SHARP ILL-POSEDNESS FOR THE NON-RESISTIVE MHD EQUATIONS IN SOBOLEV SPACES

QIONGLEI CHEN, YAO NIE, WEIKUI YE

Abstract. In this paper, we prove a sharp ill-posedness result for the incompressible non-resistive MHD equations. In any dimension $d \geq 2$, we show the ill-posedness of the non-resistive MHD equations in $H^{\frac{d}{2}-1}(\mathbb{R}^d)\times H^{\frac{d}{2}}(\mathbb{R}^d)$, which is sharp in view of the results of the local well-posedness in $H^{s-1}(\mathbb{R}^d)\times H^{s}(\mathbb{R}^d)(s>\frac{d}{2})$ established by Fefferman et al.(Arch. Ration. Mech. Anal., **223** (2), 677-691, 2017). Furthermore, we generalize the ill-posedness results from $H^{\frac{d}{2}-1}(\mathbb{R}^d)\times H^{\frac{d}{2}}(\mathbb{R}^d)$ to Besov spaces $B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d)\times B^{\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ and $\dot{B}^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d)\times \dot{B}^{\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ for $1\leq p\leq\infty, q>1$. Different from the ill-posedness mechanism of the incompressible Navier-Stokes equations in $\dot{B}^{-1}_{\infty,q}$ [3, 22], we construct an initial data such that the paraproduct terms (low-high frequency interaction) of the nonlinear term make the main contribution to the norm inflation of the magnetic field.

1. Introduction

Magneto-hydrodynamics (MHD) is concerned with the study of the mutual interaction between magnetic fields and electrically conducting fluids. In this paper, we investigate the Cauchy problem for the incompressible non-resistive MHD equations in $\mathbb{R}^d$ for $d \geq 2$:

$$\begin{cases} \partial_t u - \Delta u + \nabla P = b\cdot\nabla b - u\cdot\nabla u, \\ \partial_t b + u\cdot\nabla b = b\cdot\nabla u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (u,b)|_{t=0} = (u_0,b_0). \end{cases} \tag{1.1}$$

Here the initial data (u_0,b_0) is divergence-free, $u=(u^1,u^2,\cdots,u^d)$ represents the velocity field, $b=(b^1,b^2,\cdots,b^d)$ denotes the magnetic field and P is the scalar pressure. The non-resistive MHD system (1.1) can be applied to describe strong collisional plasmas, or plasmas with extremely small resistivity due to these collisions [4, 18].

Compared with the viscous and resistive MHD system, the study of well-posedness of the non-resistive MHD system (1.1) becomes much more difficult owning to the hyperbolic type of magnetic equation. In recent years, there has been some significant progress in the global well-posedness of the system (1.1) with smooth initial data that is close to some nontrivial steady state (see [1, 19, 21, 23, 26] and the references therein). For the

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local-in-time existence of solutions to the non-resistive MHD system (1.1), Jiu and Niu [14] firstly showed the local well-posedness result in $H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ with $s\geq 3$ via viscous approximations. By means of a new commutator estimate, Fefferman et al. [11] proved the local-in-time existence and uniqueness of strong solutions in $H^s(\mathbb{R}^d)\times H^s(\mathbb{R}^d)$ for $s>\frac{d}{2}, d=2,3$. Later, relying on maximal regularity estimates for the Stokes equation, the authors in [12] established the local-in-time existence and uniqueness of solutions in $H^{s-1+\varepsilon}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)$ for $s>\frac{d}{2}, 0<\varepsilon<1, d=2,3$. Indeed, via time-space mixed Besov spaces $L^1_T(B^{\frac{d}{2}+1}_{2,1})$, one can generalize well-posedness result to $H^{s-1}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)$ for $s>\frac{d}{2}$. With respect to Besov spaces, Chemin et al. [8] made generalisation of the main result in [11] and obtained the local existence with initial data $(u_0,b_0)\in B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\times B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$, $d=2,3$. Meanwhile, the authors in [8] proposed an interesting problem whether or not the solution for the Cauchy problem of the system (1.1) exists locally in time and is unique in corresponding homogeneous Besov spaces. Subsequently, Li, Tan and Yin [17] solved this problem by showing the local existence and uniqueness of the solution in $\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)\times\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ for $1\leq p\leq 2d$ and $d\geq 2$. Ye, Luo and Yin [24] generalized the local existence result in $\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)\times\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ from $1\leq p\leq 2d$ to $1\leq p<\infty$ and proved that the solution map from the initial data (u_0,b_0) to solution (u,b) is continuous from $\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1}$ to $C([0,T];\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1})$ for $d\geq 2, 1\leq p\leq 2d$, which combined with the result in [17] shows the local well-posedness of the system (1.1) in $\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1}$ if $1\leq p\leq 2d$ for $d\geq 2$. Unfortunately, whether the system (1.1) is well-posed or not in $\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1}(2d<p<\infty)$ is still open.

When $b=0$, the system (1.1) is reduced to the classical incompressible Navier-Stokes equations. Well-posedness issues of the Navier-Stokes equations in different types of the critical spaces have attracted attention of many researchers and there have been many relevant results until now (e.g. [3, 5, 7, 13, 15, 16, 20, 22, 25]). In the context of the critical Besov spaces $\dot{B}^{\frac{d}{p}-1}_{p,q}(p<\infty, q\leq\infty)$, the well-posedness of global strong solution with small initial data was established by Planchon [20], Cannone [5] and Chemin [7]. By showing the solution map is discontinuous at origin, Bourgain-Pavlović [3], Yoneda [25] and Wang [22] verified the ill-posedness of the Navier-Stokes equations in $\dot{B}^{-1}_{\infty,q}(1\leq q\leq\infty)$. These results imply the ill-posedness of the non-resistive MHD system (1.1) in $\dot{B}^{-1}_{\infty,q}\times\dot{B}^{0}_{\infty,q}$ for $q\geq 1$.

Throughout the current literature, Fefferman et al. in [11] established the local-in-time existence and uniqueness of strong solutions in $H^s(\mathbb{R}^d)$ for $s>\frac{d}{2}$ to the system (1.1) and suspected that it seems ill-posed in $H^{\frac{d}{2}}(\mathbb{R}^d)$. Subsequently, they in [12] proved that this system is local well-posedness in $H^{s+\epsilon-1}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)$ for $s>\frac{d}{2}$ and concluded that this result is nearly optimal in the scale of Sobolev spaces. In view of their interest on the sharp well-posedness result of this system in Sobolev spaces, we are focused on the problem *whether this system is well-posed or not in* $H^{\frac{d}{2}-1}(\mathbb{R}^d)\times H^{\frac{d}{2}}(\mathbb{R}^d)$. Furthermore, the problem whether the non-resistive MHD system (1.1) is well-posed or not in $\dot{B}^{\frac{d}{p}-1}_{p,q}\times\dot{B}^{\frac{d}{p}}_{p,q}$ if $1\le p<\infty, q>1$ remains unsolved. Therefore, we are interested in the following question:

Question:

Is the system (1.1) *well-posed or ill-posed in* $H^{\frac{d}{2}-1}(\mathbb{R}^d)\times H^{\frac{d}{2}}(\mathbb{R}^d)$ *and* $\dot{B}^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d)\times\dot{B}^{\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ *for* $1\le p<\infty, q>1$*?*

In the important work [3] of Bourgain-Pavlović, they showed that the worst contribution on regularity of solution to the Navier-Stokes equations in a short time stems from the remainder terms (high-high frequency interaction), which is a part of the Bony decomposition of the convection term $u\cdot\nabla u$. This core idea that remainder terms in the convection term prevent from the continuity of solution mapping has also been applied to other ill-posedness results of Navier-Stokes equations [22, 25].

Different from the mechanism of the Navier-Stokes equations in Besov spaces, it is the *paraproduct terms* (low-high frequency interaction) not the remainder terms of the nonlinear term $b\cdot\nabla u$ of the system (1.1) that may lead to the discontinuous solution map of the magnetic field b in the context of $\dot{B}^{\frac{d}{p}}_{p,q}$. This observation forces us to construct an example to saturate the paraproduct operator from $\dot{B}^{\frac{d}{p}}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,q}$ to $\dot{B}^{\frac{d}{p}}_{p,q}$ for $q>1$. More precisely, the key ingredient is to construct two Schwartz functions f,g such that

$$\big\|\mathscr{F}^{-1}\big(\widehat{f}_{2^{\frac{N}{2}}\le|\xi|\le 2^{\frac{4N}{5}}}\big)\mathscr{F}^{-1}\big(\widehat{g}_{2^{N-1}\le|\xi|\le 2^{N+1}}\big)\big\|_{\dot{B}^{\frac{d}{p}}_{p,q}}\sim\|f\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|g\|_{\dot{B}^{\frac{d}{p}}_{p,q}}.$$

Furthermore, f,g satisfy the following estimates: for $q>1$ and $\frac{1}{q}<\alpha<1$,

$$\|f\|_{\dot{B}^{\frac{d}{p}}_{p,q}}+\|g\|_{\dot{B}^{\frac{d}{p}}_{p,q}}\le(\ln\ln N)^{-1},\ \ \|f\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\sim N^{1-\alpha},\ \ \|f\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|g\|_{\dot{B}^{\frac{d}{p}}_{p,q}}\sim(\ln\ln N)^{-1}N^{1-\alpha}.$$

Based on such scalar functions f and g, we construct a special initial data b_0 whose frequency is supported in a family of cuboids in different dyadic annuli. Distinct from the structure of b_0, the frequency of u_0 is higher and locates in one cuboid with different direction from that of b_0. By using the Lagrangian coordinates in the equation of the magnetic field and the asymmetric structure between u_0 and b_0, we can show that the frequency of the second approximation concentrates on the superposition of many cuboids and thereby the norm inflation phenomenon occurs for a short time.

