# On the Number of Steps of CyclePopping in Weakly Inconsistent U(1)-Connection Graphs 

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April 24, 2024


#### Abstract

A $\mathrm{U}(1)$-connection graph $G$ is a graph in which each oriented edge is endowed with a unit complex number, the latter being conjugated under orientation flip. We consider cycle-rooted spanning forests (CRSFs), a particular kind of spanning subgraphs of $G$ that have recently found computational applications as randomized spectral sparsifiers. In this context, CRSFs are drawn from a determinantal measure. Under a condition on the connection, Kassel and Kenyon gave an elegant algorithm, named CyclePopping, to sample from this distribution. The algorithm is an extension of the celebrated algorithm of Wilson that uses a loop-erased random walk to sample uniform spanning trees. In this paper, we give an alternative, elementary proof of correctness of CyclePopping for CRSF sampling; we fill the gaps of a proof sketch by Kassel, who was himself inspired by Marchal's proof of the correctness of Wilson's original algorithm. One benefit of the full proof $\grave{a}$ la Marchal is that we obtain a concise expression for the law of the number of steps to complete the sampling procedure, shedding light on practical situations where the algorithm is expected to run fast. Furthermore, we show how to extend the proof to more general distributions over CRSFs, which are not determinantal. The correctness of CyclePopping is known even in the non-determinantal case from the work of Kassel and Kenyon, so our merit is only to provide an alternate proof. One interest of this alternate proof is again to provide the distribution of the time complexity of the algorithm, in terms of a Poisson point process on the graph loops, or equivalently as a Poisson process on pyramids of cycles, a combinatorial notion introduced by Viennot. Finally, we strive to make the connections to loop measures and combinatorial structures as explicit as possible, to provide a reference for future extensions of the algorithm and its analysis.


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## 1 Introduction

Given a graph $G$, graph sparsification is the task of building a small subgraph of $G$ that preserves some target property. Spectral sparsifiers, for instance, come with a controlled approximation of the quadratic form associated with the Laplacian. This makes them useful preconditioners in Laplacian linear systems [Vishnoi, 2013]. In a seminal paper, Spielman and Srivastava [2011] defined random spectral sparsifiers by i.i.d. sampling edges of $G$. Since then, non-i.i.d. distributions over edges have been investigated, with spanning trees becoming the state-of-the-art in randomized spectral sparsification [Fung et al., 2011; Kyng and Song, 2018; Kaufman et al., 2022]. The popularity of random spanning trees rests on the availability of efficient sampling algorithms, such as the celebrated random walk-based algorithm of [Wilson, 1996]. Many proofs of the correctness of Wilson's algorithm have been given, from the concise proof in the original paper [Wilson, 1996] to the detailed analysis of Wilson's loop-erased random walk by Marchal [1999], the so-called Diaconis-Fulton or stack-of-cards representation, see e.g. [Lyons and Peres, 2017, Section 4.1], or more recently as an instance of partial rejection sampling [Guo, Jerrum, and Liu, 2019; Jerrum, 2021]. Among these proofs, the analysis of Marchal [1999] stands out as the easiest to follow step by step, and yields the law of the running time of the algorithm as an immediate corollary. In comparison, other proofs are typically more elegant, but invariably include a high-level statement that is not so easily checked by a non-expert mechanical reader.

We are interested in the spectral sparsification of graphs that are further endowed with a $U(1)$-connection, that is, a map $x y \mapsto \phi_{x y}$ which associates to each oriented edge $x y$ a unit modulus complex number $\phi_{x y}$ such that $\phi_{y x}=\phi_{x y}^{*}$. Let a cycle-rooted spanning forest (CRSF) be a spanning subgraph, in which each connected component has exactly one cycle. Additionally considering a weight $w_{e}$ on each edge $e$ of the oriented graph $G$, Kenyon [2011] introduced a probability measure on unoriented CRSFs,

$$
\begin{equation*}
\mu_{\mathrm{CRSF}}\left(\mathcal{F}_{\mathrm{un}}\right) \propto \prod_{e \in \mathcal{F}_{\mathrm{un}}} w_{e} \prod_{\substack{\text { non-oriented } \\ \text { cycle } c \subseteq \mathcal{F}_{\mathrm{un}}}}(2-2 \cos \theta(c)), \tag{1}
\end{equation*}
$$

which favors cycles with large inconsistency $1-\cos \theta(c)$, where

$$
\exp (-\mathrm{i} \theta(c))=\prod_{e \in c} \phi_{e}
$$

In line with the case of spanning trees, we have proved in [Fanuel and Bardenet, 2022] that CRSFs drawn from (1) provide spectral sparsifiers of connection graphs. This motivates the search for efficient sampling algorithms.

As noted by Kenyon [2011], the measure (1) is a determinantal point process (DPP; Macchi, 1972; Soshnikov, 2000) over the edges of the graph; and the generic DPP sampler of Hough et al. [2006] thus applies, with a complexity that is cubic in the number of edges in the original graph. In the case of a weakly inconsistent $\mathrm{U}(1)$-connection graph, that is, when $\theta(c) \in[-\pi / 2, \pi / 2]$ for all cycles $c$, Kassel and Kenyon [2017] introduced an alternative algorithm that is a variant of Wilson's algorithm for spanning trees. To say it briefly, this algorithm relies on a simple random walk which pops (erases) any cycle $c$ it makes along its path with probability $\cos \theta(c)$. We refer to this algorithm as CyclePopping. The proof of correctness in [Kassel and Kenyon, 2017] uses a stack-of-cards representation à la Diaconis-Fulton; we believe that transferring CRSF techniques to computational scientists would be further accelerated with a more pedestrian proof à la Marchal [1999], which introduces the objects needed to describe and compute the probability of computational events like the algorithm terminating. Kassel [2015] actually briefly sketched such a proof; our first contribution is to fill the technical details to give a complete elementary proof of the correctness of Wilson's algorithm for CRSFs. Like Marchal [1999] for spanning trees, we obtain for free the law of the running time of the algorithm, which complements our results in [Fanuel and Bardenet, 2022], by giving intuition on what kind of connection graphs are easy to sparsify.

Our second contribution is to prove the correctness of CyclePopping for a generic cycle weight in (1), when the measure is not necessarily determinantal anymore. It is not obvious that this extension is possible from reading [Marchal, 1999], but the result is known to hold by the stacks-of-cards proof of Kassel and Kenyon [2017]. We provide a Marchal-type proof of this extension, inspired by the work of Lawler and Werner [2004]; Le Jan [2011]; Sznitman [2012]; Kassel and Lévy [2021] on Poissonian loop ensembles. In particular, we derive the distribution of the running time

$$
\begin{equation*}
T \stackrel{\text { (law) }}{=} n+\sum_{[\gamma] \in \mathcal{X}}|\gamma| \tag{2}
\end{equation*}
$$

where $n$ is the number of vertices of $G,|\gamma|$ is the length of the oriented loop $[\gamma], \mathcal{X}$ is the Poisson point process on loops of $G$ with intensity $m_{\alpha}$, and $m_{\alpha}([\gamma])$ is proportional both to the probability of the simple random walk following $[\gamma]$ and to the weight of any cycle in $[\gamma]$. Finally, an additional contribution is to provide an alternative expression for $T$ using a Poisson process of pyramids of (popped) oriented cycles. This approach describes the loop formed by popped oriented cycles as a heap of cycles with a unique maximal element, a pyramid in the sense of Viennot [2006].

In conclusion, we do not claim novelty on the correctness of CyclePopping for connection graphs, since the result even for general weights is known from [Kassel and Kenyon, 2017]. Yet we contribute a Marchaltype proof that applies to general weights, has the merit of being elementary, yields the distribution of the running time dear to computational scientists, and explicits Poisson point processes that are by-products of Wilson's construction.

### 1.1 Organization

The paper is organized as follows. To give context, we review in Section 2 known results for the time complexity of CyclePopping when cycles are always popped, i.e., when sampling spanning trees and spanning forests. A formalization of CyclePopping for sampling CRSFs - including the necessary variables to describe the probability of accepting cycles - is given in Section 3. Armed with these tools, Section 4 gives the proof of correctness and the law of the sampling time $T$ when the CRSF measure is determinantal. Thanks to the introduction of loop measures, Section 5 gives an alternative derivation of the law of $T$ for generic cycle weights $\alpha(c) \in[0,1]$ in (1), when the measure is not necessarily determinantal. Empirical evaluations of the expectation and variance of $T$ are given in Section 6 for a few random $\mathrm{U}(1)$-connection graphs, in order to illustrate the results.

### 1.2 Notations

For ease of reference, we gather here our main notations.
Laplacian. Consider a connected undirected graph $G$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, with $n=|\mathcal{V}|>1$ and $m=|\mathcal{E}|$. Often, we implicitly identify $\mathcal{V}$ with $\{1, \ldots, n\}$. When two vertices $x, y \in \mathcal{V}$ are connected by an edge, we write $x \sim y$. A $\mathrm{U}(1)$-connection endows each oriented edge $x y$ with a complex phase $\phi_{x y}=\exp (-\mathrm{i} \vartheta(x y))$ such that $\vartheta(y x)=-\vartheta(x y) \in[0,2 \pi)$. Let $\Phi$ be the $n \times n$ matrix such that $(\Phi)_{x y}=\phi_{x y}$ for each $x \sim y$ and $(\Phi)_{x y}=0$ otherwise. Similarly, let W be the symmetric $n \times n$ matrix such that $(\mathrm{W})_{x y}=w_{x y}>0$ for each $x \sim y$ and $(\mathrm{W})_{x y}=0$ otherwise. For $x \in \mathcal{V}$, let $\operatorname{deg}(x)=\sum_{y: y \sim x} w_{x y}$. The magnetic Laplacian is the $n \times n$ matrix defined by

$$
\Delta=\mathrm{D}-\mathrm{W} \odot \Phi
$$

where D is the diagonal matrix such that $\mathrm{D}_{x x}=\operatorname{deg}(x)>0$ for all $x \in \mathcal{V}$, and where $\mathrm{W} \odot \Phi$ denotes the entrywise product. In particular, writing $\Lambda=\mathrm{D}-\mathrm{W}$ for the usual (combinatorial) Laplacian, we see that $\Delta$ and $\Lambda$ coincide when the phase is trivial, i.e. when $\vartheta \equiv 0$. We refer to e.g. [Lieb and Loss, 1993; Colin de Verdière, 1998; Colin de Verdière et al., 2011; Berkolaiko, 2013; Fanuel et al., 2018] for references on the magnetic Laplacian and a motivation for the adjective magnetic.

Transition matrix. Furthermore, to each oriented edge $x y$, we associate a transition probability $p_{x y} \triangleq$ $\frac{w_{x y}}{\operatorname{deg}(x)}$, which will be used to define Markov chains on the graph. As we did for the weight matrix W, we also define an $n \times n$ transition matrix $\mathrm{P}=\mathrm{D}^{-1} \mathrm{~W}$, so that $(\mathrm{P})_{x y}=p_{x y}$ for $x \sim y$ and $(\mathrm{P})_{x y}=0$ otherwise. Finally, for $\mathcal{S} \subseteq\{1, \ldots, \ell\}$ and M an $\ell \times \ell$ matrix, $\mathrm{M}_{\overline{\mathcal{S}}}$ is the matrix obtained by removing the rows and columns of M which are indexed by $\mathcal{S}$.

Loops. In what follows, we denote an oriented path in $G$ by $\left(x_{0}, \ldots, x_{k}\right)$, where consecutive nodes are linked by an edge. A based loop $\gamma$ is an oriented path $\gamma=\left(x_{0}, \ldots, x_{k}\right)$ in the graph $G$, for some integer $k \geq 2$, and with $x_{k}=x_{0}$. Also, we say that the based loop $\gamma=\left(x_{0}, \ldots, x_{k}=x_{0}\right)$ has $x_{0}$ as base node. A based loop of the form ( $x_{0}, x_{1}, x_{0}$ ) is called a backtrack. Let $\bar{\gamma}$ be the based loop $\gamma$ with opposite orientation. For convenience, we also consider based loops of zero length, consisting only of one (base) point. The number of edges in a based loop $\gamma$ is denoted by $|\gamma|$. Two based loops $\gamma=\left(x_{0}, x_{1}, \ldots, x_{k}\right)$ and $\gamma^{\prime}=\left(x_{0}, x_{1}^{\prime}, \ldots, x_{k^{\prime}}^{\prime}\right)$ based at $x_{0}$ can be concatenated to yield $\gamma \circ \gamma^{\prime}=\left(x_{0}, x_{1}, \ldots, x_{k-1}, x_{0}, x_{1}^{\prime}, \ldots, x_{k^{\prime}}^{\prime}\right)$. Naturally, $\gamma^{m}$ denotes the $m$-th power of $\gamma$ w.r.t. concatenation. Contrary to [Le Jan, 2011], all based loops considered here are discrete.

Next, we define an equivalence relation between based loops by identifying $\gamma=\left(x_{0}, \ldots, x_{k}\right)$ with any based loop obtained by a $j$-shift of the form

$$
\begin{equation*}
\left(x_{0}, \ldots, x_{k-1}, x_{0}\right) \mapsto\left(x_{j}, x_{j+1}, \ldots, x_{k-1}, x_{0}, \ldots, x_{j-1}, x_{j}\right) \tag{3}
\end{equation*}
$$

for $0 \leq j \leq k-1$; see Lawler and Limic [2010, Section 9.1]. The corresponding equivalence class [ $\gamma$ ] is simply a based loop, of which we forget the base node: we call the equivalence class an unbased loop, or sometimes simply a loop. By definition, any representative $\gamma$ of $[\gamma]$ has the same number of edges $|\gamma|$, so that we write $|\gamma|$ for $|[\gamma]|$ without ambiguity. In particular, we also consider the trivial loop of only one point as an unbased loop with zero edge. Finally, the number of representatives in the class $[\gamma]$ is denoted by $\mathcal{N}_{[\gamma]}$; note that it is not necessarily equal to $|\gamma|$.

Cycles. A based loop $c$ is called an oriented cycle if it is minimal, i.e., if it does not include another loop. In particular, an oriented cycle of length 2 - called a backtrack is also considered as an oriented cycle. Denote by $n_{c}$ the number of oriented cycles of the graph. A non-oriented cycle is simply an equivalence class of oriented cycles of length larger than 2 under orientation flip.

## 2 CyclePopping for spanning trees and forests

In this section, we outline the results of Marchal [1999] on the law of the number of steps of CyclePopping for spanning trees. The case of spanning forests follows immediately.

A spanning tree (ST) of $G$ is a connected spanning subgraph of $G$ without cycle. Identifying an ST with its edges, STs can be endowed with a determinantal probability measure [Pemantle, 1991]

$$
\begin{equation*}
\mu_{\mathrm{ST}}(\mathcal{F})=\frac{1}{\operatorname{det}\left(\Lambda_{\bar{r}}\right)} \prod_{e \in \mathcal{F}} w_{e} \tag{4}
\end{equation*}
$$

where $\Lambda=\mathrm{D}-\mathrm{W}$ is the combinatorial Laplacian of the graph and $r \in \mathcal{V}$ is any of its nodes. ${ }^{1}$
A spanning forest (SFs) is a spanning subgraph without cycle. In other words, an SF is a spanning graph, of which each connected component is a tree. Let $q>0$. A notable determinantal measure [Avena and Gaudillière, 2018; Pilavcı et al., 2021] on SFs is

$$
\begin{equation*}
\mu_{\mathrm{SF}}(\mathcal{F})=\frac{q^{\rho(\mathcal{F})}}{\operatorname{det}(\Lambda+q \mathbb{I})} \prod_{e \in \mathcal{F}} w_{e} \tag{5}
\end{equation*}
$$

where $\rho(\mathcal{F})=n-|\mathcal{F}|$ is the number of connected components of $\mathcal{F}$.
CyclePopping refers to an algorithm to sample from (4) and (5).

### 2.1 CyclePopping for spanning trees

A classical approach to sampling from (4) is Wilson's original algorithm [Wilson, 1996], which sequentially builds a growing rooted tree. In Wilson's algorithm, one arbitrary node $r \in \mathcal{V}$, called the root, is fixed, as well as an ordering of the remaining nodes. We initialize the tree as containing only the root node. Starting from the lowest-order node not already in the tree, a random walker performs a loop-erased random walk (LERW) with edge weights $w_{e}$, until reaching the tree. Upon reaching the tree, the walker's trajectory is added to the tree, forming a new branch. Next, the walker restarts the LERW from the lowest-order node in the complement of the tree, and the process continues until the tree is spanning. Forgetting the root and the order of the nodes, we obtain a tree drawn from (4). Since the classical procedure to draw a trajectory from a LERW involves erasing loops as they appear in a simple random walk, the resulting algorithm - Wilson's algorithm followed by outputting only an unrooted tree - is called CyclePopping. The complexity of the algorithm is the total number of steps made by the involved simple random walks, i.e. the number of calls to the Markov kernel of the simple random walk.

We start by quoting a fundamental result on the law of the total number of steps for sampling uniform spanning trees, which we shall progressively generalize.

Proposition 1 (Proposition 1 in [Marchal, 1999]). Let $G=(V, E)$ be a connected graph without self-loops, and $r \in V$. Let $T_{r}$ be the random number of steps to complete Wilson's algorithm for sampling a spanning tree rooted at $r$. Let $\mathrm{P}=\mathrm{D}^{-1} \mathrm{~W}$, and $\mathrm{P}_{\bar{r}}$ be the matrix obtained by deleting the $r$-th row and the $r$-th column of P .
(i) For all $t \in(0,1)$, we have $\mathbb{E}\left[t^{T_{r}}\right]=t^{n-1} \frac{\operatorname{det}\left(\mathbb{I}-\mathrm{P}_{\overline{\bar{r}}}\right)}{\operatorname{det}\left(\mathbb{I}-t \mathrm{P}_{\bar{r}}\right)}$.
(ii) It holds that $\mathbb{E}\left[T_{r}\right]=\operatorname{Tr}\left(\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)=n-1+\operatorname{Tr}\left(\mathrm{P}_{\bar{r}}\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)$.

