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Hamiltonian simulation of minimal holographic sparsified SYK model

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Abstract

The circuit complexity for Hamiltonian simulation of the sparsified SYK model with N Majorana fermions and $q = 4$ (quartic interactions) which retains holographic features (referred to as 'minimal holographic sparsified SYK') with $k \ll N^3/24$ (where k is the total number of interaction terms times $1/N$) using second-order Trotter method and Jordan-Wigner encoding is found to be $\widetilde{\mathcal{O}}(k^p N^{3/2} \log N (\mathcal{J}t)^{3/2} \varepsilon^{-1/2})$ where t is the simulation time, ε is the desired error in the implementation of the unitary $U = \exp(-iHt)$, $\mathcal{J}$ is the disorder strength, and $p < 1$. This complexity implies that with less than a hundred logical qubits and about 10^6 gates, it will be possible to achieve an advantage in this model and simulate real-time dynamics up to scrambling time.

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I. INTRODUCTION

The quest to find simple quantum many-body systems with holographic behavior is an important research direction that *might* be one day also helpful to perform lab experiments and understand the effects of quantum gravity. It would have been better if this quantum gravity was classically Newtonian gravity but it still has merit as a toy model. The simplest quantum model (yet known) with such a feature is the Sachdev-Ye-Kitaev (SYK) model. As we move towards the era of quantum computing, we would like to explore the possibility of studying these holographic models using quantum computers. We are far from this at the moment as there is no problem in quantum many-body physics that *can be* studied on a quantum computer, but *cannot* be matched using a classical computer. Most likely, this is expected to change in the next decade. Due to its importance, one would like to study the SYK model and compute observables with minimum resources and go beyond the regime yet accessed using classical computers (referred to as 'advantage'). One such observable is the out-of-time-ordered (OTOC) correlation function used in the study of quantum chaos. For the computation of this observable, we need to implement the time evolution whose complexity is partially determined by the structure of the Hamiltonian. The SYK model with all-to-all quartic interaction results in a dense and non-local Hamiltonian. Though this is not *very hard* to study with quantum computers, a natural question is the degree to which the model can be sparsified (i.e., how many terms in the dense Hamiltonian can be ignored and set to zero) to maintain holographic behavior such that it becomes slightly *easier*. This was first studied in Ref. [1, 2] and recent analysis [3] suggests that one can go pretty far (but not below some k_{critical}) in sparsifying the model and still have signs of holographic behavior, measured in the saturation of Lyapunov exponent in the low-temperature ($\beta\mathcal{J} \gg 1$) limit. In this paper, we estimate the resources required to simulate this model in the noisy and fault-tolerant era and find that the dependence of resources on sparsity is sub-linear in k. We find the leading order circuit complexity using the second-order Trotter formula is $\widetilde{\mathcal{O}}(k^{p<1}N^{3/2}\log N(\mathcal{J}t)^{3/2}\varepsilon^{-1/2})$ with Jordan-Wigner mapping of the fermions to qubits. This can possibly be improved to $\widetilde{\mathcal{O}}(k^{p<1}N\log^2 N(\mathcal{J}t)^{3/2}\varepsilon^{-1/2})$ using Bravyi-Kitaev encoding but we do not pursue it here.

II. SYK HAMILTONIAN AND THE SPARSIFIED VERSION

The Hamiltonian for the dense SYK model [4] is given by:

