QUASIMORPHISMS ON FREE RACKS AND FREE QUANDLES

MASAMITSU AOKI

Abstract. We show that the second bounded cohomology of finitely generated free racks is infinite dimensional by constructing rack quasimorphisms using homogenous group quasimorphisms. Similar construction works in the case of free quandles. We show that Rolli's generalization to free products also works in the case of racks.

1. Introduction

A rack is a set together with a binary operation which comes from *rack*ing the group operation and remaining the conjugacy [6]. From the viewpoint of knots and braids, the axioms of a rack correspond to Reidemeister moves II and III. A quandle is introduced by [8] and [11] independently. The additional axiom corresponds to Reidemeister move I. Those algebraic objects are studied not only in knot theory, but also in the theory of set-theoretic Yang-Baxter equations [4], pointed Hopf algebra [1], and so on.

The study of bounded cohomology of groups started from Gromov's pioneering paper [7]. While the cohomology of a free group is trivial, its second bounded cohomology is known to be infinite-dimensional. One way to show this is by constructing quasimorphisms on a given free group and relating the second bounded cohomology to an infinite-dimensional space via these quasimorphisms (see, for example, [3]).

We can construct the cohomology of racks and quandles (see, for example, [12]). The bounded cohomology and quasimorphisms of racks and quandles are introduced by Kędra. In [10], it was observed that free quandles and free racks are unbounded in the sense of rack metric on a connected component and that the second bounded cohomology of an unbounded rack is nontrivial. These observations imply that the second bounded cohomology of free racks and free quandles are nontrivial, while the cohomology of free racks and free quandles are trivial ([5]), similar to the case of free groups. Thus, our next question is whether its second bounded cohomology is infinite dimensional.

In this article, we give an affirmative answer to this question.

Theorem 1. If $2 \leq |S| < \infty$, then the second bounded rack cohomology of a free rack on S is infinite dimensional.

Most of the discussions including the above theorem can be directly applied even in the case of free quandles. We will explain additional details for quandles if necessary.

We conclude this article by generalizing Rolli's method to construct quasimorphisms on the free product of groups to the free product of racks. In [13], Rolli

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showed that the similar construction of Rolli quasimorphisms works for the free product of groups. We show that the analogy holds for the free product of racks.

In the first two sections 2 and 3, we recall the definition of a rack and its (bounded) cohomology, and collect several facts. In section 4 we recall the facts about quasimorphisms on a free group and show that a homogenous quasimorphism on a free group provides a quasimorphism on a free rack. Then we prove our main theorem in section 5. We conclude this article by investigating the construction on the free product of racks in section 6.

2. Racks

We start by recalling the definition and examples of a rack. A *rack* is a set X together with a binary operation $\lhd\colon X\times X\to X$ satisfying the following axioms:

(1) the *rack identity*: $(x\lhd y)\lhd z=(x\lhd z)\lhd(y\lhd z)$ for any $x,y,z\in X$, and
(2) a map $\psi_y\colon X\to X$ defined by $\psi_y(x)=x\lhd y$ is bijective for any $y\in X$.

A *quandle* is a rack $(X,\lhd)$ satisfying

(3) $x\lhd x=x$ for any $x\in X$.

We write $\psi_y^n(x)=x\lhd^n y$ for any n.

A *rack homomorphism* between racks is a map $f\colon X\to Y$ with $f(x\lhd y)=f(x)\lhd f(y)$. In the case of quandles, it is called a *quandle homomorphism*.

Example 1. A set X together with the operation $x\lhd y=x$ is a rack, called the *trivial rack*.

Example 2. Let G be a group. A 'set' G together with the operation $g\lhd h=h^{-1}gh$ is a rack, called the *conjugacy rack* of G.

These racks are also, in fact, quandles. Of course, there is an example of a rack but not a quandle.

Example 3. Let n be a positive integer. The *cyclic rack* C_n *of order* n is the set $\mathbb{Z}/n\mathbb{Z}$ together with the operation $x\lhd y=x+1$. When $n\geq 2$, C_n is not a quandle.

The following constructions are the main subjects of our argument.

Example 4 ([6], [5])**.** Let S be a set. We write the free group on S by $F^G(S)$ and its identity by 1. A *free rack* on S is a set $F^R(S)=S\times F^G(S)$ together with the operation

$$(s,g)\lhd(t,h)=(s,gh^{-1}th). \tag{1}$$

Each generator $s\in S$ is identified with $(s,1)\in S\times F^G(S)$.