As noted in Marchal [1999, Eq (8)], the expected number of steps $\mathbb{E}\left[T_{r}\right]$ to complete Wilson's algorithm is equal to the expectation of the commute time between a $\pi$-random $i$ and the root $r$, where $\pi(i)=$ $\operatorname{deg}(i) / \sum_{j} \operatorname{deg}(j)$; see also Wilson [1996, Thm 2]. In particular, even if CyclePopping forgets the root, the expected complexity does depend on $r$. Actually, the diagonal element $\left(\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)_{i i}$ is the inverse of the probability that a random walker starting from $i$ hits the root $r$ before coming back to $i$; see e.g. [Levin and Peres, 2017, Eq (9.12) and (9.17)]. This suggests that, in order to minimize $\mathbb{E}\left[T_{r}\right]=\operatorname{Tr}\left(\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)$, the root has to be chosen at a central position in the graph.

Since Proposition 1 yields the cumulant generating function for $T_{r}$, we also note the following consequence.

[^0]Corollary 1. With the notations of Proposition 1, $\operatorname{var}\left[T_{r}\right]=\operatorname{Tr}\left(\mathrm{P}_{\bar{r}}\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)+\operatorname{Tr}\left(\mathrm{P}_{\bar{r}}\left(\mathbb{I}-\mathrm{P}_{\bar{r}}\right)^{-1}\right)^{2}$.
The results in Proposition 1 and Corollary 1 indicate that the random number of steps $T_{r}$ only depends on the eigenvalues of transition matrix $\mathrm{P}_{\bar{r}}$ of the walk absorbed at $r$; see Section 1. Intuitively, $T_{r}$ is likely to be small if the largest eigenvalue of $\mathrm{P}_{\bar{r}}$ is much smaller than one.

### 2.2 CyclePopping for spanning forests

Marchal's result in Proposition 1 can be directly extended to the distribution (5) on spanning forests. CyclePopping for (5) also corresponds to running Wilson's algorithm, but on an auxiliary graph $G_{r}$, defined as $G$ where each node is additionally connected to a new node $r$ with a constant edge weight $q>0$. Then, a random ST $\mathcal{F}$ rooted at $r$ is sampled using Wilson's algorithm on $G_{r}$. Forgetting $r$ in $\mathcal{F}$ and all edges that connects to $r$, the $S T$ in $G_{r}$ becomes an SF in $G$, with distribution (5) [Avena and Gaudillière, 2018].

Proposition 2. Let $q>0$ and let $T$ denote the number of steps to sample an $S F$ with Wilson's algorithm according to (5). We have

$$
\mathbb{E}[T]=\operatorname{Tr}\left(\mathrm{D}(\Lambda+q \mathbb{I})^{-1}\right)+q \operatorname{Tr}(\Lambda+q \mathbb{I})^{-1} \quad \text { and } \operatorname{var}[T]=\operatorname{Tr}\left(\mathrm{M}_{q}\left(\mathbb{I}-\mathrm{M}_{q}\right)^{-1}\right)+\operatorname{Tr}\left(\mathrm{M}_{q}\left(\mathbb{I}-\mathrm{M}_{q}\right)^{-1}\right)^{2}
$$

with $\mathrm{M}_{q}=(\mathrm{D}+q \mathbb{I})^{-1} \mathrm{~W}$.
Proof. The combinatorial Laplacian $\Lambda^{\prime}$ of $G_{r}$ can be written in block form

$$
\Lambda^{\prime}=\left(\begin{array}{cc}
\Lambda+q \mathbb{I} & -q \mathbb{1} \\
-q 1^{\top} & n q
\end{array}\right)=\mathrm{D}^{\prime}-\mathrm{W}^{\prime}, \quad \text { where } \mathrm{D}^{\prime}=\left(\begin{array}{cc}
\mathrm{D}+q \mathbb{I} & 0 \\
0^{\top} & n q
\end{array}\right), \quad \mathrm{W}^{\prime}=\left(\begin{array}{cc}
\mathrm{W} & -q 1 \\
-q 1^{\top} & 0
\end{array}\right)
$$

and $\mathbf{1}$ is the all-ones vector. Upon noting that $\mathrm{D}_{\bar{r}}^{\prime}=\mathrm{D}+q \mathbb{I}$ and $\mathrm{W}_{\bar{r}}^{\prime}=\mathrm{W}$, the result follows from Proposition 1 and Corollary 1.

Notice how the expectation of $T$ in Proposition 2 includes two terms, one for the number of steps of the walker within $G$, and the other for the number of steps to the auxiliary root of $G_{r}$. As a side note, the latter is also the number of roots in the determinantal point process formed by the roots of this random SF ; see [Avena and Gaudillière, 2018].

## 3 Formalizing CyclePopping for cycle-rooted spanning forests

Fix an orientation of $G$, and associate to each oriented cycle $c$ of the graph a weight $\alpha(c) \in[0,1]$. Kassel and Kenyon [2017] gave a variant of Wilson's algorithm to sample oriented CRSFs w.r.t.

$$
\begin{equation*}
\mu_{\mathrm{CRSF}, \alpha}(\mathcal{F})=\frac{1}{Z_{\alpha}} \prod_{e \in \mathcal{F}} p_{e} \times \prod_{\substack{\text { oriented } \\ \text { cycle } c \subseteq \mathcal{F}}} \alpha(c) \tag{6}
\end{equation*}
$$

with $Z_{\alpha}>0$ a normalization factor, and $p_{e}=w_{i j} / \operatorname{deg}(i)$ for $e=i j$. By analogy with the cases of spanning trees and forests in Section 2, the algorithm is still called CyclePopping. To describe the algorithm, we first introduce a Markov chain and a few stopping times.

### 3.1 The Markov chain

Let $\left(X_{n}\right)_{n \geq 0}$ be a simple random walk (SRW) on the graph nodes. For definiteness, we take the initial distribution $X_{0}$ to be uniform over the nodes, but we shall most often consider, for $x \in \mathcal{V}$, the law $\mathbb{Q}_{x}$ of the chain when $X_{0} \sim \delta_{x}$; we fix the transition probability to

$$
\begin{equation*}
\mathbb{Q}_{x}\left(X_{1}=y\right)=\frac{w_{x y}}{\operatorname{deg}(x)} \triangleq p_{x y} \tag{7}
\end{equation*}
$$

as announced in Section 1.2. In particular, for $k \geq 1$ and for all $x, y \in \mathcal{V}, \mathbb{Q}_{x}\left(X_{k}=y\right)=\left(\mathrm{P}^{k}\right)_{x y}$.

To each oriented cycle $c$ in the graph, we also associate a sequence of i.i.d. Bernoulli random variables $\left(B_{c, n}\right)_{n \geq 0}$, with constant success probability $\alpha(c)$. For a fixed enumeration $c_{1}, \ldots, c_{d}$ of the oriented cycles of the graph, we consider the Markov chain ${ }^{2}$

$$
Z_{n}=\left(X_{n}, B_{c_{1}, n}, \ldots, B_{c_{d}, n}\right)_{n \geq 0}
$$

We still write $\mathbb{Q}_{x}$ the law of $Z_{n}$ when $X_{0} \sim \delta_{x}$.

### 3.2 Hitting and cycle times

We now describe how the SRW can be conditioned not to intersect a subset of nodes.
Let $\mathcal{A} \subset \mathcal{V}$ be a subset of nodes and let $x \in \mathcal{V} \backslash \mathcal{A}$. It is customary to define the first hitting time as the random variable

$$
N_{\rightarrow \mathcal{A}}=\min \left\{n \geq 0 \text { such that } X_{n} \in \mathcal{A}\right\} .
$$

It is a stopping time, since the event $\left\{N_{\rightarrow \mathcal{A}}=m\right\}$ only depends on $X_{0}, X_{1}, \ldots, X_{m}$, and is thus in the $\sigma$-algebra $\sigma\left(Z_{1}, \ldots, Z_{m}\right)$. For later convenience, for $m \geq 0$, we further introduce the first hitting time after $m$ steps

$$
N_{\rightarrow \mathcal{A}}^{(\text {after } m)}=\min \left\{n \geq m \text { such that } X_{n} \in \mathcal{A}\right\}
$$

which is also non-anticipating, and satisfies $N_{\rightarrow \mathcal{A}}^{(\text {after 0) }}=N_{\rightarrow \mathcal{A}}$.
Intuitively, Bernoulli variables will correspond to cycles being accepted after they have been formed by the SRW. For $x \in \mathcal{V}$ and a fixed cycle $c$, we thus define

$$
N_{\circlearrowleft c}=\min \left\{n \geq 0 \text { such that } c \subseteq\left(X_{0}, X_{1}, \ldots, X_{n}\right) \text { and } B_{c, n}=1\right\}
$$

that is, the first time a cycle is formed and accepted. Again, $N_{\circlearrowleft c}$ is a stopping time, since $\left\{N_{\circlearrowleft c}=m\right\}$ only depends on $Z_{0}, Z_{1}, \ldots, Z_{m}$, through their components. Naturally, the value of $N_{O c}$ is at least equal to the length of $c$.

In the same way, we define the cycle time of $c$ after $m$ steps by

$$
N_{\circlearrowleft c}^{(\text {after } m)}=\min \left\{n \geq m \text { such that } c \subseteq\left(X_{m}, X_{m+1}, \ldots, X_{n}\right) \text { and } B_{c, n}=1\right\}
$$

Lastly, the first time after $m$ steps at which either a cycle is accepted or the SRW hits $\mathcal{A}$ is the stopping time ${ }^{3}$

$$
N_{\text {stop }, \mathcal{A}}^{(\text {after } m)}=N_{\rightarrow \mathcal{A}}^{(\text {after } m)} \wedge \min _{\text {cycle } c} N_{\circlearrowleft c}^{(\text {after } m)}
$$

For simplicity, we also write $N_{\text {stop }, \mathcal{A}}=N_{\text {stop }, \mathcal{A}}^{(\text {after } 0)}$. For later use, we state a lemma whose proof is elementary.
Lemma 1. The event $\left\{N_{\text {stop }, \mathcal{A}}^{(\text {after })}=k\right\}$ is in the $\sigma$-algebra generated by $Z_{m}, Z_{m+1}, \ldots, Z_{m+k}$.
Finally, the algorithm will be simpler to explain if we define as many copies of $\left(X_{n}\right)$ as there are nodes in $G$, denoting by $X_{n}^{x}$ a chain with the same Markov kernel as $\left(X_{n}\right)$, but with initial distribution $\delta_{x}, x \in G$. Similarly, we define independent streams i.i.d. Bernoullis $\left(B_{c, n}^{x}\right)_{n \geq 0}$, one stream for each $x \in \mathcal{V}$ and each cycle $c$ in $G$. Again, for $x \in \mathcal{V}$, the stopping times

$$
N_{\rightarrow \mathcal{A}}^{x,(\text { after } m)}, N_{\circlearrowleft c}^{x,(\text { after } m)}, \text { and } N_{\mathrm{stop}, \mathcal{A}}^{x,(\text { after } m)}
$$

are defined by replacing $X_{n}$ by $X_{n}^{x}$ and $B_{c, n}$ by $B_{c, n}^{x}$ in the corresponding definitions. We denote by $\mathbb{P}$ the joint law of all chains and all Bernoulli streams. We are now ready to formalize the sampling algorithm.

[^1]

Figure 1: Illustration of stages $_{\iota}(\mathcal{F})=\left(\mathcal{F}_{1}, \mathcal{F}_{2}, \mathcal{F}\right)$ given in (8) on a CRSF $\mathcal{F}$ of a $3 \times 4$ grid graph. The first three nodes in the order given by $\iota$ are colored in blue; the remainder of the ordering is not displayed. At each stage of the growing sequence, the subgraph appears in bold whereas its complement in $\mathcal{F}$ is represented with dashed lines. Here $\kappa_{\iota}(\mathcal{F})=3$.

### 3.3 Formalization of CyclePopping to sample CRSFs

$\operatorname{CyclePopping}(G, \mathrm{~W}, \alpha, \iota)$ is a modified version of Wilson's algorithm, which takes the graph $G$, the edge weights W , the cycle weights $\alpha$ and a bijection $\iota:\{0, \ldots, N-1\} \rightarrow \mathcal{V}$ as arguments. It defines a growing set of branches and cycle-rooted branches. Recall that a branch (or path graph) is a tree with exactly two nodes of degree 1, whereas all the other nodes have degree 2. Similarly, a cycle-rooted branch (or lasso) is either a cycle or a cycle to which a branch has been connected at one of its end points.

The algorithm defines an increasing sequence of graphs $\mathcal{T}_{1} \subset \mathcal{T}_{2} \subset \ldots$, with node set $\mathcal{V}_{\ell}$ at iteration $\ell$, until we reach the first $\ell$ such that $\mathcal{V}_{\ell}=\mathcal{V}$, where the algorithm terminates. The pseudocode of Cyclepop$\operatorname{PING}(G, \mathrm{~W}, \alpha, \iota)$ is given in Algorithm 1.

```
Algorithm 1 CyclePopping( \(G\) : graph, W: edge weights, \(\alpha\) : cycle weights, \(\iota\) : node ordering)
    - Initialize \(\ell=0, \mathcal{T}_{0}=\left(\mathcal{V}_{0}, \mathcal{E}_{0}\right)=(\emptyset, \emptyset)\).
```

- While $\mathcal{V} \backslash \mathcal{V}_{\ell}$ is not empty,

1. Set $x_{0}=\iota\left(k_{\star}\right)$, with $k_{\star}=\min \left\{k\right.$ such that $\left.\iota(k) \notin \mathcal{V}_{\ell}\right\}$.
2. Run $\left(X_{n}^{x_{0}}\right)_{n \geq 0}$, and erase the loops up to the first time we hit $\mathcal{T}_{\ell}$ or a cycle is accepted, that is, up to the stopping time

$$
N_{\text {stop }, T_{\ell}}^{x_{0}} N_{\rightarrow T_{\ell}}^{x_{0}} \wedge \min _{\text {cycle } c} N_{O c}^{x_{0}} .
$$

Denote by $\mathcal{J}$ the branch or lasso corresponding to $\left(X_{0}^{x_{0}}, \ldots, X_{N_{\text {stop },}, \tau_{\ell}}^{x_{0}}\right)$.
3. Build the new subgraph $\mathcal{T}_{\ell+1}=\mathcal{T}_{\ell} \cup \mathcal{J}$.
4. If $\mathcal{V} \backslash \mathcal{V}_{\ell}$ is empty, return $\mathcal{T}=\mathcal{T}_{\ell+1}$. Otherwise, increment $\ell$ by 1 .

We make an important observation for later use.
Remark 1. Any run of $\operatorname{CyclePopping}(G, \mathrm{~W}, \alpha, \iota)$ which outputs a given CRSF $\mathcal{F}$ produces the same sequence of subgraphs. In other words for a fixed ordering ८ of the nodes, we have a one-to-one function

$$
\begin{equation*}
\operatorname{stages}_{\iota}(\mathcal{F})=\left(\mathcal{F}_{1}, \mathcal{F}_{2}, \ldots, \mathcal{F}_{\kappa_{\iota}(\mathcal{F})}\right), \tag{8}
\end{equation*}
$$

with $\mathcal{F}_{1} \subset \mathcal{F}_{2} \cdots \subset \mathcal{F}_{\kappa_{\iota}(\mathcal{F})}=\mathcal{F}$ the realizations of $\mathcal{T}_{1} \subset \cdots \subset \mathcal{T}_{\kappa_{\iota}(\mathcal{F})}$ in Algorithm 1. This can be seen as follows. At the first stage, the oriented CRSF $\mathcal{F}$ naturally determines an oriented cycle-rooted tree, say $\mathcal{F}_{1}$, which connects the first node in the ordering to an oriented cycle of $\mathcal{F}$. This is true since $\mathcal{F}$ is spanning. The second node in the ordering which does not belong to $\mathcal{F}_{1}$ is the origin of another oriented subgraph which either connects to $\mathcal{F}_{1}$ or is another cycle-rooted tree. The union of this new subgraph and $\mathcal{F}_{1}$ is $\mathcal{F}_{2}$. The rest of the construction proceeds similarly. We refer to Figure 1 for an illustration.

## 4 CyclePopping for CRSFs: the determinantal case

In this section, we prove that CyclePopping outputs a sample from measure (6) in the particular case where the cycle weights derive from a $\mathrm{U}(1)$-connection $\phi$, as introduced by Kenyon [2011]. We actually flesh out a sketch of proof given by Kassel [2015], itself based on the algebraic proof for spanning trees by Marchal [1999]; see Section 2.1. A byproduct of the proof is that it allows us to investigate the complexity of CyclePopping.

### 4.1 A determinantal probability measure over CRSFs

Given a $\mathrm{U}(1)$-connection $\phi$ on the edges of $G$, we first define the holonomy of a based loop $\gamma=\left(x_{0}, \ldots, x_{k}\right)$ as the unit-modulus complex number

$$
\begin{equation*}
\operatorname{hol}(\gamma)=\phi_{x_{0} x_{1}} \ldots \phi_{x_{k-1} x_{0}} \triangleq \exp (-\mathrm{i} \theta(\gamma)) \tag{9}
\end{equation*}
$$

In words, $\theta(\gamma)$ is seen as the angular inconsistency obtained by composing the edge angles along $\gamma$. Holonomies are conjugated under orientation flip, and the holonomy of a backtrack is always equal to 1 .

