrm{aff}}(e_i)$. Conversely, if Q is a lattice polygon and if $c_1([\mathbf{D}_{e_i}]) = \mathrm{Len}_{\mathrm{aff}}(e_i)$ for $i = 1, \ldots, \ell$, then (4.1.2) becomes $\mathrm{Len}_{\mathrm{aff}}(e_i) = 2 + \frac{-\mathbb{h}(e_{i+1}) - \mathbb{h}(e_{i-1}) + \mathrm{Len}_{\mathrm{aff}}(e_i)}{\mathbb{h}(e_i)}$. Since $\mathbb{h}(e_1), \ldots, \mathbb{h}(e_\ell) \geqslant 1$, this is only possible if $\mathbb{h}(e_1) = \cdots = \mathbb{h}(e_\ell) = 1$. $\square$

4.1c T-singularities and polygon mutations

A cyclic quotient surface singularity $\frac{1}{n}(1,q)$ is a ***T*-singularity** if we have $n = mr^2$ and $q = mra - 1$ for some $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r, a) = 1$.[20] These were shown in [KS88] to be precisely those cyclic quotient surface singularities which admit $\mathbb{Q}$-Gorenstein smoothings. Here a **$\mathbb{Q}$-Gorenstein smoothing** of a normal surface X with quotient singularities is a flat family $\mathcal{X}$ over a smooth curve germ $\mathcal{S}$ such that the central fiber is X, the general fiber is smooth, and the relative canonical divisor $K_{\mathcal{X}/\mathcal{S}}$ is $\mathbb{Q}$-Cartier (see e.g. [HP10, §2.1] or [LP11, §2] and the references therein). Note that this last condition is equivalent to the total space being $\mathbb{Q}$-Gorenstein, i.e. having $\mathbb{Q}$-Cartier canonical divisor.

Although the local deformation theory of the cyclic quotient surface singularity $\frac{1}{n}(1,q)$ is quite complicated, the restriction to $\mathbb{Q}$-Gorenstein deformations is well-understood by [Kol90; KS88] (c.f. [Akh+16, §1]). Specializing to the case of T-singularities, the base of the miniversal family of $\mathbb{Q}$-Gorenstein deformations of the T-singularity $\frac{1}{mr^2}(1, mra-1)$ is isomorphic to $\mathbb{C}^{m-1}$, corresponding to the family of hypersurfaces

$$\{xy = z^{rm} + C_{m-2}z^{r(m-2)} + \cdots + C_1 z^r + C_0\} \subset \mathbb{C}^3/\mu_r^{1,-1,a} \tag{4.1.3}$$

for parameters $C_1, \ldots, C_{m-2} \in \mathbb{C}$. Note here that the central fiber is indeed isomorphic to $\frac{1}{mr^2}(1, mra-1)$ by the isomorphism

$$\mathbb{C}^2/\mu_{mr^2}^{1,mra-1} \xrightarrow{\cong} \{xy = z^{rm}\}/\mu_r^{1,-1,a}, \qquad (z_1, z_2) \mapsto (z_1^{rm}, z_2^{rm}, z_1 z_2). \tag{4.1.4}$$

The general fiber is smooth and is diffeomorphic to

$$B_{m,r,a} := \{xy = (z^r - \zeta_1)\cdots(z_r - \zeta_m)\}/\mu_r^{1,-1,a}, \tag{4.1.5}$$

for some real $0 < \zeta_1 < \cdots < \zeta_m$, i.e. $B_{m,r,a}$ is the quotient of the A_{rm-1} Milnor fiber by μ_r, and we have $H_1(B_{m,r,a}; \mathbb{Q}) = 0$ and $\dim H_2(B_{m,r,a}; \mathbb{Q}) = m - 1$. In the special case $m = 1$, $B_{1,r,a}$ is a rational homology ball and plays an important role in constructing exotic four-manifolds with small homology groups (see e.g. [FS97]).

We will refer to a vertex $\mathfrak{v} \in Q$ of a polygon $Q \subset \mathbb{M}_{\mathbb{R}}$ as a ***T*-vertex** if the corresponding toric fixed point $\mathfrak{p}_{\mathfrak{v}} \in V_Q$ is a T-singularity, and we will call Q a ***T*-polygon** if all of its vertices are T-singularities (note this includes the case of Delzant vertices).[21] Given a T-vertex $\mathfrak{v}$, there is an isomorphism $\mathbb{M}_{\mathbb{R}} \cong \mathbb{R}^2$ of integral affine manifolds which sends $\mathfrak{v}$ to $(0,0)$ with edge vectors $(0,1)$ and $(mr^2, mra-1)$, and we will refer to the image of the direction (r,a) as the[22] **eigenray emanating from $\mathfrak{v}$** (this corresponds to the eigendirection of a suitable affine monodromy in §4.2b).[23]

Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a T-polygon, and as before let V_Q denote the corresponding toric surface with T-singularities. By definition T-singularities admit local $\mathbb{Q}$-Gorenstein

[20] Notice that the singularity type depends only on $a \bmod r$.

[21] The notions of T-polygon and dual Fano polygon are independent: there are T-polygons whose fan is not Fano and there are dual Fano polygons that are not T-polygons.

[22] One can check that this definition is unambiguous, since the integral affine transformation of $\mathbb{R}^2$ which swaps $(0,1)$ and $(mr^2, mra-1)$ fixes (r,a). In particular, this singularity is equivalent to its reflection in the y-axis with edge vectors , $(0,-1), (mr^2, 1-mra)$ and eigenray $(r,-a)$.

[23] This is also often referred to as a **nodal ray** in the context of almost toric fibrations as in §4.2b.

smoothings as in (4.1.3), and according to [HP10, Prop. 3.1][24] there are no local-to-global obstructions to deformations, so in particular V_Q admits a $\mathbb{Q}$-Gorenstein smoothing. Thus we have:

Lemma 4.1.2. *For any T-polygon $Q \subset \mathbb{M}_\mathbb{R}$, there is a $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q. If in addition Q is dual Fano, then $\widetilde{V}_Q$ is Fano for any sufficiently small smoothing, and in particular rigid if $\operatorname{rk} H^2(\widetilde{V}_Q, \mathbb{Q}) \leqslant 5$.*

Following e.g. [GU10; Akh+12], there is a notion of mutation which inputs a dual Fano polygon[25] $Q \subset \mathbb{M}_\mathbb{R}$ and a choice of vertex $\mathfrak{v} \in Q$ and produces a new dual Fano polygon $\mathrm{Mut}_\mathfrak{v}(Q) \subset \mathbb{M}_\mathbb{R}$. Namely, let $f \in \mathbb{N}$ be a primitive lattice vector such that $\langle f, \mathfrak{v}\rangle = 0$, let $\mathfrak{v}_{\mathrm{prim}} = t\mathfrak{v} \in \mathbb{M}$ be primitive for some $t \in \mathbb{R}_{>0}$, and consider the piecewise-linear map

$$\psi : \mathbb{M}_\mathbb{R} \to \mathbb{M}_\mathbb{R}, \qquad u \mapsto u - \min(\langle f, u\rangle, 0)\, \mathfrak{v}_{\mathrm{prim}}. \tag{4.1.6}$$

We put $\mathrm{Mut}_\mathfrak{v}(Q) := \psi(Q) \subset \mathbb{M}_\mathbb{R}$. In the case $\mathbb{M}_\mathbb{R} = \mathbb{R}^2$, this is equivalent to

$$\psi(u) = \begin{cases} \mathbb{S}_{\mathfrak{v}_{\mathrm{prim}}}(u) & \text{if } \langle f, u\rangle \leqslant 0 \\ u & \text{if } \langle f, u\rangle > 0, \end{cases} \tag{4.1.7}$$

where $\mathbb{S}_{\mathfrak{v}_{\mathrm{prim}}}(u) : \mathbb{R}^2 \to \mathbb{R}^2$ is the **primitive shear** along $\mathfrak{v}_{\mathrm{prim}}$, defined by

$$\mathbb{S}_{\mathfrak{v}_{\mathrm{prim}}}(u) = u + \det(\mathfrak{v}_{\mathrm{prim}}, u) \cdot \mathfrak{v}_{\mathrm{prim}}, \tag{4.1.8}$$

with $\det(\mathfrak{v}_{\mathrm{prim}}, u)$ the determinant of the 2×2 matrix with columns $\mathfrak{v}_{\mathrm{prim}}$ and u. We will say that $\mathrm{Mut}_\mathfrak{v}(Q)$ is the **(primitive) mutation** of the polygon Q at the vertex $\mathfrak{v}$.

Remark 4.1.3. Strictly speaking there are two choices for f in the above (i.e. in the case $\mathfrak{v} = (x, y) \in \mathbb{R}^2$ we have $f = \pm(y, -x)$), although the choice becomes unique if we choose an orientation on $\mathbb{M}_\mathbb{R}$ and ask for $f, \mathfrak{v}_{\mathrm{prim}}$ to be an oriented basis. At any rate, it is easy to show using (4.1.7) that the two choices for f give mutations which are related by an integral affine transformation of $\mathbb{M}_\mathbb{R}$. $\lozenge$

We also define $\mathrm{Mut}_\mathfrak{v}(Q)$ in the case that $Q \subset \mathbb{M}_\mathbb{R}$ is any polygon (not necessarily dual Fano) and $\mathfrak{v} \in Q$ is a T-vertex. Note that in this case the origin in $\mathbb{M}_\mathbb{R}$ is not necessarily in distinguished position relative to Q (it may not even be contained in Q). We put

$$\mathrm{Mut}_\mathfrak{v}(Q) := \tau^{-1}(\mathrm{Mut}_{\tau(\mathfrak{v})}(\tau(Q))), \tag{4.1.9}$$

[24] To apply the hypotheses of [HP10, Prop. 3.1] it suffices to note the anticanonical divisor of a toric divisor is always big. Indeed, if $\mathbf{D}_1, \ldots, \mathbf{D}_k$ denote the toric boundary divisors, then we can find an ample divisor of the form $\sum_{i=1}^k a_i \mathbf{D}_i$ for $a_1, \ldots, a_k \in \mathbb{Z}_{\geqslant 1}$. Then for any positive integer $m \geqslant a_1, \ldots, a_k$ we have that m times the anticanonical divisor is $m \sum_{i=1}^k \mathbf{D}_i = \sum_{i=1}^k a_i \mathbf{D}_i + \sum_{i=1}^k (m - a_i)\mathbf{D}_i$, which is a sum of an ample divisor and an effective divisor and hence big by [Laz17, Cor 2.2.7].

[25] One can also describe the mutation in terms of the Fano polygon $Q^o \subset \mathbb{N}_\mathbb{R}$, although for our purposes the formula using dual Fano polygons is more succinct and more directly related to mutations of almost toric fibrations. An extension of mutations to higher dimensional Fano polytopes is also defined in [Akh+12].

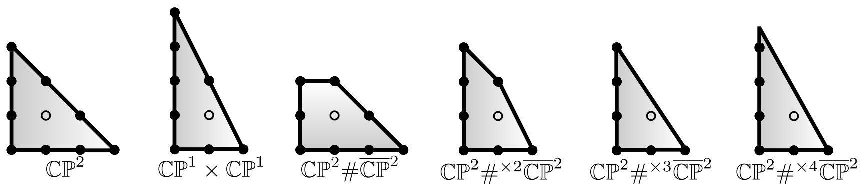


Figure 4: Dual Fano T-polygons Q such that $\widetilde{V}_Q$ is a smooth rigid del Pezzo surface. Although these are not unique (due to the possibility of mutation), each representative has a Delzant vertex and the minimal possible number of sides. Note that the polygon for $\mathbb{CP}^2 \#^{\times 4} \overline{\mathbb{CP}}^2$ is not a lattice polygon (the top left vertex is $(-1, 5/2)$.)

where $\tau : \mathbb{M}_R \to \mathbb{M}_{\mathbb{R}}$ is any translation sending a point on the eigenray emanating from $\mathfrak{v}$ to the origin (c.f. §4.1c), and $\mathrm{Mut}_{\tau(\mathfrak{v})}(\tau(Q))$ is defined as above. One can check that this coincides with the previous definition when Q is a dual Fano T-polygon (in that case the eigenray emanating from $\mathfrak{v}$ passes through the origin).

If $\mathfrak{v} \in Q$ is a T-vertex of type $\frac{1}{mr^2}(1, mra-1)$ with $m = 1$, then it is straightforward to check that the mutated polytope $\mathrm{Mut}_{\mathfrak{v}}(Q)$ no longer has a vertex at $\mathfrak{v}$, but there is a (possibly new) vertex at the point where the eigenray emanating from $\mathfrak{v}_{\mathrm{prim}}$ meets a side of Q. On the other hand, if $m > 1$ then $\mathfrak{v}$ is still a vertex of $\mathrm{Mut}_{\mathfrak{v}}(Q)$, but now with parameters $(m-1, r, a)$. Thus for all $1 \leqslant k \leqslant m$ the ***k*-fold mutation** $\mathrm{Mut}^k_{\mathfrak{v}}(Q) := \psi^k(Q)$ of Q at $\mathfrak{v}$ is well-defined, and we define the **full mutation** of Q at $\mathfrak{v}$ to be $\mathrm{Mut}^{\mathrm{full}}_{\mathfrak{v}}(Q) := \mathrm{Mut}^m_{\mathfrak{v}}(Q)$. We have:

Lemma 4.1.4. *If $\mathfrak{v}$ is a T-vertex of a polygon $Q \subset \mathbb{M}_{\mathbb{R}}$ with k sides, then $\mathrm{Mut}^{\mathrm{full}}_{\mathfrak{v}}(Q)$ is a polygon with either k or $k-1$ sides. In particular, if Q is a triangle then so is $\mathrm{Mut}^{\mathrm{full}}_{\mathfrak{v}}(Q)$.*

It is shown in [Akh+16, Lem. 7] (building on [Ilt12]) that if two dual Fano polygons $Q, Q' \subset \mathbb{M}_{\mathbb{R}}$ are mutation equivalent (i.e. Q' can be obtained from Q by a sequence of mutations), then the corresponding singular del Pezzo surfaces V_Q and $V_{Q'}$ are $\mathbb{Q}$-Gorenstein deformation equivalent (in the sense of [Akh+16, Def. 2]). We discuss an approach to this in §6.3. In particular, if Q (and hence Q') is also a T-polygon, the smoothings $\widetilde{V}_Q$ and $\widetilde{V}_{Q'}$ as in Lemma 4.1.2 are also $\mathbb{Q}$-Gorenstein deformation equivalent. Conversely, [Akh+16, Conj. A] conjectures that two dual Fano polygons $Q, Q' \subset \mathbb{M}_{\mathbb{R}}$ are mutation equivalent if the corresponding singular toric del Pezzos $V_Q, V_{Q'}$ are $\mathbb{Q}$-Gorenstein deformation equivalent. The specialization of this conjecture to dual Fano T-polygons is proved in [KNP17, Thm. 1.2]. By the classification of smooth del Pezzo surfaces, it follows that there are precisely 10 mutation equivalence classes of dual Fano T-polygons. Representatives of these equivalence classes (or rather their duals) are shown in [Akh+16, Fig. 1]. For the 6 equivalence classes corresponding to rigid smooth del Pezzo surfaces, representatives which are particularly useful for constructing ellipsoid embeddings are shown in Figure 4 (which is a reproduction of [Cri+20, Fig. 5.12]). The construction of these representatives is based on almost toric techniques as in [Via17].

If Q is a T-triangle, then by Lemma 4.1.4 it remains so under successive full mutations.

As explained in [Eva23, §H.2], if the vertices have types $\frac{1}{m_i r_i^2}(1, m_i r_i a_i - 1)$ for $i = 1, 2, 3$, then this data satisfies a generalized Markov equation

$$m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = C\sqrt{m_1 m_2 m_3}\, r_1 r_2 r_3, \tag{4.1.10}$$

where $C \in \mathbb{R}_{>0}$ is invariant under full mutations (one can check that the set $\{m_1, m_2, m_3\}$ is also preserved under full mutations). We thus have a bijection between

(a) the set of T-triangles which are full mutation equivalent to Q, and

(b) the set of triples $(r_1, r_2, r_3) \in \mathbb{Z}^3_{\geqslant 1}$ satisfying (4.1.10).

The specialization $r_1 = 1$ corresponds to triangles with a Delzant vertex (c.f. Figure 4). We will see later that these taken together encode all of the relevant unicuspidal curves for the two-stranded rigid del Pezzo infinite staircases (c.f. Proposition 5.1.3 and Proposition 6.1.11).

Example 4.1.5. When Q is the first triangle in Figure 4, (4.1.10) becomes the classical Markov equation $r_1^2 + r_2^2 + r_3^2 = 3r_1 r_2 r_3$. The solutions are well-known to form an infinite trivalent tree, with edges corresponding to Markov mutations $(r_1, r_2, r_3) \mapsto (r_1, r_2, 3r_1 r_2 - r_3)$ and their permutations (see e.g. [Aig15]). The solutions with $r_1 = 1$ correspond to pairs of consecutive odd index Fibonacci numbers. $\Diamond$

4.2 Almost toric fibrations and polygons

In this section we discuss symplectic almost toric fibrations from the point of view of T-polygons. After some preliminaries on Lagrangian torus fibrations, we define almost toric fibrations, discuss their construction from T-polygons, and compare these with the $\mathbb{Q}$-Gorenstein smoothings from the previous subsection.

4.2a Abstract almost toric fibrations and almost toric bases

By definition a closed symplectic manifold M^{2n} is toric if it carries a Hamiltonian $\mathbb{T}^n$-action. In this case the image of the corresponding moment map $\pi : M \to \mathbb{R}^n$ is a convex polytope $\pi(M)$ (see [Ati82; GS82]), such that π is a regular Lagrangian torus fibration over the interior of $\pi(M)$. Meanwhile, π has toric singularities over the boundary of $\pi(M)$, with the fiber over a point in the interior of a k-dimensional face of $\pi(M)$ being an isotropic torus $\mathbb{T}^{2k}$ of dimension $2k$. Almost toric fibrations extend this picture (in dimension four) by allowing π to have additional focus-focus singularities.

An **almost toric fibration** (see e.g. [Sym; LS10; Eva23]) is a smooth proper surjective map $\pi : M^4 \to B^2$ with connected fibers, where M^4 is a symplectic four-manifold and B^2 is a smooth two-manifold with corners, such that

- at each regular point $\mathfrak{p} \in M^4$ the kernel $\ker d_{\mathfrak{p}}\pi \subset T_{\mathfrak{p}}M$ is a Lagrangian subspace
- for each critical point $\mathfrak{p} \in M$ of π there are Darboux coordinates x_1, y_1, x_2, y_2 for M near $\mathfrak{p}$ and smooth coordinates b_1, b_2 for B near $\pi(\mathfrak{p})$ such that π has one of the following local normal forms:

1. $\pi(x_1, y_1, x_2, y_2) = (x_1, x_2^2 + y_2^2)$ (corank one elliptic)
2. $\pi(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2, x_2^2 + y_2^2)$ (corank two elliptic)
3. $\pi(x_1, y_1, x_2, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 - x_2 y_1)$ (focus-focus).

Note that the regular fibers are necessarily two-dimensional tori by the Arnold–Liouville theorem. The first two types of singularities are modeled on the singularities of a moment map of a toric symplectic manifold over an edge or vertex respectively of the moment polytope. The third type of singularity is topologically equivalent to a critical point of a Lefschetz fibration, but here the fibers are Lagrangian rather than symplectic. The images of focus-focus singularities are called **base-nodes**.

Given an almost toric fibration $\pi : M^4 \to B^2$, observe that π restricts to a regular Lagrangian torus fibration over the regular values B^{reg} of π. Thus B^{reg} inherits an integral affine structure as follows. Given $\mathfrak{p} \in M$ and $\mathfrak{b} = \pi(\mathfrak{p}) \in B^{\mathrm{reg}}$, the symplectic form on M induces a nondegenerate pairing

$$\langle -, - \rangle : T_{\mathfrak{b}} B \times T_{\mathfrak{p}}^{\mathrm{vert}} M \to \mathbb{R}, \qquad \langle u, v \rangle = \omega(\widetilde{u}, v), \tag{4.2.1}$$

where $\widetilde{u} \in T_{\mathfrak{p}} M$ is any lift of $u \in T_{\mathfrak{b}} B$ (i.e. $\pi_* \widetilde{u} = u$) and $T_{\mathfrak{p}}^{\mathrm{vert}} M = T_{\mathfrak{p}} \pi^{-1}(\mathfrak{b})$ is the vertical tangent space at $\mathfrak{p}$. In particular, for each covector in $T_{\mathfrak{b}}^* B$ there is a corresponding vector field along $\pi^{-1}(\mathfrak{b})$, and taking its time-1 flow gives an action of $T_{\mathfrak{b}}^* B$ on the fiber $\pi^{-1}(\mathfrak{b})$. The set of covectors in $T_{\mathfrak{b}}^* B$ which act trivially on $\pi^{-1}(\mathfrak{b})$ defines a lattice $\Lambda_{\mathfrak{b}}^* \subset T_{\mathfrak{b}}^* B$, with dual lattice $\Lambda_{\mathfrak{b}} \subset T_{\mathfrak{b}} B$. The corresponding lattice bundle $\Lambda^* = \bigcup_{\mathfrak{b} \in B^{\mathrm{reg}}} \Lambda_{\mathfrak{b}}^* \subset T^* B^{reg}$ gives a natural symplectomorphism

$$T^* B^{\mathrm{reg}} / \Lambda^* \cong M^{\mathrm{reg}}, \tag{4.2.2}$$

while the dual lattice bundle $\Lambda = \bigcup_{\mathfrak{b} \in B} \Lambda_{\mathfrak{b}} \subset TB^{reg}$ defines the integral affine structure on B^{reg}.