Example 5 ([8], [5])**.** A *free quandle* on S is a quandle $F^Q(S)$ defined to be the quotient $F^R(S)/{\sim}$ where the equivalence $\sim$ is generated by

$$(s,g)\sim(s,g)\lhd(s,g) \tag{2}$$

for all $(s,g)\in F^R(S)$. The operation descends to the quotient, that is,

$$[s,g]\lhd[t,h]=[s,gh^{-1}tg]. \tag{3}$$

By the definition of the equivalence, observe that $(s,g)\sim(t,h)$ if and only if $s=t$ and $h=s^ng$ for some n. Each generator $s\in S$ is identified with $[s,1]\in S\times F^G(S)/{\sim}$. As observed in [8], a free quandle $F^Q(S)$ is also viewed as the union of all conjugacy classes of $s\in S$ in $F^G(S)$, that is, $F^Q(S)=\bigcup_{s\in S}\{\,g^{-1}sg\mid g\in F^G(S)\,\}$, with the operation $\gamma\lhd\eta=\eta^{-1}\gamma\eta$.

There is a group associated with the given rack X. The *adjoint group* of X is a group $\mathrm{Ad}(X)$ presented as

$$\langle e_x \ (x \in X) \mid e_x e_y = e_y e_{x \triangleleft y} \rangle \, . \tag{4}$$

Since a rack homomorphism $f\colon X \to Y$ induces a group homomorphism $f_\sharp\colon \mathrm{Ad}(X) \to \mathrm{Ad}(Y)$, this gives rise to the functor $\mathrm{Ad}(-)$ from the category of racks to the category of groups.

Example 6. The adjoint group of a free rack $F^R(S)$ is a free group $F^G(S)$.

The adjoint group $\mathrm{Ad}(X)$ acts on X by $x \cdot e_y = x \triangleleft y$. A *connected component* in X is an orbit of this action.

3. (Bounded) cohomology of racks

The cohomology group of racks is defined similarly to that of groups. Since we will consider bounded cohomology and quasimorphisms, we restrict our attention to the trivial real coefficient $\mathbb{R}$.

For a rack X and non-negative integer n, let $C^n(X;\mathbb{R})$ be the set of functions $X^n \to \mathbb{R}$. For $n < 0$, let $C^n(X;\mathbb{R}) = 0$. Here we understand X^0 to be a one element set. The coboundary operator $\delta^n\colon C^n(X;\mathbb{R}) \to C^{n+1}(X;\mathbb{R})$ is defined by

$$\begin{aligned}\delta f(x_1, \ldots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^i \, [f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \\ - f(x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_{n+1})] \, .\end{aligned} \tag{5}$$

In case of $n \leq 0$, we define $\delta^n = 0$. Thus we obtain a cochain complex $C^*(X;\mathbb{R}) = (C^n(X;\mathbb{R}), \delta^n)$. The *rack cohomology* is the homology of this complex

$$H^n(X;\mathbb{R}) = \ker \delta^n / \operatorname{im} \delta^{n+1}. \tag{6}$$

Example 7. We demonstrate some calculations of coboundary operators in lower degrees.

$$\begin{aligned}\delta^1 f(x, y) &= f(x) - f(x \triangleleft y) \\ \delta^2 f(x, y, z) &= f(x, z) - f(x, y) - f(x \triangleleft y, z) + f(x \triangleleft z, y \triangleleft z) \\ \delta^3 f(x, y, z, w) &= f(x, z, w) - f(x, y, z) + f(x, y, z) \\ &\quad - f(x \triangleleft y, z, w) + f(x \triangleleft z, y \triangleleft z, w) - f(x \triangleleft w, y \triangleleft w, z \triangleleft w)\end{aligned} \tag{7}$$

The bounded cohomology of a rack is defined similarly to that of a group. For a function $f\colon Z \to \mathbb{R}$, we write $\|f\|_\infty = \sup\{\, |f(z)| \mid z \in Z \,\}$. For a rack X, let $C_b^n(X;\mathbb{R})$ be the submodule of functions $X^n \to \mathbb{R}$ which is bounded with respect to $\|\cdot\|_\infty$. The coboundary operators δ^n may be restricted to $C_b^n(X;\mathbb{R})$, and $C_b^*(X;\mathbb{R}) = (C_b^n(X;\mathbb{R}), \delta^n)$ forms a cochain complex. The *bounded rack cohomology* $H_b^n(X;\mathbb{R})$ is the cohomology of this complex. The inclusion $C_b^n \hookrightarrow C^n$ induces the maps $c^n\colon H_b^n(X;\mathbb{R}) \to H^n(X;\mathbb{R})$ called the *comparison maps*.