Assumption 1 (non-trivial connection). There exists at least one cycle $c$ such that $\operatorname{hol}(c) \neq 1$.
This assumption is crucial to define the determinantal measure on CRSFs studied in this paper. If Assumption 1 does not hold, the magnetic Laplacian is unitarily equivalent to the combinatorial Laplacian. An important remark is that Assumption 1 implies that $\Delta$ is nonsingular, as can be seen by considering the quadratic form associated to $\Delta$; see e.g. [Colin de Verdière et al., 2011].

For technical reasons that will become clear below, we further need to require that the "sign-flipped" connection $x y \mapsto e^{\mathrm{i} \pi} \phi_{x y}$ is non-trivial, so that its magnetic Laplacian is also non-singular.

Assumption 2 (non-trivial sign-flipped connection). There is at least one cycle $c$ such that hol $(c) \neq(-1)^{|c|}$.
Denote by $\mathcal{F}_{\text {un }}$ an unoriented CRSF. Under Assumption 1, we define a probability measure on unoriented CRSFs as

$$
\begin{equation*}
\mu_{\mathrm{CRSF}}\left(\mathcal{F}_{\mathrm{un}}\right)=\frac{1}{\operatorname{det}(\Delta)} \prod_{e \in \mathcal{F}_{\mathrm{un}}} w_{e} \times \prod_{\substack{\text { non-oriented } \\ \text { cycle } c \subseteq \mathcal{F}_{\mathrm{un}}}} 2 \cdot(1-\cos \theta(c)) . \tag{10}
\end{equation*}
$$

That (10) is well-normalized is a consequence of the generalized matrix-tree theorem of Forman [1993].
Define the weighted and oriented edge-vertex incidence matrix $\mathrm{B} \in \mathbb{C}^{m \times n}$ given by

$$
(\mathrm{B})_{e, v}= \begin{cases}\sqrt{w_{e}} & \text { if } e=v u \text { for some } u \\ -\phi_{e}^{*} \sqrt{w_{e}} & \text { if } e=u v \text { for some } u \\ 0 & \text { otherwise }\end{cases}
$$

As shown by Kenyon [2011], the probability measure (10) is determinantal over the graph edges with correlation kernel

$$
\mathrm{K}=\mathrm{B} \Delta^{-1} \mathrm{~B}^{*},
$$

that is, the measure of a subset $\mathcal{U}$ of the edges of an unoriented CRSF $\mathcal{F}_{\text {un }}$ is $\operatorname{det}\left(\mathrm{K}_{\mathcal{U}}\right)$.
A few remarks are in order. First, the measure (10) favors CRSFs that include edges with large weights, and cycles with large inconsistencies. Second, since the measure (10) is determinantal, it is thus licit to use the algebraic algorithm of Hough et al. [2006] to sample from it. In this paper, we take a different route to sampling (10) and consider a variant of CyclePopping introduced by Kassel and Kenyon [2017]. CyclePopping has several interesting features such as being completely graph-based, decentralized, and robust to numerical errors; see e.g. [Fanuel and Bardenet, 2022] for more details and an application. Third, CyclePopping actually allows to consider more general cycle weights than (12), yielding non-determinantal measures; we later treat this general case in Section 5.

Actually, since it is based on a random walk, CyclePopping samples an oriented CRSF, that is, a spanning subgraph, each connected component of which has a unique oriented cycle, towards which all edges
are directed. Accounting for the fact that each cycle may have two orientations, the probability measure of interest is given by, for $\mathcal{F}$ an oriented CRSF,

$$
\mu_{\mathrm{CRSF}}(\mathcal{F})=\frac{1}{\operatorname{det}(\Delta)} \prod_{e \in \mathcal{F}} w_{e} \times \prod_{\begin{array}{c}
\text { oriented }  \tag{11}\\
\text { cycle } c \subseteq \mathcal{F}
\end{array}}(1-\cos \theta(c))
$$

This is a special case of (6), with the cycle weights taken to be

$$
\begin{equation*}
\alpha(c)=1-\cos \theta(c) \tag{12}
\end{equation*}
$$

We further make the following assumption on cycle holonomies, which shall later allow us to interpret $\alpha(c)$ in (12) as a probability.

Assumption 3 (weakly inconsistent cycles). For all cycles c, we have $\cos \theta(c) \geq 0$.
In physical terms, this condition intuitively assumes a weak flux of the external magnetic field through the cycles.

Remark 2. If Assumption 1 and Assumption 3 hold, then Assumption 2 is necessarily satisfied.
We shall make use of the following matrix, which we think of as a connection-aware transition matrix,

$$
\begin{equation*}
\Pi=\mathrm{D}^{-1}(\mathrm{~W} \odot \Phi) \tag{13}
\end{equation*}
$$

Note that $(\Pi)_{x x}=0$ for all $x \in \mathcal{V}$.
Proposition 3. If Assumption 1 holds, the eigenvalues of $\Pi$ are in the interval $[-1,1)$. If Assumption 2 holds, the eigenvalues of $\Pi$ are in the interval $(-1,1]$.

In particular, under Assumption 1, $\mathbb{I}-\Pi$ is non-singular. The first part of the proof of Proposition 3 is inspired from Singer and Wu [2016, page 63].

Proof. The matrix $\Pi$ is related by a similarity transformation to the Hermitian matrix $\Pi^{\prime}=\mathrm{D}^{-1 / 2}(\mathrm{~W} \odot$ $\Phi) D^{-1 / 2}$. Thus, the eigenvalues of $\Pi$ and $\Pi^{\prime}$ are identical and real-valued. Next, we observe that the eigenvalues of $\mathbb{I} \pm \Pi^{\prime}$ are non-negative. Indeed, for all $v \in \mathbb{C}^{n}$, we have

$$
\mathrm{v}^{*}\left(\mathbb{I} \pm \Pi^{\prime}\right) \mathrm{v}=\sum_{x \sim y} w_{x y}\left|\frac{\mathrm{v}_{x}}{\sqrt{\operatorname{deg}(x)}} \pm \frac{\phi_{y x} \mathrm{v}_{y}}{\sqrt{\operatorname{deg}(y)}}\right|^{2} \geq 0
$$

where $w_{x y} \geq 0$. From this observation, we deduce that the eigenvalues of $\mathbb{I}+\Pi^{\prime}$ and $\mathbb{I}-\Pi^{\prime}$ are non-negative. Thus, the smallest (resp. largest) eigenvalue of $\Pi^{\prime}$ cannot be smaller (resp. larger) than than -1 (resp. 1). Since $\Pi$ and $\Pi^{\prime}$ share the same eigenvalues, the spectrum of $\Pi$ lies in $[-1,1]$.

Furthermore, since $\Pi=\mathbb{I}-D^{-1} \Delta$ with $\Delta$ non-singular due to Assumption 1, $\Pi$ cannot have an eigenvalue equal to 1 . This shows that the spectrum of $\Pi$ lies within the interval $[-1,1)$. A similar argument applies to show that $\mathbb{I}+\Pi^{\prime}$ is nonsingular under Assumption 2. We simply need to define the magnetic Laplacian for the sign-flipped connection. Hence, if Assumption 2 holds, $\Pi$ cannot have an eigenvalue equal to -1. This completes the proof.

We are ready to state the correctness of CyclePopping.
Proposition 4 (correctness). Let Assumption 1 and Assumption 3 hold and fix any ordering $\iota$ of the nodes. Let $\mathcal{F}$ be a CRSF in the support of (11), and let $\mathcal{F}_{1}, \mathcal{F}_{2}, \ldots, \mathcal{F}_{\kappa_{\iota}(\mathcal{F})}$ be the deterministic decomposition induced by $\iota$; see Remark 1. Denote by $\mathcal{T}_{\ell}$ the random subgraphs output by $\operatorname{CyClePopping}(G, \mathrm{~W}, \alpha, \iota)$. For all $1 \leq \ell \leq \kappa_{\iota}(\mathcal{F})$, we have

$$
\mathbb{P}\left(\mathcal{T}_{\ell}=\mathcal{F}_{\ell}\right)=\mathbb{P}\left(\mathcal{T}_{1}=\mathcal{F}_{1}, \ldots, \mathcal{T}_{\ell}=\mathcal{F}_{\ell}\right)=\frac{\operatorname{det}\left(\Delta \overline{\mathcal{V}_{\ell}}\right)}{\operatorname{det}(\Delta)} \times \prod_{e \in \mathcal{F}_{\ell}} w_{e} \times \prod_{\begin{array}{c}
\text { oriented }  \tag{14}\\
\text { cycle } c \subseteq \mathcal{F}_{\ell}
\end{array}}(1-\cos \theta(c))
$$

where $\mathcal{V}_{\ell}$ is the set of nodes in $\mathcal{F}_{\ell}$.

The first equality in (14) is a consequence of Remark 1. When $\ell=\kappa_{\iota}(\mathcal{F})$, the second equality shows that CyclePopping indeed produces samples from (11), upon noting that $\mathcal{F}_{\kappa_{\iota}(\mathcal{F})}=\mathcal{F}$ and det $\Delta_{\emptyset}=1$. The extra factor 2 in front of each cycle weight in (10) comes from the fact that each cycle of a CRSF can be obtained with two orientations in an oriented CRSF. Remark that backtracks are popped with probability 1. Finally, note that the resulting measure does not depend on the node ordering $\iota$. Hence, in the sequel, we often omit the dependence on $\iota$.

### 4.2 Proof of correctness in the determinantal case

This section revisits the proof sketch in [Kassel, 2015, Section 2.3], fleshing out all the technical details. A closely related approach was used by Marchal [1999] to prove the correctness of Wilson's algorithm for uniformly sampling spanning trees; see also [Avena and Gaudillière, 2018].

### 4.2.1 A last-exit decomposition

With the notation of Proposition 4,

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{T}_{1}=\mathcal{F}_{1}, \ldots, \mathcal{T}_{\ell}=\mathcal{F}_{\ell}\right)=\mathbb{E} \prod_{i=1}^{\ell} 1_{\mathcal{T}_{i}=\mathcal{F}_{i}}=\mathbb{E} \prod_{i=1}^{\ell} 1_{\mathcal{T}_{i}=\mathcal{F}_{i-1} \cup \mathcal{J}_{i}} \tag{15}
\end{equation*}
$$

where $\mathcal{F}_{1}$ is a lasso, and for $\ell \geq 2, \mathcal{F}_{i}=\mathcal{F}_{i-1} \cup \mathcal{J}_{i}, \mathcal{J}_{i}$ being either a branch or a lasso. Now, fix $1 \leq i \leq \ell$, and write $\mathcal{J}_{i}=\left(x_{0}^{i}, \ldots, x_{k_{i}-1}^{i}\right)$. On $\left\{\mathcal{T}_{i-1}=\mathcal{F}_{i-1}\right\}$, the event $\left\{\mathcal{T}_{i}=\mathcal{F}_{i-1} \cup \mathcal{J}_{i}\right\}$ decomposes as a disjoint union

$$
\begin{equation*}
\left\{\mathcal{T}_{i-1}=\mathcal{F}_{i-1}\right\} \cap\left\{\mathcal{T}_{i}=\mathcal{F}_{i-1} \cup \mathcal{J}_{i}\right\}=\left\{\mathcal{T}_{i-1}=\mathcal{F}_{i-1}\right\} \cap\left[\bigcup_{\mathbf{m} \in \mathbb{N}^{k_{i}-1}} \mathcal{E}_{\mathbf{m}}^{i}\right] \tag{16}
\end{equation*}
$$

where for $\mathbf{m} \in \mathbb{N}^{k-1}$, the event $\mathcal{E}_{\mathbf{m}}^{i}$ is defined by a last-exit decomposition; see [Lawler and Limic, 2010, Proposition 4.6.4]. In words, we walk an $m_{1}$-step loop based as $x_{0}^{i}$ without hitting $\mathcal{V}_{i-1}$, pop the loop, and walk one step to $x_{1}^{i}$, never to return to $x_{0}^{i}$; then we walk an $m_{2}$-loop based at $x_{1}^{i}$ without hitting $\mathcal{V}_{i-1}$, pop the loop, and walk one step to $x_{2}^{i}$, never to return to $x_{1}^{i}$, etc. until we accept the cycle in $\mathcal{J}_{i}$ if the latter is a lasso, or we hit $\mathcal{V}_{i-1}$ if $\mathcal{J}_{i}$ is a branch. Formally, let $m_{1}^{*}=m_{1}$ and $m_{j}^{*}=m_{j-1}^{*}+m_{j}+1$ for $j \geq 2$. We first treat the case of $\mathcal{J}_{i}=\left(x_{0}^{i}, \ldots, x_{k_{i}-1}^{i}\right)$ being a lasso, so that $k_{i} \geq 2$ and $x_{j}^{i}=x_{k_{i}}^{i}$ for some $j<k_{i}$. Denoting by $c_{i}$ the cycle of $\mathcal{J}_{i}$, we let

$$
\begin{align*}
& \mathcal{E}_{\mathbf{m}}^{i}=\overbrace{\left\{X_{m_{1}^{*}}^{x_{0}^{i}}=x_{0}^{i} \text { and } m_{1}^{*}<N_{\text {stop }, \mathcal{V}_{i-1}}^{x_{0}^{i}}\right\}}^{\text {loops of } m_{1}^{*} \text { steps based at } x_{0}^{i}} \cap \overbrace{\left\{X_{m_{1}^{*}+1}^{x_{0}^{i}}=x_{1}^{i}\right\}}^{\text {step from } x_{0}^{i}} \text { to } x_{1}^{i} \\
& \cap\left\{X_{m_{2}^{*}}^{x_{0}^{i}}=x_{1}^{i} \text { and } m_{2}^{*}<N_{\text {stop }, \mathcal{V}_{i-1} \cup\left\{x_{0}^{i}\right\}}^{x_{0}^{i}\left(\text { after } m_{1}^{*}+1\right)}\right\} \cap\left\{X_{m_{2}^{*}+1}^{x_{0}^{i}}=x_{2}^{i}\right\} \\
& \vdots \\
& \cap\{\underbrace{\left\{X_{m_{k_{i}-1}^{*}}^{x_{0}^{i}}=x_{k_{i}-2}^{i} \text { and } m_{k_{i}-1}^{*}<N_{\text {stop }, \mathcal{V}_{i-1} \cup\left\{x_{0}^{i}, \ldots, x_{k_{i}-3}^{i}\right\}}^{x_{0}^{i},\left(\text { after } m_{k_{i}}^{*},-1\right)}\right\} \cap\left\{X_{m_{k_{i}-1}^{*}+1}^{x_{0}^{i}}=x_{k_{i}-1}^{i}\right\}}_{c_{i} \text { is accepted }} \\
& x_{0}^{i} \tag{17}
\end{align*}
$$

When $\mathcal{J}_{i}$ is a branch rather than a lasso, the last row of (17) is simply omitted: we have hit $\mathcal{V}_{i-1}$ at $x_{k_{i}-1}^{i}$.
Repeatedly plugging (16) into (15), we obtain

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{T}_{1}=\mathcal{F}_{1}, \ldots, \mathcal{T}_{\ell}=\mathcal{F}_{\ell}\right)=\mathbb{E}\left[\prod_{i=1}^{\ell} \sum_{\mathbf{m} \in \mathbb{N}^{k_{i}-1}} 1_{\mathcal{E}_{\mathbf{m}}^{i}}\right] \tag{18}
\end{equation*}
$$

upon noting that, by convention, we set $\mathcal{F}_{0}=\emptyset$ and thus $\mathcal{T}_{0}=\mathcal{F}_{0}$ almost surely. By independence of the $\ell$ involved SRWs and countable additivity, it comes

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{T}_{1}=\mathcal{F}_{1}, \ldots, \mathcal{T}_{\ell}=\mathcal{F}_{\ell}\right)=\prod_{i=1}^{\ell}\left[\sum_{\mathbf{m} \in \mathbb{N}_{k_{i}-1}} \mathbb{P}\left(\mathcal{E}_{\mathbf{m}}^{i}\right)\right] \tag{19}
\end{equation*}
$$

We are thus led to computing $\mathbb{P}\left(\mathcal{E}_{\mathbf{m}}^{i}\right)$, for $1 \leq i \leq \ell$, and $\mathbf{m} \in \mathbb{N}^{k_{i}}$.
Again, we start with the case where $\mathcal{J}_{i}$ is a lasso. The event of the last row of (17) is independent from the rest. Then we use Lemma 1 and apply the Markov property to the chain

$$
Z_{n}^{x_{0}^{i}}=\left(X_{n}^{x_{0}^{i}}, B_{1, c_{1}}^{x_{0}^{i}}, \ldots, B_{1, c_{d}}^{x_{0}^{i}}\right)
$$

where $c_{1}, \ldots, c_{d}$ are the cycles of the graph $G$. Remembering that $\mathbb{Q}_{x}$ denotes the law of the SRW $\left(X_{n}\right)$ on $G$ with initial distribution $\delta_{x}$, we obtain,

$$
\begin{align*}
\mathbb{P}\left(\mathcal{E}_{\mathbf{m}}^{i}\right)= & \mathbb{Q}_{x_{0}^{i}}\left(\left\{X_{m_{1}^{*}}=x_{0}^{i} \text { and } m_{1}<N_{\text {stop }, \mathcal{V}_{i-1}}\right\}\right) \times p_{x_{0}^{i} x_{1}^{i}} \\
& \times \mathbb{Q}_{x_{1}^{i}}\left(\left\{X_{m_{2}}=x_{1}^{i} \text { and } m_{2}<N_{\text {stop }, \mathcal{V}_{i-1} \cup\left\{x_{0}^{i}\right\}}\right\}\right) \times p_{x_{1}^{i} x_{2}^{i}} \\
& \vdots \\
& \times \mathbb{Q}_{x_{k_{i}-2}^{i}}\left(\left\{X_{m_{k_{i}-1}}=x_{k_{i}-2}^{i} \text { and } m_{k_{i}-1}<N_{\text {stop }, \mathcal{V}_{i-1} \cup\left\{x_{0}^{i}, \ldots, x_{k_{i}-3}^{i}\right\}}\right\}\right) \times p_{x_{k_{i}-2}^{i} x_{k_{i}-1}^{i}} \\
& \times \alpha\left(c_{i}\right) . \tag{20}
\end{align*}
$$

The case of $\mathcal{J}_{i}$ being a branch is similar, resulting in the omission of the factor $\alpha(c)$ in (20).
Now, we recognize in (20) evaluations of the Green generating function of the chain, which we introduce before continuing the computation of (20).