Firstly, let us recall a the local-in-time existence and uniqueness of strong solutions in H^s for $s>\frac{d}{2}$ to the non-resistive MHD equations (1.1).

Theorem 1.1 ([12]). *Let $d \geq 2$. Take $s > \frac{d}{2}$ and $0 < \epsilon < 1$. Suppose that the initial conditions satisfy $u_0 \in H^{s-1+\epsilon}(\mathbb{R}^d)$ and $b_0 \in H^s(\mathbb{R}^d)$. Then there exists $T_* > 0$ such that the non-resistive MHD system (1.1) has a unique solution (u, b) with $b \in C([0, T_*); H^s(\mathbb{R}^d))$ and*

$$u \in C([0, T_*); H^{s-1+\epsilon}(\mathbb{R}^d)) \cap L^2([0, T_*); H^{s+\epsilon}(\mathbb{R}^d)) \cap L^1([0, T_*); H^{s+1}(\mathbb{R}^d)).$$

Remark 1.2. *By using the time-space mixed Besov spaces $\mathcal{L}^1_T(B^{\frac{d}{2}+1}_{2,1})$, one can show that the system (1.1) is locally well-posed in $H^{s-1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$. That is, let $d \geq 2$ and $s > \frac{d}{2}$, if initial data $(u_0, b_0) \in H^{s-1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$, there exists $T_* > 0$ such that the system (1.1) has a unique solution (u, b) with $u \in C([0, T_*); H^{s-1}(\mathbb{R}^d)) \cap \mathcal{L}^1([0, T_*); B^{s+1}_{2,2})$ and $b \in C([0, T_*); H^s(\mathbb{R}^d))$.*

This theorem guarantees that for any initial data $(u_0, b_0) \in \mathcal{S}(\mathbb{R}^d)$, there locally exists a unique solution (u, b) to the non-resistive MHD equations (1.1).

Next, we are in position to state our ill-posedness result by constructing special Schwarz functions as initial data.

Theorem 1.3. *Let $d \geq 2$. The system (1.1) is ill-posed in $H^{\frac{d}{2}-1}(\mathbb{R}^d) \times H^{\frac{d}{2}}(\mathbb{R}^d)$ in the sense of "norm inflation". More precisely, for any $\delta > 0$, there exists a solution (u, b) to the non-resistive MHD equations (1.1) such that initial data $(u_0, b_0) \in \mathcal{S}(\mathbb{R}^d)$ satisfies*

$$\|u_0\|_{H^{\frac{d}{2}-1}} + \|b_0\|_{H^{\frac{d}{2}}} \leq \delta,$$

and for some $0 < t < \delta$,

$$\|b(t, \cdot)\|_{H^{\frac{d}{2}}} > \frac{1}{\delta}.$$

Remark 1.4. *Note that the system (1.1) is locally well-posed in $H^{s-1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$, Theorem 1.3 shows the sharp ill-posedness for (1.1) in $H^{\frac{d}{2}-1}(\mathbb{R}^d) \times H^{\frac{d}{2}}(\mathbb{R}^d)$. As it is known, the Navier-Stokes equations are locally well-posed in the critical Sobolev space $H^{\frac{d}{2}-1}(\mathbb{R}^d)$. On the other hand, Theorem 1.3 shows that the non-resistive MHD equations is ill-posed under the framework of the critical Sobolev space $H^{\frac{d}{2}-1}(\mathbb{R}^d) \times H^{\frac{d}{2}}(\mathbb{R}^d)$. Interestingly, in our example, the "norm inflation" happens to the magnetic field not the flow field (see(3.36))*, which reflects that the velocity field plays a more important role than the magnetic field in the interaction between the two fields of the non-resistive MHD system.

We are indeed able to prove a stronger statement than Theorem 1.3. More precisely, we can show the following main result:

Theorem 1.5 (Main result). *Let $d \geq 2$, $1 \leq p \leq \infty$ and $q > 1$. For any $\delta > 0$, there exists a solution (u, b) to the non-resistive MHD equations (1.1) such that initial* *data* $(u_0, b_0) \in \mathcal{S}(\mathbb{R}^d)$ *satisfies that the Fourier transforms of* (u_0, b_0) *are supported on an annulus and*

$$\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,q}} + \|b_0\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \le \delta,$$

and for some $0 < t < \delta$*,*

$$\|b(t,\cdot)\|_{\dot{B}^{\frac{d}{p}}_{p,q}} > \frac{1}{\delta}.$$

Because the Fourier transforms of (u_0, b_0) in Theorem 1.5 are supported on an annulus, one obtains that $\|u_0\|_{H^{\frac{d}{2}-1}} \approx \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}}$ and $\|b_0\|_{H^{\frac{d}{2}}} \approx \|b_0\|_{\dot{B}^{\frac{d}{2}}_{2,2}}$. Taking advantage of $H^{\frac{d}{2}} \hookrightarrow \dot{H}^{\frac{d}{2}}$, we immediately conclude Theorem 1.3 by Theorem 1.5. Moreover, one can immediately show the ill-posedness in the corresponding nonhomogeneous Besov spaces by the proof of Theorem 1.5.

Corollary 1.6. *The system* (1.1) *is ill-posed in* $B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d) \times B^{\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ *for* $1 \le p \le \infty, q > 1$*.*

Remark 1.7. *Recall that the Navier-Stokes equations are locally well-posed in* $\dot{B}^{\frac{d}{p}-1}_{p,q}$ $(p < \infty, q < \infty)$ *for large initial data and whether the Navier-Stokes equations are well-posed or not for large initial data in* $\dot{B}^{\frac{d}{p}-1}_{p,\infty}$ $(p < \infty)$ *remains open. Different from the Navier-Stokes equations, Theorem 1.5 shows that the non-resistive MHD system* (1.1) *is ill-posed in* $\dot{B}^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d) \times \dot{B}^{\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ *with* $1 \le p \le \infty, q > 1$*.*

Below we list the local well-posedness/ill-posdeness results of the non-resistive MHD system in the homogeneous Besov spaces $\dot{B}^{\frac{d}{p}-1}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$.

Local well-posedness/ill-posedness of the system (1.1) in $\dot{B}^{\frac{d}{p}-1}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$		
Results	Range	Category
[17, 24]	$q = 1,\ 1 \le p \le 2d$	Local well-posedness
[24]	$q = 1,\ 2d < p < \infty$	Local existence
[3, 22, 25]	$q \ge 1,\ p = \infty$	Ill-posedness
Theorem 1.5	$q > 1,\ 1 \le p \le \infty$	Ill-posedness

As is shown in the table, our result completes the well-posedness and ill-posedness of the non-resistive MHD equations in critical homogeneous Besov spaces except for the case $2d < p < \infty, q = 1$.

2. Preliminaries

To begin with, we review briefly the so-called Littlewood-Paley decomposition theory introduced e.g., in [2, 6]. Suppose (χ, φ) be a couple of smooth functions with values in $[0,1]$, where $\operatorname{supp}\chi \subset \left\{\xi \in \mathbb{R}^d \big| |\xi| \leq \frac{4}{3}\right\}$ and $\operatorname{supp}\varphi \subset \left\{\xi \in \mathbb{R}^d \big| \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$. Moreover, we assume that φ satisfies

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1,\ \forall \xi \in \mathbb{R}^d \backslash \{0\}, \qquad \text{where } \varphi_j(\xi) := \varphi(2^{-j}\xi).$$

Let us define the homogeneous localization operators as follows.

$$\dot{\Delta}_j u = \varphi_j(D)u = 2^{dj}\int_{\mathbb{R}^3} g(2^j y)u(x-y)\,\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$
$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{dj}\int_{\mathbb{R}^3} h(2^j y)u(x-y)\,\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$

where $g = \mathscr{F}^{-1}\varphi$ and $h = \mathscr{F}^{-1}\chi$. The nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j u = 0,\ \text{if}\ j \leq -2, \qquad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^3} h(y)u(x-y)\,\mathrm{d}y,$$
$$\Delta_j u = \varphi_j(D)u = 2^{dj}\int_{\mathbb{R}^3} g(2^j y)u(x-y)\,\mathrm{d}y, \qquad \forall j \geq 0.$$

Definition 2.1 ([2] Homogeneous Besov spaces)**.** *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, The homogeneous Besov space $\dot{B}^s_{p,q}$ consists of all tempered distributions $u \in \mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,q}} \overset{def}{=} \left\|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Definition 2.2 ([2] Nonhomogeneous Besov spaces)**.** *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The nonhomogeneous Besov space $B^s_{p,q}$ consists of all tempered distributions u such that*

$$\|u\|_{B^s_{p,q}} \overset{def}{=} \left\|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} < \infty.$$

In the context of this paper, we often use the following mixed type time-spatial space.