$$H = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l, \tag{1}$$

where χ are the Majorana fermions satisfying $\{\chi_i, \chi_j\} = 2\delta_{ij}$, $\chi_i^2 = \mathbb{1}$. N Majorana fermions can be represented by $N/2$ fermions (complex) which has a local two-dimensional Hilbert space and hence can be represented by one qubit. Therefore, for SYK with N Majorana fermions, we need $n = N/2$ qubits with the Hamiltonian, H, of size $2^n \times 2^n$. The random disorder coupling J_{ijkl} are taken from a Gaussian distribution with mean $\overline{J_{ijkl}} = 0$ and variance equal to $\overline{J_{ijkl}^2} = 6\mathcal{J}^2/N^3$ [4] where $\mathcal{J}$ has dimension of energy. For this dense H with exactly four fermions (quartic) interacting in each term of the Hamiltonian, there is a total of $\binom{N}{4}$ terms which grows like $\sim N^4/4!$ for large N. In previous work, we found the gate complexity for implementing time evolution using first-order Trotter (for fixed t and error ε) scaled like $\sim N^5$ unlike $\sim N^{10}$ discussed in Ref. [5]. For interesting holographic interpretation and computation of out-of-time order correlators (OTOC), we need a large N limit and sufficiently many Trotter steps to ensure that the Lyapunov exponent can be reliably computed. Therefore, we seek a simplified version of the SYK model such that the complexity can be improved further. Such a model is known as the 'sparse SYK' model and was first discussed in [1, 2]. The average number of terms in the Hamiltonian is $p\binom{N}{4}$. The average degree of the interaction hypergraph k (defined as the number of hyperedges divided by the number of vertices, i.e., N) in the large N limit is $p\frac{N^4}{24}\frac{1}{N} = \frac{pN^3}{24}$. If we take $k = \frac{N^3}{24}$, then $p = 1$ and we have the dense (or the usual) SYK model. The minimum value of k such that one can still have holographic features in the model is an open question, and the most recent estimate argues that $k \geq 8.7$ [3], while it was argued in [1] that a value smaller $k \approx 4$ would also suffice. The Hamiltonian of the sparsified SYK model is:

$$H = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} p_{ijkl} J_{ijkl}\chi_i\chi_j\chi_k\chi_l, \tag{2}$$

where now we have an altered variance equal to $\overline{J_{ijkl}^2} = \frac{6\mathcal{J}^2}{pN^3}$. Note that due to the random number that determines the removal of terms (i.e., sparsifying with some probability), the average number of terms is not fixed like the dense SYK but varies, and, therefore, the corresponding gate costs. For the dense model, $p_{ijkl} = 1$ for all $\{i, j, k, l\}$. The terms in sparse SYK are removed with probability $(1 - p)$ and retained with probability p, i.e., $p_{ijkl} = 0$ if a random uniform number between 0 and 1 is less than $(1 - p)$. This drastically alters the sparseness of the Hamiltonian and makes it *easier* for quantum simulations. To quantum simulate this model, we use the standard Jordan-Wigner mapping to qubits where one needs $N/2$ qubits to describe a model with N Majorana fermions. Using the 2×2 Pauli matrices and identity matrix $\mathbb{1}$, we obtain:

$$\begin{aligned} \chi_{2k-1} &= Z_1 \cdots Z_{k-1} X_k \mathbb{1}_{k+1} \cdots \mathbb{1}_{N/2}, \\ \chi_{2k} &= Z_1 \cdots Z_{k-1} Y_k \mathbb{1}_{k+1} \cdots \mathbb{1}_{N/2}, \end{aligned} \tag{3}$$

where k runs from $1, \cdots, N/2$ and the Kronecker product between 2×2 matrices is not written explicitly. Once the Hamiltonian is written in terms of Pauli operators, we use the Trotter approach [6] to simulate the model. Though popular, the Trotter method is not the only method to do the time evolution of a quantum system. Many better algorithms exist with the additional requirements of constructing specific oracles. For example, the Hamiltonian simulation algorithm in the fault-tolerant era based on qubitization argues that complexity to leading order for dense SYK model is $\widetilde{\mathcal{O}}(N^{7/2}t)$ [7]. This is several orders of magnitude less than the first proposal using Trotter based approach of Ref. [5] using $\sim N^{10}$ gates per step. This was recently improved to $\sim N^5$ in Ref. [8] using the commuting clusters of terms in the Hamiltonian and evolving the cluster together by its diagonalizing circuit. In this work, we will only consider the Trotter-based methods due to their inherent simplicity and non-requirement of ancilla qubits or specific oracles. To do such a Hamiltonian simulation, one decomposes the sparse SYK Hamiltonian into strings of Pauli operators as $H = \sum_{j=1}^{m} H_j$ where each H_j is a tensor product of Pauli operators.