### 4.2.2 The Green generating function

We begin by computing the probability that $\left(X_{n}\right)$, starting from node $x$, makes a based loop of $k$ steps ( $k \geq 2$ ), while staying in the complement of a subset of nodes $\mathcal{A}$ and popping (i.e. erasing, rejecting) all the cycles along its path,

$$
\begin{equation*}
g_{k}(x, \mathcal{A}) \triangleq \mathbb{Q}_{x}\left(X_{k}=x \text { and } k<N_{\rightarrow \mathcal{A}} \wedge \min _{\text {cycle } c} N_{\circlearrowleft c}\right) \tag{21}
\end{equation*}
$$

with $x \in \mathcal{V} \backslash \mathcal{A}$. For definiteness, we write $g_{1}(x, \mathcal{A})=0$ and $g_{0}(x, \mathcal{A})=1$. For an oriented based loop $\gamma=\left(x_{0}, x_{1}, \ldots, x_{k-1}, x_{0}\right)$, where $k \geq 2$, write

$$
\begin{equation*}
q(\gamma)=p_{x_{0} x_{1}} \ldots p_{x_{k-1} x_{0}} \tag{22}
\end{equation*}
$$

that is, the probability that the simple random walk starting at $x_{0}$ walks exactly on this closed path. ${ }^{4}$ In particular, $q(\gamma)=q(\bar{\gamma})$ for any based loop $\gamma$, in light of the definition of $p_{x y}$ in (7). It is conventional to set $q(\gamma)=1$ when $\gamma$ consists of only one point. Note that based loops are not necessarily cycles, and may contain cycles, backtracks being popped almost surely in Algorithm 1.
Notation 1 (set of based loops). Let $\mathcal{A}$ be a subset of $\mathcal{V}$ and $x \in \mathcal{V} \backslash \mathcal{A}$. Denote by $\operatorname{BL}(\overline{\mathcal{A}}, x)$ the set of oriented based loops in $\mathcal{V} \backslash \mathcal{A}$ whose base is $x$, including trivial loops of only one point.
Notation 2 (cycles within a based loop). Let $\gamma$ be a loop based at $x_{0}$. We now go over the oriented path defined by $\gamma$ and chronologically erase all the cycles encountered. Note that these (oriented) cycles may occur more than once. We collect the cycles, with multiplicity, in a multiset ${ }^{5}$ we call cycles( $\gamma$ ). Figure 2 illustrates a based loop, for which the multiset contains 3 cycles, one of which has multiplicity two. ${ }^{6}$

[^2]

Figure 2: Based loop as a heap of cycles. In Figure 2a, we display a loop $\gamma$ based at $x$ where we visualized the steps by a small vertical shift. Colors allow to visualize $\operatorname{cycles}(\gamma)=\left\{c_{1},\left(c_{2}\right)^{2}, c_{3}\right\}$. On the right-hand side, in Figure 2b, we display the heap associated with $\gamma$ which is constituted of the following pieces: $c_{1}, c_{2}$ and $c_{3}$. Double arrows indicate multiplicity 2 .


Figure 3: Two loops based at $x_{0}$, namely, $\gamma, \gamma^{\prime} \in \operatorname{BL}\left(\overline{\mathcal{A}}, x_{0}\right)$. We have $\operatorname{cycles}(\gamma)=\left\{c_{0}, c_{1}\right\}$ with $c_{0}=\left(x_{0}, x_{1}, x_{2}, x_{0}\right)$ and $c_{1}=\left(x_{0}, x_{4}, x_{5}, x_{0}\right)$. Similarly, $\operatorname{cycles}\left(\gamma^{\prime}\right)=\left\{\bar{c}_{0}, c_{1}\right\}$ The loop weights are $q(\gamma)=q\left(\gamma^{\prime}\right)$ and their holonomies satisfy $\operatorname{hol}(\gamma)+\operatorname{hol}\left(\gamma^{\prime}\right)=\operatorname{hol}\left(c_{1}\right) \operatorname{Rehol}\left(c_{0}\right)$.

Thanks to this notation, we can write the holonomy (9) of a based loop as

$$
\begin{equation*}
\operatorname{hol}(\gamma)=\prod_{c \in \operatorname{cycles}(\gamma)} \operatorname{hol}(c)=\prod_{c \in \operatorname{cycles}(\gamma)} \exp (-\mathrm{i} \theta(c)) \tag{23}
\end{equation*}
$$

Now, recall that when a cycle $c$ is formed at time $n$, it is popped independently from the past, with probability $\mathbb{P}\left(B_{c, n}=0\right)=\cos \theta(c)$. Armed with these definitions, we find that the probability (21) that the SRW starting at $x$ is back at $x$ in $k$ steps $(k \geq 2)$, before hitting $\mathcal{A}$ and before accepting any cycle along its path, is

$$
\begin{align*}
g_{k}(x, \mathcal{A}) & =\sum_{\substack{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x) \\
|\gamma|=k}} q(\gamma) \prod_{c \in \operatorname{cycles}(\gamma)} \cos \theta(c) \\
& =\sum_{\substack{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x) \\
|\gamma|=k}} q(\gamma) \prod_{c \in \operatorname{cycles}(\gamma)} \exp (-\mathrm{i} \theta(c)) . \tag{24}
\end{align*}
$$

The equality (24) is obtained by realizing that, for an oriented based loop $\gamma$ of length $k$ containing an oriented simple cycle $c \in \operatorname{cycles}(\gamma)$, there is a distinct oriented based loop $\gamma^{\prime}$ of length $k$ containing the cycle $\bar{c}$ (of opposite orientation) and keeping the same orientation for the remaining simple cycles. Thus, the imaginary part of $\operatorname{hol}(c)$ and $\operatorname{hol}(\bar{c})$ simply cancel out by summing the terms corresponding to $\gamma$ and $\gamma^{\prime}$; see Figure 3 for an illustration. This also trivially applies when $c$ is a backtrack since $\cos \theta(c)=1$ in this case. Finally,
we replace each term by its definition in (24), and we obtain

$$
\begin{aligned}
g_{k}(x, \mathcal{A}) & =\sum_{x_{1}, \ldots, x_{k-1} \in \mathcal{V} \backslash \mathcal{A}} p_{x x_{1}} \ldots p_{x_{k-1} x} \cdot \phi_{x x_{1}} \ldots \phi_{x_{k-1} x} \\
& =\left(\left(\Pi_{\overline{\mathcal{A}}}\right)^{k}\right)_{x x}
\end{aligned}
$$

We collect the contributions of the popped loops based at $x$ of all possible lengths, into the so-called Green generating function

$$
\begin{equation*}
G(t, x, x ; \mathcal{A}) \triangleq \sum_{k=0}^{+\infty} t^{k} g_{k}(x, \mathcal{A}), \quad t \in(0,1] \tag{25}
\end{equation*}
$$

where $g_{k}(x, \mathcal{A})$ is defined in (21). Note that, to be consistent with most references, like [Marchal, 1999], we use here the notation $G\left(t, x, x^{\prime} ; \mathcal{A}\right)$, though we will always take $x^{\prime}=x$. In particular, by using (24),

$$
\begin{equation*}
G(t, x, x ; \mathcal{A})=\sum_{k=0}^{+\infty} t^{k}\left(\left(\Pi_{\overline{\mathcal{A}}}\right)^{k}\right)_{x x}=\left(\left(\mathbb{I}-t \Pi_{\overline{\mathcal{A}}}\right)^{-1}\right)_{x x} \tag{26}
\end{equation*}
$$

where the convergence of the Neumann series is guaranteed for all $t \in(0,1]$ under the conjunction of Assumption 1 and Assumption 2 since the spectrum of $\Pi$ lies in $(-1,1)$ in this case as a consequence of Proposition 3. The series also converges if Assumption 1 and Assumption 3 hold, as a consequence of Remark 2.

Note that the term $k=0$ above is equal to one, and the term $k=1$ vanishes since $(\Pi)_{x x}=0$ for all $x \in \mathcal{V}$. We refer to Pitman and Tang [2018] for an interpretation of the Green generating function in terms of spanning forests.

Now, under Assumption 1 and Assumption 3, we consider in more detail the value of the Green generating function in $t=1$. Applying (13) to (26),

$$
\begin{equation*}
G(1, x, x ; \mathcal{A})=\operatorname{deg}(x)\left(\left(\Delta_{\overline{\mathcal{A}}}\right)^{-1}\right)_{x x} \tag{27}
\end{equation*}
$$

As a side remark, (27) is the average number of visits of $x$ that CyclePopping does before accepting any cycle and before hitting $\mathcal{A}$, as can be seen by summing the probability in (21).

### 4.2.3 Putting it all together

Armed with the Green generating function (27), we continue the computation of (20), which becomes

$$
\begin{align*}
& \sum_{\mathbf{m} \in \mathbb{N}^{k_{i}-1}} \mathbb{P}\left(\mathcal{E}_{\mathbf{m}}^{i}\right) \\
& \quad=\alpha\left(c_{i}\right) \prod_{j=0}^{k_{i}-2} p_{x_{j}^{i} x_{j+1}^{i}} G\left(1, x_{j}^{i}, x_{j}^{i} ; \mathcal{V}_{i-1} \cup\left\{x_{0}^{j}, \ldots, x_{j-1}^{i}\right\}\right)  \tag{28}\\
& \quad=\alpha\left(c_{i}\right)\left[\operatorname{deg}\left(x_{0}^{i}\right)\left(\Delta_{\overline{\mathcal{V}_{i-1}}}\right)_{x_{0}^{i} x_{0}^{i}}^{-1}\right] p_{x_{0}^{i} x_{1}^{i}} \ldots\left[\operatorname{deg}\left(x_{k_{i}-2}\right)\left(\Delta_{\overline{\mathcal{V}_{i-1} \backslash\left\{x_{0}^{i}, \ldots, x_{k_{i}-3}^{i}\right\}}}\right)_{x_{k_{i}-2} x_{k_{i}-2}}^{-1}\right] p_{x_{k_{i}-2} x_{k_{i}-1}} \tag{29}
\end{align*}
$$

for a lasso, and the same equality without $\alpha\left(c_{i}\right)$ for a branch.
Finally, we note that, for $\mathcal{A} \subset \mathcal{V}$ and $x \in \mathcal{V} \backslash \mathcal{A}$,

$$
\left(\left(\Delta_{\overline{\mathcal{A}}}\right)^{-1}\right)_{x x}=\frac{\operatorname{det}\left(\Delta_{\overline{\mathcal{A} \cup\{x\}}}\right)}{\operatorname{det}\left(\Delta_{\overline{\mathcal{A}}}\right)} .
$$

In particular, (29) is a telescopic product, and

$$
\sum_{\mathbf{m} \in \mathbb{N}^{k_{i}-1}} \mathbb{P}\left(\mathcal{E}_{\mathbf{m}}^{i}\right)=w_{x_{0}^{i} x_{1}^{i}} \ldots w_{x_{k_{i}-2}^{i} x_{k_{i}-1}^{i}} \times \frac{\operatorname{det} \Delta_{\overline{\mathcal{V}_{i-1} \cup\left\{x_{0}^{i}, \ldots, x_{k_{i}-2}^{i}\right\}}}}{\operatorname{det} \Delta_{\overline{\mathcal{V}_{i-1}}}} \times \alpha\left(c_{i}\right)
$$

for a lasso, and the same without $\alpha\left(c_{i}\right)$ for a branch. Plugging back into the decomposition (19), we obtain (14), which concludes the proof.

### 4.3 Law of the number of steps to complete CyclePopping

Let $T$ be the number of steps to complete CyclePopping. We characterise the law of $T$ through its moment generating function (MGF).

Proposition 5. Let $t \in[-1,1]$. If Assumption 1 and Assumption 3 hold, we have

$$
\begin{equation*}
\mathbb{E}\left[t^{T}\right]=t^{n} \operatorname{det}\left(\mathbb{I}+(1-t) \Pi(\mathbb{I}-\Pi)^{-1}\right)^{-1} \tag{30}
\end{equation*}
$$

where $\Pi$ is defined in (13).
Proof. In light of the proof of correctness in Section 4.2, this result is simply obtained as follows: the weight $w_{e}$ of each edge followed by the SRW is formally replaced by $t w_{e}$. This accounts for one power of $t$ for each step taken by the SRW. Then, we have

$$
\begin{align*}
\mathbb{E}\left[t^{T}\right] & =\sum_{\substack{\mathcal{F} \text { CRSF } \\
\text { oriented }}} \frac{\prod_{e \in \mathcal{F}} t w_{e} \times \prod_{c \in \mathcal{F}}(1-\cos \theta(c))}{\operatorname{det}(\mathrm{D}) \operatorname{det}\left(\mathbb{I}-t \mathrm{D}^{-1}(\mathrm{~W} \odot \Phi)\right)} \\
& =\sum_{\substack{\mathcal{F} \text { CRSF } \\
\text { non-oriented }}} \frac{\prod_{e \in \mathcal{F}} t w_{e} \times \prod_{c \in \mathcal{F}} 2(1-\cos \theta(c))}{\operatorname{det}(\mathrm{D}-t \mathrm{~W} \odot \Phi)} \\
& =\frac{t^{n}<}{\operatorname{det}(\mathrm{D}-t \mathrm{~W} \odot \Phi)} \operatorname{det}(\mathrm{D}-\mathrm{W} \odot \Phi)=\frac{t^{n}}{\operatorname{det}(\mathbb{I}-t \Pi)} \operatorname{det}(\mathbb{I}-\Pi), \tag{31}
\end{align*}
$$

where we used the generalized matrix-tree theorem of Forman [1993] at the next-to-last equality, which is simply the normalization of (10). Note that $\mathbb{I}+\Pi$ is invertible in the light of Remark 2 and Proposition 3. This completes the proof.

Proposition 6. Under Assumption 1 and Assumption 3, it holds that

$$
\mathbb{E}\left[t^{T}\right]=t^{n} \exp \sum_{k=1}^{+\infty} \frac{(1-t)^{k}}{k} \operatorname{Tr}\left(\Pi(\mathbb{I}-\Pi)^{-1}\right)^{k}
$$

In particular, we have the following identities

$$
\mathbb{E}[T]=n+\operatorname{Tr}\left(\Pi(\mathbb{I}-\Pi)^{-1}\right) \quad \text { and } \operatorname{var}[T]=\operatorname{Tr}\left(\Pi(\mathbb{I}-\Pi)^{-1}\right)+\operatorname{Tr}\left(\Pi(\mathbb{I}-\Pi)^{-1}\right)^{2}
$$

as well as a formula for the expected parity $\mathbb{E}\left[(-1)^{T}\right]=\operatorname{det}(\Pi-\mathbb{I}) / \operatorname{det}(\Pi+\mathbb{I})$ of the number of steps.
Proof. These identities are merely obtained by differentiating the cumulant generating function $\log \mathbb{E}\left[e^{\alpha T}\right]=$ $\sum_{k \geq 1} \kappa_{k}(T) \alpha^{k} / k!$ with respect to $\alpha$ such that $\alpha<0$. This generating function is obtained by taking the logarithm of (30) evaluated at $t=e^{\alpha}$, which gives $\log \mathbb{E}\left[e^{\alpha T}\right]=n \alpha-\operatorname{Tr} \log \left(\mathbb{I}+\left(1-e^{\alpha}\right) \Pi(\mathbb{I}-\Pi)^{-1}\right)$. Lastly, the formula for $\mathbb{E}\left[(-1)^{T}\right]$ is obtained by taking $t=-1$ in Proposition 5.

Recall that $\mathbb{I}-\Pi$ is non-singular in light of Proposition 3. Moreover, letting $\Delta_{N}=D^{-1 / 2} \Delta D^{-1 / 2}$ be the so-called normalized magnetic Laplacian, we see that $\mathbb{E}[T]=\operatorname{Tr} \Delta_{N}{ }^{-1}$. The expected running time is thus intuitively low if the least eigenvalue of the normalized magnetic Laplacian is not close to zero. The variational formula for this least eigenvalue,

$$
\lambda_{\min }\left(\Delta_{\mathrm{N}}\right)=\min _{\|\mathrm{v}\|_{2}=1} \mathrm{v}^{*} \Delta_{\mathrm{N}} \mathrm{v}=\min _{\|\mathrm{v}\|_{2}=1} \sum_{x \sim y} w_{x y}\left|\frac{\mathrm{v}_{x}}{\sqrt{\operatorname{deg}(x)}}-\frac{\phi_{y x} \mathrm{v}_{y}}{\sqrt{\operatorname{deg}(y)}}\right|^{2}
$$

shows that $\lambda_{\min }\left(\Delta_{N}\right)$ relates to the inconsistency the $U(1)$-connection graph, see [Fanuel and Bardenet, 2022].

### 4.4 Multi-type spanning forests case

A multi-type spanning forest $\mathcal{F}$ (MTSF) is a spanning graph where each connected component is either a tree or a cycle-rooted tree, as defined by Kenyon [2019], who also proves that the measure on MTSFs given by

$$
\begin{equation*}
\mu_{\operatorname{MTSF}}(\mathcal{F})=\frac{q^{\rho(\mathcal{F})}}{\operatorname{det}(\Delta+q \mathbb{I})}\left(\prod_{e \in \mathcal{F}} w_{e}\right) \prod_{\substack{\text { non-oriented } \\ \text { cycle } c \subseteq \mathcal{F}}}(2-2 \cos \theta(c)) \tag{32}
\end{equation*}
$$

is determinantal. This measure (32) has been used to build sparsifiers of the regularized magnetic Laplacian in [Fanuel and Bardenet, 2022], and we are thus interested in the computational complexity of sampling from (32). As a simple extension of the previous discussions, Proposition 7 below gives the mean and variance of the time to sample from (4) with a variant of CyclePopping, denoted by CyclePopping ${ }_{q}$ to emphasize ${ }^{7}$ the dependence of the parameter $q \geq 0$. It is briefly described below and we refer the reader to [Fanuel and Bardenet, 2022].