Definition 2.3. *Let $T > 0$, $s \in \mathbb{R}$ and $(p,q) \in [1,\infty]^2$. The mixed time-spatial Besov space $\mathcal{L}^r_T\dot{B}^s_{p,q}$ consists of all $u \in \mathcal{S}'_h$ such that*

$$\|u\|_{\mathcal{L}^r_T\dot{B}^s_{p,q}} \overset{def}{=} \left\|(2^{js}\|\dot{\Delta}_j u\|_{L^r_T L^p})_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Lemma 2.4 ([2])**.** *Let $1 \leq p \leq p_1 \leq \infty$ and $s \in (-d\min\{\frac{1}{p_1}, 1-\frac{1}{p}\}, 1+\frac{d}{p_1}]$. Let v be a vector field such that $\nabla v \in L^1_T(\dot{B}^{\frac{d}{p_1}}_{p_1,1}(\mathbb{R}^d))$. There exists a constant C depending on p, s, p_1 such that all solutions $f \in \mathcal{L}^\infty_T(\dot{B}^s_{p,1}(\mathbb{R}^d))$ of the transport equation*

$$\partial_t f + v\cdot\nabla f = g, \quad f(0,x) = f_0(x).$$

with initial data $f_0 \in \dot B^s_{p,1}(\mathbb{R}^d)$ *and* $g \in L^1_{loc}(\mathbb{R}^+; \dot B^s_{p,1}(\mathbb{R}^d))$*, we have, for* $t \in [0,T]$*,*

$$\|f\|_{\mathcal{L}^\infty_T(\dot B^s_{p,1})} \le e^{CV_{p_1}(t)}\Big(\|f_0\|_{\dot B^s_{p,1}} + \int_0^t e^{-CV_{p_1}(\tau)}\|g(\tau)\|_{\dot B^s_{p,1}}\,\mathrm{d}\tau\Big),$$

where $V_{p_1}(t) = \int_0^t \|\nabla v\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}(\mathbb{R}^d)}\,\mathrm{d}s$.

Lemma 2.5 ([10])**.** *Let* $s \in \mathbb{R}$ *and* $1 \le r_1, r_2, p, q \le \infty$ *with* $r_2 \le r_1$*. Consider the heat equation*

$$\partial_t u - \Delta u = f, \qquad u(0,x) = u_0(x).$$

Assume that $u_0 \in \dot B^s_{p,q}(\mathbb{R}^d)$ *and* $f \in \mathcal{L}^{r_2}_T(\dot B^{s-2+\frac{2}{r_2}}_{p,q}(\mathbb{R}^d))$*. Then the above equation has a unique solution* $u \in \mathcal{L}^{r_1}_T(\dot B^{s+\frac{2}{r_1}}_{p,q}(\mathbb{R}^d))$ *satisfying*

$$\|u\|_{\mathcal{L}^{r_1}_T(\dot B^{s+\frac{2}{r_1}}_{p,q}(\mathbb{R}^d))} \le C\big(\|u_0\|_{\dot B^s_{p,q}(\mathbb{R}^d)} + \|f\|_{\mathcal{L}^{r_2}_T(\dot B^{s-2+\frac{2}{r_2}}_{p,q}(\mathbb{R}^d))}\big).$$

Lemma 2.6 ([2])**.** *Assume that* u *is a smooth vector field. And* $\Phi(t,x)$ *satisfies*

$$\Phi(t,x) = x + \int_0^t u(s,\Phi(s,x))\,\mathrm{d}s. \tag{2.1}$$

Then, for all $t \in \mathbb{R}^+$*, the flow* $\Phi(t,x)$ *is a* C^1 *diffeomorphism over* $\mathbb{R}^d$*, and we have*

$$\|D\Phi^{\pm}(t)\|_{L^\infty_x} \le \exp\big(\int_0^t \|Du(s)\|_{L^\infty_x}\,\mathrm{d}s\big).$$

Lemma 2.7 ([23])**.** *Let* $u \in \mathcal{S}(\mathbb{R}^d)$ *with* $\operatorname{div} u = 0$*. The flow* Φ *is defined by* u *in* (2.1)*. Then* Φ *and the inverse* Φ^- *are* C^1 *measure-preserving global diffeomorphism over* $\mathbb{R}^d$*. There holds that*

$$\|u\circ\Phi\|_{\dot B^s_{p,q}} \le C\exp\Big(\int_0^t \|Du(s)\|_{L^\infty_x}\,\mathrm{d}s\Big)\|u\|_{\dot B^s_{p,q}}, \quad s \in (-1,1), \quad p,q \in [1,\infty]^2.$$

Proof. Using Lemma 2.7 in Chapter 2 of [2], we infer that for $1 \le p \le \infty$ and any $j,k \in \mathbb{Z}$,

$$\begin{aligned}\|\dot\Delta_j\big((\dot\Delta_k u)\circ\Phi\big)\|_{L^p} \le& C\|\dot\Delta_k u\|_{L^p}\min\Big\{2^{k-j}\|D\Phi^-\|_{L^\infty}, 2^{j-k}\|D\Phi\|_{L^\infty}\Big\}\\ \le& C\|\dot\Delta_k u\|_{L^p}(\|D\Phi\|_{L^\infty} + \|D\Phi^-\|_{L^\infty})\min\big\{2^{k-j}, 2^{j-k}\big\}.\end{aligned}$$

Therefore, for $s \in (-1,1)$ and $u \in \dot B^s_{p,q}(\mathbb{R}^d)$, we have

$$\begin{aligned}&2^{js}\|\dot\Delta_j(u\circ\Phi)\|_{L^p}\\ \le& 2^{js}\Big(\sum_{k\le j}\|\dot\Delta_j\big((\dot\Delta_k u)\circ\Phi\big)\|_{L^p} + \sum_{k>j}\|\dot\Delta_j\big((\dot\Delta_k u)\circ\Phi\big)\|_{L^p}\Big)\\ \le& C(\|D\Phi\|_{L^\infty} + \|D\Phi^-\|_{L^\infty})(\sum_{k\le j}2^{k-j} + \sum_{k>j}2^{j-k})2^{js}\|\dot\Delta_k u\|_{L^p}\end{aligned}$$

$$=C(\|D\Phi\|_{L^\infty}+\|D\Phi^{-}\|_{L^\infty})(\sum_{k\le j}2^{(k-j)(1-s)}2^{ks}\|\dot{\Delta}_k u\|_{L^p}+\sum_{k>j}2^{(j-k)(1+s)}2^{ks}\|\dot{\Delta}_k u\|_{L^p}).$$

Then taking ℓ^q norm on both sides of the above inequality, thanks to $s\in(-1,1)$, one obtains that

$$\|u\circ\Phi\|_{\dot{B}^s_{p,q}}\le C(\|D\Phi\|_{L^\infty}+\|D\Phi^{-}\|_{L^\infty})\|u\|_{\dot{B}^s_{p,q}}.$$

Combining with Lemma 2.6, we complete the proof of this lemma. □

3. Proof of Theorem 1.5

First of all, we introduce the parameters in this section. Let N be a large enough integer defined later. For any $q>1$, α is a constant satisfies that $\frac{1}{q}<\alpha<1$.