If X is a quandle, the *quandle cochain complex* of X is the quotient of the rack cochain complex of X by the subcomplex D^* defined by

$$D^n = \{\, f \in C^n \mid f(x) = 0 \text{ for each } x \text{ such that } x_i = x_{i+1} \text{ for some } i \,\} \, . \tag{8}$$

for $n \geq 2$ and $D^n = 0$ for $n \leq 1$. The *quandle cohomology* of X is a cohomology of this complex and the *bounded quandle cohomology* is the bounded cohomology in this sense. Since we treat the (bounded) rack cohomology and the (bounded) quandle cohomology in parallel in our argument, we also write the (bounded) quandle cochain group and the (bounded) quandle cohomology by the same notations C^*, C_b^*, H^*, H_b^*.

4. Quasimorphisms

In this section, we first recall quasimorphisms on groups and then show that homogenous quasimorphisms on a free group provide quasimorphisms on a free rack and a free quandle.

A *group quasimorphism* on a group G is a function $\phi\colon G \to \mathbb{R}$ satisfying

$$D(\phi) := \sup_{g,h\in G} |\phi(g) + \phi(h) - \phi(gh)| < \infty. \tag{9}$$

The constant $D(\phi)$ is called the *defect* of ϕ. In terms of group cohomology, ϕ is a group quasimorphism if and only if $\delta\phi$ is bounded.

A group quasimorphism $\phi\colon G \to \mathbb{R}$ is *homogenous* if it satisfies $\phi(g^n) = n \cdot \phi(g)$ for any $g \in G$ and $n \in \mathbb{Z}$. It is known that a homogenous group quasimorphism is constant on conjugacy classes, that is,

$$\phi(h^{-1}gh) = \phi(g) \tag{10}$$

for any $g, h \in G$.

Let S be a set with $2 \leq |S|$. The following quasimorphisms are known to be homogenous.

Example 8 ([2], [3])**.** Suppose that S is symmetric, that is, $s \in S \Leftrightarrow s^{-1} \in S$. Let w be a reduced word in S. A map $\phi_w\colon F^G(S) \to \mathbb{R}$ defined by

$$\begin{aligned} \phi_w(g) = \# &\text{ of copies of } w \text{ in the reduced representative of } g \\ - \# &\text{ of copies of } w^{-1} \text{ in the reduced representative of } g \end{aligned} \tag{11}$$

is a group quasimorphism, called a *Brooks* or (*big*) *counting* (*group*) quasimorphism.

Example 9 ([13])**.** Each non-trivial element $g \in F^G$ has a unique shortest factorization by powers

$$g = s_1^{n_1} s_2^{n_2} \cdots s_l^{n_l}, \tag{12}$$

where $s_1, \ldots, s_l \in S$ with $s_i \neq s_{i+1}$ and $n_1, \ldots, n_l \in \mathbb{Z} - \{0\}$. Let $\lambda\colon \mathbb{Z} \to \mathbb{R}$ be an odd bounded real sequence, i.e. $\lambda(-n) = -\lambda(n)$. A map $\phi_\lambda\colon F^G(S) \to \mathbb{R}$ defined by

$$\phi_\lambda(g) = \sum_{i=1}^{l} \lambda(n_i), \tag{13}$$

where the factorization $g = s_1^{n_1} s_2^{n_2} \cdots s_l^{n_l}$, called a *Rolli* (*group*) *quasimorphism*.

A *rack quasimorphism* on a rack X is a function $\phi\colon X \to \mathbb{R}$ satisfying

$$\sup_{x,y\in X} |\phi(x) - \phi(x \lhd y)| < \infty. \tag{14}$$

When X is a quandle, such ϕ is called a *quandle quasimorphism*. As in the case of group quasimorphisms, ϕ is a rack quasimorphism if and only if $\delta\phi$ is bounded by the definition of the rack coboundary map. In the case of quandles, $\delta\phi(x, x) = 0$ comes from the axiom 3.

Lemma 1 (5.2 in [10])**.** If ϕ is a rack quasimorphism then $\delta\phi$ is a rack 2-cocycle. The bounded rack 2-cohomology class $[\delta\phi]_b$ is in the kernel of the comparison map. If f is unbounded on a connected component then $\delta\phi$ is nontrivial.