As in Section 2.2, define the auxilliary graph $G_{r}$ where each node of $G$ is connected to an auxiliary root $r \notin \mathcal{V}$, with an edge weight $q>0$. Then, under Assumption 1 and Assumption 3, CyclePopping $q$ proceeds as CyclePopping on $G_{r}$ (see Section 3.3), by adding a branch or lasso at each iteration, but starting from $\mathcal{F}_{0}=\{r\}$. This way, the SRWs are also absorbed upon reaching $r$. Finally, the spanning subgraph of $G_{r}$ produced by CyclePopping ${ }_{q}$ is postprocessed by removing $r$ and all the edges of which $r$ is an endpoint. The resulting subgraph of $G$ is then an MTSF distributed as (4).

Proposition 7. Let Assumption 1 and Assumption 3 hold. Let $q>0$ and let $T$ denote the number of steps to sample a MTSF with CYclePopping $q$ according to (32). We have

$$
\mathbb{E}[T]=\operatorname{Tr}\left((\mathrm{D}+q \mathbb{I})(\Delta+q \mathbb{I})^{-1}\right) \quad \text { and } \operatorname{var}[T]=\operatorname{Tr}\left(\mathrm{M}_{q}^{(\Phi)}\left(\mathbb{I}-\mathrm{M}_{q}^{(\Phi)}\right)^{-1}\right)+\operatorname{Tr}\left(\mathrm{M}_{q}^{(\Phi)}\left(\mathbb{I}-\mathrm{M}_{q}^{(\Phi)}\right)^{-1}\right)^{2}
$$

with $\mathrm{M}_{q}^{(\Phi)}=(\mathrm{D}+q \mathbb{I})^{-1}(\mathrm{~W} \odot \Phi)$.
The proof of Proposition 7 follows the same lines as the proof of Proposition 2, and we thus omit it.

## 5 CyclePopping for CRSFs: beyond the determinantal case

We now extend the proof that Algorithm 1 samples from the measure (6), by removing any assumption on the form of the cycle weight function $\alpha$. In particular, we do not assume (12), and the probability measure (6) is not necessarily determinantal over the graph edges. Kassel and Kenyon [2017] already proved the correctness of Algorithm 1 in that case using a stack-of-cards argument. By adapting Section 4.2, we provide a different proof, more in line with the seminal proof of Marchal [1999] for spanning trees. In particular, a by-product of this new proof is the law of the running time of the algorithm.

### 5.1 Loop measures

We follow Lawler and Limic [2010] in defining so-called loop measures, with a slight adaptation to allow for cycles to be accepted with non-zero probability.

For each oriented cycle $c$ of the graph, we have a weight $\alpha(c) \in[0,1]$, which is interpeted as the probability to accept $c$. To avoid trivial measures, we take the following assumption which implies Assumption 1 when $\alpha(\cdot)=1-\cos \theta(\cdot)$.

Assumption 4 (Non-trivial cycle weight). There is at least one cycle $c_{\star}$ such that $\alpha\left(c_{\star}\right)>0$.
We first define a measure on based loops by

$$
\mu_{\alpha}(\gamma)= \begin{cases}q(\gamma) \prod_{c \in \operatorname{cycles}(\gamma)}(1-\alpha(c)) & \text { if }|\gamma| \geq 2  \tag{33}\\ 1 & \text { if }|\gamma|=0\end{cases}
$$

[^3]where $q(\gamma)$ is the product of the transition probabilities along $\gamma$, as defined in (22), and cycles $(\gamma)$ is the multiset of popped cycles defined in Notation 2. Note that $\mu_{\alpha}$ is not necessarily a probability measure. An immediate property of the measure (33) is that it behaves well under concatenation.

Lemma 2. Let $\gamma_{1}$ and $\gamma_{2}$ be two loops based at $x \in \mathcal{V}$, and $\gamma_{1} \circ \gamma_{2}$ the concatenation of $\gamma_{1}$ and $\gamma_{2}$. Then $\mu_{\alpha}\left(\gamma_{1} \circ \gamma_{2}\right)=\mu_{\alpha}\left(\gamma_{1}\right) \mu_{\alpha}\left(\gamma_{2}\right)$.

Let $x \in \mathcal{V} \backslash \mathcal{A}$, define the Green generating function by

$$
G_{\alpha}(t, x, x ; \mathcal{A})=\sum_{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x)} t^{|\gamma|} \mu_{\alpha}(\gamma), \quad t \in(0,1]
$$

where the sum goes over loops based at $x$ and included in the complement of $\mathcal{A}$. Under Assumption 3, this definition reduces to the Green function defined in Section 4.2 .2 when $\alpha(\cdot)=1-\cos \theta(\cdot)$.

Lemma 3. Let $\mathcal{A} \subset \mathcal{V}$ be a subset of nodes and let $x \in \mathcal{V} \backslash \mathcal{A}$. Under Assumption $4, G_{\alpha}(t, x, x ; \mathcal{A})<+\infty$.
Proof. Consider first the case $\mathcal{A}=\emptyset$ and $t=1$. Let $c_{\star}$ such that $\alpha\left(c_{\star}\right)>0$. By (21), it comes

$$
\begin{aligned}
G_{\alpha}(1, x, x ; \emptyset) & =\sum_{k=0}^{+\infty} \mathbb{Q}_{x}\left(X_{k}=x \text { and } k<\min _{\text {cycle } c} N_{\circlearrowleft c}\right) \\
& \leq \sum_{k=0}^{+\infty} \mathbb{Q}_{x}\left(X_{k}=x \text { and } k<N_{\circlearrowleft c_{\star}}\right) .
\end{aligned}
$$

The quantity on the right-hand side equals

$$
1+\mathbb{Q}_{x}\left(N_{\rightarrow\{x\}}^{(\text {after 1) }}<N_{\circlearrowleft c_{\star}}\right)
$$

which is finite. To conclude, it is readily checked that $G_{\alpha}(t, x, x ; \mathcal{A}) \leq G_{\alpha}(1, x, x ; \mathcal{A}) \leq G_{\alpha}(1, x, x ; \emptyset)$ since $t \in(0,1]$ and $\operatorname{BL}(\overline{\mathcal{A}}, x)$ is a subset of $\operatorname{BL}(\mathcal{V}, x)$.

We shall show that non-determinantality does not prevent to use a telescoping argument in the vein of (29).

Notation 3 (number of revisits of based point). For a loop $\gamma$ based at $x$, denote by $d(\gamma)$ the number of times $\gamma$ comes back to $x$ after its start.

We start with two lemmas. For example, $d((x, y, x, y, x))=2$.
Lemma 4. Let $\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x)$ and $d(\gamma)$ the number of visits of $x$ as in Notation 3. Let $t \in(0,1]$. Under Assumption 4, for any integer $k \geq 1$, we have

$$
\begin{equation*}
\sum_{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x): d(\gamma)=k} t^{|\gamma|} \mu_{\alpha}(\gamma)=\left(\sum_{\gamma \in \mathrm{BL}(\overline{\mathcal{A}}, x): d(\gamma)=1} t^{|\gamma|} \mu_{\alpha}(\gamma)\right)^{k} \tag{34}
\end{equation*}
$$

Proof. This follows from distributing the $k$ sums on the right-hand side and using Lemma 2.
To understand the impact of forgetting the base point of a based loop, we need to be precise about sets of unbased loops and their cardinality. Let L be the set of (unbased) loops in the graph $G$. The marginal measure on L corresponding to (33) is

$$
\begin{equation*}
\mu_{\alpha}([\gamma])=\sum_{\gamma \in[\gamma]} \mu_{\alpha}(\gamma)=\mathcal{N}_{[\gamma]} \mu_{\alpha}(\gamma) \tag{35}
\end{equation*}
$$

where the multiplicative factor $\mathcal{N}_{[\gamma]}$ is the number of representatives of the equivalence class $[\gamma]$. Note that $\mathcal{N}_{[\gamma]}$ is not necessarily equal to $|\gamma|$. For instance, if we take a loop $\gamma=[(x, y, x, y, x)]$ made of two identitical
backtracks, we have $\mathcal{N}_{[\gamma]}=2$ by counting the number of cyclic permutations. However, the length of this loop is $|[\gamma]|=4=|\gamma|$.

To clarify the meaning of $\mathcal{N}_{[\gamma]}$, we introduce the following definition: a based loop is primitive if it cannot be written as $\gamma_{0}^{m}$ for any integer $m \geq 2$ and any other based loop $\gamma_{0}$. Also, we say that an unbased loop is primitive if one of its based representatives is primitive. For any representative $\gamma$ of a loop $[\gamma]$, we have $\gamma=\gamma_{0}^{m}$ for some $m \geq 1$ and a primitive based loop $\gamma_{0}$, possibly equal to $\gamma$. Upon defining mult $(\gamma)=m$, we have

$$
\begin{equation*}
\mathcal{N}_{[\gamma]}=|\gamma| / \operatorname{mult}(\gamma), \tag{36}
\end{equation*}
$$

i.e., the number of representatives of $[\gamma]$ is the length of $\gamma_{0}$.

Lemma 5 (Number of $x$-based representatives of an unbased loop). Let $[\gamma]$ be an unbased loop containing $x$. Denote by $d_{x}([\gamma])$ the number of visits of $x$ after its start by any based representative of $[\gamma]$ based at $x$. Let $\#_{[\gamma], x}$ be the number of representatives of $[\gamma]$ based at $x$. For any representative $\gamma$ of $[\gamma]$ that is based at $x$, we have
(i) $d_{x}([\gamma])=d(\gamma)$ and
(ii) $\#_{[\gamma], x}=\mathcal{N}_{[\gamma]} d_{x}([\gamma]) /|\gamma|$.

Proof. Let $\gamma$ be a representative of $[\gamma]$ based at $x$ and recall $d(\gamma)$ is the number of times $\gamma$ comes back to $x$ after its start as in Notation 3. The proof of $(i)$ is straightforward. To show (ii), we consider two cases. (a) If $\gamma$ is not primitive, i.e., if $\gamma$ can be written has $\gamma_{0}^{m}$ for some $m \geq 2$ and a primitive based loop $\gamma_{0}$, there is only one representative of $[\gamma]$ based at $x$, namely $\#_{[\gamma], x}=1$. Furthermore, the number of visits of $x$ by the based loop $\gamma$ after its start is $d_{x}([\gamma])=m$, whereas the number of representatives of $[\gamma]$ is $\mathcal{N}_{[\gamma]}=|\gamma| / m$; see (36). Thus, by a direct substitution, we also have $\mathcal{N}_{[\gamma]} d_{x}([\gamma]) /|\gamma|=1$, and this yields (ii). (b) Otherwise, $\gamma$ is primitive and we have $|\gamma|=\mathcal{N}_{[\gamma]}$ as a consequence of (36), whereas the number of representatives of $[\gamma]$ based at $x$ is equal to the number of visits of $x$ by $\gamma$ after its start, namely $\#_{[\gamma], x}=d_{x}([\gamma])$. Recalling that $\mathcal{N}_{[\gamma]} /|\gamma|=1$, we have shown (ii). This completes the proof.

### 5.2 An exponential formula for the Green generating function

Define the measure on nontrivial unbased loops

$$
m_{\alpha}([\gamma])= \begin{cases}\mu_{\alpha}([\gamma]) /|\gamma| & \text { if }|\gamma| \geq 2  \tag{37}\\ 0 & \text { if }|\gamma|=0\end{cases}
$$

As anticipated in Section 1, the expression (37) can be simplified as follows: using (36), we have that $m_{\alpha}([\gamma])=\mu_{\alpha}(\gamma) / \operatorname{mult}(\gamma)$, where $\gamma$ is any representative of $[\gamma]$. The presence of loop multiplicity is emphasized in Le Jan [2022, section 2.2].

Lemmas 4 and 5 imply that the Green generating function can be expressed as an exponential involving this measure $m_{\alpha}$.

Proposition 8 (Green exponential formula). Let $t \in(0,1]$ and $\mathcal{A} \subset \mathcal{V}$. Let $x \in \mathcal{V} \backslash \mathcal{A}$. Denote by $\mathrm{L}(\overline{\mathcal{A}})$ the set of unbased loops in the complement of a subgraph $\mathcal{A}$ of $G$. Under Assumption 4,

$$
G_{\alpha}(t, x, x ; \mathcal{A})=\exp \left(\sum_{[\gamma] \in \mathrm{L}(\overline{\mathcal{A}}): x \in[\gamma]} t^{|\gamma|} m_{\alpha}([\gamma])\right) .
$$

Proposition 8 may be seen as a generalization of [Lawler and Limic, 2010, Lemma 9.3.2] to measures involving cycle weights; the proof technique is similar.

Proof. We begin by considering the sum of the measures of loops based at $x$ that visit $x$ once after their start. The quantity

$$
\begin{equation*}
\sum_{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x): d(\gamma)=1} \mu_{\alpha}(\gamma)=\mathbb{Q}_{x}\left(N_{\rightarrow\{x\}}^{(\text {after 1) }}<N_{\rightarrow \mathcal{A}} \wedge \min _{\text {cycle } c} N_{\circlearrowleft c}\right)<1 \tag{38}
\end{equation*}
$$

is the probability of coming back to $x$ before hitting $\mathcal{A}$ and before accepting any cycle. Now, with the help of (34), we calculate the sum of the measures of loops based at $x$ that visit $x$ exactly $k$ times after their start. By summing (34) over all possible numbers $k$ of visits, including the case $k=0$ for which no loop occurs, we find

$$
\begin{equation*}
G_{\alpha}(t, x, x ; \mathcal{A})=\left(1-\sum_{\gamma \in \operatorname{BL}(\overline{\mathcal{A}}, x): d(\gamma)=1} t^{|\gamma|} \mu_{\alpha}(\gamma)\right)^{-1} \tag{39}
\end{equation*}
$$

Armed with this identity, we now give the expression of $\log G_{\alpha}(t, x, x ; \mathcal{A})$. Recalling that $\mu_{\alpha}([\gamma])=\mathcal{N}_{[\gamma]} \mu_{\alpha}(\gamma)$ by definition, and using Lemma 5 for turning a sum over unbased loops into a sum over its based representatives, we find

$$
\sum_{\substack{[\gamma] \in \mathrm{L}(\overline{\mathcal{A}}) \\ x \in[\gamma]}} t^{|\gamma|} \frac{\mu_{\alpha}([\gamma])}{|\gamma|}=\sum_{\gamma \in \mathrm{BL}(\overline{\mathcal{A}}, x)} t^{|\gamma|} \frac{\mu_{\alpha}(\gamma)}{d(\gamma)}=\sum_{k=1}^{+\infty} \frac{1}{k} \sum_{\substack{\gamma \in \mathrm{BL}(\overline{\mathcal{A}}, x) \\ d(\gamma)=k}} t^{|\gamma|} \mu_{\alpha}(\gamma)=\sum_{k=1}^{+\infty} \frac{1}{k}\left(\sum_{\substack{\gamma \in \mathrm{BL}(\overline{\mathcal{A}}, x) \\ d(\gamma)=1}} t^{|\gamma|} \mu_{\alpha}(\gamma)\right)^{k}
$$

Note that we used (34) to establish the last equality. The result follows by using the Taylor series $\log (1-$ $s)^{-1}=\sum_{k=1}^{+\infty} s^{k} / k$ for $s \in[0,1)$.

### 5.3 Proof of correctness beyond the determinantal case

In the proof of Proposition 4, we first used the form of the weights when rewriting (28) as (29). We thus start from (28), and note that, at that stage, we had in particular proven the following. Let $\mathcal{F}$ in the support of (6), and $x_{1}, \ldots, x_{n}$ enumerate the nodes of $G$, in the unique possible order they were added by Algorithm 1 to construct $\mathcal{F}$, see Remark 1. Then plugging (28) into (19), we have

$$
\begin{equation*}
\mathbb{P}(\mathcal{T}=\mathcal{F})=p_{x_{1} x_{2}} \ldots p_{x_{n-1} x_{n}} \prod_{i=1}^{n} G_{\alpha}\left(1, x_{i}, x_{i} ; \mathcal{B}_{i}\right) \prod_{c \in \mathcal{F}} \alpha(c) \tag{40}
\end{equation*}
$$

where we introduced the growing size subsets of nodes $\mathcal{B}_{1}=\emptyset, \mathcal{B}_{2}=\left\{x_{1}\right\}, \mathcal{B}_{3}=\left\{x_{1}, x_{2}\right\}, \ldots, \mathcal{B}_{n}=$ $\left\{x_{1}, \ldots, x_{n-1}\right\}$. Note that $G_{\alpha}\left(1, x_{n}, x_{n} ; \mathcal{B}_{n}\right)=1$.