Before constructing initial data (u_0,b_0), we introduce two smooth functions $\widehat{\psi}(\xi),\widehat{\phi}(\xi)$ satisfies that

$$\begin{cases}\operatorname{supp}\widehat{\psi}(\xi)=\big\{\xi\in\mathbb{R}^d\big|\ 1\le\xi_2\le 2,(\ln\ln N)^{-1}\le\xi_i\le 2(\ln\ln N)^{-1},i\ne 2\big\}:=A,\\ \widehat{\psi}(\xi)\equiv 1,\ \forall\xi\in\big\{\xi\in\mathbb{R}^d\big|\ \frac54\le\xi_2\le\frac74,\frac54(\ln\ln N)^{-1}\le\xi_i\le\frac74(\ln\ln N)^{-1},i\ne 2\big\}:=B,\end{cases}\tag{3.1}$$

and

$$\begin{cases}\operatorname{supp}\widehat{\phi}(\xi)=\big\{\xi\in\mathbb{R}^d\big|\ 1\le\xi_1\le 2,(\ln\ln N)^{-1}\le\xi_i\le 2(\ln\ln N)^{-1},i\ge 2\big\}:=C,\\ \widehat{\phi}(\xi)\equiv 1,\ \forall\xi\in\big\{\xi\in\mathbb{R}^d\big|\ \frac54\le\xi_1\le\frac74,\frac54(\ln\ln N)^{-1}\le\xi_i\le\frac74(\ln\ln N)^{-1},i\ge 2\big\}:=D.\end{cases}\tag{3.2}$$

We construct initial data (u_0,b_0) as follows:

$$\begin{cases}u_0=\Big(-\mathscr{F}^{-1}\Big(\frac{\xi_2}{\xi_1}\frac{2^N\widehat{\psi}(2^{-N}\xi)}{2^{Nd}(\ln\ln N)^{3+d}}\Big),\mathscr{F}^{-1}\Big(\frac{2^N\widehat{\psi}(2^{-N}\xi)}{2^{Nd}(\ln\ln N)^{3+d}}\Big),0,\dots,0\Big),\\ b_0=\Big(-\sum_{\frac N2\le j\le\frac{4N}{5}}\mathscr{F}^{-1}\Big(\frac{\xi_2}{\xi_1}\frac{\widehat{\phi}(2^{-j}\xi)}{2^{jd}j^\alpha}\Big),\sum_{\frac N2\le j\le\frac{4N}{5}}\mathscr{F}^{-1}\Big(\frac{\widehat{\phi}(2^{-j}\xi)}{2^{jd}j^\alpha}\Big),0,\dots,0\Big),\end{cases}\tag{3.3}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation. Actually, the initial data (u_0,b_0) depends on N from the above definition. To begin with, we need to verify that the initial data (u_0,b_0) is small in $\dot{B}^{\frac dp-1}_{p,q}(\mathbb{R}^d)\times\dot{B}^{\frac dp}_{p,q}(\mathbb{R}^d)$ for all $q>1$.

3.1 Estimates of initial data (u_0,b_0).

By the definition of (u_0,b_0) in (3.3), we have

$$\begin{cases}\widehat{u}_0(\xi)=\Big(-\frac{\xi_2}{\xi_1}\frac{2^N\widehat{\psi}(2^{-N}\xi)}{2^{Nd}(\ln\ln N)^{3+d}},\frac{2^N\widehat{\psi}(2^{-N}\xi)}{2^{Nd}(\ln\ln N)^{3+d}},0,\dots,0\Big),\\ \widehat{b}_0(\xi)=\Big(-\frac{\xi_2}{\xi_1}\sum_{\frac N2\le j\le\frac{4N}{5}}\frac{\widehat{\phi}(2^{-j}\xi)}{2^{jd}j^\alpha},\sum_{\frac N2\le j\le\frac{4N}{5}}\frac{\widehat{\phi}(2^{-j}\xi)}{2^{jd}j^\alpha},0,\dots,0\Big).\end{cases}\tag{3.4}$$

It is easy to verify that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Assume that $\widetilde{\psi}(\xi) \in C_c^\infty(\mathbb{R}^d)$, $\widetilde{\psi}(\xi) \equiv 1$ on A and that $\widetilde{\psi}$ is supported in an annulus $\widetilde{A}$, where

$$\widetilde{A} = \Big\{ \xi \in \mathbb{R}^d \big| \ \tfrac{1}{2} \le \xi_2 \le 3, \tfrac{1}{2}(\ln\ln N)^{-1} \le \xi_i \le 3(\ln\ln N)^{-1}, i \neq 2 \Big\}.$$

Moreover, $\widetilde{\psi}$ satisfies that $\|D^k \widetilde{\psi}\|_{L^\infty} \le C(\ln\ln N)^k, \forall k \ge 0$.

Noting the supp $\widehat{\psi}(\xi) \subset \widetilde{A}$, we obtain

$$\widehat{u}_0(\xi) = \frac{2^N}{2^{Nd}(\ln\ln N)^{3+d}} \big(-\tfrac{\xi_2}{\xi_1} \widetilde{\psi}(2^{-N}\xi)\widehat{\psi}(2^{-N}\xi), \widehat{\psi}(2^{-N}\xi), 0, \dots, 0 \big).$$

For u_0^1, we have $u_0^1 = \frac{2^N}{(\ln\ln N)^{3+d}} K * \psi(2^N \cdot)$, where

$$K(x) = -\,(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \tfrac{\xi_2}{\xi_1} \widetilde{\psi}(2^{-N}\xi) e^{ix\cdot\xi} \,\mathrm{d}\xi = -(2\pi)^{-\frac{d}{2}} 2^{Nd} \int_{\mathbb{R}^d} \tfrac{\xi_2}{\xi_1} \widetilde{\psi}(\xi) e^{i2^N x\cdot\xi} \,\mathrm{d}\xi.$$

Let $M = \lfloor 1 + \frac{d}{2} \rfloor$[1], we have

$$\begin{aligned}
(1 + |2^N x|^2)^M |K(x)| =& -\,(2\pi)^{-\frac{d}{2}} 2^{Nd} \Big| \int_{\mathbb{R}^d} \big((\mathrm{Id} - \Delta_\xi)^M e^{i2^N x\cdot\xi} \big) \tfrac{\xi_2}{\xi_1} \widetilde{\psi}(\xi) \,\mathrm{d}\xi \Big| \\
=& -\,(2\pi)^{-\frac{d}{2}} 2^{Nd} \Big| \int_{\mathbb{R}^d} \big((\mathrm{Id} - \Delta_\xi)^M \tfrac{\xi_2}{\xi_1} \widetilde{\psi}(\xi) \big) e^{i2^N x\cdot\xi} \,\mathrm{d}\xi \Big| \\
=& -\,(2\pi)^{-\frac{d}{2}} 2^{Nd} \Big| \sum_{|\alpha|+|\beta| \le 2M} c_{\alpha,\beta} \int_{\widetilde{A}} e^{i2^N x\cdot\xi} \partial^\alpha \widetilde{\psi}(\xi) \Big(\partial^\beta \frac{\xi_2}{\xi_1} \Big) \,\mathrm{d}\xi \Big| \\
\le& C 2^{Nd} \sum_{|\alpha|+|\beta| \le 2M} c_{\alpha,\beta} (\ln\ln N)^{|\alpha|+|\beta|+1-(d-1)} \\
\le& C 2^{Nd} (\ln\ln N)^{2M}.
\end{aligned}$$

Due to $M > \frac{d}{2}$, we can infer from the above inequality that

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(x)| \,\mathrm{d}x \le& C(\ln\ln N)^{2M} 2^{Nd} \int_{\mathbb{R}^d} (1 + |2^N x|^2)^{-M} \,\mathrm{d}x \\
\le& C(\ln\ln N)^{2M} \int_{\mathbb{R}^d} (1 + |x|^2)^{-M} \,\mathrm{d}x \\
\le& C(\ln\ln N)^{2+d}.
\end{aligned}$$

Therefore, with the aid of Young's inequality, one has

$$\begin{aligned}
\|u_0^1\|_{\dot{B}^{\frac{d}{p}-1}_{p,q}} \le& \frac{C 2^N}{(\ln\ln N)^{3+d}} \|K\|_{L^1} \|\psi(2^N \cdot)\|_{\dot{B}^{\frac{d}{p}-1}_{p,q}} \\
\le& \frac{C(\ln\ln N)^{2+d} 2^N}{(\ln\ln N)^{3+d}} \|\psi(2^N \cdot)\|_{B^{\frac{d}{p}-1}_{p,q}} \le \frac{C}{\ln\ln N},
\end{aligned} \tag{3.5}$$

$$\|u_0^2\|_{\dot{B}^{\frac{d}{p}-1}_{p,q}} = \frac{2^N}{(\ln\ln N)^{3+d}} \|\psi(2^N \cdot)\|_{B^{\frac{d}{p}-1}_{p,q}} \le \frac{C}{(\ln\ln N)^{3+d}}. \tag{3.6}$$

[1] $\lfloor x \rfloor$ denotes the floor function.

In terms of b_0, noting the support of $\widehat{\phi}(\xi)$, it is easy to verify that $\frac{\xi_2}{\xi_1} \sim (\ln\ln N)^{-1}$. In the same way as estimating of $\|u_0\|_{B^{\frac{d}{p}-1}_{p,q}}$, we obtain that

$$\|b_0\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \sim \Big(\sum_{\frac{N}{2}\leq j\leq \frac{4N}{5}} 2^{\frac{d}{p}jq} \frac{\|\phi(2^j x)\|^q_{L^p}}{j^{\alpha q}} \Big)^{\frac{1}{q}} \sim N^{\frac{1}{q}-\alpha}. \tag{3.7}$$

Similarly, it is easy to verify that for N large enough and all $1 \leq p \leq \infty$, we have

$$\begin{cases} \|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq C(\ln\ln N)^{-1}, \ \|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \leq 2^N, \quad \|u_0\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \leq 2^{2N}, \\ \|b_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \leq N^{1-\alpha}, \qquad\qquad \|b_0\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \leq 2^N, \ \ \|b_0\|_{\dot{B}^{\frac{d}{p}+2}_{p,1}} \leq 2^{2N}. \end{cases} \tag{3.8}$$

3.2 Local well-posedness for (u_0, b_0) with the form of (3.4).

We begin to establish a locally well-posed result for the system (1.1) with the initial data constructed in (3.4).