Then using the first-order product formula (Trotter1) we can bound the error as follows:

$$\left\|e^{-iHt} - \Big(\prod_{j=1}^{m} e^{-iH_j t/r}\Big)^r\right\| \leq \frac{\mathcal{J}^2 t^2}{2r} \sum_{p=1}^{m} \left\|\Big[\sum_{q=p+1}^{m} H_q, H_p\Big]\right\|, \tag{4}$$

where we denote the unitary invariant spectral norm by $\|\cdots\|$. One can also use a second-order formula, in which case the error bounds are given by [9]:

$$\begin{aligned}\left\|e^{-iHt} - \Big(\prod_{j=m}^{1} e^{-iH_j t/2r} \prod_{j=1}^{m} e^{-iH_j t/2r}\Big)^r\right\| &\leq \frac{\mathcal{J}^3 t^3}{12r^2} \sum_{p=1}^{m} \left\|\Big[\sum_{r=p+1}^{m} H_r, \Big[\sum_{q=p+1}^{m} H_q, H_p\Big]\Big]\right\| \\ &+ \frac{\mathcal{J}^3 t^3}{24r^2} \sum_{p=1}^{m} \left\|\Big[H_p, \Big[H_p, \sum_{r=p+1}^{m} H_r\Big]\Big]\right\|. \end{aligned} \tag{5}$$

For example, if we assume that m is reasonably small compared to the total number of terms in the Hamiltonian as discussed in [8] and assuming that the operator norms are bounded by unity (this can be done by redefining the H by including a factor of $1/N$), then the second-order product formula reads:

$$\left\|e^{-iHt} - \Big(\prod_{j=m}^{1} e^{-iH_j t/2r} \prod_{j=1}^{m} e^{-iH_j t/2r}\Big)^r\right\| \leq \mathcal{O}(\mathcal{J}^3 t^3/r^2). \tag{6}$$

If we want to bound this error by ε, then we would need to perform about $(\mathcal{J}t)^{3/2}/\sqrt{\varepsilon}$ Trotter steps. We show the cost per Trotter step in terms of two-qubit gates and the cost in fault-tolerant era gates i.e., Clifford and T-gates in Table I for $N \leq 250$ for various N. The complexity using Jordan-Wigner encoding for first-order Trotter is $\widetilde{\mathcal{O}}(k^{p<1} N^{3/2} \log N (\mathcal{J}t)^2 \varepsilon^{-1})$ while for second-order Trotter is $\widetilde{\mathcal{O}}(k^{p<1} N^{3/2} \log N (\mathcal{J}t)^{3/2} \varepsilon^{-1/2})$.

III. SUMMARY

We have computed the gate complexity for simulating the sparse SYK model on near-term and fault-tolerant devices by calculating the number of two-qubit gates and Clifford+T gates required using the Trotter approach. The resource estimates clearly show that while dense SYK will not be *easy*, controlled sparsification with $k < 10$ will be possible to study in the coming decade on quantum devices. There are possibly other effective ways of sparsifying the Hamiltonian based on the commutativity of Pauli string representation of terms rather than just random deletion such that complexity can be minimized without loss of any holographic feature. But, even with reduced complexity for the time evolution, the calculation of the Lyapunov exponent for some $\beta\mathcal{J} \gg 1$ might still be hard due to the challenges of computing four-point correlations over thermal quantum states at finite temperatures. The extent to which the SYK model can be sparsified is promising because this hints at the fact that there might be other quantum many-body systems at low temperatures which admit holographic features and for which the Hamiltonian simulation also scales favorably in the degrees of freedom. It would be interesting to find such models and study them alongside sparse SYK on quantum hardware in the coming decades. This is *much easier* than other 0+1-dimensional holographic models, such as BFSS (Banks-Fischler-Shenker-Susskind) matrix

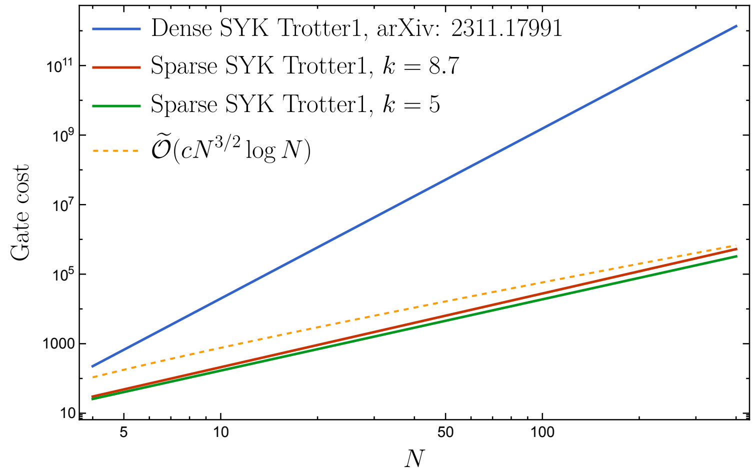

Figure 1: The circuit complexity to simulate the sparse SYK model with two choices of sparsification of the model. The gate costs are evaluated based on arranging the terms in Hamiltonian in commuting clusters and evolving each cluster by finding its diagonalizing circuit [8]. We denote the sub-linear dependence on k with $c \sim k^p$ where $p < 1$. We find that leading circuit complexity per Trotter step is $\widetilde{\mathcal{O}}(k^p N^{3/2}\log N)$.