Consider now the mutually disjoint sets of loops $\mathrm{E}_{1}=\left\{[\gamma]\right.$ s.t. $\left.x_{1} \in[\gamma]\right\}, \mathrm{E}_{2}=\left\{[\gamma]\right.$ s.t. $x_{1} \notin[\gamma]$ and $x_{2} \in$ $[\gamma]\} \mathrm{E}_{3}=\left\{[\gamma]\right.$ s.t. $x_{1} \notin[\gamma]$ and $x_{2} \notin[\gamma]$ and $\left.x_{3} \in[\gamma]\right\}, \ldots$, whereas $\mathrm{E}_{n}$ is the trivial loop containing $x_{n}$. In the more compact notation introduced in Section 5.1 , for $1 \leq i \leq n$, we let

$$
\begin{equation*}
\mathrm{E}_{i}=\left\{[\gamma] \in \mathrm{L}\left(\overline{\mathcal{B}_{i}}\right) \text { such that } x_{i} \in[\gamma]\right\} \text { with } \mathcal{B}_{i}=\left\{x_{1}, \ldots, x_{i-1}\right\} \tag{41}
\end{equation*}
$$

In particular, $\mathrm{E}_{1}, \ldots, \mathrm{E}_{n}$ are a partition of the entire set of loops L. For simplicity, by slightly generalizing (37), we define another measure over loops including the dependence on $t \in(0,1]$, as

$$
m_{\alpha, t}([\gamma])=t^{|\gamma|} m_{\alpha}([\gamma])
$$

and we write $m_{\alpha, t}(\mathrm{E})=\sum_{[\gamma] \in \mathrm{E}} m_{\alpha, t}([\gamma])$ for any subset of unbased loops E . Using $n$ times the exponential formula of Proposition 8, and the additivity of the measure $m_{\alpha, t}$, it comes

$$
\begin{equation*}
G_{\alpha}\left(t, x_{1}, x_{1} ; \mathcal{B}_{1}\right) \ldots G_{\alpha}\left(t, x_{n}, x_{n} ; \mathcal{B}_{n}\right)=\exp \left(m_{\alpha, t}\left(\mathrm{E}_{1}\right)+\cdots+m_{\alpha, t}\left(\mathrm{E}_{n}\right)\right)=\exp m_{\alpha, t}(\mathrm{~L}) \tag{42}
\end{equation*}
$$

Equation (42) generalizes the telescoping product of determinants of Section 4.2.3. In particular, for $t=1$, (42) applied to (40) shows that $\mathbb{P}(\mathcal{T}=\mathcal{F})$ is (6), and that the normalizing factor in (6) is

$$
\begin{equation*}
Z_{\alpha}^{-1}=\exp m_{\alpha}(\mathrm{L}) \tag{43}
\end{equation*}
$$

This concludes the proof of correctness.

### 5.4 Interpretation as a Poisson point process on unbased oriented loops

The proof of Proposition 5 applies mutatis mutandis for general weights, generalizing its result to the identity

$$
\begin{equation*}
\mathbb{E}\left[t^{T}\right]=t^{n} \exp \left(\sum_{[\gamma] \in \mathrm{L}}\left(t^{|\gamma|}-1\right) \frac{\mu_{\alpha}([\gamma])}{|\gamma|}\right) \text { for } t \in(0,1] \tag{44}
\end{equation*}
$$

Interestingly, (44) is a Laplace transform w.r.t. a Poisson point process. Indeed, under the (inhomogeneous) Poisson process on L with intensity $m_{\alpha}$, for any non-negative function $f$ on L ,

$$
\begin{equation*}
\mathbb{E}_{\mathcal{X} \sim \operatorname{Poisson}\left(m_{\alpha}, \mathrm{L}\right)} \exp \left(-\sum_{[\gamma] \in \mathcal{X}} f([\gamma])\right)=\exp \left(\sum_{[\gamma] \in \mathrm{L}}\left(e^{-f([\gamma])}-1\right) m_{\alpha}([\gamma])\right) \tag{45}
\end{equation*}
$$

see $^{8}$ e.g. [Last and Penrose, 2017, Thm 3.9]. Letting $t \in(0,1]$ and applying (45) to $f:[\gamma] \mapsto \log \left(t^{-|\gamma|}\right)$ yields

$$
\mathbb{E}_{\mathcal{X} \sim \operatorname{Poisson}\left(m_{\alpha}, \mathrm{L}\right)}\left[t^{\left.n+\sum_{[\gamma] \in \mathcal{X}}|\gamma|\right]}\right]=\mathbb{E}\left[t^{T}\right]
$$

where the right-hand side is given by (44). This justifies our initial claim in (2), which we rephrase here.
Corollary 2 (law of the running time of CyclePopping). Let $T$ be the number of (Markov chain) steps to complete Algorithm 1. Then, under Assumption 4,

$$
T \stackrel{(l a w)}{=} n+\sum_{[\gamma] \in \mathcal{X}}|\gamma| \text { with } \mathcal{X} \sim \operatorname{Poisson}\left(m_{\alpha}, \mathrm{L}\right)
$$

where $m_{\alpha}$ is the measure over loops given in (37).
In the light of Corollary 2, we now briefly discuss the complexity of CyclePopping in comparison with HKPV algorithm for sampling the determinantal measure (10).

- On the one hand, the law of $T$ given in Corollary 2 indicates that if the cycles are very consistent, i.e., $\cos \theta(c) \approx 1$, the intensity $m_{\alpha}([\gamma])$ of the Poisson process of loops is large. In other words, the number of popped loops is expected to be large which tends to slow down CyclePopping. In the determinantal case, we can derive from Proposition 6 the simple upper bound $\mathbb{E}[T] \leq n / \lambda_{\min }\left(\Delta_{N}\right)$ with $\Delta_{N}=D^{-1 / 2} \Delta D^{-1 / 2}$. Incidentally, it is interesting to observe at this point the appearance of the least eigenvalue of the normalized magnetic Laplacian which plays a central role in the Cheeger inequality [Bandeira et al., 2013] which controls the frustration of the connection graph.
- On the other hand, as discussed in [Fanuel and Bardenet, 2022], if the cycles are very consistent, the least eigenvalue of the Laplacian $\lambda_{\min }(\Delta)$ - or of its normalized version $\lambda_{\min }\left(\Delta_{N}\right)$ - is expected to be close to zero. In this case, the algebraic HKPV sampler of [Hough, Krishnapur, Peres, and Virág, 2006] is expected to be error prone since the computation of the correlation kernel of the determinantal process essentially necessitates to solve a linear system of the type $\Delta \boldsymbol{x}=\boldsymbol{b}$. This matrix being illconditioned, we expect numerical errors on the correlation kernel to affect HKPV. The output of this algorithm is not guaranteed to be a CRSF, in contrast with CyclePopping.

As we discussed under Proposition 1, the characterization by Marchal [1999] of the law of the running time of Wilson's algorithm for spanning trees has practical consequences on the choice of the root. We now examine a similar consequence of Corollary 2 for cycle-rooted spanning forests.

[^4]
### 5.5 Expectation of the number of steps and cycle times

We formalize here the intuition that CyclePopping is fast when cycles of large weights are well-spread in the graph. In view of (44), the expectation of the number of steps to complete CyclePopping is

$$
\begin{equation*}
\mathbb{E}[T]=\sum_{x \in \mathcal{V}} G_{\alpha}(1, x, x ; \emptyset) \tag{46}
\end{equation*}
$$

Expression (46) - which naturally reduces to $\mathbb{E}[T]=\operatorname{Tr}\left((\mathbb{I}-\Pi)^{-1}\right)$ in the determinantal case - has a clear probabilistic interpretation as the sum over all the nodes $x$ of the inverse of the probability that the walker starting from $x$ accepts a cycle no later than its first return to $x$, or in other words, an escape-to-a-cycle probability. This is in complete analogy with the rooted spanning tree case of Section 2.1 where the role of the roots is now played by the inconsistent cycles of the graph.
Proposition 9 (Escape-to-a-cycle probability). Let $x \in \mathcal{V}$. Under the assumptions of Corollary 2, it holds that $G_{\alpha}(1, x, x ; \emptyset)=1 / \rho_{x}$, where

$$
\begin{equation*}
\rho_{x}=\mathbb{Q}^{x}\left(\min _{c} N_{\circlearrowleft c} \leq N_{\rightarrow\{x\}}^{(\text {after } 1)}\right)=1-\sum_{\gamma \in \operatorname{BL}(\mathcal{V}, x): d(\gamma)=1} \mu_{\alpha}(\gamma) . \tag{47}
\end{equation*}
$$

Here, $\operatorname{BL}(\mathcal{V}, x)$ is the set of based loops based at $x, d(\gamma)$ is the number of visits of the base point of $\gamma$ after its start, and $\mu_{\alpha}(\gamma)$ is the probability (33) that the SRW $\left(X_{n}^{x}\right)$ walks along the based loop $\gamma$ and pops all cycles in cycles $(\gamma)$.

An intuitive consequence of (46) and Proposition 9 is that CyclePopping is expected to be fast in a $\mathrm{U}(1)$-connection graph where the set of inconsistent cycles can be hit from any node in the graph with a large probability.
Proof. By Definitions (21) and (25), and dominated convergence, the Green function reads

$$
G_{\alpha}(1, x, x ; \emptyset)=\sum_{k=0}^{+\infty} \mathbb{Q}^{x}\left(X_{k}=x \text { and } k<\min _{c} N_{\circlearrowleft c}\right)=\mathbb{E} R_{x},
$$

where

$$
R_{x}=\sum_{k=0}^{+\infty} 1_{\left\{X_{k}^{x}=x \text { and } k<\min _{c} N_{0 C}^{x}\right\}}
$$

is the number of returns of $\left(X_{k}^{x}\right)$ to its starting point before a cycle is accepted. At this point, we recall from Section 3.1 the definition of the Markov chain

$$
Z_{n}^{x}=\left(X_{n}^{x}, B_{c_{1}, n}^{x}, \ldots, B_{c_{d}, n}^{x}\right) .
$$

Remark that $\mathbb{Q}^{x}\left(\min _{c} N_{O c}>N_{\rightarrow\{x\}}^{(\text {after 1) })}\right.$ ) is the probability of first return before a cycle is accepted, i.e., a special case of (38) with $\mathcal{A}=\emptyset$. Denote the probability of the complementary event by $\rho_{x}=\mathbb{Q}^{x}\left(\min _{c} N_{O c} \leq\right.$ $N_{\rightarrow\{x\}}^{(\text {after 1) })}$. Using the Markov property for $Z_{n}^{x}$, we have the following law

$$
\mathbb{P}\left(R_{x}=k\right)=\left(1-\rho_{x}\right)^{k} \rho_{x}, \quad k=0,1, \ldots,
$$

so that $R_{x}$ is a geometric random variable with success parameter $\rho_{x}$, whose expectation is $\left(1-\rho_{x}\right) / \rho_{x}$. Thus, we find $G_{\alpha}(1, x, x ; \emptyset)=1+\left(1-\rho_{x}\right) / \rho_{x}=1 / \rho_{x}$. Now, by using (39) with $\mathcal{A}=\emptyset$ and $t=1$, we also have

$$
G_{\alpha}(1, x, x ; \emptyset)=\left(1-\sum_{\gamma \in \operatorname{BL}(\mathcal{V}, x): d(\gamma)=1} \mu_{\alpha}(\gamma)\right)^{-1}
$$

Recalling the definition of $q(\gamma)$ in (22) in terms of the product of the transition probabilities along the edges of $\gamma$, it is easy to see that

$$
\mathbb{Q}^{x}\left(N_{\rightarrow\{x\}}^{(\text {after } 1)}<\min _{c} N_{\circlearrowleft c}\right)=\sum_{\gamma \in \operatorname{BL}(\mathcal{V}, x): d(\gamma)=1} q(\gamma) \prod_{c \in \operatorname{cycles}(\gamma)}(1-\alpha(c))=\sum_{\gamma \in \operatorname{BL}(\mathcal{\nu}, x): d(\gamma)=1} \mu_{\alpha}(\gamma),
$$

in the light of the definition of $\mu_{\alpha}(\gamma)$ in (33). This completes the proof.

### 5.6 A verbose CyclePopping also samples a Poisson point process on loops

In this subsidiary section, we seize the opportunity to write down the details of an intriguing construction that [Le Jan, 2011, Section 8.3] defined with fewer details and in the absence of U(1)-connection; see also [Le Jan, 2022, Remark 18] for continuous loops. This is not directly related to the analysis of the running time of Wilson's algorithm, but it is more of a side observation that, as hinted by Corollary 2 and upon suitably randomizing the popped based loops, CyclePopping also implicitly gives a sample of a Poisson process of unbased loops. The key to understand this construction is how the based loops popped by CyclePopping are randomly partitioned to yield a Poisson sample with intensity (37).

Consider a sample output of Algorithm 1 and let $x_{1}, \ldots, x_{n}$ be the vertices of the graph as they are visited by the algorithm. The loops erased by Algorithm 1 form a tuple of - possibly empty - based loops

$$
\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}\right) \in \operatorname{BL}\left(\overline{\mathcal{B}_{1}}, x_{1}\right) \times \operatorname{BL}\left(\overline{\mathcal{B}_{2}}, x_{2}\right) \times \cdots \times \operatorname{BL}\left(\overline{\mathcal{B}_{n}}, x_{n}\right)
$$

where $\mathcal{B}_{1}=\emptyset, \mathcal{B}_{2}=\left\{x_{1}\right\}, \mathcal{B}_{3}=\left\{x_{1}, x_{2}\right\}, \ldots, \mathcal{B}_{n}=\left\{x_{1}, \ldots, x_{n-1}\right\}$ in the notations of (40). Note that the $n$-th entry $\gamma_{n}$ is always the trivial loop containing only $x_{n}$, which satisfies $\mu_{\alpha}\left(\gamma_{n}\right)=1$. It is easy to see that collecting the popped loops and the output of Algorithm 1 defines a random pair $(\Gamma, \mathcal{T})$, where

$$
\begin{equation*}
\Gamma=\left(\Gamma_{1}, \ldots, \Gamma_{n}\right) \tag{48}
\end{equation*}
$$

is a random tuple of loops in $\operatorname{BL}\left(\overline{\mathcal{B}_{1}}, x_{1}\right) \times \cdots \times \operatorname{BL}\left(\overline{\mathcal{B}_{n}}, x_{n}\right)$ and $\mathcal{T}$ is a random CRSF. Their joint law ${ }^{9}$ is

$$
\begin{equation*}
\mathbb{P}\left(\Gamma=\left(\gamma_{1}, \ldots, \gamma_{n}\right), \mathcal{T}=\mathcal{F}\right)=\mu_{\alpha}\left(\gamma_{1}\right) \ldots \mu_{\alpha}\left(\gamma_{n}\right) \times \prod_{e \in \mathcal{F}} p_{e} \prod_{c \in \mathcal{F}} \alpha(c) \tag{49}
\end{equation*}
$$

By marginalizing, i.e., by summing the first factor of (49) over all possible based loops, we recover the normalization (43), namely,

$$
\sum_{\gamma_{1} \in \operatorname{BL}\left(\overline{\mathcal{B}_{1}}, x_{1}\right)} \ldots \sum_{\gamma_{n} \in \operatorname{BL}\left(\overline{\mathcal{B}_{n}}, x_{n}\right)} \mu_{\alpha}\left(\gamma_{1}\right) \ldots \mu_{\alpha}\left(\gamma_{n}\right)=\prod_{i=1}^{n} G_{\alpha}\left(1, x_{i}, x_{i} ; \mathcal{B}_{i}\right)=\exp m_{\alpha}(\mathrm{L})=Z_{\alpha}^{-1}
$$

where we used Lemma 3 to express the Green function as a sum over based loops, and the notation for the normalization (43) of the measure on CRSFs. In other words, by forgetting the erased loops, we indeed find that Algorithm 1 yields $\mathcal{F}$ with probability $\mathbb{P}(\mathcal{T}=\mathcal{F})$ as given in (40).

Now, by inserting $e^{-m_{\alpha}(\mathrm{L})} e^{m_{\alpha}(\mathrm{L})}=1$ in (49) and by using the partition of the set of unbased loops in (42), we write

$$
\begin{equation*}
\mathbb{P}\left(\Gamma=\left(\gamma_{1}, \ldots, \gamma_{n-1}\right), \mathcal{T}=\mathcal{F}\right)=\mu_{\alpha}\left(\gamma_{1}\right) e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)} \ldots \mu_{\alpha}\left(\gamma_{n}\right) e^{-m_{\alpha}\left(\mathrm{E}_{n}\right)} \times Z_{\alpha}^{-1} \prod_{e \in \mathcal{F}} p_{e} \prod_{c \in \mathcal{F}} \alpha(c) \tag{50}
\end{equation*}
$$

where one factor is trivial since $\mu_{\alpha}\left(\gamma_{n}\right)=1$ by definition of trivial loops and $m_{\alpha}\left(\mathrm{E}_{n}\right)=0$ by (37). We shall see that a randomization of $\Gamma$ actually yields a Poisson point process of loops with intensity measure $m_{\alpha}$.

Proposition 10 (Coupling between Poisson process and CRSFs). Under Assumption 4, let ( $\Gamma, \mathcal{T}$ ) be the output of verbose CyclePopping. Denote by $\mathcal{L}=\left\{\left[\delta_{1}\right]^{\nu_{1}}, \ldots,\left[\delta_{\ell}\right]^{\nu_{\ell}}\right\}$ a set of loops with multiplicity and $\mathcal{F}$ a CRSF. Let $\mathcal{X} \sim \operatorname{Poisson}\left(m_{\alpha}, \mathrm{L}\right)$ We have

$$
\mathbb{P}(\text { forgetorder }(\text { forgetbase }(\text { randomsplit }(\Gamma)))=\mathcal{L}, \mathcal{T}=\mathcal{F})=\mathbb{P}(\mathcal{X}=\mathcal{L}) \times \mu_{\mathrm{CRSF}, \alpha}(\mathcal{F})
$$

where the measure over CRSFs is given in (6).
The precise definition of randomsplit is given in the proof of Proposition 10.
Proof. Consider the first factor in (50) with the based loop $\gamma_{1} \in \mathrm{BL}\left(\overline{\mathcal{B}_{1}}, x_{1}\right)$. To avoid trivialities, we assume $\left|\gamma_{1}\right| \geq 2$. We now describe how the based loop $\gamma_{1}$ can be randomly split into a set of unbased loops that

[^5]provides a Poisson sample of unbased loops in $\mathrm{E}_{1}$, as suggested by Le Jan [2011, Section 8.3] or Lawler and Limic [2010, Section 9.4] in a slightly different setting.