Proposition 3.1. *Let initial data (u_0, b_0) be defined by (3.4). Given $1 \leq r \leq 2d$, there exist constants C_0 and N_0 such that for $N > N_0$ and $T = (\ln N)^{-1}2^{-2N}$, the system (1.1) has a unique local solution (u, b) associated with initial data (u_0, b_0) satisfying*

$$\begin{aligned} &u \in C([0,T], \dot{B}^{\frac{d}{r}-1}_{r,1} \cap \dot{B}^{\frac{d}{r}+1}_{r,1}) \cap \mathcal{L}^1([0,T], \dot{B}^{\frac{d}{r}+1}_{r,1} \cap \dot{B}^{\frac{d}{r}+3}_{r,1}), \\ &b \in C([0,T], \dot{B}^{\frac{d}{r}}_{r,1} \cap \dot{B}^{\frac{d}{r}+2}_{r,1}), \end{aligned}$$

and the following estimates hold for all $t \leq T$:

$$\|u\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}-1}_{r,1})} + \|u\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{r}+1}_{r,1})} \leq 2C_0(\ln\ln N)^{-1}, \tag{3.9}$$

$$\|u\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}}_{r,1})} + \|u\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{r}+2}_{r,1})} \leq 2C_0 2^N, \tag{3.10}$$

$$\|u\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}+1}_{r,1})} + \|u\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{r}+3}_{r,1})} \leq 2C_0 2^{2N}, \tag{3.11}$$

$$\|b\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}}_{r,1})} \leq 2C_0 N^{1-\alpha}, \tag{3.12}$$

$$\|b\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}+1}_{r,1})} \leq 2C_0^3 N^{1-\alpha} 2^N, \tag{3.13}$$

$$\|b\|_{\mathcal{L}^\infty_t(\dot{B}^{\frac{d}{r}+2}_{r,1})} \leq 2C_0^4 N^{1-\alpha} 2^{2N}. \tag{3.14}$$

Proof. According to the local well-posedness theory of system (1.1) in [18], there exists a positive time $\bar{T}_0$ such that the system (1.1) possesses a unique solution $(u, b) \in \dot{B}^{\frac{d}{r}-1}_{r,1} \times \dot{B}^{\frac{d}{r}}_{r,1}$ with initial data (u_0, b_0) satisfying

$$\begin{aligned} &u \in C([0,\bar{T}_0], \dot{B}^{\frac{d}{r}-1}_{r,1}) \cap \mathcal{L}^1([0,\bar{T}_0], \dot{B}^{\frac{d}{r}+1}_{r,1}), \\ &b \in C([0,\bar{T}_0], \dot{B}^{\frac{d}{r}}_{r,1}). \end{aligned}$$

Furthermore, the uniform estimates in [18] shows that, for any small enough η, there exist $C_0>1$ and a $0<T_\eta\le \bar{T}_0$, where T_η depends on u_0 and b_0, such that for any $0<T_0\le T_\eta$,

$$\begin{cases}\|u(t)\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}-1}_{r,1})}+\|b(t)\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}}_{r,1})}\le C_0(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}}),\\ \|u\|_{L^2([0,T_0],\dot{B}^{\frac{d}{r}}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+1}_{r,1})}\le\eta.\end{cases}\tag{3.15}$$

Since $(u_0,b_0)\in\mathcal{S}(\mathbb{R}^d)$, one can deduce that for short time T_0,

$$u\in C([0,T_0],\dot{B}^{\frac{d}{r}-1}_{r,1}\cap\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+1}_{r,1}\cap\dot{B}^{\frac{d}{r}+3}_{r,1}),$$
$$b\in C([0,T_0],\dot{B}^{\frac{d}{r}}_{r,1}\cap\dot{B}^{\frac{d}{r}+2}_{r,1}).$$

Indeed, using Lemma 2.1–Lemma 2.2, one has that

$$\begin{aligned}&\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}\\
\le&\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}+\int_0^{T_0}\|(u\cdot\nabla u,\ b\cdot\nabla b)\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}\,\mathrm{d}t,\\
\le&\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}+C\|u\|_{L^2([0,T_0],\dot{B}^{\frac{d}{r}}_{r,1})}\|u\|_{L^2([0,T_0],\dot{B}^{\frac{d}{r}+2}_{r,1})}+C\int_0^{T_0}\|b(t)\|_{\dot{B}^{\frac{d}{r}}_{r,1}}\|b(t)\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}\,\mathrm{d}t,\\
\le&\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}+C\eta\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}\\
&+C(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})T_0\|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}.\end{aligned}\tag{3.16}$$

Taking ∇ on E.q.$(1.1)_2$ and using Lemma 2.2, we obtain that

$$\begin{aligned}&\|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\le\|b_0\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}\\
&+C\int_0^{T_0}\|u(t)\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}\|b(t)\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}+\|u(t)\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}\|b(t)\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}+\|u(t)\|_{\dot{B}^{\frac{d}{r}+3}_{r,1}}\|b(t)\|_{\dot{B}^{\frac{d}{r}}_{r,1}}\,\mathrm{d}t\\
\le&\|b_0\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}+C\eta\|b(t)\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\\
&+C\sqrt{T_0}\|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}\\
&+C\sqrt{T_0}(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}\\
&+C(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})\|u(t)\|_{\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}.\end{aligned}$$

Plugging estimate (3.16) into the above inequality yields that

$$\begin{aligned}\|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\le&\|b_0\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}+C(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}\\
&+C(\eta+(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}+\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})T_0)\|b(t)\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\\
&+C\sqrt{T_0}\|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}\end{aligned}$$

$$+ C(\sqrt{T_0}+\eta)(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}} + \|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})}. \tag{3.17}$$

Collecting the estimates (3.16) and (3.17) together, for taking η and T_0 small enough, by continuity argument, we have

$$\begin{aligned}
&\|u\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+1}_{r,1})\cap\mathcal{L}^1([0,T_0],\dot{B}^{\frac{d}{r}+3}_{r,1})} + \|b\|_{\mathcal{L}^\infty([0,T_0];\dot{B}^{\frac{d}{r}+2}_{r,1})}\\
\le& C(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}}, \|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}})(\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}} + \|b_0\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}}).
\end{aligned}$$

Now we need to show that T_0 can be extended to $(\ln N)^{-1}2^{-2N}$. To prove this, we are focused on showing a priori estimates (3.9)-(3.14) on $[0, 2(\ln N)^{-1}2^{-2N})$.

With the aid of Lemma 2.4, we have

$$\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})} \le \exp(C\|u\|_{L^1_T(B^{\frac{d}{r}+1}_{r,1})})(\|b_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}} + \|b\|_{L^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|u\|_{L^1_T(B^{\frac{d}{r}+1}_{r,1})}). \tag{3.18}$$

Taking advantage of Lemma 2.5, one can deduce that

$$\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}-1}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{r}+1}_{r,1})} \le C(\|u_0\|_{\dot{B}^{\frac{d}{r}-1}_{r,1}} + \|u\|^2_{\mathcal{L}^2_T(\dot{B}^{\frac{d}{r}}_{r,1})} + T\|b\|^2_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}). \tag{3.19}$$

Similarly, we can obtain that

$$\begin{aligned}
\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+1}_{r,1})} \le& \exp(C\|u\|_{\mathcal{L}^1_T(B^{\frac{d}{r}+1}_{r,1})})(\|b_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}}\\
&+ \|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|u\|_{\mathcal{L}^1_T(B^{\frac{d}{r}+2}_{r,1})} + \|b\|_{\mathcal{L}^\infty_T\dot{B}^{\frac{d}{r}+1}_{r,1}}\|u\|_{\mathcal{L}^1_T(B^{\frac{d}{r}+1}_{r,1})}).
\end{aligned} \tag{3.20}$$

With the aid of Lemma 2.5, one can infer that

$$\begin{aligned}
&\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{r}+2}_{r,1})}\\
\le& C(\|u_0\|_{\dot{B}^{\frac{d}{r}}_{r,1}} + \sqrt{T}\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|u\|_{\mathcal{L}^2_T(\dot{B}^{\frac{d}{r}+1}_{r,1})} + T\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+1}_{r,1})}),
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+1}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{r}+3}_{r,1})}\\
\le& C(\|u_0\|_{\dot{B}^{\frac{d}{r}+1}_{r,1}} + \sqrt{T}\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|u\|_{\mathcal{L}^2_T(\dot{B}^{\frac{d}{r}+2}_{r,1})}\\
&+ T\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}}_{r,1})}\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+2}_{r,1})} + T\|b\|^2_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+1}_{r,1})}).
\end{aligned} \tag{3.22}$$