From the above lemma, one way to obtain a nontrivial bounded rack 2-cohomology class is to construct a rack quasimorphism that is unbounded on a connected component. We may construct such rack quasimorphism on a free rack $F^R(S)$ using a homogenous group quasimorphism on the free group $F^G(S)$. Recall that a free rack $F^R(S)$ on S is a set $S \times F^G(S)$ together with the operation

$$(s,g) \lhd (t,h) = (s, gh^{-1}th). \tag{15}$$

Proposition 1. Let S be a set with $2 \leq |S| < \infty$, and $\phi\colon F^G(S) \to \mathbb{R}$ a group quasimorphism. If ϕ is homogenous, then a map $\hat{\phi}\colon F^R \to \mathbb{R}$ defined by

$$\hat{\phi}(s,g) = \phi(g) \tag{16}$$

is a rack quasimorphism. Moreover, this is unbounded.

Proof. Since ϕ is homogenous, $\hat{\phi}$ is unbounded.

Next, we show that $\hat{\phi}$ is a rack quasimorphism. We have

$$\begin{aligned} |\hat{\phi}(s,g) - \hat{\phi}((s,g) \lhd (t,h))| &= |\phi(g) - \phi(gh^{-1}th) + (\phi(h^{-1}th) - \phi(h^{-1}th))| \\ &\leq D(\phi) + |\phi(t)| < \infty \end{aligned} \tag{17}$$

since $|S| < \infty$ and then the values at generators $\phi(t)$ are finite. $\square$

Example 10. In the case of Rolli group quasimorphisms ϕ_λ, we have $D(\phi_\lambda) = 3\|\lambda\|_\infty$ and $|\phi_\lambda(t)| \leq \|\lambda\|_\infty$. Thus We have

$$|\hat{\phi}_\lambda(s,g) - \hat{\phi}_\lambda((s,g) \lhd (t,h))| \leq 4\|\lambda\|_\infty. \tag{18}$$

Remark 1. A similar construction works in the case of free quandle. By the construction of a free quandle, each equivalence class $[s,g]$ consists of elements in $F^R(S)$ of the form

$$(s, s^n g) \text{ for all } n \in \mathbb{Z}. \tag{19}$$

Thus we can choose the unique representative (s,g) such that the reduced word of g has the prefix $s_1^{\pm 1}$ which cannot be cancelled with s. Using such expression, a map $\hat{\phi}\colon F^Q(S) \to \mathbb{R}$ defined by

$$\hat{\phi}([s,g]) = \phi(g) \tag{20}$$

is a quandle quasimorphism by the same argument in the case of free racks.

Remark 2. The construction of Rolli group quasimorphisms can be generalized directly. With the analogy of the case of free groups, for $(s,g) \in F^R(S)$ with the power factorization $g = s_1^{n_1} \cdots s_l^{n_l}$, we have the 'power factorization'

$$(s,g) = (((s,1) \lhd^{n_1} (s_1,1)) \lhd \cdots) \lhd^{n_l} (s_l,1) \tag{21}$$

since the generators $s \in F^R(S)$ can be seen as $(s,1)$ in $S \times F^G(S)$. Such factorization is also unique [14]. Thus, for a bounded odd sequence λ, a map $\psi_\lambda\colon F^R(S) \to \mathbb{R}$ defined by

$$\psi_\lambda(s,g) = \sum_{i=1}^{l} \lambda(n_i) \tag{22}$$

is a rack quasimorphism which coinsides to the rack quasimorphism $\hat{\phi}_\lambda$ obtained from a Rolli group quasimorphism ϕ_λ.

We conclude this section with remarks about rather natural constructions, which fail to obtain nontrivial rack quasimorphisms.

Remark 3. For a rack X and its adjoint group $\mathrm{Ad}(X)$, there is a map $\eta_X : X \to \mathrm{Ad}(X)$, $x \mapsto e_x$ (this is not necessarily injective). Thus we might obtain a rack quasimorphism by composing a group quasimorphism $\phi\colon \mathrm{Ad}(X) \to \mathbb{R}$ with η_X. However, this rack quasimorphism is trivial if ϕ is homogenous since η_X maps $x \triangleleft y$ to the conjugate $e_y^{-1} e_x e_y$ and a homogenous quasimorphism is constant on each conjugacy class.