This integral affine structure on B^{reg} gives rise to an **affine monodromy** map $mon : \pi_1(B^{\mathrm{reg}}, *) \to \mathrm{Aut}(H_1(\pi^{-1}(*); \mathbb{Z}))$ which measures how the integral affine structure twists as we go around loops in B^{reg} (here $* \in B^{\mathrm{reg}}$ is a basepoint). By picking a basis we can also view this as a map $\pi_1(B^{\mathrm{reg}}) \to \mathrm{GL}_2(\mathbb{Z})$ which is defined up to global conjugation by an element of $\mathrm{GL}_2(\mathbb{Z})$.

For a small loop γ in B^{reg} surrounding a base-node $\mathfrak{b}_0$, the corresponding affine monodromy $mon(\gamma) \in \mathrm{GL}_2(\mathbb{Z})$ is conjugate to the matrix $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ with eigenvalue 1, where k is the number of focus-focus critical points in the fiber $\pi^{-1}(\mathfrak{b}_0)$. Thus for any $\mathfrak{b}' \neq \mathfrak{b}_0$ in a small neighborhood of $\mathfrak{b}_0$ in B there is a well-defined eigenline in $T_{\mathfrak{b}'} B$ for the affine monodromy around $\mathfrak{b}_0$, and these limit to a line $\mathbb{L}_{\mathfrak{b}_0} \subset T_{\mathfrak{b}_0} B$, which we call the **eigenline at $\mathfrak{b}_0$**.

Recall that there exists a closed smooth toric symplectic manifold with moment polygon $Q \subset \mathbb{R}^n$ if and only if Q is a Delzant polytope ([Del88]). In dimension four we expect to associate almost toric fibrations $\pi : M^4 \to B^2$ to more general polygons with non-Delzant vertices (even though M is assumed to be smooth). However, some care is needed in formulating the almost toric analogue of the moment polygon, since there is no global torus action and thus no global moment map.

Given an almost toric fibration $\pi : M^4 \to B^2$, the set of regular values $B^{\mathrm{reg}} \subset B$ naturally inherits an integral affine structure, and this extends over the toric critical values to $B \setminus \{\mathfrak{b}_1, \ldots, \mathfrak{b}_\ell\}$, where $\mathfrak{b}_1, \ldots, \mathfrak{b}_\ell \in B$ are the base-nodes. Thus we have a **nodal integral affine surface**, i.e. a triple $(B, \{\mathfrak{b}_i\}, \mathcal{A})$, where B is a smooth two-dimensional manifold with corners equipped with a subset $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_\ell\} \subset B$ (the base-nodes) and an integral affine structure $\mathcal{A}$ on $B \setminus \{\mathfrak{b}_1, \ldots, \mathfrak{b}_\ell\}$, such that for each $i = 1, \ldots, \ell$ the affine monodromy around a small loop γ_i surrounding $\mathfrak{b}_i$ is conjugate to $\left(\begin{smallmatrix} 1 & k_i \\ 0 & 1 \end{smallmatrix}\right)$ for some nonzero $k_i \in \mathbb{Z}$. Given an almost toric fibration $\pi : M^4 \to B^2$, we will refer to the associated nodal integral affine surface $(B, \{\mathfrak{b}_i\}, \mathcal{A})$ as its **almost toric base**.

We will say that two nodal integral affine surfaces $(B, \{\mathfrak{b}_i\}, \mathcal{A})$ and $(B', \{\mathfrak{b}'_i\}, \mathcal{A}')$ are isomorphic if there is a diffeomorphism $B \xrightarrow{\cong} B'$ which restricts to a bijection $\{\mathfrak{b}_i\} \xrightarrow{\cong} \{\mathfrak{b}'_i\}$ and preserves the integral affine structures on the complements of the base-nodes. It is shown in [Sym, Cor. 5.4] that if two almost toric fibrations $\pi_1 : M_1 \to B_1$ and $\pi_2 : M_2 \to B_2$ have isomorphic almost toric bases with nonempty boundaries then their total spaces are symplectomorphic.

Moreover, by [Sym, Thm. 5.2], a nodal integral affine surface $(B, \{\mathfrak{b}_i\}, \mathcal{A})$ is the base of an almost toric fibration if and only if each point $\mathfrak{b} \in B \setminus \{\mathfrak{b}_i\}$ has a neighborhood which is integral affine isomorphic to a neighborhood of a point $\mathbb{R}^2_{\geqslant 0}$ (with its standard integral affine structure). In particular, given such a nodal integral affine surface $(B, \{\mathfrak{b}_i\}, \mathcal{A})$ with B compact, there is an associated closed symplectic four-manifold which we denote by $\mathbb{A}(B, \{\mathfrak{b}_i\}, \mathcal{A})$ and which is well-defined up to symplectomorphism.

4.2b Nodal integral affine surfaces from T-polygons

We now construct a nodal integral affine surface $Q_{\mathrm{nodal}} = (B, \{\mathfrak{b}_i\}, \mathcal{A})$ from a T-polygon $Q \subset \mathbb{M}_{\mathbb{R}}$. Here B is the smooth surface with boundary given by smoothing the corners of the polygon Q, while $\mathcal{A}$ is given roughly by implanting various nodes into the integral affine structure on Q (i.e. the one induced from $\mathbb{M}_{\mathbb{R}}$). Together with the discussion in §4.2a, this construction associates to any T-polygon Q a closed symplectic four-manifold $\mathbb{A}(Q_{\mathrm{nodal}})$ which carries an almost toric fibration

$$\pi : \mathbb{A}(Q_{\mathrm{nodal}}) \to Q_{\mathrm{nodal}}.$$

Strictly speaking Q_{nodal} depends on some auxiliary parameters giving the locations of the base-nodes (varying these is called a **nodal slide**), but using Moser's stability theorem $\mathbb{A}(Q_{\mathrm{nodal}})$ is independent of these choices up to symplectomorphism.

In more detail, we assume $\mathbb{M} = \mathbb{Z}^2$ for simplicity, and let $Q \subset \mathbb{R}^2$ be a T-polygon with vertices $\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell$ ordered counterclockwise. Since each vertex $\mathfrak{v}_i$ is a T-vertex, the corresponding toric fixed point has type $\frac{1}{m_i r_i^2}(m_i r_i^2, m_i r_i a_i - 1)$ for some $m_i \in \mathbb{Z}_{\geqslant 1}$ and coprime $r_i, a_i \in \mathbb{Z}_{\geqslant 1}$. For $i = 1, \ldots, \ell$, let $-h_i \in \mathbb{M}$ denote the primitive integral vector pointing in the direction of the eigenray emanating from $\mathfrak{v}_i$ (as in §4.1c). Note that by our conventions $-h_i$ points inward towards Q.

For $i = 1, \ldots, \ell$, pick $\varepsilon_i > 0$ sufficiently small, and let $\gamma_i \subset Q$ be the line segment $\{\mathfrak{v}_i - th_i \mid t \in [0, \varepsilon_i]\}$. Pick $\varepsilon = t_i^1 > \cdots > t_i^{m_i} > 0$, and let $\mathfrak{b}_i^1, \ldots, \mathfrak{b}_i^{m_i}$ be the corresponding points along γ_i (ordered towards $\mathfrak{v}_i$) given by $\mathfrak{b}_i^j := \mathfrak{v}_i - t_i^j h_i$. Let $\mathbb{S}_i \in \mathrm{GL}_2(\mathbb{Z})$ be the primitive shear along $-h_i$ as in §4.1c, i.e. $\mathbb{S}_i(u) = u + \det(-h_i, u) \cdot (-h_i)$, and let $\tau_i : \mathbb{R}^2 \to \mathbb{R}^2$ be any translation sending a point on the eigenray emanating from $\mathfrak{v}_i$ to the origin. Let $s_i^1, \ldots, s_i^{m_i}$ be the components of $\gamma_i \smallsetminus \{\mathfrak{b}_i^1, \ldots, \mathfrak{b}_i^{m_i}\}$ ordered towards $\mathfrak{v}_i$.

We now modify the integral affine surface $Q \smallsetminus \{\mathfrak{b}_i^j\}$ to obtain a new one by cutting Q along s_i^j and regluing via the tranformation $\tau_i \circ \mathbb{S}_i^j \circ \tau_i^{-1} \in \mathrm{GL}_2(\mathbb{Z})$ for $i = 1, \ldots, \ell$ and $j = 1, \ldots, m_i$ (here $\mathbb{S}_i^j$ denotes the jth power of $\mathbb{S}_i$). More precisely, if $(s_i^j)^-, (s_i^j)^+$ are the boundary segments of $Q \smallsetminus \bigcup_{i=1}^{\ell} \gamma_i$ (in counterclockwise order) arising from cutting along s_i^j, then the gluing identifies $u \in (s_i^j)^+$ with $(\tau_i \circ \mathbb{S}_i^j \circ \tau_i^{-1})(u) \in (s_i^j)^-$. Since $\mathbb{S}_i^j$ fixes γ_i pointwise, the resulting topological space B_{nodal} is naturally identified with $Q \smallsetminus \{\mathfrak{b}_i^j\}$, but by construction it carries an integral affine structure $\mathcal{A}$ which has affine monodromy around each $\mathfrak{b}_i^j$ conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 4.2.1. *The glued integral affine structure $\mathcal{A}$ on $Q \smallsetminus \{\mathfrak{b}_i^j\}$ is locally isomorphic to $\mathbb{R}^2$ near any interior point $\mathfrak{b} \in \operatorname{Int} Q \smallsetminus \{\mathfrak{b}_i^j\}$, and it is locally isomorphic to a boundary point of $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$ near any point in ∂Q. In particular, the smooth structure on B_{nodal} is such that there are no corner points.*

To complete the construction of Q_{nodal}, we let B be the smooth surface with boundary given by filling in the punctures of B_{nodal}. Note that B is diffeomorphic to a two-dimensional closed disk.

Proof of Lemma 4.2.1. It suffices to analyze the integral affine structure near a vertex $\mathfrak{v}_i$. After an integral affine transformation we can assume that $\mathfrak{v}_i = (0,0)$ with incoming edge vector $(0,-1)$ and outgoing edge vector $(m_i r_i^2, m_i r_i a_i - 1)$, and $-h_i = (r_i, a_i)$. Let $C \subset \mathbb{R}^2_{\geqslant 0}$ be the cone spanned by $(0,1), (m_i r_i^2, m_i r_i a_i - 1)$, and let $C^-, C^+ \subset C$ be the subcones spanned by $(0,1), (r_i, a_i)$ and $(r_i, a_i), (m_i r_i^2, m_i r_i a_i - 1)$ respectively. Then we have an integral affine isomorphism from a neighborhood of $\mathfrak{v}_i$ in B_{nodal} to a neighborhood of $(0,0)$ in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$, given by

$$u \mapsto \begin{cases} u & u \in C^- \\ \mathbb{S}_i^{m_i}(u) & u \in C^+. \end{cases}$$

Indeed, it suffices to check that the vector $(m_i r_i^2, m_i r_i a_i - 1)$ gets sent to $(0,-1)$, i.e. $\mathbb{S}_i^{m_i}(m_i r_i^2, m_i r_i a_i - 1) = (0,-1)$, and this is the content of the following lemma. ☐

Lemma 4.2.2. *If $\mathbb{S} \in \mathrm{GL}_2(\mathbb{Z})$ is the primitive shear along (r,a), we have $\mathbb{S}^m(mr^2, mra - 1) = (0,-1)$.*

Proof. Using (4.1.8) we have $\mathbb{S}^m(x,y) = (x,y) + m\det((r,a),(x,y)) \cdot (r,a)$, from which the result directly follows. ☐

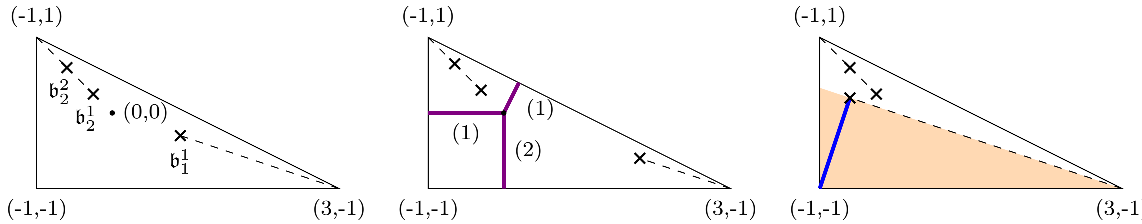


Figure 5: Left: Q_{nodal}, where Q is the triangle with vertices $\mathfrak{v} = (3,-1), \mathfrak{v}_2 = (-1,1), \mathfrak{v}_3 = (-1,-1)$. Middle: a tropical representation of a curve intersecting each toric divisor once as in Corollary 5.2.3. Right: a visible symplectic unicuspidal curve as in §5.1a. The outer corner obstruction given by this curve implies that the shaded triangle represents an optimal ellipsoid embedding.

Remark 4.2.3. Gluing in a node at a Delzant vertex is called a **nodal trade** and it does not change the resulting symplectic four-manifold up to symplectomorphism (see e.g. [Sym, Thm. 6.5] or [Eva23, §8.2]). In the above construction of Q_{nodal} we have chosen to glue in nodes at all of the Delzant vertices of Q (i.e. those of type $\frac{1}{mr^2}(1, mra-1)$ with $m = r = 1$) only for uniformity of exposition, but we can equally well leave the integral affine structure alone near some or all of the Delzant vertices. When constructing sesquicuspidal curves it will be beneficial to have one Delzant vertex without a nodal trade. ♢

Remark 4.2.4. For an almost toric fibration $\pi : \mathbb{A}(Q_{\text{nodal}}) \to Q_{\text{nodal}}$ as above, the polygon Q decorated by its base-nodes $\{\mathfrak{b}_i^j\}$ and the eigenrays at its vertices is sometimes called an **almost toric base diagram**. Our approach here is to keep track of only the T-polygon $Q \subset \mathbb{M}_{\mathbb{R}}$, since the eigenrays and number of base-nodes are uniquely determined by Q and the locations of the base-nodes are immaterial up to symplectomorphism of the total space.

In §6, we construct (p,q)-unicuspidal curves in $\mathbb{A}(Q_{\text{nodal}})$ in T-polygons that have one smooth vertex. The types (p,q) of their cusps do not depend simply on the types $\frac{1}{q^2}(1, pq-1)$ of the T-singularities of the vertices of Q (where p is only well-defined mod q), but rather on the relation between the eigenrays at the vertices of Q and the smooth corner at the origin. For a vertex of type $\frac{1}{q^2}(1, pq-1)$ on the y-axis the adjacent edges of Q have directions $(0,-1), (q^2, 1-pq)$ with eigenray $(q,-p)$, while in the quadrilaterals considered in Lemma 6.5.3 a vertex of the same type on the x-axis is taken to have adjacent edges in directions $(-1,0), (1+\widetilde{p}q, q^2)$ with eigenray $(\widetilde{p}, q)$, where $\widetilde{p} := p - 6q$. Recall also from §4.1c that eigenrays are invariant under the reflection that interchanges its two adjacent edges. ♢

Example 4.2.5. Figure 5 left illustrates Q_{nodal} in the case that Q is a triangle with vertices $\mathfrak{v}_1 = (3,-1), \mathfrak{v}_2 = (-1,1), \mathfrak{v}_3 = (-1,-1)$. The vertices are:

- smooth at $(3,-1)$, with a nodal trade
- type $\frac{1}{2}(1,1)$ at $(-1,1)$

- smooth at $(-1,-1)$, without a nodal trade.

Here we have $-h_1 = (-3,1)$ and $-h_2 = (1,-1)$. In this example the symplectic four-manifold $\mathbb{A}(Q_{\mathrm{nodal}})$ is symplectomorphic to $\mathbb{CP}^1(2) \times \mathbb{CP}^1(2)$. $\Diamond$

We can equivalently describe $\mathbb{A}(Q_{\mathrm{nodal}})$ by starting with the symplectic toric orbifold with moment map Q and performing a cut-and-paste operation near each toric fixed point. Namely, near the ith toric fixed point we excise a neighborhood which is symplectomorphic to a neighborhood in $\frac{1}{m_i r_i^2}(1, m_i r_i a_i - 1) \cong \{xy = z^{r_i m_i}\}/\mu_{r_i}^{1,-1,a_i}$ and we glue in a neighborhood in $B_{m_i,r_i,a_i} = \{xy = (z^{r_i} - \zeta_1)\cdots(z^{r_i} - \zeta_{m_i})\}/\mu_{r_i}^{1,-1,a_i}$ (c.f. §4.1c). Since the group actions above are unitary, these spaces inherit symplectic forms from the standard one on affine space. Here $B_{m,r,a}$ carries the Auroux-type almost toric fibration

$$\pi_{\mathrm{Aur}} : B_{m,r,a} \to \mathbb{C}, \qquad \pi_{\mathrm{Aur}}(x,y,z) = \left(|z|^2, \tfrac{1}{2}|x|^2 - \tfrac{1}{2}|y|^2\right), \tag{4.2.3}$$

which has m_i focus-focus critical points mapping to distinct base-nodes. Note that π_{Aur} this does indeed descend to the quotient by $\mu_r^{(1,-1,a)}$ (see [Eva23, §7.4] and §6.2 below).

By comparing the local description of $\mathbb{Q}$-Gorenstein smoothings of T-singularities as in (4.1.3) with the above cut-and-paste description of $\mathbb{A}(Q_{\mathrm{nodal}})$, we have the following.

Proposition 4.2.6. *Let Q be a T-polygon. For any sufficiently close $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q, with integrable almost complex structure $\widetilde{J}$, there exists a diffeomorphism $\Phi : \widetilde{V}_Q \to \mathbb{A}(Q_{\mathrm{nodal}})$ such that $\Phi_*(\widetilde{J})$ tames the symplectic form on $\mathbb{A}(Q_{\mathrm{nodal}})$.*

4.2c Mutations of almost toric fibrations

Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a T-polygon. For almost toric fibrations of the form $\mathbb{A}(Q_{\mathrm{nodal}})$ as in §4.2b, mutating Q at a vertex $\mathfrak{v}$ as in §4.1c recovers the familiar notion of mutation for almost toric fibrations (see e.g. [Eva23, §8.4]). One can check that, up to nodal slides (i.e. moving the base-nodes), $(\mathrm{Mut}_{\mathfrak{v}}(Q))_{\mathrm{nodal}}$ and Q_{nodal} are isomorphic nodal integral affine polygons, differing in only their presentations in terms of polygons with branch cuts (roughly speaking, a primitive mutation at a vertex $\mathfrak{v}$ corresponds to rotating the branch cut at one of the nearby base-nodes by 180 degrees). In particular, we have:

Proposition 4.2.7. *If $Q, Q' \subset \mathbb{M}_{\mathbb{R}}$ are mutation equivalent T-polygons, then the symplectic four-manifolds $\mathbb{A}(Q_{\mathrm{nodal}})$ and $\mathbb{A}(Q'_{\mathrm{nodal}})$ are symplectomorphic.*

5 Singular algebraic curves in almost toric manifolds I

In this section, we first discuss in §5.1 various geometric features of a symplectic four-manifold which are "visible" from the base of an almost toric fibration, such as (singular) symplectic and Lagrangian subspaces and ellipsoid embeddings. Then, in §5.2 we construct some explicit rational holomorphic curves with prescribed cusp singularities in (singular) toric surfaces. Finally, in §5.3, we explain how to push these holomorphic curves into symplectic rigid del Pezzo surfaces in order to construct inner corner curves and prove Theorem E.

5.1 Visible geometry in almost toric fibrations

5.1a Visible Lagrangians and symplectic curves

Suppose that $\pi : M^4 \to B^2$ is a four-dimensional regular Lagrangian torus fibration, and let $C^2 \subset M^4$ be a compact two-dimensional submanifold which projects to a path $\gamma \subset B^2$. Put $C_{\mathfrak{b}} := C \cap \pi^{-1}(\mathfrak{b})$ for each $\mathfrak{b} \in B$. Let $\langle -, - \rangle$ denote the pairing from (4.2.1). The following is readily checked using the symplectomorphism (4.2.2):

- C is Lagrangian if and only if $\langle u, v \rangle = 0$ for any $\mathfrak{p} \in C$, $u \in T_{\pi(\mathfrak{p})}\gamma$, and $v \in T_{\mathfrak{p}} C_{\pi(\mathfrak{p})}$
- C is symplectic if and only if $\langle u, v \rangle \neq 0$ for any $\mathfrak{p} \in C$, $u \in T_{\pi(\mathfrak{p})}\gamma$, and $v \in T_{\mathfrak{p}} C_{\pi(\mathfrak{p})}$.

In these situations we will say that C is a **visible** Lagrangian or symplectic submanifold of M. For $\mathfrak{b} \in \gamma$, we will say that the fiber $C_{\mathfrak{b}}$ is **straight** if it is an orbit of the action of $T^*_{\mathfrak{b}} B$ on $\pi^{-1}(\mathfrak{b})$. For a visible Lagrangian C, it is easy to check that each fiber $C_{\mathfrak{b}}$ is straight, and thus each tangent vector to γ is a multiple of a lattice vector, so γ is also straight with respect to the integral affine structure on B.