Taking derivative on E.q.$(1.1)_2$, we obtain by Lemma 2.4 that

$$\begin{aligned}
\|b\|_{\mathcal{L}^\infty_T(\dot{B}^{\frac{d}{r}+2}_{r,1})} \le& \exp(C\|u\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{r}+1}_{r,1})})(\|b_0\|_{\dot{B}^{\frac{d}{r}+2}_{r,1}} + C\|b\|_{\mathcal{L}^\infty_T\dot{B}^{\frac{d}{r}}_{r,1}}\|u\|_{\mathcal{L}^1_T\dot{B}^{\frac{d}{r}+3}_{r,1}}\\
&+ CT\|u\|_{\mathcal{L}^\infty_T\dot{B}^{\frac{d}{r}+1}_{r,1}}\|b\|_{\mathcal{L}^\infty_T\dot{B}^{\frac{d}{r}+2}_{r,1}} + C\sqrt{T}\|b\|_{\mathcal{L}^\infty_T\dot{B}^{\frac{d}{r}+1}_{r,1}}\|u\|_{\mathcal{L}^2_T\dot{B}^{\frac{d}{r}+2}_{r,1}}).
\end{aligned} \tag{3.23}$$

Since initial data (u_0, b_0) satisfies (3.8), let constant $C_0 > \max\{2C^2, 16\}$, we can infer from the above estimates that there exist a positive time T_1 such that for $t \le T_1$, estimates (3.9)-(3.14) hold. We define

$$T^* := \sup\big\{0 < t \le 2^{-2N+1}(\ln N)^{-1}\big|(3.9) - (3.14) \text{ hold on the time interval } [0,t]\big\}.$$

If $T^* = 2^{-2N+1}(\ln N)^{-1}$, we complete the proof. Otherwise, for $t < T^* < 2^{-2N+1}(\ln N)^{-1}$, combining (3.9) with (3.18), we have

$$\|b\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}}_{r,1})} \le \exp(2CC_0(\ln\ln N)^{-1})(N^{1-\alpha} + 4C_0^2 N^{1-\alpha}(\ln\ln N)^{-1}) \le C_0 N^{1-\alpha},$$

where the last inequality holds for large enough N. Utilizing (3.19) and the above inequality, one has

$$\begin{aligned}\|u\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}-1}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot B^{\frac{d}{r}+1}_{r,1})} \le& C(C(\ln\ln N)^{-1} + 4C_0(\ln\ln N)^{-2} + 2^{-4N}(\ln N)^{-2}C_0^2 N^{2-2\alpha})\\ \le& C_0(\ln\ln N)^{-1}.\end{aligned}$$

In the same way as deriving the above inequality, plugging (3.9), (3.12) (3.13) and (3.10) into (3.20) yields that for large N,

$$\begin{aligned}\|b\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}+1}_{r,1})} \le& \exp(2CC_0(\ln\ln N)^{-1})(2^N + 4C_0^2 N^{1-\alpha}2^N + 4C_0^4 N^{1-\alpha}2^N(\ln\ln N)^{-1})\\ \le& C_0^3 N^{1-\alpha}2^N.\end{aligned}$$

Similarly, owing to (3.10), (3.12) and (3.13), we have

$$\|u\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot B^{\frac{d}{r}+2}_{r,1})} \le C(2^N + 8C_0^2(\ln N)^{-\frac12}2^N + 8C_0^4(\ln N)^{-1}N^{2-2\alpha}2^{-N}) \le C_0 2^N,$$

and we can deduce by (3.10)-(3.14) that

$$\begin{aligned}\|u\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}+1}_{r,1})} + \|u\|_{\mathcal{L}^1_T(\dot B^{\frac{d}{r}+3}_{r,1})} \le& C(2^{2N} + 8C_0^2(\ln N)^{-\frac12}2^{2N} + 8C_0^2(\ln N)^{-1}2^{2N}\\ &+ 8C_0^5 N^{2-2\alpha}(\ln N)^{-1}N^{1-\alpha} + 8C_0^6(\ln N)^{-1}N^{2-2\alpha})\\ \le& C_0 2^{2N}.\end{aligned}$$

Finally, plugging the above inequality, (3.9), (3.12)-(3.14) into (3.23), one obtains that

$$\begin{aligned}\|b\|_{\mathcal{L}^\infty_T(\dot B^{\frac{d}{r}+2}_{r,1})} \le& \exp(2CC_0(\ln\ln N)^{-1})(2^{2N} + 4CC_0^2 N^{1-\alpha}2^{2N} + 16CC_0^5(\ln N)^{-1}N^{1-\alpha}2^{2N})\\ \le& C_0^4 N^{1-\alpha}2^{2N}.\end{aligned}$$

The above estimates contradict to the definition of T^*. Therefore, $T^* = 2^{-2N+1}(\ln N)^{-1}$.

□

3.3 Norm inflation.

In this section, we show a norm inflation phenomenon for the magnetic field $b(t,x)$. Before doing this, we define the flow map $\Phi(x,t)$ by

$$\begin{cases} \dfrac{\mathrm{d}\Phi(x,t)}{\mathrm{d}t} = u(t,\Phi(x,t)), \\ \Phi(x,t)|_{t=0} = x, \end{cases} \tag{3.24}$$

In the following, $T=(\ln N)^{-1}2^{-2N}$. We rewrite the magnetic field $b(t,x)$ on $[0,T]$ as follows:

$$b(t,\Phi(x,t)) = b_0(x) + \int_0^t (b\cdot\nabla u)(s,\Phi(x,s))\,\mathrm{d}s.$$

Based on the above equality, we decompose $b(T,\Phi(x,T))$ into the following three parts:

$$\begin{aligned} b(T,\Phi(x,T)) =& b_0(x) + \underbrace{\int_0^T (b_0\cdot\nabla e^{t\Delta}u_0)(t,x)\,\mathrm{d}t}_{I^{\mathrm{B}}} \\ &+ \underbrace{\int_0^T (b\cdot\nabla u)(s,\Phi(x,s))\,\mathrm{d}s - \int_0^T (b_0\cdot\nabla e^{t\Delta}u_0)(t,x)\,\mathrm{d}t}_{I^{\mathrm{S}}}. \end{aligned}$$

Now, we aim to estimate the lower bound of $\|I^{\mathrm{B}}\|_{\dot B^0_{\infty,q}}$ and the upper bound of $\|I^{\mathrm{S}}\|_{\dot B^0_{\infty,q}}$.
Estimates of $\|I^{\mathrm{B}}\|_{\dot B^0_{\infty,q}}$. Due to $\dot B^0_{\infty,q}\hookrightarrow \dot B^0_{\infty,\infty}$ and $\|f\|_{L^\infty}\ge |f(0)| = |\int_{\mathbb{R}^d}\hat f(\xi)\,\mathrm{d}\xi|$, we have

$$\begin{aligned} \|I^{\mathrm{B}}\|_{\dot B^0_{\infty,q}} \ge& C\Big\|\dot\Delta_N \int_0^T b_0\cdot\nabla e^{s\Delta}u_0^2\,\mathrm{d}s\Big\|_{L^\infty} \\ \ge& C\Big|\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d}\widehat\varphi(2^{-N}\xi)\widehat{b}_0(\eta)\cdot(\xi-\eta)e^{-s|\xi-\eta|^2}\widehat{u}_0^2(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}\xi\Big| \\ \ge& C\Big|\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d}\widehat\varphi(2^{-N}\xi)\widehat{b}_0^2(\eta)(\xi_2-\eta_2)e^{-s|\xi-\eta|^2}\widehat{u}_0^2(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}\xi\Big| \\ &- C\Big|\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d}\widehat\varphi(2^{-N}\xi)\widehat{b}_0^1(\eta)(\xi_1-\eta_1)e^{-s|\xi-\eta|^2}\widehat{u}_0^2(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}\xi\Big| \\ :=& I_1^{\mathrm{B}} - I_2^{\mathrm{B}}. \end{aligned}$$

For I_1^{B}, by the definitions of b_0 and u_0, taking change of variables, one yields that

$$\begin{aligned} I_1^{\mathrm{B}} =& C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widehat\varphi(2^{-N}\xi)\widehat{b}_0^2(\eta)(\xi_2-\eta_2)\frac{1-e^{-T|\xi-\eta|^2}}{|\xi-\eta|^2}\widehat{u}_0^2(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\Big| \\ =& C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widehat\varphi(2^{-N}\xi)\sum_{\frac{N}{2}\le j\le\frac{4N}{5}}\frac{\widehat\phi(2^{-j}\eta)}{2^{jd}j^\alpha}(\xi_2-\eta_2)\frac{1-e^{-T|\xi-\eta|^2}}{|\xi-\eta|^2}\frac{2^N\widehat\psi(2^{-N}(\xi-\eta))}{2^{Nd}(\ln\ln N)^{3+d}}\,\mathrm{d}\eta\,\mathrm{d}\xi\Big| \end{aligned}$$

$$=C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widehat{\varphi}(\tilde\xi)\sum_{\frac{N}{2}\le j\le\frac{4N}{5}}\frac{\widehat{\phi}(\tilde\eta)}{j^\alpha}(\tilde\xi_2-2^{j-N}\tilde\eta_2)\frac{1-e^{-T2^{2N}|\tilde\xi-2^{j-N}\tilde\eta|^2}}{|\tilde\xi-2^{j-N}\tilde\eta|^2}\frac{\widehat{\psi}(\tilde\xi-2^{j-N}\tilde\eta)}{(\ln\ln N)^{3+d}}\,\mathrm{d}\tilde\eta\,\mathrm{d}\tilde\xi\Big|.$$