In particular, while the adjoint group of a free rack $F^R(S)$ is a free group $F^G(S)$, we cannot obtain nontrivial rack quasimorphism by composing a homogenous group quasimorphism on F^G with $\eta\colon F^R \to \mathrm{Ad}(F^R) = F^G$.

Remark 4. Kabaya [9] introduced a chain map $\kappa_n\colon C_n(X) \to C_n(\mathrm{Ad}(X))$ from the natural map $\eta\colon X \to \mathrm{Ad}(X)$. It is just $\kappa_1(x) = [e_x]$, and

$$\kappa_2(x, y) = [e_x|e_y] - [e_y|e_{x\triangleleft y}]. \tag{23}$$

We might obtain rack quasimorphisms by pulling back group quasimorphisms via induced cochain map $\kappa^*\colon C^n(\mathrm{Ad}(X), \mathbb{R}) \to C^n(X, \mathbb{R})$. However, for the same reason as the above remark, the pullback of a homogenous group quasimorphism by κ^* gives rise to a trivial rack quasimorphism.

Remark 5. A free quandle may also be constructed as the subset of a free group. However, we can obtain only trivial quandle quasimorphisms by just restricting group quasimorphisms on this subset since the operation here is conjugation.

5. Proof of Main Theorem

Since the space of homogenous quasimorphisms on a free group, $Q^h(F^G)$, is infinite dimensional, it is sufficient to show the following.

Proposition 2 (cf. Proposition 2.2 in [13])**.** If $2 \leq |S| < \infty$, then the linear map $Q^h(F^G(S)) \to H_b^2(F^R(S); \mathbb{R})$, $\phi \mapsto [\delta^1\hat{\phi}]_b$ is injective.

Proof. Linearity is clear by construction.

To show injectivity, assume $[\delta^1\hat{\phi}]_b = 0$. That is, there exists a bounded function $\beta \in C_b^1(F^R; \mathbb{R})$ such that $\delta\beta = \delta\hat{\phi}$. Thus, $f = \hat{\phi} - \beta\colon F^R \to \mathbb{R}$ is a rack homomorphism where $\mathbb{R}$ is endowed with a trivial rack structure. Since $f(x \triangleleft y) = f(x) \triangleleft f(y) = f(x)$, f is constant on each connected component. Therefore, $\hat{\phi} = f + \beta$ is bounded. Since ϕ is homogenous, we have

$$\hat{\phi}(s, g^n) = \phi(g^n) = n\phi(g) \tag{24}$$

for any $s \in S$, $g \in F^G(S)$ and $n \in \mathbb{Z}$. Since $\hat{\phi}$ is bounded, this implies $\phi = 0$. $\square$

Remark 6. For a quandle quasimorphism ϕ on a quandle, $\delta\phi$ provides a quandle 2-cocycle since

$$\delta\phi(x, x) = \phi(x) - \phi(x \triangleleft x) = 0. \tag{25}$$

The above argument does not affect our choice of the representatives. Therefore, the same argument also works in the case of free quandles.

6. Free Product

We can define the free product of racks (cf. [6]). For a family of racks X_s $(s \in S)$, the *free product* is a rack $\ast_{s\in S} X_s$ consists of elements of the form (x, g) where $x \in X_t$ for some $t \in S$ and $g \in \ast_{s\in S} \mathrm{Ad}(X_s)$ under the equivalence generated by

$$(x, gk) \sim (y, k) \tag{26}$$

where $x, y \in X_t$, $g \in \mathrm{Ad}(X_t)$ with $x \cdot g = y$ in X_t and $k \in \ast \mathrm{Ad}(X_s)$ for each $t \in S$. The operation is

$$(x, g) \triangleleft (y, h) = (x, gh^{-1}yh). \tag{27}$$

This is well-defined by the definition of the adjoint group.

Observe that for each $t \in S$, if $x \in X_t$ and $g = g_0 \cdot g_1 \cdots g_n \in \ast \mathrm{Ad}(X_s)$ with $g_i \in \mathrm{Ad}(X_{s_i})$ $(s_i \neq s_{i+1})$, then the element (x, g) can be written in the form

$$(x \cdot g_0, g_1 \cdots g_n) \tag{28}$$

if $g_0 \in \mathrm{Ad}(X_t)$. Therefore, we can assume that each element in $\ast X_s$ has the form

$$(x, g_1 \cdots g_n) \tag{29}$$

where $x \in X_t$ and $g_1 \in \mathrm{Ad}(X_s)$ with $s \neq t$. In such an expression the factorization $g = g_1 \cdots g_n$ is unique whereas x varies within the defining relations of X_t.