For a visible symplectic curve C the fibers $C_{\mathfrak{b}}$ need not be straight, but at each point $\mathfrak{b} \in \gamma$ there is a unique (up to sign) primitive covector $\alpha_{\mathfrak{b}} \in \Lambda^*_{\mathfrak{b}}$ such that $C_{\mathfrak{b}}$ is homologous to an orbit of the vector field along $\pi^{-1}(\mathfrak{b})$ corresponding to $\alpha_{\mathfrak{b}}$ (c.f. the discussion after (4.2.1).) . Thus for a visible symplectic curve C there is a section α of the pullback of Λ^* along γ, such that $\alpha_{\mathfrak{b}}(v) \neq 0$ for every $\mathfrak{b} \in \gamma$ and $0 \neq v \in T_{\mathfrak{b}}\gamma$. We will refer to the pair (γ, α) as a **covector-decorated path**. For future reference, we note that the symplectic area of the visible symplectic curve C is naturally computed by the integrating the covector field α along γ:

$$\mathrm{area}(C) = \int_t \alpha(\gamma'(t))dt. \tag{5.1.1}$$

Example 5.1.1. Suppose that B is $\mathbb{R}^2$ with its standard integral affine structure and coordinates x_1, x_2, and let $\gamma : (a, b) \mapsto \mathbb{R}^2$ be the straight line segment $t \to (pt, qt)$ for some primitive lattice vector $(p, q) \in \mathbb{Z}^2$. Then we can take α to be the constant covectorfield $pdx_1 + qdx_2$, and the area of the corresponding visible symplectic cylinder $C_{(\gamma,\alpha)} \subset T^*\mathbb{T}^2$ is $(b - a)(p^2 + q^2)$. $\Diamond$

One can extend the above discussion to define visible Lagrangian and symplectic submanifolds in an almost toric fibration $\pi : M^4 \to B^2$. Over B^{reg} the situation is identical, and with some care we can also allow paths in B which end on toric boundary points or base-nodes. Roughly, as $\mathfrak{b}' \in B^{\mathrm{reg}}$ approaches an interior point $\mathfrak{b}$ of an edge e of a moment polygon, the torus fibers $\pi^{-1}(\mathfrak{b}')$ collapse to circles along a direction determined by e, and, similarly, as $\mathfrak{b}'$ approaches a Delzant vertex $\mathfrak{v}$ (having no base-nodes) the torus fibers collapse to a point.[26] Meanwhile, as $\mathfrak{b}' \in B^{\mathrm{reg}}$ approaches a base-node $\mathfrak{b}$, the torus fibers $\pi^{-1}(\mathfrak{b}')$ get pinched along a circle which depends on the eigenline $\mathbb{L}_{\mathfrak{b}} \in T_{\mathfrak{b}}B$. In slightly more detail, we have the following building blocks:

[26] Note that a general corank one or two elliptic value $\mathfrak{b} \in B$ is locally integral affine isomorphic to one of these situations.

(i) a smooth Lagrangian disk over any straight line segment γ ending on a base-node $\mathfrak{b}$, provided that γ is tangent to the eigenline $\mathbb{L}_{\mathfrak{b}}$ at $\mathfrak{b}$

(ii) a symplectic disk over any covector-decorated path (γ, α) ending perpendicularly to an interior point $\mathfrak{b}$ of an edge e, provided that the covector $\alpha_{\mathfrak{b}} \in T^*_{\mathfrak{b}}B$ vanishes on the tangent space $T_{\mathfrak{b}}e$ to the edge

(iii) a symplectic disk over any covector-decorated path (γ, α) ending on a base-node $\mathfrak{b}$ perdendicularly to its eigenline $\mathbb{L}_{\mathfrak{b}}$, provided that the covector $\alpha_{\mathfrak{b}}$ vanishes on $\mathbb{L}_{\mathfrak{b}}$

(iv) a (p,q)-unicuspidal symplectic disk over any covector-decorated path (γ, α) that has one end on a Delzant vertex $\mathfrak{v}$, where p and q are obtained by evaluating α on the primitive tangent vectors to the edges adjacent to $\mathfrak{v}$ (see also (3.1.1) below).

The symplectic discs in (ii), (iii) are smooth if the paths are straight near their endpoints. Other local models for visible singular Lagrangians are also discussed e.g. in [ES18, §5.1 and §6.4].

Let us now specialize to the case of an almost toric fibration $\pi : \mathbb{A}(Q_{\text{nodal}}) \to Q_{\text{nodal}}$ associated to a T-polygon $Q \subset \mathbb{R}^2$ as in §4.2b. Here we assume that the vertex $\mathfrak{v}_i$ of Q is of type $\frac{1}{m_i r_i^2}(1, m_i r_i a_i - 1)$. Note that a path in Q_{nodal} is smooth in the usual sense in $\mathbb{R}^2$, except that whenever it crosses some γ_i it bends by the appropriate shear. In this case we have for example:

(a) a visible Lagrangian two-sphere $\mathbb{L}_{[\mathfrak{b}_i^j, \mathfrak{b}_i^{j+1}]}$ over the line segment $[\mathfrak{b}_i^j, \mathfrak{b}_i^{j+1}] \subset Q_{\text{nodal}}$ for each $i = 1, \dots, \ell$ and $j = 1, \dots, m_i - 1$ (c.f. [Eva23, Fig. 7.8])

(b) a visible **Lagrangian (r_i, a_i)-pinwheel** $\mathbb{L}_{[\mathfrak{b}_i^{m_i}, \mathfrak{v}_i]}$ (see [Kho13, Def. 3.1]) over the line segment $[\mathfrak{b}_i^{m_i}, \mathfrak{v}_i]$ for each $i = 1, \dots, \ell$ (c.f. again [Eva23, Fig. 7.8])

(c) a visible symplectic two-sphere C with $c_1(C) = 2$ over any straight line segment in $Q_{\text{nodal}} \setminus \{\mathfrak{b}_i^j\}$ with both ends ending perpendicularly on interior points of edges (see the violet path in Figure 6 left)

(d) a visible nonsingular symplectic two-sphere C with $c_1(C) = 1$ over any straight line segment γ in Q_{nodal} with one end ending perpendicularly on an interior point of an edge and the other end ending on a base-node perpendicularly to the associated eigenline (see the blue path in Figure 6 right)

(e) a visible (p,q)-unicuspidal symplectic two-sphere $C_{p,q}$ with $c_1(C) = p + q$ over any straight line segment γ in Q_{nodal} of slope p/q with one end at a Delzant vertex (having no base-nodes) and the other end ending on a base-node perpendicularly to the associated eigenray (see the blue path in Figure 5 right).

The above claims about the first Chern class $c_1(C)$ are justified by work of Symington, quoted in Lemma 5.1.7 below. As illustrated in Figure 6 below (see also Figure 3), the normal crossing resolution $\widetilde{C}_{p,q}$ of the cuspidal curve $C_{p,q}$ is a curve of type (d) that is visible in an appropriate blowup of $\mathbb{A}(Q_{\text{nodal}})$ at its smooth point. Note also that a Lagrangian (r,a)-pinwheel is homeomorphic to the closed unit two-disk $\overline{\mathbb{D}}^2$ with its boundary quotiented out by the equivalence relation $z \sim e^{2\pi a\sqrt{-1}/r}z$ for $z \in \partial\overline{\mathbb{D}}^2$. In particular, a Lagrangian $(1,1)$-pinwheel is just an embedded Lagrangian disk and a Lagrangian $(2,1)$-pinwheel is an embedded Lagrangian real projective plane $\mathbb{RP}^2$. The visible Lagrangians (a) and (b) will be useful for describing the symplectic form on $\mathbb{A}(Q_{\text{nodal}})$ in the sequel, while the visible symplectic curves (c), (d), and (e) give symplectic analogues of some of the algebraic curves which we construct in §5 and §6 respectively.

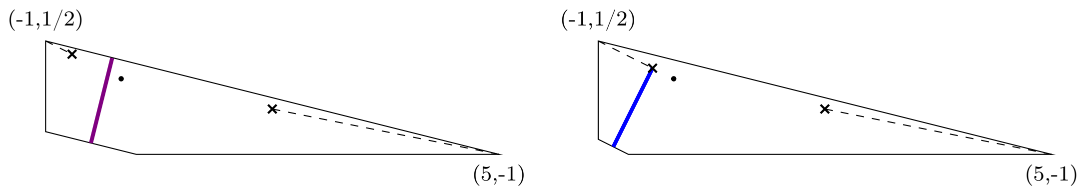


Figure 6: Some visible symplectic curves in almost toric fibrations. The curve $\widetilde{C}_{1,2}$ in the right diagram is the normal crossing resolution of the visible cuspidal curve $C_{1,2}$

5.1b Visible ellipsoid embeddings

Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a T-polygon, and let $\pi : \mathbb{A}(Q_{\text{nodal}}) \to Q_{\text{nodal}}$ denote the corresponding almost toric fibration as in §4.2b. Noting that the base-nodes of Q_{nodal} can be pushed arbitrarily close to the boundary by nodal slides (and recalling Remark 4.2.3), we have:

Proposition 5.1.2 ([Cri+20, Prop. 2.35])**.** *Let $\mathfrak{v}$ be a Delzant vertex of Q such that the adjacent edges have affine lengths a and b. Then for any $\varepsilon > 0$ we have a symplectic embedding $E(\frac{a}{1+\varepsilon}, \frac{b}{1+\varepsilon}) \overset{s}{\hookrightarrow} \mathbb{A}(Q_{\text{nodal}})$.*

We will refer to an embedding as in Proposition 5.1.2 as a **visible ellipsoid embedding**. Note that if Q has a smooth corner $\mathfrak{v}_{\text{sm}}$, that the same is true of its mutations $\text{Mut}_{\mathfrak{v}}(Q)$ at vertices $\mathfrak{v} \neq \mathfrak{v}_{\text{sm}}$. Since by Proposition 4.2.7 Q and $\text{Mut}_{\mathfrak{v}}(Q)$ encode symplectic four-manifolds which are symplectomorphic, this gives a mechanism for constructing many ellipsoid embeddings into a fixed target space. Indeed, the ellipsoid embeddings corresponding to the inner corner points of the rigid del Pezzo infinite staircases can all be described in this way:

Proposition 5.1.3 ([Cri+20, §5])**.** *Let M be a rigid del Pezzo surface, and let Q be the corresponding dual Fano T-polygon (either a triangle or a quadrilateral) in Figure 4. Then each inner corner point of the infinite staircase in the ellipsoid embedding function $c_M(x)$ corresponds to a visible ellipsoid embedding after applying a sequence of full mutations to Q.*

If Q is a triangle then the ellipsoidal embedding in Proposition 5.1.3 fills the entire volume of M and hence is clearly maximal. We will explain in §6.1 (see in particular Proposition 6.1.11) why this is still the case when Q is a quadrilateral via the notion of visible symplectic obstructions. It will follow the numerics of these staircases can be completely understood in terms of structures that are visible in suitable families of almost toric bases.

5.1c Cohomology class of the symplectic form

For Q a Delzant polygon with edges $e_1,\dots,e_\ell$, recall that the associated toric surface V_Q carries a natural Kähler form ω for which Q is the moment polygon of a Hamiltonian $\mathbb{T}^2$-action (see e.g. [Da 03, §6.6]). Using (4.1.1) together with the fact that the toric divisor $\mathbf{D}_{e_i}$ has symplectic area $\mathrm{Len}_{\mathrm{aff}}(e_i)$ for $i=1,\dots,\ell$, this characterizes the cohomology class $[\omega] \in H^2(V_Q;\mathbb{R})$.

We seek to extend this to almost toric fibrations of the type $\pi : \mathbb{A}(Q_{\mathrm{nodal}}) \to Q_{\mathrm{nodal}}$ constructed in §4.2b. Under the homeomorphic identification $Q \cong Q_{\mathrm{nodal}}$, the vertices $\mathfrak{v}_1,\dots,\mathfrak{v}_\ell$ of Q now correspond to corank one elliptic values in Q_{nodal}. In particular, the edge preimages $\pi^{-1}(e_i) \subset \mathbb{A}(Q_{\mathrm{nodal}})$ are symplectic annuli rather than two-spheres, so they do not a priori represent homology classes. However, note that r_i times the circle $\pi^{-1}(\mathfrak{v}_i)$ bounds the Lagrangian pinwheel $\mathbb{L}_{[\mathfrak{b}_i^{m_i},\mathfrak{v}_i]}$ discussed in §5.1a. Therefore the rational cycle

$$\widetilde{\mathbf{D}}_{e_i} := \pi^{-1}(e_i) - \tfrac{1}{r_i}\cdot \mathbb{L}_{[\mathfrak{b}_i^{m_i},\mathfrak{v}_i]} + \tfrac{1}{r_{i+1}}\cdot \mathbb{L}_{[\mathfrak{b}_{i+1}^{m_{i+1}},\mathfrak{v}_{i+1}]} \tag{5.1.2}$$

defines a homology class $[\widetilde{\mathbf{D}}_{e_i}] \in H_2(\mathbb{A}(Q_{\mathrm{nodal}});\mathbb{Q})$.

Using the cut-and-paste discussion in §4.2b, we have:

Lemma 5.1.4. *The rational homology group $H_2(\mathbb{A}(Q_{\mathrm{nodal}});\mathbb{Q})$ is generated by $[\widetilde{\mathbf{D}}_{e_i}]$ for $i=1,\dots,\ell$ and $[\mathbb{L}_{[\mathfrak{b}_i^j,\mathfrak{b}_i^{j+1}]}]$ for $i=1,\dots,\ell$ and $j=1,\dots,m_i-1$.*

Corollary 5.1.5. *A homology class $A \in H_2(\mathbb{A}(Q_{\mathrm{nodal}});\mathbb{R})$ is Poincaré dual to the cohomology class of the symplectic form if and only if we have $A\cdot[\widetilde{\mathbf{D}}_{e_i}] = \mathrm{Len}_{\mathrm{aff}}(e_i)$ for $i=1,\dots,\ell$ and $A\cdot[\mathbb{L}_{[\mathfrak{b}_i^j,\mathfrak{b}_i^{j+1}]}] = 0$ for $i=1,\dots,\ell$ and $j=1,\dots,m_i-1$.*

By Lemma 4.1.1, if Q is a Delzant polygon the toric symplectic four-manifold V_Q is monotone with monotonicity constant 1 (i.e. $[\omega]=\mathrm{PD}(c_1)$) if and only if Q is dual Fano. This extends to almost toric manifolds as follows:

Proposition 5.1.6. *If Q is a dual Fano T-polygon, then the symplectic four-manifold $\mathbb{A}(Q_{\mathrm{nodal}})$ is monotone with monotonicity constant 1.*

Before beginning the proof, we recall the following result.

Lemma 5.1.7 ([Sym, Prop. 8.2])**.** *For a general T-polygon Q with edges $e_1,\dots,e_\ell$, the full boundary preimage $\pi^{-1}(\partial Q_{\mathrm{nodal}}) = \bigcup_{i=1}^{\ell} \pi^{-1}(e_i)$ is Poincaré dual to $c_1(\mathbb{A}(Q_{\mathrm{nodal}}))$.*

Proof of Proposition 5.1.6. By [KNP17, Thm. 1.2] there are precisely 10 mutation equivalence classes of dual Fano T-polygons, corresponding to the 10 topological types of smooth del Pezzo surfaces. Since mutations imply symplectomorphisms (see Proposition 4.2.7), it suffices to check the result for one representative in each of the 10 equivalence classes.

In fact, [Via17] shows that 8 of the 10 mutation equivalence classes of dual Fano T-polygons have a triangular representative. Moreover, the two exceptions, $\mathbb{CP}^1(3)\#\overline{\mathbb{CP}}^2(1)$ and $\mathbb{CP}^1(3)\#^{\times 2}\overline{\mathbb{CP}}^2(1)$, instead have Delzant representatives, for which the result follows directly by Lemma 4.1.1 since in the smooth case $\mathbb{A}(Q) = V_Q$.

Now suppose that $Q \subset \mathbb{R}^2$ is a dual Fano T-triangle. Note that the boundary preimage $[\pi^{-1}(\partial Q_{\text{nodal}})]$ together with the visible Lagrangian spheres $[\mathbb{L}_{[\mathfrak{b}_i^j,\mathfrak{b}_i^{j+1}]}]$ for $i = 1,\ldots,\ell$ and $j = 1,\ldots,m_i - 1$ form a basis for $H_2(\mathbb{A}(Q_{\text{nodal}});\mathbb{Q})$, and since $c_1([\mathbb{L}_{[\mathfrak{b}_i^j,\mathfrak{b}_i^{j+1}]}]) = [\mathbb{L}_{[\mathfrak{b}_i^j,\mathfrak{b}_i^{j+1}]}] \cdot [\pi^{-1}(\partial Q_{\text{nodal}})] = 0$, it follows that $\mathbb{A}(Q_{\text{nodal}})$ is monotone.

To see that the monotonicity constant is 1, we will define a test homology class in $H_2(\mathbb{A}(Q_{\text{nodal}});\mathbb{R})$ and check that its symplectic area agrees with its Chern number. Let e_1, e_2, e_3 denote the edges of Q, where e_i has primitive outward normal vector $(p_i, q_i) \in \mathbb{Z}^2$ and affine length $\ell_i \in \mathbb{R}_{>0}$, so that we have $\sum_{i=1}^{3} \ell_i \cdot (p_i, q_i) = (0,0)$. Let $\gamma_i \subset \mathbb{R}^2$ denote a straight line segment which starts at the origin and ends on the interior of the edge e_i, let α_i denote the covector field along γ_i with constant value $p_i dx_1 + q_i dx_2$, and let $C_i := C_{(\gamma_i,\alpha_i)} \subset \mathbb{A}(Q_{\text{nodal}})$ denote the corresponding visible symplectic disk as in §5.1a. Using (5.1.1) and the fact that Q is dual Fano, the symplectic area of C_i is 1. Also, note that ∂C_i is a circle in the Lagrangian torus fiber $\pi^{-1}(\vec{0})$ for $i = 1,2,3$, and we have

$$\sum_{i=1}^{3} \ell_i [\partial C_i] = 0 \in H_1(\pi^{-1}(\vec{0});\mathbb{R}).$$

Thus we can find a Lagrangian $\mathbb{R}$-chain $\mathcal{L}$ in $\pi^{-1}(\vec{0})$ such that $\sum_{i=1}^{3} \ell_i C_i + \mathcal{L}$ is an $\mathbb{R}$-cycle in $\mathbb{A}(Q_{\text{nodal}})$, say representing a class $A \in H_2(\mathbb{A}(Q_{\text{nodal}});\mathbb{R})$. The symplectic area of A is $\sum_{i=1}^{3} \ell_i \operatorname{area}(C_i) = \ell_1 + \ell_2 + \ell_3$, and by Lemma 5.1.7 its Chern number is also $A \cdot [\pi^{-1}(\partial Q_{\text{nodal}})] = \ell_1 + \ell_2 + \ell_3$. □

5.2 Rational curves in toric surfaces

We now turn our attention to the construction of rational algebraic curves in toric surfaces, typically of strictly positive index. Our approach here is to write down explicit formulas for rational curves in the dense complex torus $(\mathbb{C}^*)^2 \subset V_Q$, and then take their closures to obtain rational curves in V_Q. This could be seen as a step in the direction of a tropical-to-holomorphic correspondence as in [Mik05], but here we focus primarily on those curves which we will need in §5.3, keeping the discussion as elementary and explicit as possible.

As before, let $\mathbb{M}$ be a rank two lattice with dual lattice $\mathbb{N}$, and put $\mathbb{M}_{\mathbb{R}} = \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{N}_{\mathbb{R}} = \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}$. In the following, we will say that a polygon $Q \subset \mathbb{M}_{\mathbb{R}}$ is **rational** if all of its vertices lie in $\mathbb{M}_{\mathbb{Q}} := \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that any rational polygon becomes a lattice polygon after scaling by some positive integer.

Proposition 5.2.1. *Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a rational polygon with edges $e_1, \ldots, e_\ell$ and corresponding primitive inward normals $\vec{n}_1, \ldots, \vec{n}_\ell \in \mathbb{N}$. Suppose that we are given a set (possibly empty) of positive integers $J_{e_i} = \{k_1^i, \ldots, k_{s_i}^i\}$ for each $i = 1, \ldots, \ell$, such that the following balancing condition holds*

$$\sum_{i=1}^{\ell} \sum_{j=1}^{s_i} k_j^i \vec{n}_i = \vec{0}, \tag{5.2.1}$$

and such that the quantities $d_i := \sum_{j=1}^{s_i} k_j^i$ satisfy $\gcd(d_1, \ldots, d_\ell) = 1$. Then there exists a rational algebraic curve C in V_Q such that, for $i = 1, \ldots, \ell$, C intersects $\mathbf{D}_{e_i}$ in its interior in precisely s_i local branches, with contact orders $k_1^i, \ldots, k_{s_i}^i$, and C is otherwise disjoint from $\mathbf{D}_{e_i}$.

Example 5.2.2. Figure 5 middle gives a tropical representation of the curve C in Proposition 5.2.1 in the case that Q is a triangle with vertices $(-1,-1), (-1,1), (3,-1)$, with $J_{e_1} = \{2\}$, $J_{e_2} = \{1\}$, $J_{e_3} = \{1\}$. $\Diamond$

Let Σ denote the punctured Riemann surface $\mathbb{CP}^1 \smallsetminus \{w_1, \ldots, w_\varkappa\}$ for some pairwise distinct $w_1, \ldots, w_\varkappa \in \mathbb{CP}^1$. Given $\vec{\mathfrak{o}} = (\mathfrak{o}_1, \ldots, \mathfrak{o}_\varkappa) \in \mathbb{Z}^\varkappa$, there is a unique (up to choice of constant $A \in \mathbb{C}^*$) holomorphic function $f_{\Sigma,\vec{\mathfrak{o}}} : \Sigma \to \mathbb{C}^*$ with zero of order $\mathfrak{o}_i$ (i.e. pole of order $-\mathfrak{o}_i$) at the puncture z_i for $i = 1, \ldots, \varkappa$ and no other zeros or poles, given explicitly by

$$f_{\Sigma,\vec{\mathfrak{o}}}(z) = A(z - w_1)^{\mathfrak{o}_1} \cdots (z - w_\varkappa)^{\mathfrak{o}_\varkappa}. \tag{5.2.2}$$

For simplicity we will usually take $A = 1$.