Suppressing indices for a moment, say that a loop $\gamma$ based at $x$ visits its base point exactly $n_{x}$ times after its start, i.e., $d(\gamma)=n_{x}$ as defined in Notation 3. Thus, this oriented loop is the concatenation of $n_{x}$ loops,

$$
\begin{equation*}
\gamma=\eta_{1} \circ \cdots \circ \eta_{n_{x}} \tag{51}
\end{equation*}
$$

each of them visiting $x$ only once of its start: $d\left(\eta_{i}\right)=1$ for $1 \leq i \leq n_{x}$.
Trivial multiplicities. For simplicity, we assume that these loops are all distinct; the case of general multiplicities is treated in the next paragraph. The main idea that we describe below is to define a decomposition of $\gamma$ into loops thanks to a random partition of the set of basic $n_{x}$ loops which is identified to $\left\{1, \ldots, n_{x}\right\}$.

Recall that a partition of $\left\{1, \ldots, n_{x}\right\}$ is an unordered collection of disjoint sets, called blocks, such that their union is $\left\{1, \ldots, n_{x}\right\}$. Let $\left\{\mathcal{S}_{1}, \ldots, \mathcal{S}_{k}\right\}$ be a partition of $\left\{1, \ldots, n_{x}\right\}$. Naturally, the block sizes satisfy $\left|\mathcal{S}_{1}\right|+\cdots+\left|\mathcal{S}_{k}\right|=n_{x}$. The blocks can be put in order of appearance: the smallest element of $\mathcal{S}_{1}$ is smaller than the smallest element of $\mathcal{S}_{2}$, etc. Let $\mathcal{S}$ be a random partition with law

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{S}=\left\{\mathcal{S}_{1}, \ldots, \mathcal{S}_{k}\right\}\right)=\frac{\left(\left|\mathcal{S}_{1}\right|-1\right)!\ldots\left(\left|\mathcal{S}_{k}\right|-1\right)!}{n_{x}!} \tag{52}
\end{equation*}
$$

and denote by $|\boldsymbol{\mathcal { S }}|=k$ its number of blocks. ${ }^{10}$ Define the random ordered partition $\mathcal{S}^{\boldsymbol{o}}$ by taking the blocks in order of appearance, and then uniformly permuting the $k$ blocks. Formally, the probability to obtain a given ordered partition is

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{S}^{o}=\left(\mathcal{S}_{1}^{o}, \ldots, \mathcal{S}_{k}^{o}\right)\right)=\frac{1}{k!} \mathbb{P}\left(\mathcal{S}=\left\{\mathcal{S}_{1}^{o}, \ldots, \mathcal{S}_{k}^{o}\right\}\right) \tag{53}
\end{equation*}
$$

At this point, we define the "exchangeable random order" of block sizes in the terminology of [Pitman, 2006, Section 2]. Let $M$ be a random tuple whose entries are the sizes of the blocks of $\mathcal{S}^{\boldsymbol{o}}$. Its law is obtained by summing (53) over all possible ordered partitions, say of given sizes $\left(m_{1}, \ldots, m_{k}\right)$. By inspection of (52), all these partitions have the same probability. Hence, this summation amounts to multiply (53) by a multinomial factor as follows

$$
\begin{equation*}
\mathbb{P}\left(M=\left(m_{1}, \ldots, m_{k}\right)\right)=\frac{n_{x}!}{m_{1}!\ldots m_{k}!} \frac{1}{k!} \frac{\left(m_{1}-1\right)!\ldots\left(m_{k}-1\right)!}{n_{x}!}=\frac{1}{k!} \frac{1}{m_{1} \cdots m_{k}} . \tag{54}
\end{equation*}
$$

That (54) is well-normalized is a consequence of [Lawler and Limic, 2010, Exersise 9.1]. Next, draw M according to (54) and concatenate the based loops $\eta_{1}, \ldots, \eta_{n_{x}}$ in (51) to obtain

$$
\gamma=\overbrace{\delta_{1}}^{m_{1}} \stackrel{\text { visits of } x}{x}
$$

with number of visits of $x$ respectively equal to $d\left(\delta_{1}\right)=m_{1}, \ldots, d\left(\delta_{k}\right)=m_{k}$; e.g., the first loop being $\delta_{1}=\eta_{1} \circ \cdots \circ \eta_{m_{1}}$. The corresponding random function reads randomsplit : $\gamma \mapsto\left(\delta_{1}, \ldots, \delta_{k}\right)$. Considering a nontrivial based loop $\gamma$, (54) gives

$$
\mathbb{P}\left(\text { randomsplit }(\gamma)=\left(\delta_{1}, \ldots, \delta_{k}\right)\right)=\frac{1}{k!} \frac{1}{m_{1} \cdots m_{k}}
$$

For any ordered sequence of based loops $\left(\delta_{1}, \ldots, \delta_{k}\right)$, we also define the function

$$
\text { forgetbase : }\left(\delta_{1}, \ldots, \delta_{k}\right) \mapsto\left(\left[\delta_{1}\right], \ldots,\left[\delta_{k}\right]\right)
$$

which simply maps any list of based loop to the most of their unbased equivalence class.

[^6]Coming back to the measure (50), we are ready to write the law of forgetbase(randomsplit $\left(\Gamma_{1}\right)$ ). Let $\gamma_{1}$ be a loop based at the common base $x_{1}$ of every loop in $\Gamma_{1}$. Using the definition of the unbased loop measures (35) and (37), as well as the random tuple of popped based loops (48), we find

$$
\begin{align*}
& \mathbb{P}\left(\text { forgetbase }\left(\text { randomsplit }\left(\Gamma_{1}\right)\right)=\left(\left[\delta_{1}\right], \ldots,\left[\delta_{k}\right]\right)\right)  \tag{55}\\
&=\mathbb{P}\left(\operatorname{randomsplit}\left(\Gamma_{1}\right) \in \text { forgetbase }^{-1}\left(\left(\left[\delta_{1}\right], \ldots,\left[\delta_{k}\right]\right)\right)\right)  \tag{56}\\
&=\sum_{\delta_{1}^{\prime}, \ldots, \delta_{k}^{\prime}} \frac{e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)}}{k!} \prod_{i=1}^{k} \frac{\mu_{\alpha}\left(\delta_{i}\right)}{d\left(\delta_{i}\right)} 1_{\delta_{i}^{\prime} \in \operatorname{BL}\left(\mathcal{V}, x_{1}\right)} 1_{\left[\delta_{i}^{\prime}\right]=\left[\delta_{i}\right]}  \tag{57}\\
&=\frac{e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)}}{k!} \prod_{i=1}^{k} \#_{\left[\delta_{i}\right], x_{1}} \frac{\mu_{\alpha}\left(\delta_{i}\right)}{d\left(\delta_{i}\right)}  \tag{58}\\
&=\frac{e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)}}{k!} \prod_{i=1}^{k} m_{\alpha}\left(\left[\delta_{i}\right]\right) \tag{59}
\end{align*}
$$

where $\#_{[\gamma], x_{1}}$ is be the number of representatives of $[\gamma]$ based at $x_{1}$; see Lemma 5. Hence, if $\mathcal{X}_{1} \sim$ Poisson $\left(m_{\alpha}, \mathrm{E}_{1}\right)$, the final probability is obtained by multiplying by the number of permutations of $k$ distinct objects,

$$
\begin{align*}
\left.\left.\mathbb{P}\left(\text { forgetorder(forgetbase(randomsplit }\left(\Gamma_{1}\right)\right)\right)=\left\{\left[\delta_{1}\right], \ldots,\left[\delta_{k}\right]\right\}\right) & =m_{\alpha}\left(\left[\delta_{1}\right]\right) \ldots m_{\alpha}\left(\left[\delta_{k}\right]\right) e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)} \\
& =\mathbb{P}\left(\mathcal{X}_{1}=\left\{\left[\delta_{1}\right], \ldots,\left[\delta_{k}\right]\right\}\right) \tag{60}
\end{align*}
$$

Repeating the construction for each factor in the joint distribution (50), we recognize the law of the Poisson process Poisson $\left(m_{\alpha}, \mathrm{L}\right)$.

General multiplicities. In order to derive the previous results, we assumed that the loops in the decomposition (51) where pairwisely distinct. We can relax this assumption. Denote by $\left\{\left[\delta_{1}\right]^{\nu_{1}}, \ldots,\left[\delta_{\ell}\right]^{\nu_{\ell}}\right\}$ a multiset where $\left[\delta_{1}\right], \ldots,\left[\delta_{\ell}\right]$ appear respectively $\nu_{1}, \ldots, \nu_{\ell}$ times with $\nu_{1}+\cdots+\nu_{\ell}=k$. Now, the change in the derivation happens at the "forgetorder" step between (59) and (60), where one has to multiply (59) by the number of ordered sequences yielding the same unordered sequence, i.e., $\frac{k!}{\nu_{1}!\ldots \nu_{\ell}!}$. Thus, the corresponding generalization of (59) reads

$$
\begin{aligned}
\mathbb{P}\left(\text { forgetorder }\left(\text { forgetbase }\left(\text { randomsplit }\left(\Gamma_{1}\right)\right)\right)=\left\{\left[\delta_{1}\right]^{\nu_{1}}, \ldots,\left[\delta_{\ell}\right]^{\nu_{\ell}}\right\}\right) & =\frac{m_{\alpha}\left(\left[\delta_{1}\right]\right)^{\nu_{1}} \ldots m_{\alpha}\left(\left[\delta_{\ell}\right]\right)^{\nu_{\ell}}}{\nu_{1}!\ldots \nu_{\ell}!} e^{-m_{\alpha}\left(\mathrm{E}_{1}\right)} \\
& =\mathbb{P}\left(\mathcal{X}_{1}=\left\{\left[\delta_{1}\right]^{\nu_{1}}, \ldots,\left[\delta_{\ell}\right]^{\nu_{\ell}}\right\}\right)
\end{aligned}
$$

The last equality can be shown as follows. Denote by $\mathcal{N}_{\mathcal{S}}(\mathcal{X})$ the random number of points (loops) of $\mathcal{X} \sim \operatorname{Poisson}\left(m_{\alpha}, \mathrm{L}\right)$ in the set $\mathcal{S} \subseteq$ L. By using the properties of Poisson Point Processes, we have

$$
\begin{aligned}
& \mathbb{P}\left(\mathcal{N}_{\left\{\left[\delta_{1}\right]\right\}}\left(\mathcal{X}_{1}\right)=\nu_{1} \text { and } \ldots \text { and } \mathcal{N}_{\left\{\left[\delta_{\ell}\right]\right\}}\left(\mathcal{X}_{1}\right)=\nu_{k} \text { and } \mathcal{N}_{\mathrm{E}_{1} \backslash\left\{\left[\delta_{1}\right], \ldots,\left[\delta_{\ell}\right]\right\}}\left(\mathcal{X}_{1}\right)=0\right) \\
& =\frac{m_{\alpha}\left(\left[\delta_{1}\right]\right)^{\nu_{1}} e^{-m_{\alpha}\left(\left[\delta_{1}\right]\right)}}{\nu_{1}!} \ldots \frac{m_{\alpha}\left(\left[\delta_{\ell}\right]\right)^{\nu_{k}} e^{-m_{\alpha}\left(\left[\delta_{\ell}\right]\right)}}{\nu_{\ell}!} e^{-m_{\alpha}\left(\mathrm{E}_{1} \backslash\left\{\left[\delta_{1}\right], \ldots,\left[\delta_{\ell}\right]\right\}\right)},
\end{aligned}
$$

which yields the desired result since $\left\{\left[\delta_{1}\right]\right\} \cup \cdots \cup\left\{\left[\delta_{\ell}\right]\right\} \cup \mathrm{E}_{1} \backslash\left\{\left[\delta_{1}\right], \ldots,\left[\delta_{\ell}\right]\right\}=\mathrm{E}_{1}$. This concludes the proof of Proposition 10.

### 5.7 Interpretation of CyclePopping with heaps of cycles

One of our initial motivations was to understand the normalization constant (43) as expressed in somewhat compact form " $\sum_{\mathcal{C}} \mathbb{P}(\operatorname{pop} \mathcal{C})$ " in [Kassel and Kenyon, 2017, p938], where $\mathcal{C}$ denotes all possible collections of cycles which can be popped. In this section, we explain how one can fall back onto the expression of Kassel and Kenyon [2017] using the combinatorial notion of heaps of pieces [Viennot, 2006], which are here simply heaps of oriented cycles [Viennot, Chapter 5b] popped by CyclePopping.

Definition 1 (concurrent cycles). Let $[c]$ and $\left[c^{\prime}\right]$ be two unbased cycles. We say that $[c]$ is concurrent to $\left[c^{\prime}\right]$, namely $[c] \mathcal{R}\left[c^{\prime}\right]$, if $[c]$ and $\left[c^{\prime}\right]$ share at least one node.

The definition of a heap abstracts the definition of $\operatorname{cycles}(\gamma)$ in Notation 2. Loosely speaking, nodedisjoint consecutive cycles commute in a heap. Although, as emphasized by Viennot, one possible definition follows from the construction of the partially commutative monoid of Cartier and Foata [1969], we give below a less formal definition.

Definition 2 (labelled heap). A labelled heap of cycles $H$ is a partially ordered set of unbased cycles endowed with an order index, namely $\left\{\left(\left[c_{1}\right], k_{1}\right), \ldots,\left(\left[c_{h}\right], k_{h}\right)\right\}$ with $k_{1}, \ldots, k_{h}$ positive integers and $\left\{\left[c_{1}\right], \ldots,\left[c_{h}\right]\right\}$ is a multiset of cycles. The set of pieces of H [Krattenthaler, 2006, Definition 2.2] is simply the set of distinct cycles in $\left\{\left[c_{1}\right], \ldots,\left[c_{h}\right]\right\}$. The partial order is defined as follows. If $[c] \mathcal{R}\left[c^{\prime}\right]$ and $k \leq k^{\prime}$, we write $([c], k) \preceq_{0}\left(\left[c^{\prime}\right], k^{\prime}\right)$. This partial order is extended by transitivity for nonconcurrent cycles. First, when $[c] \mathcal{R}\left[c^{\prime}\right]$, then $\preceq$ is given by $\preceq_{0}$. Second, for $[c] \mathcal{R}\left[c^{\prime}\right]$, we write $([c], k) \preceq\left(\left[c^{\prime}\right], k^{\prime}\right)$ if there is a sequence $\left(\left[c_{i}\right], k_{i}\right)$ for $1 \leq i \leq j$ for some integer $j \geq 1$ such that $([c], k) \preceq_{0}\left(\left[c_{1}\right], k_{1}\right) \preceq_{0} \cdots \preceq_{0}\left(\left[c_{j}\right], k_{j}\right) \preceq_{0}\left(\left[c^{\prime}\right], k^{\prime}\right)$. For compactness, we often leave the index of $([c], k)$ implicit and simply write $[c]$.

Also, the empty heap associated with the empty set of pieces is denoted by $\emptyset$. The following definitions formalize the intuition that several labellings correspond to the same heap.

Definition 3 (isomorphism of labelled heaps). Two heaps of cycles $H_{1}=\left\{\left(\left[c_{1}\right], k_{1}\right), \ldots,\left(\left[c_{h}\right], k_{h}\right)\right\}$ and $H_{2}=\left\{\left(\left[c_{1}\right], \ell_{1}\right), \ldots,\left(\left[c_{h}\right], \ell_{h}\right)\right\}$ are isomorphic if they share the same multiset of cycles and for all cycles $[c]$ and $\left[c^{\prime}\right]$ such that $[c] \mathcal{R}\left[c^{\prime}\right]$, we have $k \leq k^{\prime}$ if and only if $\ell \leq \ell^{\prime}$.

Definition 4 (heap). A heap of cycles is an equivalence class of labelled heaps of cycles under isomorphism.
The construction of Viennot also includes a formal definition of putting a heap on top of another heap that we briefly sketch here for conciseness. Given two heaps $H_{1}$ and $H_{2}$, superposition $H_{1} \circ H_{2}$ of $H_{2}$ on $H_{1}$ has for set of pieces the disjoint union of the pieces of $H_{1}$ and $H_{2}$, and the partial order is given by an intuitive composition rule inheriting the partial order of $H_{1}$ and $H_{2}$; see [Krattenthaler, 2006, Definition 2.5] or [Viennot, 2006, Definition 2.7] for a rigorous definition.

Definition 5 (pyramid). The element $[c]$ of a heap is maximal if there is no $\left[c^{\prime}\right]$ such that $[c] \preceq\left[c^{\prime}\right]$. A heap is a pyramid if it contains exactly one maximal piece.

We refer to Figure 2 b for an illustration of a pyramid where $c_{3}$ is the maximal piece. For further use, denote by $\mathcal{P}(\overline{\mathcal{A}}, x)$ the set of pyramids with pieces in the complement of $\mathcal{A}$ and such that node $x$ belongs to the maximal piece. Note that an empty heap is not a pyramid. Hence, it is convenient to define

$$
\begin{equation*}
\mathcal{P}_{\emptyset}(\overline{\mathcal{A}}, x)=\{\emptyset\} \cup \mathcal{P}(\overline{\mathcal{A}}, x), \tag{61}
\end{equation*}
$$

where $\emptyset$ is the empty heap.

### 5.7.1 MGF of $T$ as a ratio of sums over heaps

Define the weight of a heap as $w(H)=\prod_{[c] \in H} w([c])$ where $w([c]) \geq 0$ is a weight function over unbased oriented cycles. Furthermore, we set the weight of the empty heap to be $w(\emptyset)=1$. In what follows, we are going to use e.g. $w([c])=\mu_{\alpha}(c)$ with $c$ any based representative of $[c]$ and where the based loop measure is given in (33). This is licit since the expression of $\mu_{\alpha}(c)$ does not depend on the base point.