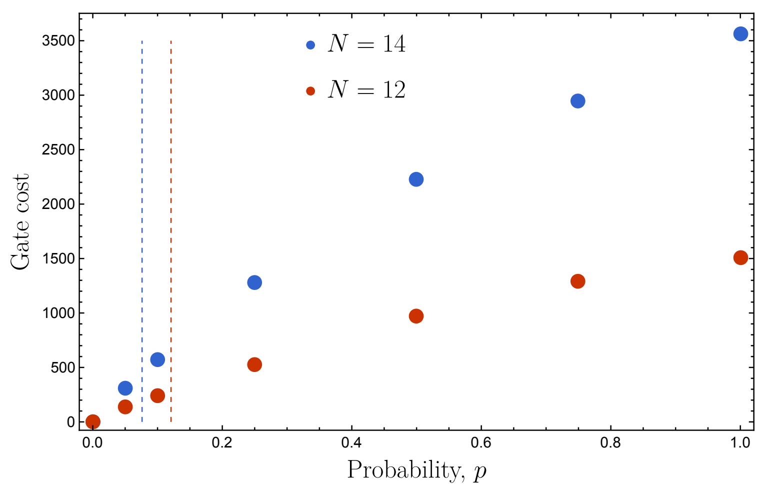

Figure 2: The dependence of circuit complexity on probability p for $N = 12, 14$ SYK. The standard SYK model with no pruning is at $p = 1$. As we remove the terms, the complexity decreases. The dashed lines denote the result from Ref. [3] corresponding to minimum p, which retains holographic features for $N = 12, 14$.

model [10] with large gauge groups that require resources (estimated by the number of logical qubits and number of gates) far greater than required for factoring 2048-bit RSA integers [11].

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N	Two-qubit gates	Clifford+T gates	T-gates
6	29 ± 1	126 ± 3	50 ± 2
8	74 ± 2	349 ± 12	150 ± 7
10	165 ± 5	645 ± 12	253 ± 6
12	294 ± 6	1110 ± 22	430 ± 11
14	440 ± 10	1593 ± 30	609 ± 12
16	596 ± 16	2053 ± 56	763 ± 23
18	771 ± 27	2478 ± 91	880 ± 38
20	967 ± 18	3097 ± 77	1104 ± 38
30	2528 ± 77	6955 ± 165	2198 ± 41
40	4462 ± 59	10939 ± 157	3072 ± 53
50	6943 ± 85	15844 ± 140	4062 ± 37
60	9878 ± 150	21129 ± 311	4970 ± 79
70	14316 ± 116	29420 ± 239	6449 ± 79
80	17780 ± 201	35370 ± 306	7287 ± 59
90	22736 ± 236	43799 ± 475	8484 ± 108
100	27602 ± 221	51839 ± 378	9415 ± 70
110	33176 ± 412	60820 ± 677	10443 ± 97
120	39972 ± 501	71618 ± 868	11614 ± 142
130	47841 ± 176	83651 ± 342	12544 ± 117
140	54893 ± 661	95233 ± 1179	13829 ± 222
150	64050 ± 526	109482 ± 919	15059 ± 136
200	113727 ± 702	184621 ± 1179	19951 ± 162
250	190643 ± 1416	300888 ± 2327	25500 ± 200

Table I: The gate counts for the sparse SYK with $k = 8.7$ in terms of two-qubit CNOT gates, Clifford+T-gates, and T-gates for $N \leq 250$ using first-order Trotter approach. The T-gate count can be further optimized but we do not discuss it here. The average gate count is computed over at least ten instances of the sparse model ($k = 8.7$). The two-qubit costs for the dense SYK model was discussed previously in Ref. [8].

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