By **ordered toric degree** we will mean a tuple $\boldsymbol{\delta} = (\vec{v}_1, \ldots, \vec{v}_\varkappa)$ for some $\varkappa \in \mathbb{Z}_{\geqslant 2}$, with $\vec{v}_1, \ldots, \vec{v}_\varkappa \in \mathbb{N} \smallsetminus \{\vec{0}\}$ such that $\sum_{i=1}^{\varkappa} \vec{v}_i = \vec{0}$. Let $T_{\mathbb{N}} \subset V_Q$ denote the complex torus $\mathbb{N} \otimes_{\mathbb{Z}} \mathbb{C}^*$, which is identified with $(\mathbb{C}^*)^n$ after choosing a basis $\vec{b}_1, \ldots, \vec{b}_n$ for $\mathbb{N}$. Given an ordered toric degree $\boldsymbol{\delta} = (\vec{v}_1, \ldots, \vec{v}_\varkappa)$, we have the holomorphic function $f_{\Sigma,\boldsymbol{\delta}} : \Sigma \to T_{\mathbb{N}}$ whose jth component with respect to the chosen basis is $f_{\Sigma,(v_1^j,\ldots,v_\varkappa^j)}$, where we put $\vec{v}_i = \sum_{j=1}^n v_i^j \vec{b}_j$ for $i = 1, \ldots, \varkappa$. More explicitly, in the case $\mathbb{N}_{\mathbb{R}} = \mathbb{R}^2$ with $\vec{b}_1, \vec{b}_2$ the standard basis, we put

$$f_{\Sigma,\boldsymbol{\delta}}(z) = (f_x(z), f_y(z)) = \left((z - w_1)^{v_1^x} \cdots (z - w_\ell)^{v_\ell^x}, (z - w_1)^{v_1^y} \cdots (z - w_\ell)^{v_\ell^y}\right), \tag{5.2.3}$$

with $\vec{v}_i = (v_i^x, v_i^y)$ for $i = 1, \ldots, \varkappa$.

Proof of Proposition 5.2.1. Put $\boldsymbol{\delta} := (k_1^1 \vec{n}_1, \ldots, k_{s_1}^1 \vec{n}_1, \ldots, k_1^\ell \vec{n}_\ell, \ldots, k_{s_\ell}^\ell \vec{n}_\ell)$, let Σ be $\mathbb{CP}^1$ minus $\sum_{i=1}^{\ell} \sum_{j=1}^{s_i} k_j^i$ punctures, and let C be the closure of the image of $f_{\Sigma,\boldsymbol{\delta}} : \Sigma \to (\mathbb{C}^*)^2 \subset V_Q$.

We will show that at the first puncture, w_1, C intersects $\mathbf{D}_{e_1}$ in its interior with contact order k_1^1, the situation being similar at the other punctures by symmetry. We may assume that our basis for $\mathbb{N}$ is chosen such that $\vec{n}_1 = (0,1)$. Let $\sigma = \mathbb{R}_{\geqslant 0}\langle \vec{n}_1 \rangle \subset \mathbb{N}_{\mathbb{R}}$ be the cone generated by $\vec{n}_1$, with the dual cone $\sigma^\vee \subset \mathbb{M}_{\mathbb{R}}$ generated by $(0,1), (1,0), (-1,0)$. The corresponding affine toric variety U_σ is identified with $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_3 = 1\}$, with $\mathbf{D}_{e_1} \cap U_\sigma = \{(z_1, z_2, z_3) \in U_\sigma \mid z_2 = 0\}$ and with the inclusion map $\iota : (\mathbb{C}^*)^2 \to U_\sigma$ given by $(x, y) \mapsto (x, y, x^{-1})$. We thus have $(\iota \circ f_{\Sigma,\boldsymbol{\delta}})(z) = (f_x(z), f_y(z), f_x(z)^{-1})$, and therefore

$$\begin{aligned}
\lim_{z \to w_1} (\iota \circ f_{\Sigma,\boldsymbol{\delta}})(z) &= \lim_{z \to w_1} (A(z - w_1)^{k_1^1 n_1^x}, B(z - w_1)^{k_1^1 n_1^y}, A^{-1}(z - w_1)^{-k_1^1 n_1^x}) \\
&= \lim_{z \to w_1} (A(z - w_1)^{k_1^1 \cdot 0}, B(z - w_1)^{k_1^1 \cdot 1}, A^{-1}(z - w_1)^{-k_1^1 \cdot 0}) \\
&= (A, 0, A^{-1})
\end{aligned}$$

for some constants $A, B \in \mathbb{C}^*$. The corresponding contact order with $\mathbf{D}_{e_1}$ is given by the vanishing order of $f_y(z)$ as $z \to w_1$, which is k_1^1. $\square$

Corollary 5.2.3. *Assume that $Q \subset \mathbb{M}_{\mathbb{R}}$ is a lattice polygon such that the edge affine lengths $\mathrm{Len}_{\mathrm{aff}}(e_1), \dots, \mathrm{Len}_{\mathrm{aff}}(e_\ell) \in \mathbb{Z}_{\geqslant 1}$ are coprime. Then there exists a rational algebraic curve C in V_Q such that C intersects $\mathbf{D}_{e_i}$ in a single point in its interior with multiplicity $\mathrm{Len}_{\mathrm{aff}}(e_i)$ for $i = 1, \dots, \ell$.*

Proof. Put $J_{e_i} = \{\mathrm{Len}_{\mathrm{aff}}(e_i)\}$ for $i = 1, \dots, \ell$. Since Q is a closed polygon, we have $\sum_{i=1}^{\ell} e_i = \vec{0}$, which implies the balancing condition (5.2.1). $\square$

The following is an algebraic counterpart of the visible symplectic curves over straight lines discussed in §5.1a:

Corollary 5.2.4. *Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a rational polygon having two parallel edges e_+, e_-. Then there exists a nonsingular rational algebraic curve C in V_Q which intersects each of $\mathbf{D}_{e_+}, \mathbf{D}_{e_-}$ transversely in one point and is disjoint from the other toric divisors.*

We will also need to know that the curves constructed above are suitably robust under deformations of the complex structure. Let $\mathcal{M}(J_{e_1}, \dots, J_{e_\ell})$ denote the (uncompactified) moduli space of curves C as in Proposition 5.2.1. Here we view curves in $\mathcal{M}(J_{e_1}, \dots, J_{e_\ell})$ as holomorphic maps $\mathbb{CP}^1 \to V_Q$ (modulo biholomorphic reparametrization) having specified intersection pattern with the toric divisors $\mathbf{D}_{e_1}, \dots, \mathbf{D}_{e_\ell}$.

Lemma 5.2.5. *The moduli space $\mathcal{M}(J_{e_1}, \dots, J_{e_\ell})$ is regular.*

Proof. Let $u : \mathbb{CP}^1 \to V_Q$ be a curve in $\mathcal{M}(J_{e_1}, \dots, J_{e_\ell})$. Then u is regular by automatic transversality (see [Wen10, Thm. 1]) provided that we have $\mathrm{ind}_{\mathbb{R}}(C) > 2Z(du) - 2$, where $Z(du)$ is the total (complex) vanishing order of the derivative of u at all of its critical points. Note that we have $\mathrm{ind}_{\mathbb{R}}(u) = 2\varkappa - 2$, where $\varkappa = \sum_{i=1}^{\ell} s_i$ is the number of punctures of Σ. Meanwhile, we have

$$\frac{f_x'(z)}{f_x(z)} = \frac{d}{dz}\log(f_x(z)) = \sum_{i=1}^{\varkappa} \frac{v_i^x}{z - w_i} = \frac{P(z)}{(z-w_1)\cdots(z-w_\varkappa)},$$

where $P(z)$ is a polynomial of degree at most $\varkappa - 1$ (actually at most $\varkappa - 2$ since $\sum_{i=1}^{\varkappa} v_i^x = 0$), so we have $Z(du) \leqslant \varkappa - 1$ and thus $\mathrm{ind}_{\mathbb{R}}(u) = 2\varkappa - 2 > 2\varkappa - 4 \geqslant 2Z(du) - 2$. $\square$

Remark 5.2.6. In the case $\varkappa = 3$, $f_{\Sigma,\boldsymbol{\delta}}$ is in fact an immersion, i.e. $Z(du) = 0$. Indeed, without loss of generality we can take $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$, so that (5.2.3) becomes

$$f_{\Sigma,\boldsymbol{\delta}}(z) = \left(z^{v_1^x}(z-1)^{v_2^x}, z^{v_1^y}(z-1)^{v_2^y}.\right).$$

Observe that we have $f_{\Sigma,\boldsymbol{\delta}} = \Phi \circ g$, where $g : \Sigma \to (\mathbb{C}^*)^2$, $g(z) = (z, z-1)$ is a parametrization of the (nonsingular) pair of pants $\{x = y+1\} \subset (\mathbb{C}^*)^2$ and $\Phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$, $\Phi(x,y) = (x^{v_1^x}y^{v_2^x}, x^{v_1^y}y^{v_2^y})$ is a degree $|\det(\vec{v}_1, \vec{v}_2)|$ holomorphic covering map (c.f. [Eva23, §G.2]). $\lozenge$

5.3 Inflatable sesquicuspidal curves and the inner corners

The goal of this subsection is to prove the following result, which we then use to deduce Theorem E. Recall that we associate to a T-polygon Q an almost toric fibration $\pi : \mathbb{A}(Q_{\mathrm{nodal}}) \to Q_{\mathrm{nodal}}$ as in §4.2b.

Theorem 5.3.1. *Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a dual Fano lattice T-polygon with a Delzant vertex $\mathfrak{v}_{\mathrm{sm}}$. Let e, e' be the edges adjacent to $\mathfrak{v}_{\mathrm{sm}}$, and put $\mathrm{Len}_{\mathrm{aff}}(e) = kp$ and $\mathrm{Len}_{\mathrm{aff}}(e') = kq$, where $k, p, q \in \mathbb{Z}_{\geqslant 1}$ satisfy $\gcd(p,q) = 1$ and $p \geqslant q$. Assume also that the affine lengths of the edges of Q are coprime. Then there exists a rational J_{int}-holomorphic curve C in $\mathbb{A}(Q_{\mathrm{nodal}})$, where:*

- *C has a cusp with Puiseux characteristic $(q;p)$ if $k = 1$ and $(kq; kp, kp+1)$ if $k \geqslant 2$*
- *C is Poincaré dual to the symplectic form on $\mathbb{A}(Q_{\mathrm{nodal}})$*
- *$[C] \cdot [C] \geqslant k^2 pq$*
- *J_{int} is a tame integrable almost complex structure on $\mathbb{A}(Q_{\mathrm{nodal}})$ such that $(\mathbb{A}(Q_{\mathrm{nodal}}), J_{\mathrm{int}})$ is biholomorphic to a $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q.*

Moreover, when Q is a triangle we can assume that C is unicuspidal.

Note that in the symplectic category we can easily perturb C to make it sesquicuspidal (i.e. positively immersed away from the cusp), although this is not guaranteed as a J_{int}-holomorphic curve. In particular, after such a perturbation C satisfies the hypotheses of

Theorem A with $c = 1$, i.e. inflating along C gives a symplectic embedding $E(\frac{kq}{c'}, \frac{kp}{c'}) \overset{s}{\hookrightarrow} M$ for any $c' > 1$. In other words, any visible ellipsoid embedding (in the sense of Proposition 5.1.2) can be obtained by symplectic inflation along a sesquicuspidal rational symplectic curve. Theorem E upgrade this to algebraic curves in the case of rigid del Pezzo surfaces.

Proof of Theorem E. Let M be a rigid del Pezzo surface and let $(x = p/q, y)$ be an inner corner point on the graph of the corresponding ellipsoid embedding function $c_M(x)$. According to Proposition 5.1.3, there exists a dual Fano T-polygon $Q \subset \mathbb{R}^2$ such that

- $\mathbb{A}(Q_{\text{nodal}})$ is symplectomorphic to M
- Q has a Delzant vertex $\mathfrak{v}_{\text{sm}}$ with adjacent edges e, e' satisfying $\text{Len}_{\text{aff}}(e) = \frac{1}{y}$ and $\text{Len}_{\text{aff}}(e') = \frac{p}{qy}$
- Q is a triangle in the cases $M = \mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^2, \mathbb{CP}^2 \#^{\times j} \overline{\mathbb{CP}}^2$, $j = 3, 4$, and Q is a quadrilateral in the cases $M = \mathbb{CP}^2 \#^{\times j} \overline{\mathbb{CP}}^2$, $j = 1, 2$.

Let $s \in \mathbb{R}_{>0}$ be minimal scaling factor such that $s \cdot Q \subset \mathbb{R}^2$ is a lattice polygon. Then $s \cdot Q$ satisfies the hypotheses of Theorem 5.3.1, with $\text{Len}_{\text{aff}}(s \cdot e) = kq$ and $\text{Len}_{\text{aff}}(s \cdot e') = kp$ for $k := \frac{s}{qy} \in \mathbb{Z}_{\geqslant 1}$. Let C be the resulting curve in $\mathbb{A}(s \cdot Q_{\text{nodal}})$. Here $\mathbb{A}(s \cdot Q_{\text{nodal}})$ is naturally identified as a symplectic manifold with $\mathbb{A}(Q_{\text{nodal}})$ after scaling the symplectic form by s. In particular, C corresponds to a curve C' in $\mathbb{A}(Q_{\text{nodal}})$ which is Poincaré dual to s times the symplectic form. Note that $(\mathbb{A}(Q_{\text{nodal}}), J_{\text{int}})$ is a rigid Fano complex surface and hence is necessarily biholomorphic to M. Also, observe when Q is a triangle we must have $k = 1$, since then evidently $s = qy$ is minimal such that $s \cdot Q$ is a lattice triangle. Thus the curve C' verifies Thereom E. $\square$

Our proof of Theorem 5.3.1 will proceed roughly in the following steps:

1) construct a rational curve in a weighted blowup of V_Q using the results in §5.2
2) blow down to get a curve with a distinguished cusp in V_Q
3) push this curve into a $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q
4) identify $\widetilde{V}_Q$ diffeomorphically with $\mathbb{A}(Q_{\text{nodal}})$.

We now proceed with the details. Let Q as in Theorem 5.3.1, and let $\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell$ denote the vertices of Q, with corresponding edges $e_i = [\mathfrak{v}_i, \mathfrak{v}_{i+1}]$ for $i = 1, \ldots, \ell$ (here i is taken modulo ℓ). Here we take $\mathfrak{v}_1 = \text{sm}$ to be the Delzant vertex, with adjacent edges $e = e_1$ and $e' = e_\ell$.

Lemma 5.3.2. *With Q as above, there exists a rational algebraic curve C in V_Q such that*

- *for $i = 2, \ldots, \ell-1$, C intersects $\mathbf{D}_{e_i}$ in a single point in its interior with multiplicity* $\text{Len}_{\text{aff}}(e_i)$

- *$C \cap \mathbf{D}_{e_1} = C \cap \mathbf{D}_{e_\ell} = \mathbf{D}_{e_1} \cap \mathbf{D}_{e_\ell}$*
- *C has a cusp at the point $\mathbf{D}_{e_1} \cap \mathbf{D}_{e_\ell}$ with Puiseux characteristic $(q;p)$ if $k = 1$ and $(kq; kp, kp+1)$ if $k \geqslant 2$*
- *$[C] \cdot [C] \geqslant k^2 pq$.*

Moreover, when Q is a triangle (i.e. $\ell = 3$) we can assume that C is unicuspidal.

Proof of Lemma 5.3.2. Let Q' be the polygon with vertices $\mathfrak{v}_1', \mathfrak{v}_2, \ldots, \mathfrak{v}_\ell, \mathfrak{v}_1''$, where $\mathfrak{v}_1' = \frac{1}{2}\mathfrak{v}_1 + \frac{1}{2}\mathfrak{v}_2$ and $\mathfrak{v}_1'' = \frac{1}{2}\mathfrak{v}_\ell + \frac{1}{2}\mathfrak{v}_1$ (i.e. Q' is obtained from Q by "chopping off" the vertex $\mathfrak{v}_1$). Denote the corresponding edges of Q' by $e_1', e_2, \ldots, e_{\ell-1}, e_\ell', e_{\text{slant}}$, where $e_1' = [\mathfrak{v}_1', \mathfrak{v}_2]$, $e_\ell' = [\mathfrak{v}_\ell, \mathfrak{v}_1'']$, and $e_{\text{slant}} = [\mathfrak{v}_1'', \mathfrak{v}_1']$. Let $\mathbf{D}_{e_1'}, \mathbf{D}_{e_2}, \ldots, \mathbf{D}_{e_{\ell-1}}, \mathbf{D}_{e_\ell'}, \mathbf{D}_{e_{\text{slant}}}$ denote the corresponding toric boundary divisors of the associated toric surface $V_{Q'}$, which is a (p,q)-weighted blowup of V_Q. By Proposition 5.2.1, we can find a rational algebraic curve $C' \subset V_{Q'}$ such that

- C' intersects $\mathbf{D}_{e_{\text{slant}}}$ in a single point in its interior with multiplicity k
- for $i = 2, \ldots, \ell-1$, C' intersects $\mathbf{D}_{e_i}$ in a single point in its interior with multiplicity $\text{Len}_{\text{aff}}(e_i)$
- C' is disjoint from $\mathbf{D}_{e_1'}$ and $\mathbf{D}_{e_\ell'}$.

Note that, in the case $\ell = 3$, C' is nonsingular as in Corollary 5.2.4.

We now consider the image of C under the weighted blowdown $V_{Q'} \to V_Q$ along $\mathbf{D}_{e_{\text{slant}}}$. More explicitly, we first consider the iterated toric blowup V^{res} of $V_{Q'}$ which minimally resolves the singularities of $V_{Q'}$ at the toric fixed points corresponding to $\mathfrak{v}'$ and $\mathfrak{v}''$. Let C^{res} denote the proper transform of C' in V^{res}. These blowups result in a collection of negative self-intersection spheres $\mathbb{F}_1, \ldots, \mathbb{F}_L$, where $\mathbb{F}_L$ is the proper transform of $\mathbf{D}_{e_{\text{slant}}}$, such that C^{res} intersects $\mathbb{F}_L$ in a single point with multiplicity k and is disjoint from $\mathbb{F}_1, \ldots, \mathbb{F}_{L-1}$. In the case $k = 1$, the collection $\mathbb{F}_1, \ldots, \mathbb{F}_L, C^{\text{res}}$ has precisely the same intersection pattern as in the normal crossing resolution of a (p,q) cusp (c.f. §3.1), whence we can blow down along $\mathbb{F}_L, \ldots, \mathbb{F}_1$ to obtain a curve C in V_Q with a (p,q) cusp. Similarly, in the case $k \geqslant 2$, a comparison with Example 3.3.4 shows that the blown down curve C has a cusp with Puiseux characteristic $(kq; kp, kp+1)$.

To establish $[C] \cdot [C] \geqslant k^2 pq$, note that $c_1([C^{\text{res}}])$ is given by the homological intersection number of $[C^{\text{res}}]$ with the toric boundary divisor of V^{res}. This is evidently at least 2, so using the adjunction formula we have $[C^{\text{res}}] \cdot [C^{\text{res}}] \geqslant 0$, which by Lemma 3.3.5 is equivalent to $[C] \cdot [C] \geqslant k^2 pq$.

Finally, the last sentence of the lemma follows by Corollary 5.2.4. □

Proof of Theorem 5.3.1. Let C be a rational algebraic curve in V_Q as constructed by Lemma 5.3.2. In particular, C has a distinguished cusp and satisfies $[C] \cdot [C] \geqslant k^2 pq$. We view C as having a J-holomorphic parametrization $u : \mathbb{CP}^1 \to V_Q$ with a prescribed cusp singularity at $\mathbf{D}_{e_1} \cap \mathbf{D}_{e_\ell}$, where J is the preferred integrable almost complex structure on V_Q. Since C is regular (see Lemma 5.2.5) and disjoint from the singularities of

V_Q, it deforms to a nearby curve $\widetilde{C}$ with the same type of cusp in a sufficiently small $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q.[27] By Proposition 4.2.6, there is a diffeomorphism $\Phi : \widetilde{V}_Q \to \mathbb{A}(Q_{\text{nodal}})$ such that $\Phi_*(\widetilde{J})$ tames the symplectic form on $\mathbb{A}(Q_{\text{nodal}})$ (here $\widetilde{J}$ is the integrable almost complex structure on $\widetilde{V}_Q$). Put $\widetilde{C}' := \Phi(\widetilde{C})$. We can assume that the smoothing $\widetilde{V}_Q$ is such that $[\widetilde{C}'] \cdot [\widetilde{\mathbf{D}}_{e_i}] = [C] \cdot [\mathbf{D}_{e_i}]$ for $i = 1, \ldots, \ell$ (here $[\widetilde{\mathbf{D}}_{e_1}], \ldots, [\widetilde{\mathbf{D}}_{e_\ell}] \in H_2(\mathbb{A}(Q_{\text{nodal}}); \mathbb{Q})$ are the homology classes from §5.1c), and hence $\widetilde{C}'$ is Poincaré dual to the symplectic form of $\mathbb{A}(Q_{\text{nodal}})$ by Corollary 5.1.5. Note also that we have $[\widetilde{C}'] \cdot [\widetilde{C}'] = [\widetilde{C}] \cdot [\widetilde{C}] = [C] \cdot [C] \geqslant k^2 pq$. Thus $\widetilde{C}'$ satisfies all of the conclusions of Theorem 5.3.1 with $J_{\text{int}} := \Phi_*(\widetilde{J})$. □

6 Singular algebraic curves in almost toric manifolds II

In this section we develop techniques to construct index zero unicuspidal rational algebraic curves in $\mathbb{Q}$-Gorenstein smoothings of singular toric surfaces. These are closely parallel to visible symplectic curves in almost toric fibrations which pass through a focus-focus singularity (as in §5.1a). In particular, we prove Theorem 6.1.7, which is a (slight strengthening of a) restatement of Theorem D. The main technique is an explicit construction of $\mathbb{Q}$-Gorenstein pencils associated to polygon mutations as in [Ilt12; Akh+16].