Noting the fact that $\widehat{\varphi}$, $\widehat{\phi}$ and $\widehat{\psi}$ support on an annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^d|\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$, and $|2^{j-N}\tilde\eta|\le 2^{-\frac{N}{5}}|\tilde\eta|\ll 1$ for large enough N, we obtain that there exist constants c_0, c_1 such that

$$c_0\le|\tilde\xi-2^{j-N}\tilde\eta|^2\le c_1,\quad \tilde\xi_2-2^{j-N}\tilde\eta_2\sim 1,\quad \text{if } \tilde\xi\in\operatorname{supp}\widehat{\varphi}(\tilde\xi),\ \tilde\eta\in\operatorname{supp}\widehat{\phi}(\tilde\eta). \tag{3.25}$$

Therefore, we can easily deduce from $T=2^{-2N}(\ln N)^{-1}$ that

$$I_1^{\mathrm{B}}\ge C_0(1-e^{-c_0(\ln N)^{-1}})(\ln\ln N)^{2-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}.$$

Owning to

$$\frac{\tilde\eta_2}{\tilde\eta_1}\sim(\ln\ln N)^{-1},\ \text{ if } \tilde\eta\in\operatorname{supp}\widehat{\phi}(\tilde\eta);\quad \tilde\xi_1\sim(\ln\ln N)^{-1},\ \text{ if } \tilde\xi\in\operatorname{supp}\widehat{\psi}(\tilde\xi),$$

combining with (3.25), we have

$$\begin{aligned}I_2^{\mathrm{B}}=&C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widehat{\varphi}(2^{-N}\xi)\sum_{\frac{N}{2}\le j\le\frac{4N}{5}}\frac{\eta_2}{\eta_1}\frac{\widehat{\phi}(2^{-j}\eta)}{2^{jd}j^\alpha}(\xi_1-\eta_1)\frac{1-e^{-T|\xi-\eta|^2}}{|\xi-\eta|^2}\frac{2^N\widehat{\psi}(2^{-N}(\xi-\eta))}{2^{Nd}(\ln\ln N)^{3+d}}\,\mathrm{d}\eta\,\mathrm{d}\xi\Big|\\ =&C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widehat{\varphi}(\tilde\xi)\sum_{\frac{N}{2}\le j\le\frac{4N}{5}}\frac{\tilde\eta_2}{\tilde\eta_1}\frac{\widehat{\phi}(\tilde\eta)}{j^\alpha}(\tilde\xi_1-2^{j-N}\tilde\eta_1)\frac{1-e^{-T2^{2N}|\tilde\xi-2^{j-N}\tilde\eta|^2}}{|\tilde\xi-2^{j-N}\tilde\eta|^2}\frac{\widehat{\psi}(\tilde\xi-2^{j-N}\tilde\eta)}{(\ln\ln N)^{3+d}}\,\mathrm{d}\tilde\eta\,\mathrm{d}\tilde\xi\Big|\\ \le&C_1(1-e^{-c_1(\ln N)^{-1}})(\ln\ln N)^{-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}.\end{aligned}$$

Hence, we can deduce from the above two estimates that

$$\begin{aligned}\big\|I^{\mathrm{B}}\big\|_{\dot B^0_{\infty,q}}\ge&C_0(1-e^{-c_0(\ln N)^{-1}})(\ln\ln N)^{2-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}\\ &-C_1(1-e^{-c_1(\ln N)^{-1}})(\ln\ln N)^{-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}.\end{aligned}\tag{3.26}$$

Choosing N large enough such that $c_0(\ln N)^{-1}\le\frac{1}{2}$, utilizing the following inequality

$$1-e^{-x}\ge\frac{x}{2}\ \text{for } x\in[0,\tfrac{1}{2}],\qquad 1-e^{-x}<x\ \text{for } x>0,$$

one can infer from (3.26) that

$$\begin{aligned}\big\|I^{\mathrm{B}}\big\|_{\dot B^0_{\infty,q}}\ge&\frac{C_0c_0}{2}(\ln N)^{-1}(\ln\ln N)^{2-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}\\ &-C_1c_1(\ln N)^{-1}(\ln\ln N)^{-2d}(\ln\ln N)^{-3-d}N^{1-\alpha}\\ \ge&C(\ln N)^{-1}(\ln\ln N)^{-1-3d}N^{1-\alpha}.\end{aligned}\tag{3.27}$$

Estimates of $\|I^{\mathrm{S}}\|_{\dot B^0_{\infty,q}}$. We decompose $\|I^{\mathrm{S}}\|_{\dot B^0_{\infty,q}}$ into the following three parts:

$$\|I^{\mathrm{S}}\|_{\dot B^0_{\infty,q}}\le\Big\|\int_0^T(b\circ\Phi-b_0)\cdot((\nabla u)\circ\Phi)\,\mathrm{d}t\Big\|_{\dot B^0_{\infty,q}}$$

$$
\begin{aligned}
&+\Big\|\int_0^T b_0\cdot((\nabla u)\circ\Phi-(\nabla u)(x))\,\mathrm{d}t\Big\|_{\dot B^0_{\infty,q}}\\
&+\Big\|\int_0^T b_0\cdot\nabla(u-e^{t\Delta}u_0)(x)\,\mathrm{d}t\Big\|_{\dot B^0_{\infty,q}}\\
:=&I_1^{\mathrm{S}}+I_2^{\mathrm{S}}+I_3^{\mathrm{S}}.
\end{aligned}
$$

For I_1^{S}, using $\dot B^{\frac{d}{p}}_{p,1}\hookrightarrow \dot B^0_{\infty,q}$ and Lemma 2.7, we obtain for $d<p\le 2d$ that

$$
\begin{aligned}
I_1^{\mathrm{S}}\le&CT\|b\circ\Phi-b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|(\nabla u)\circ\Phi\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\\
\le&C\exp\Big(\int_0^T\|Du(s)\|_{L^\infty_x}\,\mathrm{d}s\Big)T\|b\cdot\nabla u\|_{L^1_T\dot B^{\frac{d}{p}}_{p,1}}\|u\|_{L^\infty_T\dot B^{\frac{d}{p}+1}_{p,1}},
\end{aligned}\tag{3.28}
$$

With the aid of Proposition 3.1, we have for $T\le T_0$,

$$
\begin{aligned}
\exp\Big(\int_0^T\|Du(s)\|_{L^\infty_x}\,\mathrm{d}s\Big)\le&\exp\big(T\|u\|_{\mathcal{L}^\infty([0,T],\dot B^{\frac{d}{p}+1}_{p,1})}\big)\\
\le&\exp(2^{-2N}(\ln N)^{-1}\cdot 2C_0 2^{2N})\le C.
\end{aligned}\tag{3.29}
$$

Combining the above inequality with (3.28), (3.11) and (3.12), one obtains that

$$
I_1^{\mathrm{S}}\le CT^2\|b\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|u\|^2_{L^\infty_T\dot B^{\frac{d}{p}+1}_{p,1}}\le C2^{-4N}(\ln N)^{-2}N^{1-\alpha}2^{4N}\le C(\ln N)^{-2}N^{1-\alpha}.\tag{3.30}
$$

In terms of I_2^S, taking advantage of Newton Leibniz formula, one can deduce that

$$
\begin{aligned}
I_2^S\le&C\sqrt{T}\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|(\nabla u)\circ\Phi-(\nabla u)(x)\|_{L^2_T\dot B^{\frac{d}{p}}_{p,1}}\\
=&C\sqrt{T}\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\Big\|\int_0^1\partial_\theta((\nabla u)(\theta\Phi+(1-\theta)x))\,\mathrm{d}\theta\Big\|_{L^2_T\dot B^{\frac{d}{p}}_{p,1}}\\
=&C\sqrt{T}\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\Big\|\int_0^1(D^2u)(\theta\Phi+(1-\theta)x)\cdot(\Phi-x)\,\mathrm{d}\theta\Big\|_{L^2_T\dot B^{\frac{d}{p}}_{p,1}}\\
\le&C\sqrt{T}\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|\Phi-x\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\int_0^1\|(D^2u)(\theta\Phi+(1-\theta)x)\|_{L^2_T\dot B^{\frac{d}{p}}_{p,1}}\,\mathrm{d}\theta.
\end{aligned}
$$