Proposition 11 (decomposition of a heap in pyramids). Let $H$ be a heap of cycles of a graph of nodes. Let $x_{1}, \ldots, x_{n}$ be an ordering of the nodes. There exists a unique sequence $\left(P_{x_{1}}, \ldots, P_{x_{n-1}}\right)$, indexed by the nodes, of heaps with disjoint sets of elements - which are either the empty heap or a pyramid - such that $H=P_{x_{1}} \circ \cdots \circ P_{x_{n-1}}$ and such that if $P_{x_{i}}(1 \leq i \leq n-1)$ is a pyramid then $x_{i}$ belongs to its maximal piece.

Proof. The proof simply gives a partition of $H$ as a composition of disjoint pyramids or empty heaps. Consider the first node in the ordering $x_{1}$. If no cycle in the heap contains $x_{1}$, then $P_{x_{1}}=\emptyset$. Otherwise, denote the maximum element in the heap containing $x_{1}$ by $\left[c\left(x_{1}\right)\right]$, which is necessarily unique since two
elements of the heap containing $x_{1}$ are always related by the partial order " $\preceq$ ", i.e., cycles containing $x_{1}$ are concurrent; see Definition 1. There exists a pyramid in the heap $H$ with $\left[c\left(x_{1}\right)\right]$ as its maximal element, which includes all $\left[c^{\prime}\right]$ so that $\left[c^{\prime}\right] \preceq\left[c\left(x_{1}\right)\right]$. Denote by $P_{x_{1}}$ this pyramid. By design, any piece containing $x_{1}$ should be in this pyramid, since this cycle is related by the partial order to $\left[c\left(x_{1}\right)\right]$. Now, in the wording of Krattenthaler [2006, proof of Thm 4.4], we push $P_{x_{1}}$ downwards, namely, we subtract it from $H$ to yield $H_{1}$. Next, consider the second node $x_{2}$ in the ordering. If no cycle of the heap contains $x_{2}$, then $P_{x_{2}}$ is the empty heap. Otherwise, let $\left[c\left(x_{2}\right)\right]$ be the unique maximum piece containing $x_{2}$ in $H_{1}$. Let $P_{x_{2}}$ be the pyramid with $\left[c\left(x_{2}\right)\right]$ as maximum element and containing all pieces in $H_{2}$ dominated by $\left[c\left(x_{2}\right)\right]$. By construction, this pyramid does not contain $x_{1}$. Going on over the nodes in the ordering, we build the desired ordered sequence of pyramids.

There is a connection between the cycles popped by CyclePopping and heaps of cycles à la Viennot, which was presumably already summarized in an oral presentation of Marchal [2001] in the case of Wilson's algorithm, i.e., when all the cycles are popped with probability 1. To our knowledge, a similar connection between pyramids and loops was pointed out by Giscard [2021] and Helmuth [2016, Appendix A]. An illustration is given in Figure 4.
Definition 6 (chronological labelling). Consider a loop based at x. A labelled heap with chronological labelling is defined as follows. The oriented loop is visited by starting at $x$ along the oriented edges. The first encountered (based) cycle $c$ defines element $([c], 1)$, i.e., the first element of the heap. Now, let $([\tilde{c}], \ell)$ be the last added element to the growing heap. Let $c^{\prime}$ be the next cycle in the visit of the loop. If $c^{\prime}$ has no common node with any cycle of the heap, then we add the element $\left(\left[c^{\prime}\right], 1\right)$ to the heap. Otherwise, let $\left(\left[c_{0}\right], \ell_{0}\right)$ be the maximum element with a common node with $c^{\prime}$. Define the index of $c^{\prime}$ to be $\ell^{\prime}=\ell_{0}+1$. Put $\left(\left[c^{\prime}\right], \ell^{\prime}\right)$ into the heap. The construction terminates when the loop is entirely visited.
Lemma 6 (Pyramid and based loops). Let $\gamma$ be a nontrivial loop based at $x$.
(i) The cycles determined by the loop erasure defined in Notation 2 form a labelled pyramid where the maximal element contains $x$ and for which the labelling is given by the chronological order of popping in Definition 6.
(ii) A pyramid together with a node $x$ in its maximal element determines a loop based at $x$.

Proof. Part ( $i$ ) of the lemma is trivial in the light of Definition 6. For (ii), we construct the based loop by adding edges iteratively to a growing path. We start this path from $x$ and follow the oriented edge emanating from $x$ in a minimal cycle containing $x$ (corresponding to the first edge visited by the loop). This cycle is unique up to multiplicity since any two concurrent cycles are ordered. The orientation of this cycle determines the edge $x x_{1}$ which is added to the path; see Figure 4 b for an illustration. We go on with $x_{1}$, and we consider the minimal cycle in the pyramid which has not yet been completely visited. The orientation of this cycle determines the edge $x_{1} x_{2}$. This procedure will eventually visit all the cycles since all concurrent cycles are ordered. The maximal cycle started at $x$ will also be completely visited at last.

Let $\mathcal{A}$ be a proper subset of the nodes and let $x \in \overline{\mathcal{A}}$. We define the weight of an unbased cycle as

$$
w_{t, \alpha}([c])=t^{|c|} \mu_{\alpha}(c),
$$

for $t \in(0,1]$ and with $c$ any representative of $[c]$. Similarly the weight of a heap reads $w_{t, \alpha}(H)=$ $\prod_{[c] \in H} w_{t, \alpha}([c])$. Thus, by using Lemma 6, the Green generating function of Lemma 3 is rewritten as a sum over pyramids

$$
G_{\alpha}(t, x, x ; \mathcal{A})=\sum_{P \in \mathcal{P}_{\emptyset}(\overline{\mathcal{A}}, x)} w_{t, \alpha}(P),
$$

where the sum is over the empty heap (corresponding to the trivial loop at $x$ with $\mu_{\alpha}$-measure 1 ) and any pyramid with pieces in the complement of $\mathcal{A}$ and $x$ in its maximal piece; see (61). Note that this series converges by construction. Now, let $x_{0}, \ldots, x_{n-1}$ be an ordering of the nodes and define $\mathcal{B}_{0}=\emptyset, \mathcal{B}_{1}=\left\{x_{0}\right\}$, $\ldots, \mathcal{B}_{n-1}=\left\{x_{0}, x_{1}, \ldots, x_{n-2}\right\}$. We calculate the product of Green generating functions as in (42)

$$
\begin{equation*}
\prod_{\ell=0}^{n-1} G_{\alpha}\left(t, x_{\ell}, x_{\ell} ; \mathcal{B}_{\ell}\right)=\sum_{P_{0} \in \mathcal{P}_{\emptyset}\left(\overline{\mathcal{B}_{0}}, x_{0}\right)} \ldots \sum_{P_{n-1} \in \mathcal{P}_{\emptyset}\left(\overline{\mathcal{B}_{n-1}}, x_{n-1}\right)} w_{t, \alpha}\left(P_{0} \circ \cdots \circ P_{n-1}\right) \tag{62}
\end{equation*}
$$



Figure 4: Loop $\gamma$ based at $x$ (Left-hand side) and the corresponding pyramid (Right-hand side) with $x$ contained in the maximal element. Following the oriented edges of the loops, the orange backtrack $c_{1}$ is popped first and the green backtrack $c_{2}$ is popped second, whereas the magenta cycle $c_{3}$ is popped third. The labelled heap given by the chronological order is $\left\{\left(c_{3}, 2\right),\left(c_{2}, 1\right),\left(c_{1}, 1\right)\right\}$. In this heap, the two consecutive backtracks are not concurrent and, thus, $\left(c_{2}, 1\right)$ cannot be compared to $\left(c_{1}, 1\right)$. Nonetheless, the maximum element $\left(c_{3}, 2\right)$ is such that $\left(c_{3}, 2\right) \succ\left(c_{1}, 1\right)$ and $\left(c_{3}, 2\right) \succ\left(c_{2}, 1\right)$. An isomorphic labelled heap is $\left\{\left(c_{3}, 4\right),\left(c_{2}, 1\right),\left(c_{1}, 1\right)\right\}$; see Definition 3.

By Proposition 11, for a fixed node ordering, any heap can be written as a unique superposition of pyramids and empty heaps $P_{0} \circ \cdots \circ P_{n-1}$. Hence, the sum at the right-hand side of (62) has exactly one term for each possible heap. Therefore, we conclude that

$$
\begin{equation*}
\prod_{\ell=0}^{n-1} G_{\alpha}\left(t, x_{\ell}, x_{\ell} ; \mathcal{B}_{\ell}\right)=\sum_{\text {heap } H} w_{t, \alpha}(H) \tag{63}
\end{equation*}
$$

is independent of the chosen ordering. By taking $t=1$ in (63), we recover the expression of the normalization $Z_{\alpha}^{-1}=\sum_{\text {heap } H} w_{1, \alpha}(H)$ which was written " $\sum_{\mathcal{C}} \mathbb{P}(\operatorname{pop} \mathcal{C})$ " in [Kassel and Kenyon, 2017, p938]. Now we use the same strategy as for the derivation of (44). The moment generating function (MGF) of the sampling time $T$ is then given by

$$
\begin{equation*}
\mathbb{E}\left[t^{T}\right]=t^{n} \frac{\sum_{\text {heap } H} w_{t, \alpha}(H)}{\sum_{\text {heap } H} w_{1, \alpha}(H)} \tag{64}
\end{equation*}
$$

### 5.7.2 MGF of $T$ as a ratio of determinants

At this point, we need to define trivial heaps in order to write the expression of (64) in a form analogous to a ratio of determinants.

Definition 7 (Trivial heap of cycles). A trivial heap of cycles is a heap whose pieces are all unbased oriented cycles which are not concurrent.

In other words, a trivial heap of cycles $\mathcal{C}$ is a set of non-concurrent cycles, and the cardinality of this set is necessarily finite. Note that the empty heap is a trivial heap. Now, we leverage a key result relating the generating function of heaps of pieces to trivial heaps, namely Corollary 4.5 in [Krattenthaler, 2006], Lemma 5 in [Jerrum, 2021] or Remark 9 in [Fredes and Marckert, 2023],

$$
\begin{equation*}
\sum_{\text {heap } H} w_{t, \alpha}(H)=\left(\sum_{\substack{\text { trivial heap } \\ \mathcal{C}}}(-1)^{|\mathcal{C}|} w_{t, \alpha}(\mathcal{C})\right)^{-1} \tag{65}
\end{equation*}
$$

where $|H|$ denotes the total number of pieces of $H$. The reader can find an example of derivation of the identity (65) in the case of heaps with a set of pieces of cardinality 4 at page 6 of [Jerrum, 2021]. By a direct substitution of (65) into (64), we have

$$
\begin{equation*}
\mathbb{E}\left[t^{T}\right]=t^{n} \frac{\sum_{\mathcal{C}}(-1)^{|\mathcal{C}|} \prod_{[c] \in \mathcal{C}} \mu_{\alpha}(c)}{\sum_{\mathcal{C}}(-1)^{|\mathcal{C}|} \prod_{[c] \in \mathcal{C}} t^{|c|} \mu_{\alpha}(c)}, \tag{66}
\end{equation*}
$$

where $|c|$ denotes here the number of edges in the cycle $c$. Noticeably, when $\mu_{\alpha}$ corresponds to the determinantal case, the expression (66) reduces to the ratio of determinants (31), as a consequence of the expansion of the determinant over permutations and the factorization of permutations over cycles.

### 5.7.3 MGF of $T$ as a Poisson process of pyramids

In order to draw a parallel with the Poisson point process of loops of Section 5.4, we rephrase (66) thanks to a well-known combinatorial result of Viennot [2006, proposition 5.10]

$$
\log \sum_{\text {heap } H} w(H)=\sum_{P \text { pyramid }} \frac{1}{|P|} w(P)
$$

for any weight $w(H)=\prod_{c \in H} w(c)$ such that $w(c) \geq 0$, and where $|P|$ is the number of elements of the pyramid $P$. Consequently, (64) reads

$$
\begin{equation*}
\mathbb{E}\left[t^{T}\right]=t^{n} \exp \left(\sum_{P \text { pyramid }}\left(t^{n_{e}(P)}-1\right) \frac{w_{1, \alpha}(P)}{|P|}\right) \tag{67}
\end{equation*}
$$

with $n_{e}(P)=\sum_{c \in P}|c|$ being the number of edges in the pyramid of cycles. The same reasoning as in Section 5.4 yields

$$
T \stackrel{\text { law) }}{=} n+\sum_{P \in \mathcal{P}} n_{e}(P)
$$

where $\mathcal{P}$ is a Poisson process over pyramids of cycles with intensity measure $w_{1, \alpha}(P) /|P|$.

## 6 Numerical simulations

The mean and variance of the number of steps to complete CyclePopping can simply be estimated in a simple $U(1)$-connection graph. We define here the following random graph. The Erdős-Rényi unicycle model, denoted by $\mathrm{ER}_{\mathrm{u}}(n, p, \eta)$, is a $\mathrm{U}(1)$-connection graph of $n$ nodes where there is an edge $e=u v$ with probability $p$, independently from other edges, and where only one noisy edge sampled uniformly comes an angle $\vartheta=\eta \pi / 2$ whereas all the other edges are endowed with a vanishing angle. Hence, for $0<\eta \leq 1$, all cycles containing this noisy edge will have an holonomy satisfying $\cos \theta(c) \geq 0$, whereas all cycles not containing the noisy edge have a unit holonomy. Assumption 1 and Assumption 3 are then guaranteed by construction. Also, any CRSF sample from (10) in this random graph will be connected with probability one.

Thus, we sample a $\mathrm{ER}_{\mathrm{u}}(n, p, \eta)$ graph with $n=100, p=0.8$ and for each $\eta \in\{0.5,0.6, \ldots, 1\}$. For each random graph, we sample 1000 CRSFs thanks to CyclePopping. ${ }^{11}$ In Figure 5a, we observe that the empirical estimates of $\mathbb{E}[T]$ and $\operatorname{var}[T]$ given in Proposition 6, are good approximations. Notice that we represent the standard deviation as an error bar in Figure 5a. The mean and variance of $T$ naturally decrease as the noise parameter increases since a large $\eta$ intuitively promotes a large value of the least eigenvalue of $\Delta$.

The same simulation is repeated by sampling 1000 MTSFs with CyclePopping $_{q}$ for $q=5 \cdot 10^{-3}$ and the comparison of $\mathbb{E}[T]$ and $\operatorname{var}[T]$ with their empirical estimates is given in Figure 5 b.

## 7 Conclusion

In this paper, we gave a proof of the correctness of CyclePopping for connection graphs, using only elementary random walk arguments, as Marchal [1999] did for Wilson's original algorithm. From a computer scientist's point of view, on top of being easy to follow and adapt to more sophisticated variants of the algorithm, the proof yields the distribution of the running time of the algorithm. From a more probabilistic point of view, the construction sheds light on other point processes built while running the algorithm, and in

[^7]

Figure 5: Number of steps $(T)$ to complete CyclePopping $_{q}$ for sampling CRSFs (left, $q=0$ ) and MTSFs (right, $q=5 \cdot 10^{-3}$ ) w.r.t. (32) in an $\mathrm{ER}_{\mathrm{u}}(n, p, \eta)$ random $\mathrm{U}(1)$-connection graph with $n=100, p=0.8$ as a function of the noise parameter $\eta$. We report $\mathbb{E}[T]$ and $\sqrt{\operatorname{var}[T]}$ (error bar) in red whereas their estimators over 1000 Monte Carlo runs are displayed in blue. The red and blue curves almost overlap.
particular an intriguing coupling already discussed by Le Jan [2011] between cycle-rooted spanning forests and a Poisson point process of popped loops, and a related Poisson point process on Viennot pyramids.

Acknowledgements. We acknowledge support from ERC grant BLACKJACK (ERC-2019-STG-851866) and ANR AI chair BACCARAT (ANR-20-CHIA-0002). MF thanks Simon Barthelmé for instructive discussions and Adrien Kassel for his comments on a preliminary version of the paper as well as drawing our attention to the reference [Pitman and Tang, 2018]. The authors also thank Martin Rouault for commenting on an early version of this manuscript.

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[^0]:    ${ }^{1}$ It turns out that the normalization factor $\operatorname{det} \Lambda_{\bar{r}}$ does not depend on $r$. This is a consequence of the fact that the rows of $\Lambda$ sum to zero.

[^1]:    ${ }^{2}$ When completing this work, we realized that the description of a similar Markov process was given in the PhD thesis of Constantin [2023, Chapter 5], who studied limits of determinantal CRSF measures.
    ${ }^{3}$ In this paper, we write for simplicity $a \wedge b=\min \{a, b\}$.

[^2]:    ${ }^{4}$ For measures over loops in graphs and loop soups, we refer to [Sznitman, 2012; Le Jan, 2011; Lawler and Werner, 2004; Symanzik, 1969].
    ${ }^{5}$ A multiset is a set endowed with a function endoding element multiplicity.
    ${ }^{6}$ Combinatorists might prefer defining a heap [Krattenthaler, 2006], as shown in Figure 2b, but we do not need the additional sophistication for the moment.

[^3]:    ${ }^{7}$ Note that CyclePopping is simply CyclePopping ${ }_{0}$.

[^4]:    ${ }^{8}$ We note that a result analogous to (45) is obtained in [Le Jan, 2011, Eq 4.1] for continuous loop measures, for which each node also comes with a continuous holding time.

[^5]:    ${ }^{9}$ In reference to physics, Le Jan [2011, Section 8] describes this coupling as a supersymmetry.

[^6]:    ${ }^{10}$ The reader may recognize the law on partitions that arises in the so-called Chinese restaurant process.

[^7]:    ${ }^{11}$ https://github.com/For-a-few-DPPs-more/MagneticLaplacianSparsifier.jl/tree/counting_steps