More specifically, in §6.1 we introduce the notion of visible ellipsoid obstructions, which are carried by symplectic and in fact algebraic unicuspidal curves, and we observe that all obstructions for the rigid del Pezzo infinite staircases are of this type. After a brief interlude on the Auroux model in §6.2, we discuss $\mathbb{Q}$-Gorenstein pencils in §6.3, and we use these to construct explicit algebraic curves in $\mathbb{Q}$-Gorenstein smoothings in §6.4. Finally, in §6.5 we discuss the classification of index zero unicuspidal rational algebraic curves in the first Hirzebruch surface and prove Theorem F.

6.1 Unicuspidal curves from T-polygons and the outer corners

6.1a Visible ellipsoid obstructions and unicuspidal symplectic curves

We begin by discussing ellipsoid embedding obstructions which come from visible unicuspidal symplectic curves in the base of an almost toric fibration. Fix $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r, a) = 1$. Note that we do not assume $a < r$, but we can uniquely write $a = a' + \varsigma r$ for some $a' \in \{1, \ldots, r-1\}$ and $\varsigma \in \mathbb{Z}_{\geqslant 0}$. Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a T-polygon with a Delzant vertex $\mathfrak{v}$ adjacent to vertices $\mathfrak{u}$ and $\mathfrak{w}$, where $\mathfrak{w}$ has type $\frac{1}{mr^2}(1, mra-1)$, such that the eigenray emanating from $\mathfrak{w}$ intersects the line segment $[\mathfrak{u}, \mathfrak{v}]$ in a point $\mathfrak{p}$. We will assume that $\mathbb{M}_{\mathbb{R}} = \mathbb{R}^2$, $\mathfrak{v} = (0,0)$, $\mathfrak{w}$ lies on the positive y-axis, $\mathfrak{u}$ lies on the positive x-axis, and the edges emanating from $\mathfrak{w}$ are $(0,-1)$ and $(mr^2, 1 - mra)$. In particular, the eigenray emanating from $\mathfrak{w}$ points in the direction $(r, -a)$.

[27] Strictly speaking Lemma 5.2.5 says that a resolution $C' \subset V_{Q'}$ is regular (without any cusp condition), which suffices for our purposes since we can readily pass between curves with prescribed cusps and their resolutions (c.f. [McS23, §4.3]).

Put $\ell_1 := \mathrm{Len}_{\mathrm{aff}}([\mathfrak{v},\mathfrak{w}])$ and $\ell_2 := \mathrm{Len}_{\mathrm{aff}}([\mathfrak{v},\mathfrak{p}])$. Note that we have $\frac{\ell_1}{\ell_2} = \frac{a}{r}$, and, because the nodal ray from $\mathfrak{w}$ meets the side $[\mathfrak{u},\mathfrak{v}]$, there is a visible ellipsoid embedding in the sense of §5.1b (c.f. Figure 5 right[28])

$$E(\tfrac{\ell_1}{c'}, \tfrac{\ell_2}{c'}) \overset{s}{\hookrightarrow} \mathbb{A}(Q_{\mathrm{nodal}}) \tag{6.1.1}$$

for any $c' > 1$.

Proposition 6.1.1. *Let Q be a T-polygon as above such that the eigenray emanating from $\mathfrak{w}$ is in direction $(r,-a)$. Then there is an index zero rational (r,a)-unicuspidal symplectic curve C in $\mathbb{A}(Q_{\mathrm{nodal}})$ with area $r\ell_1 = a\ell_2$.*

Proof. We have $\mathfrak{v} = (0,0)$, $\mathfrak{w} = (0,\ell_1)$ and $\mathfrak{p} = (\ell_2, 0)$, with the eigenray emanating from $\mathfrak{w}$ pointing in the direction $(r,-a)$. Let (γ,α) be the covector-decorated path where:

- γ is the straight line segment starting on $\mathfrak{v}$ and ending at a point (ta,tr) on $[\mathfrak{w},\mathfrak{p}]$ for some $t \in \mathbb{R}_{>0}$
- α is the lattice covectorfield along γ which vanishes on the eigenray direction $(r,-a)$, i.e. $\alpha = a dx_1 + r dx_2$.

After a nodal slide, we can assume that Q_{nodal} has a base-node at (ta,tr). Let C be the corresponding visible (r,a)-unicuspidal symplectic curve in $\mathbb{A}(Q_{\mathrm{nodal}})$ as in item (e) in §5.1a. By (5.1.1), the symplectic area of C is given by $\int_\gamma \alpha = \langle (ta,tr),(a,r)\rangle = \langle \mathfrak{w},(a,r)\rangle = r\ell_1$. □

Remark 6.1.2. Note that the cusp type $(r,a) = (r, a'+\varsigma r)$ appearing in Proposition 6.1.1 depends not just on the vertex type $\frac{1}{mr^2}(1, mra-1) = \frac{1}{mr^2}(1, mra'-1)$ of $\mathfrak{w}$ but also on the parameter ς, which controls how the eigenray of $\mathfrak{w}$ meets the Delzant vertex $\mathfrak{v}$. One can also relax the assumption that $\mathfrak{w}$ is adjacent to $\mathfrak{v}$, provided that the line segment joining $\mathfrak{v}$ perpendicularly to the eigenray of $\mathfrak{w}$ is contained in Q. ◊

Remark 6.1.3. Using the generators from §5.1c, the homology class $[C] \in H_2(\mathbb{A}(Q_{\mathrm{nodal}});\mathbb{Q})$ in Proposition 6.1.1 is characterized by $[C]\cdot[\widetilde{\mathbf{D}}_{[\mathfrak{v},\mathfrak{w}]}] = a$, $[C]\cdot[\widetilde{\mathbf{D}}_{[\mathfrak{u},\mathfrak{v}]}] = r$, and $[C]\cdot[\widetilde{\mathbf{D}}_e] = 0$ for all other edges e of Q. In particular, we have $c_1([C]) = r + a$. ◊

Together with the obstruction provided by Theorem 1.3.1, Proposition 6.1.1 immediately gives:

Corollary 6.1.4. *Let Q be a T-polygon as above such that the eigenray emanating from $\mathfrak{w}$ meets the edge $[\mathfrak{u},\mathfrak{v}]$. Then the embedding (6.1.1) is optimal in the sense that there is no such embedding for $c' < 1$ (even after stabilizing by $\mathbb{C}^{N\geqslant 1}$).*

We will refer to an obstruction as in Corollary 6.1.4 as a **visible ellipsoid obstruction**, since it is read off visually from the polygon Q as in Figure 5 right. Note that

[28] Strictly speaking the roles of the x and y axes are swapped in Figure 5 right compared with our conventions in this section. There is another (unshaded) ellipsoid obstruction which is visible in this same figure, which is associated with the eigenray emanating from the other vertex (at $(-1,1)$).

it follows that for any $\mathfrak{p}' \in [\mathfrak{p}, \mathfrak{v}]$ the visible ellipsoid embedding corresponding to the triangle with vertices $\mathfrak{v}, \mathfrak{p}', \mathfrak{w}$ is also optimal. In particular, this determines the ellipsoid embedding function for $M = \mathbb{A}(Q_{\mathrm{nodal}})$ on an appropriate closed interval as follows:[29]

Corollary 6.1.5. *Let Q be a T-polygon as above with $\mathbb{A}(Q_{\mathrm{nodal}})$ symplectomorphic to M, and put $\ell_3 := \mathrm{Len}_{\mathrm{aff}}([\mathfrak{p}, \mathfrak{u}])$. We have:*

(A) $c_M(x) = x/\ell_1$ *for all* $x \in [\frac{\ell_1}{\ell_2+\ell_3}, \frac{\ell_1}{\ell_2}]$ *if* $\ell_1 > \ell_2 + \ell_3$;

(B) $c_M(x) = \ell_1$ *for all* $x \in [\frac{\ell_2}{\ell_1}, \frac{\ell_2+\ell_3}{\ell_1}]$ *if* $\ell_1 < \ell_2$.

Proof. Both of these claims hold by considering the family of embeddings represented by the visible triangles in Q_{nodal} with vertices $\mathfrak{w}, \mathfrak{v}$, and $\mathfrak{u}'$ where $\mathfrak{u}'$ lies on the line between $\mathfrak{p}$ and $\mathfrak{u}$. The fact that the embedding with $\mathfrak{u}' = \mathfrak{p}$ is optimal (by Corollary 6.1.4) implies that all the embeddings with $\mathfrak{u}'$ on the line $[\mathfrak{p}, \mathfrak{u}]$ are also optimal. If $\ell_1 < \ell_2$ and $\ell_1 x \in [\ell_2, \ell_2 + \ell_3]$, these give rise to optimal embeddings of $E(\ell_1, \ell_1 x)$ into $\mathbb{A}(Q_{\mathrm{nodal}})$, which implies that $c_M(x)$ is constant over the corresponding interval as in (B). On the other hand, if $\ell_2 \leqslant z \leqslant \ell_2 + \ell_3 < \ell_1$ we obtain optimal embeddings of $E(z, \ell_1)$ into $\mathbb{A}(Q_{\mathrm{nodal}})$. Setting $x = \frac{\ell_1}{z} > 1$ this translates to an optimal embedding of $E(\frac{\ell_1}{x}, \ell_1)$ into $\mathbb{A}(Q_{\mathrm{nodal}})$, which implies (A). $\square$

We will see in Proposition 6.1.11 below that each of the rigid del Pezzo infinite staircases may be entirely described in this way, with (A) accounting for the sloped line preceding an outer corner and (B) accounting for the horizontal line following an outer corner.

Remark 6.1.6. Let us explain briefly why the obstruction in Corollary 6.1.4 holds from the perpsective of exceptional homology classes. By Proposition 6.1.1 there is a visible (p, q)-unicuspidal symplectic curve $C_{p,q}$ in $\mathbb{A}(Q_{\mathrm{nodal}})$. After a sequence of blowups at the toric fixed point corresponding to the origin, we arrive at the normal crossing resolution $\widetilde{C}_{p,q}$, which is an exceptional curve. Recall that an exceptional class in a closed symplectic four-manifold (M, ω) has a symplectic representative for every symplectic form ω' that is deformation equivalent to ω, and must always have positive symplectic area. Now consider the visible ellipsoid embedding (6.1.1) for $c' > 1$, which corresponds to the subtriangle $T' \subset T$ which is a $\frac{1}{c'}$-scaling of the shaded triangle in Figure 5 right. The symplectic area of the exceptional class $[\widetilde{C}_{p,q}]$ is a function of the distance between this slant edge of T' and the eigenray emanating from $\mathfrak{w}$. Since this tends to zero as $c' \to 1$, it follows that the embedding (6.1.1) is optimal. $\Diamond$

6.1b Visible unicuspidal algebraic curves and outer corner curves

The primary goal of this section is to show that visible ellipsoid obstructions in fact come from *algebraic* curves. We prove the following Theorem in §6.4.

[29] The slightly awkward phrasing division into cases comes from the fact that $c_M(x)$ is defined in terms of ellipsoids $E(1, x)$ with $x \geqslant 1$. The case $\ell_2 \leqslant \ell_1 \leqslant \ell_2 + \ell_3$ does not occur in nontrivial cases.

Theorem 6.1.7. *Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a T-polygon which contains consecutive edges pointing in the directions $(-mr^2, mra-1), (0,-1), (1,0)$ for $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r,a) = 1$. Then there is an index zero (r,a)-unicuspidal rational symplectic curve C in $\mathbb{A}(Q_{\mathrm{nodal}})$ which is J_{int}-holomorphic, where J_{int} is a tame integrable almost complex structure on $\mathbb{A}(Q_{\mathrm{nodal}})$ such that $(\mathbb{A}(Q_{\mathrm{nodal}}), J_{\mathrm{int}})$ is biholomorphic to a sufficiently small $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q. Furthermore, we can assume that C is (r,a)-well-placed with respect to a J_{int}-holomorphic rational nodal anticanonical divisor $\mathcal{N}$.*

Remark 6.1.8. One can check that the curve C in Theorem 6.1.7 has the same homology class as that in Remark 6.1.3, and in particular it has symplectic area $\langle \mathfrak{w}, (a,r) \rangle$, where $\mathfrak{w}$ is the vertex lying on the positive y-axis. Thus in the case that the eigenray emanating from $\mathfrak{w}$ intersects the edge of Q in direction $(1,0)$, C carries the same visible ellipsoid obstruction as in Corollary 6.1.4. $\Diamond$

Remark 6.1.9. The utility of the last part of Theorem 6.1.7 is that, at least in the rigid del Pezzo case, we can apply iteratively the generalized Orevkov twist from §2, i.e. for each such curve C we get a whole sequence of index zero unicuspidal rational algebraic curves $\Phi_M(C), \Phi_M^2(C), \Phi_M^3(C), \dots$.

Recall that the preimage of $\partial Q_{\mathrm{nodal}}$ under the almost toric fibration $\pi : \mathbb{A}(Q_{\mathrm{nodal}}) \to Q_{\mathrm{nodal}}$ is an anticanonical nodal symplectic divisor. If we assume that Q_{nodal} is constructed such that there are base-nodes at every vertex except for the Delzant vertex $\mathfrak{v}$ (c.f. Remark 4.2.3), then $\mathcal{N} := \pi^{-1}(\partial Q_{\mathrm{nodal}})$ is rational with a single node at $\mathfrak{v}$. The visible symplectic curve C in Proposition 6.1.1 is by construction (r,a)-well-placed with respect to $\mathcal{N}$, and the last part of Theorem 6.1.7 is an algebraic analogue of this. $\Diamond$

Theorem D follows quickly from Theorem 6.1.7, after a preliminary lemma.

Lemma 6.1.10. *Let V be a smooth complex projective surface which is diffeomorphic to a rigid del Pezzo surface M. Then V has an arbitrarily small deformation $\widetilde{V}$ which is biholomorphic to M.*

Proof. Observe that V is necessarily rational (see [FQ95, Cor. 0.2]) and hence by the Enriques–Kodaira classification it is an iterated blowup of $\mathbb{CP}^2$ or a Hirzebruch surface F_k. Recall that the Hirzebruch surface F_k deforms to $\mathbb{CP}^1 \times \mathbb{CP}^1$ or F_1 for any $k \in \mathbb{Z}_{\geqslant 2}$. Combining this with a generic perturbation of the blowup centers, we arrive at a smooth del Pezzo surface $\widetilde{V}$ which is diffeomorphic and hence (by the classification of del Pezzo surfaces) biholomorphic to M. $\square$

Proof of Theorem D. Let C be the (r,a)-unicuspidal J_{int}-holomorphic curve in $\mathbb{A}(Q_{\mathrm{nodal}})$ guaranteed by Theorem 6.1.7. Here $(\mathbb{A}(Q_{\mathrm{nodal}}), J_{\mathrm{int}})$ is biholomorphic to a $\mathbb{Q}$-Gorenstein smoothing $\widetilde{V}_Q$ of V_Q, and by assumption $\mathbb{A}(Q_{\mathrm{nodal}})$ is diffeomorphic to a rigid del Pezzo surface M. If $\widetilde{V}_Q$ is Fano then it is necessarily biholomorphic to M. Otherwise, by Lemma 6.1.10 we can still find an arbitrarily small deformation of $\widetilde{V}_Q$ which is Fano and hence biholomorphic to M, and similar to the proof of Theorem 5.3.1 we can push C into this del Pezzo surface. $\square$

We end this subsection by showing that the staircases in the rigid del Pezzo surfaces are entirely visible in terms of triangles in certain almost toric base polygons as detailed in Corollaries 6.1.4 and 6.1.5. In particular, we use this to give an alternative proof of Theorem B (recall that our proof in §2 was based on the generalized Orevkov twist).

Proposition 6.1.11. *For M be a rigid del Pezzo surface, and let Q be the corresponding dual Fano T-polygon in Figure 4. Then, each outer corner point (x_k, y_k) of the infinite staircase in the ellipsoid embedding function $c_M(x)$ corresponds to a visible ellipsoid obstruction in some polygon Q' obtained from Q by a sequence of full mutations. Moreover, as in Corollary 6.1.5, we may choose Q' so that this polygon Q' determines the value of $c_M(x)$ for all $x \in [x_k, x_{k+1}]$.*

Let Q be one of the polygons in Figure 4, and let J denote the number of strands of the corresponding infinite staircase $c_M(x)$, i.e. $J = 3$ for $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \#^{\times 2} \overline{\mathbb{CP}}^2$ and $J = 2$ in the remaining cases (c.f. §2.3). Let Q_k denote the result after $k \in \mathbb{Z}_{\geqslant 0}$ successive full mutations of Q, each time along eigenray emanating from its top left vertex. It is shown in [Cri+20] that this eigenray always meets the horizontal edge of Q_k, so that the non-Delzant vertices cyclically permute under successive full mutations of this kind. Let $\mathfrak{v}_1^k, \ldots, \mathfrak{v}_{J+1}^k$ denote the vertices of Q_k (ordered counterclockwise), where $\mathfrak{v}_1^k$ is the Delzant vertex with edge vectors $(1,0), (0,1)$, $\mathfrak{v}_2^k$ lies on the positive x-axis, and $\mathfrak{v}_{J+1}^k$ lies on the positive y-axis. A complete description of the vertices and eigenrays of Q_k may be found in [Cri+20, §5]. In particular, for each J-tuple of successive staircase steps there is a mutation Q_k whose vertices have T-singularities of the corresponding types. Further, putting $K = [\mathcal{N}] \cdot [\mathcal{N}] - 2$, where $[\mathcal{N}] \in H_2(\mathbb{A}(Q_{\text{nodal}}))$ is the anticanonical class (c.f. Table 1), we find that the vertex types of Q_k transform under J-full mutations by the same recursion as the generalized Orevkov twist:

Lemma 6.1.12. *Suppose that $\mathfrak{v}_i^k$ has type $\frac{1}{m_i r_i^2}(1, m_i r_i a_i - 1)$ for $i = 2, \ldots, \ell$. Then $\mathfrak{v}_i^{k+J}$ has type $\frac{1}{m_i'(r_i')^2}(1, m_i' r_i' a_i' - 1)$, where $(m_i', r_i', a_i') = (m_i, K r_i - a_i, r_i)$.*

Proof of Proposition 6.1.11. Let $(x_k, y_k) = (\frac{g_{k+J}}{g_k}, \frac{g_{k+J}}{g_k + g_{k+J}})$ be the kth outer corner point on the graph of c_M (recall §2.3). By [Cri+20, §5], the eigenray emanating from the top left vertex $\mathfrak{v}_{J+1}^k$ of Q_k points in the direction $(g_k, -g_{k+J})$ and meets the edge between $\mathfrak{v}_1^k$ and $\mathfrak{v}_2^k$. According to Proposition 6.1.1 there is a visible index zero (g_{k+J}, g_k)-unicuspidal rational symplectic curve C in M, and this gives precisely the outer corner obstruction $c_M(x_k) \geqslant y + k$. Now observe that, because Q_k is obtained by mutation along the nodal ray from the top left vertex $\mathfrak{v}_{J+1}^{k-1}$ of Q_{k-1}, this nodal ray must meet the x-axis at the vertex $\mathfrak{v}_2^k$ of Q_k so that the nodal ray of Q_k at $\mathfrak{v}_2^k$ points in the direction $(-g_{k-1}, g_{k-1+J})$. If we now apply Corollary 6.1.5 to Q_k, then because $\ell_2 > \ell_1 + \ell_3$ we can calculate c_M on the interval that ends on the kth outer corner point $x_k = \frac{g_{k+J}}{g_k}$. Similarly, by applying this result to the reflection of Q_k about the line $x = y$, we are in the case $\ell_1 < \ell_2$ and hence can deduce that c_M is constant on the interval between x_{k-1} and the corner point. □

Alternative proof of Theorem B. By Proposition 6.1.11, every outer corner in $c_M(x)$ corresponds to a visible ellipsoid obstruction in some polygon Q with $\mathbb{A}(Q_{\text{nodal}})$ symplectomorphic to M, and by Theorem D this comes from an index zero rational algebraic unicuspidal curve. $\square$

6.2 Visible curves in the Auroux-type model

As a preliminary to constructing curves in $\mathbb{Q}$-Gorenstein smoothings of singular toric surfaces, we first discuss the affine case, which is modeled on the Auroux-type system from [Eva23, §7.3]. Recall from §4.1c that the T-singularity $\frac{1}{mr^2}(1, mra-1) = \{xy = z^{rm}\}/\mu_r^{1,-1,a}$ smooths to

$$B_{m,r,a} \cong \{xy = (z^r - \zeta_1)\cdots(z^r - \zeta_m)\}/\mu_r^{1,-1,a}$$

(for any fixed $0 < \zeta_1 < \cdots < \zeta_r$), and we have the Auroux-type almost toric fibration

$$\pi_{\mathrm{Aur}} : B_{m,r,a} \to \mathbb{C}, \quad \pi_{\mathrm{Aur}}(x,y,z) = \left(|z|^2, \tfrac{1}{2}|x|^2 - \tfrac{1}{2}|y|^2\right).$$

The critical values of π_{Aur} are $(\zeta_1^{2/r}, 0), \ldots, (\zeta_m^{2/r}, 0)$. By analogy with the visible unicuspidal symplectic curves appearing in the proof of Proposition 6.1.1, we seek algebraic curves in $B_{m,r,a}$ which project via π_{Aur} to vertical rays in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ emanating from a critical value. To this end, putting $\mathring{C}_i^+ := \{y = 0\} \subset B_{m,r,a}$ and $\mathring{C}_i^- := \{x = 0\} \subset B_{m,r,a}$, note that we have indeed

$$\pi_{\mathrm{Aur}}(\mathring{C}_i^\pm) = \{(\zeta_i^{2/r}, t) \mid \pm t \in \mathbb{R}_{\geqslant 0}\}.$$

Now suppose that Q is a polygon with a T-vertex $\mathfrak{w}$ of type $\frac{1}{mr^2}(1, mra-1)$, and let $\widetilde{V}_Q$ be a $\mathbb{Q}$-Gorenstein smoothing of the singular toric surface V_Q, along with a holomorphic embedding $\iota : B_{m,r,a} \hookrightarrow \widetilde{V}_Q$. Roughly speaking, we will show below that the closure of $\iota(\mathring{C}_i^+)$ in $\widetilde{V}_Q$ is a (r,a)-unicuspidal rational algebraic curve, and this underpins Theorem 6.1.7.