Rolli [13] provided the method to construct group quasimorphisms on a free product in the following method. Let Γ be the free product of a family of groups Γ_s $(s \in S)$, and λ be a uniformly bounded family of bounded odd functions $\lambda_s \colon \Gamma_s \to \mathbb{R}$, that is, $\sup_{s\in S} \|\lambda_s\|_\infty < \infty$ and 'odd' means $\lambda_s(g^{-1}) = -\lambda_s(g)$. Each element in Γ is uniquely written in the factorization $g = g_1 \cdots g_n$ with $g_i \in \Gamma_{s_i}$ $(s_i \neq s_{i+1})$. Then a map $\phi^G_\lambda \colon \Gamma \to \mathbb{R}$ defined by

$$\phi^G_\lambda(g) = \sum_{i=0}^{n} \lambda_{s_i}(g_i) \tag{30}$$

is a group quasimorphism.

Similar to the case of a free rack, we can obtain rack quasimorphisms on a free product of racks from group quasimorphisms on a free product of groups.

Proposition 3. Let S be a set with $|S| \geq 2$. For any uniformly bounded family of bounded odd functions $\lambda = (\lambda_s)_{s\in S}$, a map $\phi^R_\lambda \colon X \to \mathbb{R}$ defined by

$$\phi^R_\lambda(x, g) = \phi^G_\lambda(g) \tag{31}$$

is a rack quasimorphism.

Proof. Let $(x, g), (y, h) \in X$ with g and h have the above factorizations. Then we have

$$\delta\phi^R_\lambda((x, g), (y, h)) = \phi^R_\lambda(x, g) - \phi^R_\lambda(x, gh^{-1}yh) = \phi^G_\lambda(g) - \phi^G_\lambda(gh^{-1}yh). \tag{32}$$

The argument in Proposition 1 works as well in this context. Therefore

$$|\delta\phi^R_\lambda((x, g), (y, h))| \leq 4 \cdot \sup \|\lambda_s\|_\infty < \infty. \tag{33}$$

Therefore ϕ^R_λ is a rack quasimorphism. □

Let $X = * \, X_s$ be the free product of racks, and $\Gamma = * \, \Gamma_s$ is the free product of the adjoint groups $\Gamma_s = \mathrm{Ad}(X_s)$. We write as $V_0(\Gamma)$ the space of uniformly bounded families of bounded odd functions.

Proposition 4. If $2 \leq |S| < \infty$, then the linear map $V_0(\Gamma) \to EH_b^2(X;\mathbb{R})$, $\lambda \mapsto [\delta^1 \phi_\lambda^R]_b$ is injective.

Proof. Assume that $[\delta\phi_\lambda^R]_b = 0$. That is, there exists a bounded function $\beta \in C_b^1(X)$ such that $\delta\beta = \delta\phi_\lambda^R$. Then ϕ_λ^R is bounded since the map $\phi_\lambda^R - \beta$ is a rack homomorphism and the number of connected components in X is finite.

For any $g \in \Gamma_s$, $h \in \Gamma_t$ $(s \neq t)$, $x \in X_u$ $(u \neq s, t)$ and $k \in \mathbb{Z} - 0$, we have

$$\phi_\lambda^R(x, (gh^{\pm 1})^k) = \phi_\lambda^G((gh^{\pm 1})^k) = k \cdot (\lambda_s(g) \pm \lambda_t(h)) \,. \tag{34}$$

Since ϕ_λ^R is bounded, $\lambda_s(g) \pm \lambda_t(h) = 0$ and then $\lambda = 0$. □

Example 11. The adjoint group of a cyclic rack C_n is $\mathbb{Z}$ (see [6]).

For the free product $C_2 * C_3$, the space $V_0(\Gamma)$ is isomorphic to the direct product of two copies of the space of bounded odd sequences $\mathbb{Z} \to \mathbb{R}$, $\ell_b^\infty(\mathbb{Z})$. Thus $V_0(\Gamma) \cong \ell_b^\infty(\mathbb{Z}) \times \ell_b^\infty(\mathbb{Z})$ is infinite-dimensional, and then $EH_b^2(C_2 * C_3;\mathbb{R})$ is also infinite-dimensional by Proposition 4.

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