With the aid of (3.24), Lemma 2.7 and (3.29), we have

$$
\begin{aligned}
I_2^S\le&C\sqrt{T}\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\Big\|\int_0^t u\circ\Phi\,\mathrm{d}s\Big\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|u\|_{L^2_T\dot B^{\frac{d}{p}+2}_{p,1}}\\
\le&C\sqrt{T}T\|b_0\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|u\|_{L^\infty_T\dot B^{\frac{d}{p}}_{p,1}}\|u\|_{L^2_T\dot B^{\frac{d}{p}+2}_{p,1}}.
\end{aligned}\tag{3.31}
$$

By (3.11) in Proposition 3.1, we have that for $T\le 2^{-2N}(\ln N)^{-1}$ and $d<p\le 2d$,

$$
\|u\|_{L^2_T\dot B^{\frac{d}{p}+2}_{p,1}}\le\|u\|_{\mathcal{L}^\infty_t\dot B^{\frac{d}{p}+1}_{p,1}}+\|u\|_{\mathcal{L}^1_t\dot B^{\frac{d}{p}+3}_{p,1}}\le 2C_0 2^{2N}.
$$

Plugging the above estimate, (3.8) and (3.10) into (3.31) yields that

$$I_2^S \le C2^{-3N}(\ln N)^{-\frac{3}{2}} \cdot 2^N \cdot 2^{2N} \le CN^{1-\alpha}(\ln N)^{-\frac{3}{2}}. \tag{3.32}$$

By Lemma 2.5, we can bound I_3^S as follows:

$$\begin{aligned} I_3^S \le & CT\|b_0\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \|u - e^{t\Delta}u_0\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}+1}} \\ \le & CT\|b_0\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}}} (\|u \cdot \nabla u\|_{L_T^1 \dot{B}_{p,1}^{\frac{d}{p}+1}} + \|b \cdot \nabla b\|_{L_T^1 \dot{B}_{p,1}^{\frac{d}{p}+1}}) \\ \le & CT\|b_0\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}}} (\sqrt{T}\|u\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \|u\|_{L_T^2 \dot{B}_{p,1}^{\frac{d}{p}+2}} \\ & + T\|b\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}}} \|b\|_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}+2}} + T\|b\|^2_{L_T^\infty \dot{B}_{p,1}^{\frac{d}{p}+1}}). \end{aligned}$$

The above inequality combined with Proposition 3.1 shows that

$$\begin{aligned} I_3^S \le & C2^{-2N}(\ln N)^{-1}N^{1-\alpha}((\ln N)^{-\frac{1}{2}}2^{2N} + (\ln N)^{-1}2^{2N} + (\ln N)^{-1}N^{2(1-\alpha)}) \\ \le & C(\ln N)^{-\frac{3}{2}}N^{1-\alpha}. \end{aligned} \tag{3.33}$$

Collecting (3.30)-(3.33) together yields that

$$\|I^{\mathrm{S}}\|_{\dot{B}^0_{\infty,q}} \le C(\ln N)^{-2}N^{1-\alpha} + C(\ln N)^{-\frac{3}{2}}N^{1-\alpha} \le C(\ln N)^{-\frac{3}{2}}N^{1-\alpha}. \tag{3.34}$$

Estimates of $\|b(T)\|_{\dot{B}^{\frac{d}{p}}_{p,q}}$ In view of (3.27) and (3.34), for $\frac{1}{q} < \alpha < 1$, we can deduce that

$$\begin{aligned} \|b(T, \Phi(x,T))\|_{\dot{B}^0_{\infty,q}} \ge & \|I^B\|_{\dot{B}^0_{\infty,q}} - \|b_0\|_{\dot{B}^0_{\infty,q}} - \|I^S\|_{\dot{B}^0_{\infty,q}} \\ \ge & C(\ln N)^{-1}(\ln\ln N)^{-1-3d}N^{1-\alpha} - CN^{\frac{1-\alpha q}{q}} - C(\ln N)^{-\frac{3}{2}}N^{1-\alpha} \\ \ge & C(\ln N)^{-1}(\ln\ln N)^{-1-3d}N^{1-\alpha}. \end{aligned}$$

Therefore, we obtain from Lemma 2.7, the above inequality and (3.29) that

$$\begin{aligned} \|b(x,T)\|_{\dot{B}^{\frac{d}{p}}_{p,q}} & \ge C\|b(T,\Phi(x,T))\|_{\dot{B}^0_{\infty,q}} \\ & \ge C(\ln N)^{-1}(\ln\ln N)^{-1-3d}N^{1-\alpha} \to \infty, \text{ as } N \to \infty. \end{aligned} \tag{3.35}$$

Meanwhile, we can deduce from (3.9) that

$$\|u(x,t)\|_{L_T^\infty(\dot{B}^{\frac{d}{p}-1}_{p,1}) \cap \mathcal{L}_T^1(\dot{B}^{\frac{d}{p}+1}_{p,1})} \le C(\ln\ln N)^{-1} \to 0, \text{ as } N \to \infty. \tag{3.36}$$

Therefore, combining (3.5), (3.7) and (3.35), we complete the proof of Theorem 1.5 by setting $\delta = (\ln\ln N)^{-1}$ for sufficiently large N.

Finally, by the construction of initial data (u_0, b_0) in (3.3), we immediately obtain the following corollary which shows that $\dot{B}^{\frac{d}{p}}_{p,q}$ with $q > 1$ is not an algebra. More precisely, for $d \geq 1$, $1 \leq p \leq \infty$ and $q > 1$, there exist f and g such that

$$\|fg\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \nleq C\|f\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \|g\|_{\dot{B}^{\frac{d}{p}}_{p,q}}.$$

Corollary 3.2. *Let $1 \leq p \leq \infty, q > 1$ and $\frac{1}{q} < \alpha < 1$. Let N be large enough integer. There exist two scalar functions f^N and g^N such that*

$$\|f^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} + \|g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \lesssim (\ln\ln N)^{-1} \to 0, \ \text{as } N \to \infty,$$

meanwhile,

$$\|f^N g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \gtrsim N^{1-\alpha}(\ln\ln N)^{-1} \to \infty, \ \text{as } N \to \infty.$$

Proof. Let f^N and g^N be defined by

$$f^N = \sum_{\frac{N}{2} \leq j \leq \frac{4N}{5}} \mathscr{F}^{-1}\Big(\frac{\widehat{\phi}(2^{-j}\xi)}{2^{jd} j^\alpha}\Big), \quad g^N = \mathscr{F}^{-1}\Big(\frac{\widehat{\psi}(2^{-N}\xi)}{2^{Nd}(\ln\ln N)}\Big),$$

where $\widehat{\phi}$ and $\widehat{\psi}$ are consistent with those in (3.1) and (3.2). From (3.6) and (3.7), one easily deduces that

$$\|f^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \leq CN^{\frac{1}{q}-\alpha}, \quad \|g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \leq C(\ln\ln N)^{-1}.$$

Therefore, for large enough N, due to $\alpha > \frac{1}{q}$, we have

$$\|f^N\| + \|g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \lesssim (\ln\ln N)^{-1}.$$

By embedding $\dot{B}^0_{\infty,q} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,q}$ and $\|h\|_{L^\infty} \geq |h(0)| = |\int_{\mathbb{R}^d} \widehat{h}(\xi)\,\mathrm{d}\xi|$, one obtains that

$$\begin{aligned}
&\|f^N g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \geq C\|f^N g^N\|_{\dot{B}^0_{\infty,q}} \geq C\|\dot{\Delta}_N(f^N g^N)\|_{L^\infty} \\
\geq & C\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{\varphi}(2^{-N}\xi)\widehat{f^N}(\eta)\widehat{g^N}(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\Big| \\
= & C\Big|\sum_{\frac{N}{2} \leq j \leq \frac{4N}{5}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{\varphi}(2^{-N}\xi)\frac{\widehat{\phi}(2^{-j}\eta)}{2^{jd}j^\alpha}\frac{\widehat{\psi}(2^{-N}(\xi-\eta))}{2^{Nd}(\ln\ln N)}\,\mathrm{d}\eta\,\mathrm{d}\xi\Big| \\
= & C\Big|\sum_{\frac{N}{2} \leq j \leq \frac{4N}{5}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{\varphi}(\tilde{\xi})\frac{\widehat{\phi}(\tilde{\eta})}{j^\alpha}\frac{\widehat{\psi}(\tilde{\xi}-2^{j-N}\tilde{\eta})}{\ln\ln N}\,\mathrm{d}\tilde{\eta}\,\mathrm{d}\tilde{\xi}\Big|.
\end{aligned}$$

We easily deduce from (3.25) that

$$\|f^N g^N\|_{\dot{B}^{\frac{d}{p}}_{p,q}} \geq C \sum_{\frac{N}{2} \leq j \leq \frac{4N}{5}} j^{-\alpha}(\ln\ln N)^{-1} \geq CN^{1-\alpha}(\ln\ln N)^{-1}.$$

Therefore, we complete this proof. □

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