6.3 Pencils from polygon mutations

As we briefly recalled in §4.1c, singular toric surfaces V_Q and $V_{Q'}$ are $\mathbb{Q}$-Gorenstein deformation equivalent if (and conjecturally only if) the corresponding dual Fano polygons are mutation equivalent. More precisely, we have:

Theorem 6.3.1 ([Akh+16, Lem. 7], following [Ilt12, Thm. 1.3])**.** *Let Q be a dual Fano polygon, and let $Q' = \mathrm{Mut}_{\mathfrak{w}}(Q)$ be its mutation at a vertex $\mathfrak{w}$. There exists a $\mathbb{Q}$-Gorenstein pencil*[30] *$\mathbb{p} : \mathcal{X} \to \mathbb{CP}^1$ such that $\mathbb{p}^{-1}(0) \cong V_Q$ and $\mathbb{p}^{-1}(\infty) \cong V_{Q'}$.*

In order to construct unicuspidal algebraic curves, we will describe an explicit model for (at least a part of) the $\mathbb{Q}$-Gorenstein pencil $\mathbb{p}$ in the analogous case that Q is a T-polygon (not necessarily dual Fano) with a Delzant vertex. We first discuss the case that Q is a triangle, and then obtain the case of a general polygon by a local birational modification.

[30] That is, a flat family whose total space is $\mathbb{Q}$-Gorenstein.

Let $Q \subset \mathbb{M}_{\mathbb{R}}$ be a triangle with a Delzant vertex $\mathfrak{v}$ adjacent to a T-vertex $\mathfrak{w}$. After an integral affine transformation we can assume that $\mathbb{M}_{\mathbb{R}} = \mathbb{R}^2$ and the vertices are $\mathfrak{v} := (0,0), \mathfrak{w} := (0, mra-1), \mathfrak{u} := (mr^2, 0)$, for some $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r,a) = 1$. In particular, V_Q is isomorphic to the weighted projective space $\mathbb{CP}(1, mra-1, mr^2)$. Similar to Remark 6.1.2, we do not assume $a < r$, but we can write $a = a' + \varsigma r$ for some $a' \in \{1, \ldots, r-1\}$ and $\varsigma \in \mathbb{Z}_{\geqslant 1}$. Then the vertex $\mathfrak{w}$ has type $\frac{1}{mr^2}(1, mra-1) = \frac{1}{mr^2}(1, mra'-1)$, and the vertex $\mathfrak{u}$ has type $\frac{1}{mra-1}(1, mr^2)$ (this is not necessarily a T-singularity).

With respect to the vertex $\mathfrak{w}$, the mutated triangle $Q' := \mathrm{Mut}^{\mathrm{full}}_{\mathfrak{w}}(Q)$ has vertices $\mathfrak{v}' := (0,0), \mathfrak{w}' := (0, mra), \mathfrak{u}' := (\frac{r}{a} \cdot (mra-1), 0)$, and we have $V_{Q'} \cong \mathbb{CP}(1, mra-1, ma^2)$. Note that $\mathfrak{v}'$ is smooth, $\mathfrak{w}'$ has type $\frac{1}{ma^2}(1, mra-1)$, and $\mathfrak{u}'$ has type $\frac{1}{mra-1}(1, ma^2)$, which is the same singularity type as $\mathfrak{u}$:

Lemma 6.3.2. *For $m, r, a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r,a) = 1$, we have $\frac{1}{mra-1}(1, mr^2) = \frac{1}{mra-1}(r, a) = \frac{1}{mra-1}(1, ma^2)$.*

Proof. Modulo $mra-1$ we have $0 \equiv (mra-1)(mra+1) = m^2r^2a^2 - 1$, and hence $mr^2 = 1/(ma^2)$. □

Now consider the weighted projective 3-space $\mathbb{CP}(1, mra-1, r, a)$ with homogeneous coordinates $[x : y : z : w]$, and consider the hypersurface

$$S_t := \{xy = \tfrac{1}{1+t} w^{mr} + \tfrac{t}{1+t} z^{ma}\} \subset \mathbb{CP}(1, mra-1, r, a) \tag{6.3.1}$$

for $t \in \mathbb{CP}^1$.

Proposition 6.3.3. *Suppose as above that the triangle Q has a Delzant vertex $\mathfrak{v}$ adjacent to a T-vertex $\mathfrak{w}$. Then the family $\{S_t\}_{t \in \mathbb{CP}^1}$ defines a $\mathbb{Q}$-Gorenstein pencil such that*

- *we have isomorphisms $S_0 \cong \mathbb{CP}(1, mra-1, mr^2)$ and $S_\infty \cong \mathbb{CP}(1, mra-1, ma^2)$*
- *for $t \neq 0, \infty$, S_t has a singularity at the point $[0:1:0:0]$ of type $\frac{1}{mra-1}(r,a)$ and is otherwise nonsingular.*

Remark 6.3.4. When Q is a T-triangle, the family $\{S_t\}$ is an algebraic counterpart of a family of almost toric fibrations interpolating between the singular toric surfaces V_Q and $V_{Q'}$, corresponding to a family of nodal integral affine structures on Q with base-nodes limiting to the vertex $\mathfrak{w}$ as $t \to 0$ and limiting to $\mathfrak{p}$ as $t \to \infty$ (here $\mathfrak{p}$ is the other point where the eigenray emanating from $\mathfrak{w}$ intersects ∂Q). ◊

Proof of Proposition 6.3.3. Let U_x denote the (singular) affine chart $\{x = 1\} \subset \mathbb{CP}(1, mra, r, a)$, with $U_y, U_z, U_w \subset \mathbb{CP}(1, mra, r, a)$ defined similarly. We will analyze the intersection of S_t with each of the charts U_x, U_y, U_z, U_w. To start, observe that, for any $t \in \mathbb{CP}^1$, $S_t \cap U_x = \{y = \frac{1}{1+t} w^{mr} + \frac{t}{1+t} z^{ma}\} \cong \mathbb{C}^2_{z,m}$ is smooth. We have also

$$S_t \cap U_y = \{x = \tfrac{1}{1+t} w^{mr} + \tfrac{t}{1+t} z^{ma}\}/\mu^{1,r,a}_{mra-1} \cong \mathbb{C}^2_{z,w}/\mu^{r,a}_{mra-1} = \tfrac{1}{mra-1}(r,a). \tag{6.3.2}$$

Next, we have

$$S_t \cap U_z = \{xy = \tfrac{1}{1+t} w^{mr} + \tfrac{t}{1+t}\}/\mu_r^{1,-1,a}.$$

Comparing with (4.1.3), we see that the family $\{S_t \cap U_z\}$ is a $\mathbb{Q}$-Gorenstein smoothing of $S_0 \cap U_z \cong \frac{1}{mr^2}(1, mra-1)$, and in particular $S_t \cap U_z \cong B_{m,r,a}$ for $t \neq 0, \infty$. Note that $S_\infty \cap U_z = \{xy = 1\}/\mu_r^{1,-1,a}$ is smooth.

Finally, we have

$$S_t \cap U_w = \{xy = \tfrac{1}{1+t} + \tfrac{t}{1+t} z^{ma}\}/\mu_a^{1,-1,r},$$

i.e. the family $\{S_t \cap U_w\}$ is a $\mathbb{Q}$-Gorenstein smoothing of $S_\infty \cap U_w \cong \frac{1}{ma^2}(1, mra-1)$, and we have $S_t \cap U_w \cong B_{m,a,r}$ for $t \neq 0, \infty$, with $S_0 \cap U_w = \{xy = 1\}/\mu_a^{1,-1,r}$ smooth. Note that we have covered the total space of the deformation $\{S_t\}$ by $\mathbb{Q}$-Gorenstein neighborhoods (each is either smooth, equivalent to the miniversal model (4.1.3), or a trivial family of cyclic quotient singularities), and hence $\{S_t\}$ is a $\mathbb{Q}$-Gorenstein deformation.

Let $[s:t:u]$ be homogeneous coordinates on $\mathbb{CP}(1, mra-1, mr^2)$, and define the map

$$\iota_0 : \mathbb{CP}(1, mra-1, mr^2) \to S_0, \qquad [s:t:u] \mapsto [s^{mr} : t^{mr} : u : st].$$

Note that this is well-defined since $\iota_0([\lambda s : \lambda^{mra-1} t : \lambda^{mr^2} u]) = [\lambda^{mr} s^{mr} : \lambda^{mr(mra-1)} t^{mr} : \lambda^{mr^2} u : \lambda^{mra} st] = [s^{mr} : t^{mr} : u : st]$, and the restriction $\mathbb{CP}(1, mra-1, mr^2) \cap U_u \to S_0 \cap U_z$ agrees with the isomorphism (4.1.4). Similarly, we define the map

$$\iota_\infty : \mathbb{CP}(1, mra-1, ma^2) \to S_\infty, \qquad [s:t:u] \mapsto [s^{ma} : t^{ma} : st : u].$$

We leave it to the reader to verify that the maps ι_0 and ι_∞ are isomorphisms. $\square$

Example 6.3.5. Let $(p_1, p_2, p_3) \in \mathbb{Z}^3_{\geqslant 0}$ be a triple which satisfies the Markov equation $p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3$, and let $(p_1, p_2, p_3' := 3p_1p_2 - p_3)$ be its Markov mutation. Put $\mathbb{M} := \{(m_1, m_2, m_3) \in \mathbb{Z}^3 \mid \sum_{i=1}^{3} p_i^2 m_i = 0\}$. Then the dual Fano polygon

$$Q = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} p_i^2 x_i = 0, x_1, x_2, x_3 \geqslant -1\} \subset \mathbb{M}_{\mathbb{R}}$$

satisfies $V_Q \cong \mathbb{CP}(p_1^2, p_2^2, p_3^2)$ and has T-vertices of types $\frac{1}{p_i^2}(p_{i+1}^2, p_{i+2}^2)$ for $i = 1,2,3$ (mod 3). Let $\mathfrak{w}$ denote the vertex of type $\frac{1}{p_3^2}(p_1^2, p_2^2)$ and put $Q' := \mathrm{Mut}^{\mathrm{full}}_{\mathfrak{w}}(Q)$, so that we have $V_{Q'} \cong \mathbb{CP}(p_1^2, p_2^2, (p_3')^2)$. Then the pencil $\mathbb{Q}$-Gorenstein $\{S_t\}_{t \in \mathbb{CP}^1}$ is given by

$$S_t := \{xy = \tfrac{1}{1+t} w^{p_3} + \tfrac{t}{1+t} z^{p_3'}\} \subset \mathbb{CP}(p_1^2, p_2^2, p_3, p_3') \tag{6.3.3}$$

satisfies $S_0 \cong \mathbb{CP}(p_1^2, p_2^2, p_3^2)$, $S_\infty \cong \mathbb{CP}(p_1^2, p_2^2, (p_3')^2)$ and $S_t \cong \mathbb{CP}^2$ for $t \neq 0, \infty$. This fits into the above framework after specializing to the case $p_1 = 1$, so that Q has a Delzant vertex $\mathfrak{v}$. $\lozenge$

Now let $Q \subset \mathbb{M}_{\mathbb{R}}$ be any polygon having a Delzant vertex $\mathfrak{v}$ adjacent to a T-vertex $\mathfrak{w}$. As in the triangle case considered above, we will assume that $\mathbb{M}_{\mathbb{R}} = \mathbb{R}^2$ with $\mathfrak{v} = (0,0)$ and $\mathfrak{w} = (0, mra-1)$, where the edge vectors at $\mathfrak{v}$ are $(1,0),(0,1)$ and the edge vectors at $\mathfrak{w}$ are $(0,-1),(mr^2, 1-mra)$, for some $m,r,a \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(r,a) = 1$. We will further assume that the eigenray emanating from $\mathfrak{w}$ hits the edge of Q which lies on the x-axis.[31] Put $Q' := \mathrm{Mut}^{\mathrm{full}}_{\mathfrak{w}}(Q)$. By reducing to the triangular case above, we now construct a $\mathbb{Q}$-Gorenstein pencil $\{\widetilde{S}_t\}_{t \in \mathbb{CP}^1}$ which interpolates between V_Q and $V_{Q'}$ and smooths the toric fixed point $\mathfrak{p}_{\mathfrak{w}}$ in a general fiber. Let Q_{tri} denote the triangle with vertices $\mathfrak{v},\mathfrak{w}$, and $\mathfrak{u} = (mr^2, 0)$, so that we have $Q \subset Q_{\mathrm{tri}}$.

Let $\Sigma_Q, \Sigma_{Q_{\mathrm{tri}}}$ denote the normal fans to Q, Q_{tri} respectively, and note that Σ_Q is a refinement of $\Sigma_{Q_{\mathrm{tri}}}$. We denote by $\tau_{[\mathfrak{v},\mathfrak{w}]}, \tau_{[\mathfrak{v},\mathfrak{u}]}, \tau_{[\mathfrak{u},\mathfrak{w}]} \subset \mathbb{N}_{\mathbb{R}}$ the rays of $\Sigma_{Q_{\mathrm{tri}}}$ spanned by the inward normals to the edges $[\mathfrak{v},\mathfrak{w}],[\mathfrak{v},\mathfrak{u}],[\mathfrak{u},\mathfrak{w}]$ respectively. Let $\sigma \subset \mathbb{N}_{\mathbb{R}}$ denote the two-dimensional cone spanned by $\tau_{[\mathfrak{v},\mathfrak{u}]}$ and $\tau_{[\mathfrak{u},\mathfrak{w}]}$, with corresponding distinguished point $\mathfrak{p}_\sigma \in U_\sigma \subset V_{Q_{\mathrm{tri}}}$ (c.f. [CLS11, §3.2]). Note that the rays of $\Sigma_Q \setminus \Sigma_{Q_{\mathrm{tri}}}$ all lie in the interior of σ. The induced toric morphism $\pi : V_Q \to V_{Q_{\mathrm{tri}}}$ is a proper birational map which restricts to an isomorphism $V_Q \setminus \pi^{-1}(\mathfrak{p}_\sigma) \to V_{Q_{\mathrm{tri}}} \setminus \mathfrak{p}_\sigma$. We denote by $U_\sigma \subset V_{Q_{\mathrm{tri}}}$ the affine toric variety associated to the cone σ, which we identify with $\frac{1}{mra-1}(r,a)$, and we put $\widetilde{U}_\sigma := \pi^{-1}(U_\sigma) \subset V_Q$. Thus V_Q is obtained from $V_{Q_{\mathrm{tri}}}$ by excising U_σ and gluing in $\widetilde{U}_\sigma$, which we could view as sequence of generalized blowups at (singular) points.

Let $\{S_t\}_{t\in\mathbb{CP}^1}$ denote the $\mathbb{Q}$-Gorenstein pencil associated to Q_{tri} and its mutation at $\mathfrak{w}$ as in Proposition 6.3.3. For $t \in \mathbb{CP}^1$, let $\jmath_t : \mathbb{C}^2_{z,w}/\mu^{r,a}_{mra-1} \xrightarrow{\cong} S_t \cap U_y$ denote the natural isomorphism $(z,w) \mapsto (\frac{1}{1+t}w^{mr} + \frac{t}{1+t}z^{ma}, 1, z, w)$. For each $t \in \mathbb{CP}^1$, let $\widetilde{S}_t \to S_t$ be the birational modification corresponding to excising the image of U_σ under $\jmath_t$ and gluing in $\widetilde{U}_\sigma$. We have, essentially by construction, the following generalization of Proposition 6.3.3.

Proposition 6.3.6.

(i) Suppose as above that $Q \subset \mathbb{M}_{\mathbb{R}}$ is a polygon having a Delzant vertex $\mathfrak{v} = (0,0)$ adjacent to a T-vertex $\mathfrak{w} = (0, mra-1)$ and such that the eigenray $(r,-a)$ emanating from $\mathfrak{w}$ hits the edge of Q which lies on the x-axis. Then the family $\{\widetilde{S}_t\}_{t\in\mathbb{CP}^1}$ described above defines a $\mathbb{Q}$-Gorenstein pencil such that

- *we have isomorphisms $\widetilde{S}_0 \cong V_Q$ and $S_\infty \cong V_{Q'}$*
- *for $t \neq 0,\infty$, the singularities of $\widetilde{S}_t$ correspond naturally to the singular toric fixed points of V_Q excluding $\mathfrak{p}_{\mathfrak{w}}$.*

(ii) If in the above situation the eigenray $(r,-a)$ emanating from $\mathfrak{w}$ hits some other edge of Q then the $\mathbb{Q}$-Gorenstein pencil $\widetilde{S}_t$ with $\widetilde{S}_0 \cong V_Q$ is defined for t close to 0 and $\widetilde{S}_t$ has singularities as described above.

[31] Without this assumption we will still have $\widetilde{S}_0 \cong V_Q$, which suffices to prove Theorem 6.1.7, but we will not generally have $\widetilde{S}_\infty \cong V_{Q'}$. Indeed, since the complex variety V_Q depends only on the normal fan of Q we can always adjust Q so as to achieve this property without changing its normal fan. However, the mutation Q' depends on Q itself and not just its normal fan.

Remark 6.3.7. In [Akh+16] the authors construct a family of hypersurfaces in a toric 3-fold $V_{\widetilde{Q}}$, where $\widetilde{Q}$ is a three-dimensional polytope admitting projections to Q and the mutated polytope Q'. This polytope has a smooth corner that projects to the smooth corners of Q and Q', and the the dually induced toric morphisms $V_Q, V_{Q'} \hookrightarrow V_{\widetilde{Q}}$ embed V_Q and $V_{Q'}$ as hypersurfaces that span a $\mathbb{Q}$-Gorenstein pencil. This construction applies whether or not the eigenray from $\mathfrak{v}$ meets the edge of Q on the x-axis, and it follows that there is a family of surfaces $(\widetilde{S}_t)_{t\in\mathbb{C}P^1}$ with $\widetilde{S}_0 = V_Q$ and $\widetilde{S}_\infty = V_{Q'}$. In this construction, each additional facet of Q gives rise to an extra facet of $\widetilde{Q}$, so that the effect of blowing up Q is to blow up $\widetilde{Q}$. However this blowup of $\widetilde{Q}$ may not correspond to a family of blowups of the $\widetilde{S}_t$. As an example, one can consider the effect of a mutation of the quadrilaterals $Q^{\mathcal{T}}$ (with vertices $\mathfrak{v}_O, \mathfrak{v}_X, \mathfrak{v}_V, \mathfrak{v}_Y$) considered in the proof of Lemma 6.5.3. Here the original quadrilateral is such a large blowup of the triangle Q_{tri} with vertices $\mathfrak{v}_O, \mathfrak{v}_{X'}, \mathfrak{v}_Y$ that the nodal ray hits the edge $\mathfrak{v}_V - \mathfrak{v}_X$. One can check that this has the effect that the one of the edges of the mutation of Q_{tri} is completely excised from the mutation of $Q^{\mathcal{T}}$, which means the latter mutation is not a blowup of the former mutation. Nevertheless, in this case one can with some effort give explicit formulas in the spirit of (6.3.1) for the hypersurface $\widetilde{S}_t \subset V_{\widetilde{Q}}$ in terms of Cox coordinates on $V_{\widetilde{Q}}$ thought of as a GIT quotient. ♢

6.4 Explicit unicuspidal algebraic curves

Let Q be a polygon as above, with associated $\mathbb{Q}$-Gorenstein pencils $\{\widetilde{S}_t\}_{t\in\mathbb{CP}^1}$. We now construct a (r,a)-unicuspidal rational algebraic curve C_t^+ in $\widetilde{S}_t$ for all $t \neq 0, \infty$. Since $\{\widetilde{S}_t\}_{t\in\mathbb{CP}^1}$ is given by a fiberwise birational modification of the pencil $\{S_t\}_{t\in\mathbb{CP}^1}$ associated to Q_{tri}, we first consider the case that $Q = Q_{\mathrm{tri}}$ is a triangle. Notice that the smooth toric fixed point of $\mathbb{CP}(1, mra-1, mr^2)$ maps by ι_0 to the point $[1:0:0:0] \in S_0$, which lies in S_t for all $t \in \mathbb{CP}^1$. The curve C_t^+ will be constructed to have a (r,a) cusp at $[1:0:0:0]$, and will furthermore be well-placed with respect to an anticanonical divisor $\mathcal{N}_t$ having a node at $[1:0:0:0]$.

Recall that for $t \neq 0, \infty$ we have $S_t \cap U_z = \{xy = \frac{1}{1+t}w^{mr} + \frac{t}{1+t}\}/\mu_r^{1,-1,a} \cong B_{m,r,a}$. Following §6.2, we consider the curves $\mathring{C}_t^+ = \{y = 0\} \subset S_t \cap U_z$ and $\mathring{C}_t^- = \{x = 0\} \subset S_t \cap U_z$ and their closures C_t^+, C_t^- in S_t. We will focus on C_t^+ since C_t^- does not pass through $[1:0:0:0]$. Strictly speaking $\mathring{C}_t^+$ has m components (since we have combined $C_1^+, \ldots, C_m^+$ from §6.2 into a single curve), so we take one of the components

$$C_t^+ := \{y = 0, w^r = \zeta z^a\} \subset S_t \tag{6.4.1}$$

for fixed $\zeta \in \mathbb{C}$ satisfying $\zeta^m = -t$.

For $t \in \mathbb{CP}^1$, let $\mathcal{N}_t \subset S_t$ denote the anticanonical divisor $\{z = 0\} \cup \{w = 0\} \subset S_t$. Here $\mathcal{N}_t$ has two components for $t \neq 0, \infty$, while $\mathcal{N}_0$ (resp. $\mathcal{N}_\infty$) is the image of the toric divisor under the map $\iota_0 : \mathbb{CP}(1, mra-1, mr^2) \to S_0$ (resp. $\iota_\infty : \mathbb{CP}(1, mra-1, ma^2) \to S_\infty$). For $t \neq 0, \infty$, the two components of $\mathcal{N}_t$ intersect transversally at the nodal point $[1:0:0:0]$ and also meet at the singular point $[0:1:0:0]$.

Lemma 6.4.1. *When Q is a triangle, $C_t^+ \subset S_t$ is (r,a)-well-placed with respect to $\mathcal{N}_t$ for $t \neq 0, \infty$, and is otherwise nonsingular. In particular, C_t^+ is (r,a)-unicuspidal.*

Proof. Note that $C_t^+ \subset U_x \cup U_z$ since $C_t^+ \cap U_y = \varnothing$ and $[0:0:0:1] \notin S_t$, and $C_t^+ \cap U_z$ is smooth by construction. In the chart $S_t \cap U_x \cong \mathbb{C}^2_{z,w}$ we have $C_t^+ \cap U_x = \{(z,w) \in \mathbb{C}^2 \mid w^r = \zeta z^a\}$, which is (r,a)-well-placed with respect to $\mathcal{N}_t \cap U_x = \{(z,w) \in \mathbb{C}^2 \mid z = 0 \text{ or } w = 0\}$. ☐

In the case of a general quadrilateral Q, let $\widetilde{C}_t^+ \subset \widetilde{S}_t$ denote the proper transform of $C_t^+ \subset S_t$ with respect to the birational map $\widetilde{S}_t \to S_t$, and let $\widetilde{\mathcal{N}}_t \subset \widetilde{S}_t$ be the total transform of $\mathcal{N}_t$. Since C_t^+ is disjoint from $[0:1:0:0]$, we have the following extension of Lemma 6.4.1

Lemma 6.4.2. *For general Q and all $t \neq 0$ sufficiently close to 0, the curve $\widetilde{C}_t^+ \subset \widetilde{S}_t$ is (r,a)-well-placed with respect to $\widetilde{\mathcal{N}}_t$ and is otherwise nonsingular.*

We are now ready to complete the proof of theorem 6.1.7.

Proof of Theorem 6.1.7. Let $\{\widetilde{S}_t\}_{t\approx 0}$ be the $\mathbb{Q}$-Gorenstein deformation of V_Q constructed in Proposition 6.3.6. Using Lemma 6.4.2, there is a rational unicuspidal curve $\widetilde{C}_t^+ \subset \widetilde{S}_t$ which is well-placed with respect to the anticanonical divisor $\widetilde{\mathcal{N}}_t$. Let $\breve{S}_t$ be a further $\mathbb{Q}$-Gorenstein deformation of $\widetilde{S}_t$ which smooths the remaining singular points. As in the proof of Theorem 5.3.1, for $|t| > 0$ sufficiently small we can assume that $\widetilde{\mathcal{N}}_t$ deforms to a rational nodal anticanonical divisor $\breve{\mathcal{N}}_t \subset \breve{S}_t$ and $\widetilde{C}_t^+$ deforms to a rational curve $\breve{C}_t \subset \breve{S}_t$ which is (r,a)-well-placed with respect to $\breve{\mathcal{N}}_t$.[32] By Proposition 4.2.6, there is a diffeomorphism $\Phi : \breve{S}_t \to \mathbb{A}(Q_{\text{nodal}})$ such that $\Phi_*(\breve{J})$ tames the symplectic form on $\mathbb{A}(Q_{\text{nodal}})$, with $\breve{J}$ the integrable almost complex structure on $\breve{S}_t$. Then the curve $\Phi(\breve{C}_t^+) \subset \mathbb{A}(Q_{\text{nodal}})$ is $\Phi_*(\breve{J})$-holomorphic and satisfies the requirements of the theorem. ☐

6.5 Classifying unicuspidal algebraic curves in the first Hirzebruch surface

Our goal here is to prove Theorem F on unicuspidal algebraic curves in the first Hirzebruch surface F_1. As a warmup, we start by discussing the analogue for $\mathbb{CP}^2$, giving a new proof based on quantitative symplectic geometry that the list in Theorem 1.2.2(d) is complete:

Lemma 6.5.1. *The curves constructed in Theorem 2.1.2 give the complete list of data (d,p,q) for unicuspidal rational algebraic plane curves with one Puiseux pair.*

Proof. Any (p,q)-unicuspidal rational algebraic curve in $\mathbb{CP}^2$ of degree d is in particular a symplectic curve, and hence according to [McS23, Thm. G] the homology class $A = d\ell$ is (p,q)-perfect. Then by [McSch12, Cor. 3.1.3], p/q must be a ratio of odd index Fibonacci numbers. ☐

[32] See [McS23, Cor.3.5.5] for a discussion of the behavior of cuspidal constraints under deformations of the (almost) complex structure J.

The corresponding classification of perfect exceptional classes in F_1 was recently worked out in [MM24; MMW22] and is much more complicated than for $\mathbb{CP}^2$. Here we give a broad self-contained overview of the classification, refering to loc. cit. for the details.

Let $\mathrm{Perf}(F_1)$ denote the set of all quadruples $(p, q, d, m) \in \mathbb{Z}^4_{\geqslant 0}$ with $p > q$ coprime such that $A = d\ell - me$ is a (p, q)-perfect exceptional class in $H_2(F_1)$ (see e.g. [McS23, Def. 4.4.2]). Equivalently, by [McS23, Thm. G] this is the set of quadruples (p, q, d, m) such that there exists an index zero (p, q)-unicuspidal rational symplectic curve C in F_1 with $[C] = d\ell - me \in H_2(F_1)$.[33] Put

$$\underline{\mathrm{Perf}}(F_1) := \{p/q \mid (p, q, d, m) \in \mathrm{Perf}(F_1) \text{ for some } d, m\}.$$

The classes in $\mathrm{Perf}(F_1)$ divide naturally into two sets: those in $\mathrm{Perf}^+(F_1)$ with $m/d > 1/3$ and those in $\mathrm{Perf}^-(F_1)$ with $m/d < 1/3$; none have $m/d = 1/3$.

One can show that the forgetful map

$$\mathrm{Perf}(F_1) \to \underline{\mathrm{Perf}}(F_1) = \underline{\mathrm{Perf}}^+(F_1) \sqcup \underline{\mathrm{Perf}}^-(F_1)$$

sending (p, q, d, m) to p/q is injective. Indeed, by definition, for $(p, q, d, m) \in \mathrm{Perf}(F_1)$ we have $d^2 - m^2 = pq - 1$ and $3d - m = p + q$, and hence

$$d_{p,q} = \tfrac{1}{8}(3p + 3q + \varepsilon t_{p,q}), \quad m_{p,q} = \tfrac{1}{8}(p + q + 3\varepsilon t_{p,q}) \tag{6.5.1}$$

where $t_{p,q} := \sqrt{p^2 - 6pq + q^2 + 8}$ and there is a unique choice $\varepsilon \in \{1, -1\}$ such that $d_{p,q}$ and $m_{p,q}$ are integers (see [MM24, §2.2]).

We define the "shift" map

$$S : (1, \infty) \to (5, 6), \qquad \frac{p}{q} \mapsto \frac{6p - q}{p},$$

and one can check that the intervals $S^k([6, \infty))$ are disjoint for $k \in \mathbb{Z}_{\geqslant 0}$, with union $\bigcup_{k \in \mathbb{Z}_{\geqslant 0}} S^k([6, \infty)) = (3 + 2\sqrt{2}, \infty)$. Note also that S fixes the accumulation point $3 + 2\sqrt{2}$ of the monotone staircase, and acts on the x-coordinates $\frac{g_{j+3}}{g_j}$ of its steps (outer corner points) by $\frac{g_{j+3}}{g_j} \mapsto \frac{g_{j+6}}{g_{j+3}}$, where $g_1, g_2, g_3, \dots$ is the sequence defined by the recursion $g_{j+6} = 6g_{j+3} - g_j$ with initial values $1, 1, 1, 1, 2, 4$ as in Table 1.

We also define the "reflection" map

$$R : (6, \infty) \to (6, \infty), \qquad \frac{p}{q} \mapsto \frac{6p - 35q}{p - 6q},$$

which is an involution fixing 7 and interchanging $(6, 7)$ with $(7, \infty)$. It turns out that both S and R are symmetries of the set $\underline{\mathrm{Perf}}(F_1)$. Further, both symmetries take the classes with $m/d > 1/3$ to those with $m/d < 1/3$, and vice versa. Thus they interchange the sets $\underline{\mathrm{Perf}}^+(F_1)$ and $\underline{\mathrm{Perf}}^-(F_1)$.

[33] If such a curve exists then it exists for any symplectic form on F_1. However, it gives an interesting obstruction to embedding ellipsoids into F_1 with symplectic form in class $\mathrm{PD}(\ell - be)$ only if $m/d \approx b$.

The following is a rough summary[34] of the set $\underline{\mathrm{Perf}}(F_1)$ (see [MMW22, Fig.2.2] for an illustration).

Theorem 6.5.2 ([MMW22; MM24]). *We have:*

- $\underline{\mathrm{Perf}}(F_1) \cap [1, 3 + 2\sqrt{2}) = \{\frac{g_{j+3}}{g_j} \mid j \in \mathbb{Z}_{\geqslant 1}\}$*;*
- *for all* $k \in \mathbb{Z}_{\geqslant 0}$ *we have* $\underline{\mathrm{Perf}}^+(F_1) \cap [2k+6, 2k+7] = \{2k+6\}$*, and there is a homeomorphism* $(2k+7, 2k+8) \cong (-\frac{1}{2}, \frac{3}{2})$ *such that the image of* $\underline{\mathrm{Perf}}^+(F_1) \cap (2k+7, 2k+8)$ *is the set of rational numbers in* $(0,1)$ *whose ternary expansion is finite and ends in* 1*;*
- $\underline{\mathrm{Perf}}(F_1) \cap (6,7) = \underline{\mathrm{Perf}}^-(F_1) \cap (6, \infty) = \{R(p/q) \mid p/q \in \underline{\mathrm{Perf}}(F_1) \cap (7, \infty)\}$*;*
- $\underline{\mathrm{Perf}}(F_1) \cap (3 + 2\sqrt{2}, 6) = \{S^i(p/q) \mid i \in \mathbb{Z}_{\geqslant 1}, p/q \in [6, \infty) \cap \underline{\mathrm{Perf}}(F_1)\}$.

Proof of Theorem F. We need to show that for every $p/q \in \underline{\mathrm{Perf}}(F_1)$ there is an index zero (p,q)-unicuspidal rational algebraic curve in F_1. Recall that $\underline{\mathrm{Perf}}(F_1) \cap (1, 3+2\sqrt{2})$ corresponds precisely to the x-coordinates of the outer corners of the infinite staircase in $c_{\mathcal{H}_1}(x)$ (i.e. the monotone symplectic form), which is covered by Theorem B.

For $p/q \in \underline{\mathrm{Perf}}(F_1) \cap [6, \infty)$, Lemma 6.5.3 below states that there exists a T-polygon Q which has a Delzant vertex adjacent to a vertex of type $\frac{1}{q^2}(1, pq-1)$, and such that $\mathbb{A}(Q_{\mathrm{nodal}})$ is diffeomorphic to F_1. Then by Theorem 6.1.7 (together with Lemma 6.1.10 as in the proof of Theorem D), F_1 contains an index zero (p,q)-unicuspidal rational algebraic curve C. This proves the theorem for $p/q \in [6, \infty)$.

Furthermore, we can assume that C above is (p,q)-well-placed with respect to a rational irreducible nodal anticanonical divisor $\mathcal{N}$. Then we can iteratively apply the generalized Orevkov twist Φ_{F_1} from Construction 2.2.1 to C. Using Proposition 2.2.1 we have that $\Phi^i_{F_1}(C)$ has a cusp of type $S^i(p/q)$ for $i \in \mathbb{Z}_{\geqslant 0}$. Since every $p/q \in \underline{\mathrm{Perf}}(F_1) \cap (3+2\sqrt{2}, \infty)$ is of the form $S^i(p/q)$ for some $i \in \mathbb{Z}_{\geqslant 0}$ and $p/q \in [6, \infty)$, this completes the proof. □

Lemma 6.5.3. *Given* $p/q \in \underline{\mathrm{Perf}}(F_1) \cap [6, \infty)$*, there exists a* T*-quadrilateral* $Q \subset \mathbb{R}^2$ *with a Delzant vertex* $\mathfrak{v}$ *that is adjacent to a vertex* $\mathfrak{w}$ *of type* $\frac{1}{q^2}(1, pq-1)$*. More precisely, we can take the edge directions at* $\mathfrak{v}$ *to be* $(1,0), (0,1)$ *and the edge directions at* $\mathfrak{w}$ *to be* $(0,-1), (q^2, 1-pq)$*. Moreover,* $\mathbb{A}(Q_{\mathrm{nodal}})$ *is diffeomorphic to* F_1*.*

Proof. N. Magill proves this lemma for the elements of $\underline{\mathrm{Perf}}^+(F_1) \cap [6, \infty)$ in [Mag22], and the forthcoming paper [Mag] extends this result to all the elements in $\underline{\mathrm{Perf}}(F_1) \cap (3 + 2\sqrt{2}, \infty)$. Since her argument is computationally intensive and proves much more than we need here, we now explain a different way to extend the result from $\underline{\mathrm{Perf}}^+(F_1) \cap [6, \infty)$ to $\underline{\mathrm{Perf}}^-(F_1) \cap [6, \infty)$.

[34] The proof in [MMW22] that this list is complete uses ideas that are rather different from those presented here. Indeed an essential input is the staircase accumulation point theorem in [Cri+20]. It would be interesting to know if there is another approach to this classification result.

The key to understanding why the elements of $\underline{\mathrm{Perf}}(F_1) \cap (3+2\sqrt{2}, \infty)$ correspond to T-quadrilaterals with a smooth corner is to note that these elements are organized into so-called generating triples $\mathcal{T} := \big(p_\lambda/q_\lambda, p_\mu/q_\mu, p_\rho/q_\rho\big)$ (see [MMW22]) whose entries are increasing and satisfy certain geometrically meaningful arithmetic identities. Moreover there is a numeric "mutation process" that generates from a given triple $\mathcal{T}$ two new triples called $x\mathcal{T} = \big(p_\lambda/q_\lambda, p_{x\mu}/q_{x\mu}, p_\mu/q_\mu\big)$ (the left mutation) and $y\mathcal{T} = \big(p_\mu/q_\mu, (p_{y\mu}/q_{y\mu}, p_\rho/q_\rho\big)$ (the right mutation); see [MMW22, §2.1]. It is shown in [MMW22, §4.3] that all the elements in $\underline{\mathrm{Perf}}^+(F_1) \cap [6, \infty)$ appear as the left entry p_λ/q_λ in some triple that is formed by a sequence of x- and y-mutations from the set of basic generating triples $(\mathcal{T}_*^n)_{n\geqslant 0}$ given by

$$\mathcal{T}_*^n = \Big(2n+6,\ \frac{4n^2+24n+29}{2n+4},\ 2n+8\Big), \quad n \geqslant 0. \tag{6.5.2}$$

One main result in [Mag22] is that each such triple corresponds to a T-quadrilateral $Q^{\mathcal{T}}$ with vertices $\mathfrak{v}_O, \mathfrak{v}_X, \mathfrak{v}_V, \mathfrak{v}_Y$, where

(i) $\mathfrak{v}_O = (0,0)$, $\mathfrak{v}_X$ lies on the positive x-axis, and $\mathfrak{v}_Y$ lies on the positive y-axis,

(ii) the edge vector $\mathfrak{v}_V - \mathfrak{v}_Y$ is positively proportional to $(q_\lambda^2, 1 - p_\lambda q_\lambda)$,

(iii) the edge vector $\mathfrak{v}_V - \mathfrak{v}_X$ is positively proportional to $(1 + \widetilde{p}_\rho q_\rho, q_\rho^2)$ for $\widetilde{p}_\rho := p_\rho - 6q_\rho$.

(iv) the eigenray at $\mathfrak{v}_X$ meets the side $\mathfrak{v}_Y - \mathfrak{v}_V$, and the mutation at $\mathfrak{v}_X$ takes the side $\mathfrak{v}_V - \mathfrak{v}_Y$ to $(1 + \widetilde{p}_\mu q_\mu, q_\mu^2)$.

Note that by Remark 4.2.4, the above conditions imply that the singularities at $\mathfrak{v}_Y, \mathfrak{v}_V$, and $\mathfrak{v}_X$ have types $\frac{1}{q_\lambda^2}(1, p_\lambda q_\lambda - 1)$, $\frac{1}{q_\mu^2}(1, p_\mu q_\mu - 1)$ and $\frac{1}{q_\rho^2}(1, p_\rho q_\rho - 1)$ respectively. If the side lengths of $Q^{\mathcal{T}}$ are such that the eigenray from $\mathfrak{v}_Y$ hits the side $\mathfrak{v}_V - \mathfrak{v}_X$, then the numeric x- and y-mutations mentioned above give new generating triples $x\mathcal{T}, y\mathcal{T}$ whose associated quadrilaterals $Q^{x\mathcal{T}}, Q^{y\mathcal{T}}$ are obtained from $Q^{\mathcal{T}}$ by the geometric mutations from $\mathfrak{v}_X$ and $\mathfrak{v}_Y$. The proof of this result is given in [Mag22, Lem. 6.5] and uses only the arithmetic identities satisfied by the entries of a generating triple $\mathcal{T}$. It follows that every element in $\underline{\mathrm{Perf}}^+(F_1) \cap [6, \infty)$ does correspond to the eigenray at $\mathfrak{v}_Y$ in some T-quadrilateral $Q^{\mathcal{T}}$ as above.

By [MMW22, §2.3], the symmetries R, S preserve the t-coordinate and take generating triples to generating triples. However note that the triple $R^\sharp(\mathcal{T})$ has entries $\big(R(p_\rho/q_\rho), R(p_\mu/q_\mu), R(p_\lambda/q_\lambda)\big)$ since R reverses orientation, which implies that R interchanges x-mutations with y-mutations. When considering the symmetry R, there is also a complication caused by the fact that $R(6) = 1/0$ does not correspond to a geometric point (though the ratio $1/0$ has numerical meaning). However, as noted in [MM24, Rmk 2.3.4(ii)], R takes the elements of the decreasing sequence $8/1 = p_\rho/q_\rho, 29/4 = p_\mu/q_\mu, 79/11 = p_{x\mu}/q_{x\mu}, \ldots$ formed from the middle entry of $\mathcal{T}_*^0$ by repeated x-mutations to the Fibonacci sequence $F_{2k+7}/F_{2k+3}, k \geqslant 0$. It follows that every element in $\underline{\mathrm{Perf}}(F_1)^- \cap (6, \infty)$ is the smallest entry p_λ/q_λ in some triple formed by mutation either from one of the basic generating triples

$$R^\sharp(\mathcal{T}_n) = \Big(\frac{12n+13}{2n+2},\ \frac{24n^2+62n+34}{4n^2+10n+5},\ \frac{12n+1}{2n}\Big), \quad n > 0 \tag{6.5.3}$$

or from one of the triples $xy^k R^\sharp(\mathcal{T}^0), k \geqslant 0$, with entries

$$\big(F_{2k+7}/F_{2k+3}, F_{2k+7}^2/F_{2k+5}^2, F_{2k+9}/F_{2k+5}\big), \quad k \geqslant 0. \tag{6.5.4}$$

We now show that each generating triple $\mathcal{T}$ in (6.5.3) and (6.5.4) has a corresponding T-quadrilateral $Q^{\mathcal{T}}$ that satisfies conditions (i), (ii), (iii) above as well as the following condition (iv′):

(iv′) the eigenray at $\mathfrak{v}_X$ meets the side $\mathfrak{v}_Y - \mathfrak{v}_V$, and the mutation at $\mathfrak{v}_X$ takes the side $\mathfrak{v}_V - \mathfrak{v}_Y$ to $(1 + \widetilde{p}_\mu q_\mu, q_\mu^2)$. Moreover the eigenray at $\mathfrak{v}_V$ is $(-1, 5)$ for the triples in (6.5.3) and $(-1, 0)$ for those in (6.5.4).

Given $\mathcal{T}$, let Q be any quadrilateral that satisfies conditions (i), (ii), (iii) above and also is such that the eigenray at $\mathfrak{v}_X$, which is in the direction $(\widetilde{p}_\rho, q_\rho)$, meets the edge $\mathfrak{v}_Y - \mathfrak{v}_V$. The mutation matrix A at $\mathfrak{v}_X$ is determined by the requirements that it fix the eigenray at X and take the vector $\mathfrak{v}_V - \mathfrak{v}_X$ to a multiple of $(1, 0)$, and hence has the formula

$$A_n = \begin{pmatrix} 1 - \widetilde{p}_\rho q_\rho & \widetilde{p}_\rho^2 \\ -q_\rho^2 & 1 + \widetilde{p}_\rho q_\rho \end{pmatrix}. \tag{6.5.5}$$

Thus for the generating triples in (6.5.3) we have

$$A_n = \begin{pmatrix} 1 - 2n & 1 \\ -4n^2 & 1 + 2n \end{pmatrix}, \tag{6.5.6}$$

and it is now straightforward to check that A_n takes the vector $\mathfrak{v}_V - \mathfrak{v}_Y = \big(-4(n+1)^2, (6n+5)(4n+5)\big)$ to $(1+\widetilde{p}_\mu q_\mu, q_\mu^2)$ and the vector $(-1, 5)$ to $(\widetilde{p}_\mu, q_\mu)$, where $\widetilde{p}_\mu = 2n+4$. Similarly, the proof for the generating triples in (6.5.4) reduces to calculating some polynomial identities between Fibonacci numbers. By [McSch12, Lem. 3.2.2, Prop. 3.2.3], it suffices to check these for three values of k, which again we leave to the reader. Note also that the identity $F_{2k+1}^2 = F_{2k+3}F_{2k-1} - 1$ implies that for these triples we have $\widetilde{p}_\mu = \widetilde{p}_\rho q_\rho - 1$.

Thus each basic generating triple $\mathcal{T}$ in (6.5.3) and (6.5.4) has a corresponding T-quadrilateral $Q^{\mathcal{T}}$ that satisfies conditions (i), (ii), (iii) and (iv′). It now follows from [Mag22, Lem. 6.5] that this correspondence extends to all generating triples obtained from these by x- and y-mutations. This completes the construction of the quadrilaterals $Q^{\mathcal{T}}$.

It remains to check that the associated symplectic four-manifold $\mathbb{A}(Q^{\mathcal{T}}_{\mathrm{nodal}})$ is diffeomorphic to F_1. The classification by Leung–Symington [LS10] of symplectic 4-manifolds admitting almost toric fibrations implies that the only other possibility is that $\mathbb{A}(Q^{\mathcal{T}}_{\mathrm{nodal}})$ is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Now, if $A = d_1\ell_1 + d_2\ell_2 \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is a (p, q)-perfect exceptional class, the identities

$$c_1(A) = 2d_1 + 2d_2 = p + q, \qquad A \cdot A = 2d_1d_2 = pq - 1 \tag{6.5.7}$$

imply that $t_{p,q} = \sqrt{p^2 - 6pq + q^2 + 8}$ is an even integer. Thus, if $\mathbb{A}(Q^{\mathcal{T}})$ were diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ then all three t-values $t_{p_\lambda,q_\lambda}, t_{p_\mu,q_\mu}$ and t_{p_ρ,q_ρ} would be even integers.

On the other hand, all three of the t-values associated to the basic generating triple $\mathcal{T}_*^n, n \geqslant 0$, in (6.5.2) are odd, and the symmetries S, R preserve t. By [MMW22, Lem. 2.1.2], for a triple $\mathcal{T}$ with t-values $(t_\lambda, t_\mu, t_\lambda)$, the x-mutated (resp. y-mutated) triple $x\mathcal{T}$ has t-values $(t_\lambda, t_\lambda t_\mu - t_\rho, t_\mu)$ (resp. $(t_\mu, t_\rho t_\mu - t_\lambda, t_\rho)$). It is now easy to check by induction that no triple that is obtained from a triple with all t-coordinates odd by a sequence of x-mutations, y-mutations and symmetry operations can have all associated t-values even. It follows that $\mathbb{A}(Q^{\mathcal{T}}_{\text{nodal}})$ cannot be diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. □

Remark 6.5.4 (Relation to $\mathbb{CP}^1 \times \mathbb{CP}^1$)**.** The analogue of Theorem 6.5.2 has not yet been fully worked out, but the works [Ush11; Far+22] suggest that a picture similar to that of F_1 should hold, in which case it is likely that our proof of Theorem F also carries over mutadis mutandis to the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$. One can check that the sequence $\frac{a_1}{a_0}, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots$ defined by $a_0 = 1, a_1 = 5, a_{k+1} = 6a_k - a_{k-1}$ lies in both $\underline{\text{Perf}}(F_1)$ and $\underline{\text{Perf}}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ (in each case giving the x-coordinates of the outer corners for one strand of the monotone infinite staircase), and these are likely the only points in common. An intriguing relation between points in $\underline{\text{Perf}}(F_1) \cap (3 + 2\sqrt{2}, \infty)$ and $\underline{\text{Perf}}(\mathbb{CP}^1 \times \mathbb{CP}^1) \cap (3 + 2\sqrt{2}, \infty)$ is suggested in [Far+22, Conj. 1.2.1]. ◊

Remark 6.5.5 (Unicuspidal curves in odd Hirzebruch surfaces)**.** For $k \in \mathbb{Z}_{\geqslant 0}$, let F_{2k+1} denote the $(2k+1)$st Hirzebruch surface [Hir51], which we view as a complex structure on $\text{Bl}^1\mathbb{CP}^2$ having an irreducible rational holomorphic curve in class $(k+1)e - k\ell \in H_2(\text{Bl}^1\mathbb{CP}^2)$ with self-intersection number $-2k-1$. Note that if $A = d\ell - me \in H_2(\text{Bl}^1\mathbb{CP}^2)$ is represented by a holomorphic curve in F_{2k+1} then by positivity of intersection we must have $\frac{m}{d} \geqslant \frac{k}{k+1}$. By [MMW22, Lem. B.5(i)], any $p/q \in \underline{\text{Perf}}(F_1) \cap (2j+6, 2j+8)$ satisfies $\frac{j+1}{j+2} < \frac{m_{p,q}}{d_{p,q}} < \frac{j+2}{j+3}$. Thus $p/q \in \underline{\text{Perf}}(F_{2k+1})$ only if $p/q > 2k+4$. Therefore we conjecture: *for all $k \in \mathbb{Z}_{\geqslant 1}$, there exists an index zero algebraic (p,q)-unicuspidal rational curve in F_{2k+1} if and only if $p/q \in \underline{\text{Perf}}^+(F_1) \cap (2k+4, \infty)$.* ◊

7 Sesquicuspidal curves and stable embeddings beyond the accumulation point

In this section, we prove first Theorem H on algebraic rational plane curves corresponding to the ghost stairs in §7.1. We then discuss stabilized ellipsoid embeddings and obstructions beyond the staircase accumulation points in §7.2.

7.1 Degree three seed curves and the ghost stairs

Our proof of Theorem H is based on the generalized Orevkov twist from §2, which boils it down to finding a single degree three seed curve.

Proof of Theorem H. The identity $\text{Fib}_{4k+6} = 7\text{Fib}_{4k+2} - \text{Fib}_{4k-2}$ shows that the Fibonacci subsequence $\text{Fib}_2, \text{Fib}_6, \text{Fib}_{10}, \ldots$ obeys the same recursive formula which is achieved by the Orevkov twist $\Phi_{\mathbb{CP}^2}$ (c.f. Proposition 2.2.6). Therefore it suffices to construct a suitable degree 3 seed curve, which is the content of the following proposition. $\square$

Proposition 7.1.1. *There exists a degree three rational algebraic curve in $\mathbb{CP}^2$ which is $(8,1)$-well-placed with respect to a nodal cubic $\mathcal{N}$.*

Remark 7.1.2. It seems plausible that the curve in Proposition 7.1.1 (and hence all of its twists) is sesquicuspidal, i.e. it has a single ordinary double point away from its distinguished cusp, but we do not prove this here. $\Diamond$

Proof of Proposition 7.1.1. As in §2.1, let $\mathcal{N}$ be a fixed nodal cubic in $\mathbb{CP}^2$ with local branches $\mathcal{B}_-, \mathcal{B}_+$ near the double point $\mathfrak{p}$. If J is an almost complex structure on $\mathbb{CP}^2$ which is integrable near $\mathfrak{p}$ and $\mathbf{D}$ is any local J-holomorphic divisor through $\mathfrak{p}$, we denote by $\mathcal{M}^J_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathbf{D}}\mathfrak{p}\gtrdot$ the moduli space of J-holomorphic degree 3 rational curves which pass through $\mathfrak{p}$ with contact order 8 to $\mathbf{D}$. The count $\#\mathcal{M}^J_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathbf{D}}\mathfrak{p}\gtrdot$ for generic J was computed in [McS21] to be 4. Note that any curve in $\mathcal{M}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ excluding $\mathcal{N}$ itself is by definition $(8,1)$-well-placed, although J_{std} is not generic.

Let $\{J_t\}_{t\in[0,1]}$ be a generic family of compatible almost complex structures on $\mathbb{CP}^2$ which are integrable near $\mathfrak{p}$ and fix $\mathcal{B}_-$ (but not $\mathcal{N}$), with $J_0 = J_{\mathrm{std}}$. For $t \in (0,1]$ we have $\mathcal{M}^{J_t}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot = \{C^1_t, C^2_t, C^3_t, C^4_t\}$, where C^k_t is a family of curves which varies smoothly in $t \in (0,1]$ for $k = 1,2,3,4$. Then each C^k_t converges to some limiting configuration $C^k_0 \in \overline{\overline{\mathcal{M}}}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ as $t \to 0$. Here $\overline{\overline{\mathcal{M}}}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ denotes the subset of the standard stable map compactification $\overline{\mathcal{M}}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathfrak{p}\gtrdot$ consisting of those configurations such that if the marked point lies on a ghost component then the nearby nonconstant components together "remember" the constraint $\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ (see [McS24, Def. 2.2.1]). In fact, since a line cannot satisfy $\lessdot\mathcal{T}^{(3)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ and a conic cannot satisfy $\lessdot\mathcal{T}^{(6)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ (due to the presence of the other branch $\mathcal{B}_+$), we can easily rule out configurations with multiple components, i.e. we have $\overline{\overline{\mathcal{M}}}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot = \mathcal{M}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ and hence $C^1_0, C^2_0, C^3_0, C^4_0 \in \mathcal{M}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,3\ell}\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$.

It remains to show that at least one of $C^1_0, C^2_0, C^3_0, C^4_0$ is distinct from $\mathcal{N}$. To see this, suppose by contradiction that $C^k_0 = C^{k'}_0 = \mathcal{N}$ for distinct $k, k' \in \{1,2,3,4\}$. Then C^k_t and $C^{k'}_t$ both approximate $\mathcal{N}$ and in particular the transversely intersecting branches $\mathcal{B}_-, \mathcal{B}_+ \subset \mathcal{N}$ for t small, and since they also both satisfy the constraint $\lessdot\mathcal{T}^{(8)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$, their intersection multiplicity satisfies

$$C^k_t \cdot C^{k'}_t \geqslant 8 + 1 + 1 > 9,$$

which is a contradiction. $\square$

Remark 7.1.3. We sometimes refer to a $(3d-1,1)$-well-placed rational plane curve as in Proposition 7.1.1 informally as a "degree d seed curve". The above argument easily extends to show that the number of degree three seed curves is precisely 3, at least if we replace algebraic curves with J-holomorphic curves where J agrees with J_{std} near $\mathcal{N}$ and is otherwise generic. We take up the problem of constructing higher degree seed curves in the forthcoming work [McS]. In general a sesquicuspidal degree d seed curve and all of its Orevkov twists have $\frac{1}{2}(d-1)(d-2)$ double points by the adjunction formula. Note that the argument given above does not easily generalize to higher degrees, due to the possibility of more complicated configurations in $\overline{\overline{\mathcal{M}}}^{J_{\mathrm{std}}}_{\mathbb{CP}^2,d\ell}\lessdot\mathcal{T}^{(3d-1)}_{\mathcal{B}_-}\mathfrak{p}\gtrdot$ involving one or more copies of $\mathcal{N}$ and its covers. $\Diamond$

7.2 On the stable folding curve

It is shown in [Hin15] that for any $a \in \mathbb{R}_{>1}$ and $N \in \mathbb{Z}_{\geqslant 1}$ there is a folding-type symplectic embedding

$$\mathring{E}(1,a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} B^4(\tfrac{3a}{a+1}) \times \mathbb{C}^N \tag{7.2.1}$$

which strengthens the construction in [Gut08]. As discussed e.g. in [MS23, §1.2], this embedding is conjectured to be sharp for all $a > \tau^4$, and proving this can be reduced to an existence problem for (p,q)-sesquicuspidal symplectic curves in $\mathbb{CP}^2$. In this subsection we discuss generalizations of this to rigid del Pezzo surfaces and to the convex toric domains considered in [Cri+20].

In the following, given a symplectic manifold M, we denote by $c \cdot M$ the same smooth manifold but with symplectic form scaled by $c \in \mathbb{R}_{>0}$. Recall that $E(1,a)$ denotes a closed ellipsoid, and we denote its interior by $\mathring{E}(1,a)$.

Proposition 7.2.1. *For any $a \in \mathbb{R}_{>1}$, there are symplectic embeddings:*

$$\mathring{E}(1,a) \times \mathbb{C} \overset{s}{\hookrightarrow} \tfrac{a}{a+1} \cdot M \times \mathbb{C} \tag{7.2.2}$$

for $M = \mathbb{CP}^2(3)\#^{\times j}\overline{\mathbb{CP}}^2(1)$ with $j = 0,1,2,3$ and $M = \mathbb{CP}^1(2) \times \mathbb{CP}^1(2)$.

The paper [Cri+20] also establishes infinite staircases for the ellipsoid embedding functions of twelve convex toric domains $X_1,\dots,X_{12}$, whose moment polygons are pictured in Figure 7. More precisely, for each $j = 1,\dots,12$ we put $X_j := \mu^{-1}_{\mathbb{C}^2}(\Omega_j) \subset \mathbb{C}^2$, where $\mu_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}, (z_1,z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$ is the moment map for the standard $\mathbb{T}^2$-action on $\mathbb{C}^2$. Note that, in contrast to closed toric symplectic manifolds, these are compact domains with piecewise smooth boundary in $\mathbb{C}^2$. As we recalled in Remark 1.2.6, for $j = 1,\dots,12$ the (unstabilized) ellipsoid embedding function c_{X_j} coincides with c_M, where M is the rigid del Pezzo surface having the same negative weight expansion as X_j.

Similar to Proposition 7.2.1, we have:

Proposition 7.2.2. *For any $a \in \mathbb{R}_{>1}$, there are symplectic embeddings*

$$\mathring{E}(1,a) \times \mathbb{C} \overset{s}{\hookrightarrow} \tfrac{a}{a+1} \cdot X_k \times \mathbb{C} \tag{7.2.3}$$

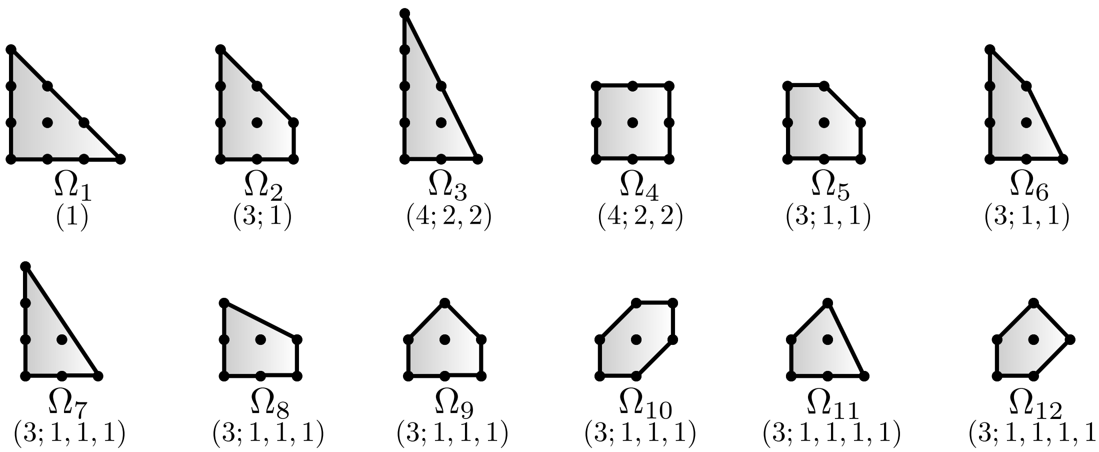
for $k = 1,2,3,4,5,6,8$.

Remark 7.2.3. In contrast to the four-dimensional (unstabilized) situation, it is not known whether the stabilized ellipsoid embedding functions of target spaces with the same negative weight expansions necessarily coincide (e.g. $c_{X_3 \times \mathbb{C}}$ versus $c_{X_4 \times \mathbb{C}}$ versus $C_{\mathbb{CP}^1(2) \times \mathbb{CP}^1(2) \times \mathbb{C}}$). $\Diamond$

Remark 7.2.4. The polygons $\Omega_1, \ldots, \Omega_{12}$ in Figure 7 correspond to twelve of the famous sixteen reflexive polygons. The remaining four reflexive polygons have the same negative weight expansion as $\mathbb{CP}^2(3) \#^{\times 5} \overline{\mathbb{CP}}^2(1)$, whose ellipsoid embedding function does not contain an infinite staircase (see [Cri+20, Rmk. 5.21]).

It is interesting to note that $M = \mathbb{CP}^2(3) \#^{\times 5} \overline{\mathbb{CP}}^2(1)$ admits an almost toric fibration $\pi : M \to Q_{\text{nodal}}$ where Q is a dual Fano T-quadrilateral with a Delzant vertex (see [Via17, Fig. 18(C_1)]). In particular, most of the results in §5 and §6 still apply in this case. Similarly, $M = \mathbb{CP}^2(3) \#^{\times 6} \overline{\mathbb{CP}}^2(1)$ admits an almost toric fibration $\pi : M \to Q_{\text{nodal}}$ where Q is a dual Fano T-pentagon with a Delzant vertex (see [Via17, Fig. 19(A_1)]) $\Diamond$

Figure 7: The convex toric domains considered in [Cri+20], along with their negative weight expansions.



Proofs of Proposition 7.2.1 and Proposition 7.2.2. We apply [CHS22, Prop. 3.1]. Specializing to the case $\mu = \frac{a}{a+1}$, we get $\lambda = 1 - \frac{\mu}{a} = \frac{a}{a+1}$, so $\mu = \lambda$. Note that $2\mu\frac{2\lambda-1}{\lambda+\mu-1} = 2\mu = \lambda + \mu$. Then f is the linear function satisfying $f(0) = 2\lambda = \frac{2a}{a+1}$ and $f(2a/(a+1)) = \lambda = \frac{a}{a+1}$. This means that there is a symplectic embedding of $(1-\delta) \cdot E(1,a) \times \mathbb{C}$ into $\frac{a}{a+1} \cdot X_{\Omega_H} \times \mathbb{C}$ for all $\delta > 0$, where $\Omega_H \subset \mathbb{R}^2_{\geqslant 0}$ is the quadrilateral having vertices $(0,0), (0,2), (2,1), (2,0)$, and $X_{\Omega_H} \subset \mathbb{C}^2$ is the corresponding four-dimensional convex toric domain. Using [PV15, Thm. 4.4], we can upgrade this to an embedding $\mathring{E}(1,a) \times \mathbb{C} \overset{s}{\hookrightarrow} \frac{a}{a+1} \cdot X_{\Omega_H} \times \mathbb{C}$.

Inspecting Figure 7, we see that Ω_H (or its reflection about the diagonal) is a subset of Ω_i for $i = 1,2,3,4,5,6,8$. Similarly, the moment polygons corresponding to $M = \mathbb{CP}^2(3), \mathbb{CP}^2(3)\#\overline{\mathbb{CP}}^2(1), \mathbb{CP}^1(2) \times \mathbb{CP}^1(2), \mathbb{CP}^2(3)\#^{\times 2}\overline{\mathbb{CP}}^2(1)$ are $\Omega_1, \Omega_2, \Omega_4, \Omega_5$ respectively, each of which directly contains Ω_H as a subset.

As for $\mathbb{CP}^2(3)\#^{\times 3}\overline{\mathbb{CP}}^2(1)$, the moment polygon is (up to an integral affine transformation) given by Ω_{10}. This does not contain Ω_H as a subset, but there is an almost toric fibration for $\mathbb{CP}^2(3)\#^{\times 3}\overline{\mathbb{CP}}^2(1)$ whose base polygon is precisely Ω_H – see [Via17, Fig. 16]. □

Remark 7.2.5. Note that Ω_H has area 3, as do the polygons $\Omega_7, \Omega_9, \Omega_{10}$, while the polygons Ω_{11}, Ω_{12} have area 5/2. In particular, by volume considerations there cannot be any four-dimensional embedding $\mathring{X}_{\Omega_H} \overset{s}{\hookrightarrow} X_k$ for $k = 11, 12$. ◊

It is natural to ask what happens in the remaining cases not covered by Propositions 7.2.1 and 7.2.2:

Question 7.2.6. *Is there a stabilized symplectic embedding*

$$\mathring{E}(1,a) \times \mathbb{C} \overset{s}{\hookrightarrow} \tfrac{a}{a+1} \cdot M \times \mathbb{C} \quad \textit{where} \quad M = \mathbb{CP}^2(3)\#^{\times 4}\overline{\mathbb{CP}}^2(1)?$$

What about $\mathring{E}(1,a) \times \mathbb{C} \overset{s}{\hookrightarrow} \frac{a}{a+1} \cdot X_k$ *for* $k = 7, 9, 10, 11, 12$*?*

Also, extending the aforementioned conjecture for stabilized embeddings of ellipsoids into the four-ball, we posit:

Conjecture 7.2.7. *The symplectic embeddings in Proposition 7.2.1 and Proposition 7.2.2 are all optimal for all* $a > a_{\mathrm{acc}}$*, where* a_{acc} *is the accumulation point of the corresponding staircase (c.f. Table 1).*

As evidence, we observe that the obstruction coming from any index zero sesquicuspidal curves is consistent with this conjecture:

Proposition 7.2.8. *Let* (M, ω_M) *be a closed symplectic manifold with* $[\omega_M] = c_1 \in H^2(M;\mathbb{R})$*, and let* C *be an index zero* (p,q)*-sesquicuspidal rational symplectic curve in* M*. Then the corresponding obstruction for a symplectic embedding* $E(1,p/q) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \lambda \cdot M \times \mathbb{C}^N$ *coming from Theorem 1.3.1 is* $\lambda \geqslant \frac{(p/q)}{(p/q)+1}$.

Proof. Since C has index zero and $c_1([C]) = p + q$, the obstruction is

$$\lambda \geqslant \tfrac{p}{[\omega_M]\cdot[C]} = \tfrac{p}{c_1([C])} = \tfrac{p}{p+q} = \tfrac{(p/q)}{(p/q)+1}.$$

□

Remark 7.2.9. According to [McS23, Thm. G], for any closed symplectic four-manifold M, the perfect exceptional homology classes in $H_2(M)$ are in bijective correspondence with index zero unicuspidal symplectic curves in M. For example, [MM24; MMW22] describes all perfect exceptional classes for the first Hirzebruch surface $\mathrm{Bl}^1\mathbb{CP}^2$ (c.f. §6.5), and by Proposition 7.2.8 these all give obstructions consistent with Conjecture 7.2.7. ◊

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