SOME INTEGRAL INEQUALITIES VIA CAPUTO AND LIOUVILLE FRACTIONAL INTEGRAL OPERATORS FOR $m-$CONVEX FUNCTIONS

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Abstract. This short study consists of two parts, firstly we obtain some inequalities on Caputo Fractional derivatives using the elementary inequalities. Secondly we establish several new inequalities including Caputo fractional derivatives for $m-$Convex functions. In general, in this work we obtain upper bounds for the left sides of Lemma 1[10] and lemma 2[20] .

1. Introduction

In mathematical analysis, we know roughly that the classical concept of derivative can be expressed in a single way as the limit of the slopes of secant lines for $\triangle x \to 0$. When it comes to Fractional Derivatives (FC),

Caputo left-sided derivative

$$ {}^{C}D_{a^{+}}^{\alpha}[f](x)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\xi)^{n-\alpha-1}\frac{d^{n}(f(\xi))}{d\xi^{n}}dt,\quad x>a $$

Caputo right-sided derivative

$$ {}^{C}D_{b^{-}}^{\alpha}[f](x)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(\xi-x)^{n-\alpha-1}\frac{d^{n}(f(\xi))}{d\xi^{n}}dt,\quad x<b $$

it has to do with the concept of tangent. As can be seen, In Caputo, she first calculated the derivative of the integer order and then the integral of the non-integer order.

Liouville left-sided derivative

$$ I_{a^{+}}^{\alpha}[f](x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}(x-\xi)^{n-\alpha-1}f(\xi)\,d\xi,\quad x>a $$

Liouville right-sided derivative

$$ I_{b^{-}}^{\alpha}[f](x)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{x}^{b}(\xi-x)^{n-\alpha-1}f(\xi)\,d\xi,\quad x<b $$

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In Liouville, the opposite of the process in Caputo is valid. That is, first the integral of the non-integer order is calculated and then the derivative of the integer order is calculated.

Since the Caputo Fractional derivative is more restritive than the Liouville one, both derivatives are defined by means of the each other.

$$ {}^{C}D_{a^+}^{\alpha}\left[f\right](x) : I_{a^+}^{n-\alpha}\left[f^{(n)}\right](x) \tag{1.1} $$

$$ {}^{C}D_{b^-}^{\alpha}\left[f\right](x) : (-1)^{n} I_{b^-}^{n-\alpha}\left[f^{(n)}\right](x) $$

Specially, $\alpha = 0$ then the left and right Caputo derivatives are equal to each other.

Today, the concept of FC dates back to Leibniz. Leibniz discussed the concept of FC with her contemporaries in 1695. Euler noticed in 1738 what a problem non-integer order derivatives (FC) pose. By 1822, Fourier gave the first definition of non-positive integers by using integral notation to define the derivative. Abel in 1826 and Liouville in 1832 gave versions of the non-integer order derivative.

Let us present the necessary definitions and preliminary information that we will use in this study.

Definition 1. [9] *The function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for all $x, y \in [0, b]$ and $t \in [0, 1]$, we have:*

$$ f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) $$

For many papers connected with m-convex and $(\alpha, m)-$convex functions see $([1-8])$ and the references therein.

We will need the modified forms of the m-convex function:

$m-$convexity of f :

$$ f\left(tx + (1-t)y\right) = f(tx + m(1-t)\frac{y}{m}) \leq tf(x) + m(1-t)f(\frac{y}{m}) $$

m-convexity of $\left|f^{(n+1)}\right|$:

$$ \left| f^{(n+1)}\left(tx + (1-t)y\right)\right| = \left|f^{(n+1)}(tx + m(1-t)\frac{y}{m})\right| \leq t\left|f^{(n+1)}\left(x\right)\right| + m(1-t)\left|f^{(n+1)}(\frac{y}{m})\right| $$

m-convexity of $\left|f^{(n+1)}\right|^{q}$. :

$$ \left| f^{(n+1)}\left(tx + (1-t)y\right)\right|^{q} = \left|f^{(n+1)}(tx + m(1-t)\frac{y}{m})\right|^{q} \leq t\left|f^{(n+1)}\left(x\right)\right|^{q} + m(1-t)\left|f^{(n+1)}(\frac{y}{m})\right|^{q} $$

Definition 2. [11] *Let $\alpha \geq 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^{n}[a, b]$, the space of functions having $n-$ th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:*

$$ ({}^{C}D_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_{a}^{x} \frac{f^{(n)}\left(t\right)}{\left(x-t\right)^{\alpha-n+1}} dt, \quad x > a \tag{1.2} $$

and

$$ ({}^{C}D_{b^-}^{\alpha} f)(x) = \frac{(-1)^{n}}{\Gamma\left(n-\alpha\right)} \int_{x}^{b} \frac{f^{(n)}\left(t\right)}{\left(t-x\right)^{\alpha-n+1}} dt, \quad x < b $$

If $n=1$ and $\alpha=0$, we have $({}^{C}D_{a+}^{0}f)(x)=({}^{C}D_{b-}^{0}f)(x)=f(x)$. For many papers connected fractional operators see ([12 − 25])

We will also use the well-known Hölder inequality in the literature : let be $p>1$ and $p^{-1}+q^{-1}=1$, If f and g reel functions on $[a,b]$ such that $|f|^{p}$ *and* $|f|^{q}$ are integrable on $[a,b]$.

Then

$$\int_{a}^{b}|f(x)\,g(x)|\,dx\leq\left(\int_{a}^{b}|f(x)|^{p}\,dx\right)^{\frac{1}{p}}\left(\int_{a}^{b}|g(x)|^{q}\,dx\right)^{\frac{1}{q}}.$$

In [10] ,Farid et al. established the following identity for Caputo fractional operators.

Lemma 1. *In* [10] *Let* $f:[a,b]\to\mathbb{R}$, *be a differentiable mapping on (a, b) with* $a<b$. *If* $f^{(n+1)}\in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^{C}D_{a+}^{\alpha}f)(b)+(-1)^{n}({}^{C}D_{b-}^{\alpha}f)(a)\right]$$

$$=\frac{b-a}{2}\int_{0}^{1}\left[(1-t)^{n-\alpha}-t^{n-\alpha}\right]f^{(n+1)}(tx+(1-t)y)\,dt$$

Lemma 2. *In* [20] *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I *,where* $a,b\in I$ *with* $t\in[0,1]$. *If* $f^{(n+1)}\in L[a,b]$, *Then for all* $a\leq x<y\leq b$ *and* $\alpha>0$ *we have*

$$\frac{1}{y-x}f^{(n)}(y)-\frac{(-1)^{n}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left(C_{D_{y-}^{\alpha}}f\right)(x)=\int_{0}^{1}(1-t)^{n-\alpha}f^{(n+1)}(tx+(1-t)y)\,dt.$$

This work is a continuation of my work in [20]. Özdemir et al. constructed an identity for left sided Caputo derivatives in Lemma 2. In this study, we constructed differently a few inequalities for both right and left sided Caputo derivatives. The aim of this paper is to establish new upper bounds. To do this, we used some classical inequalities.

2. The Results

Theorem 1. *Let* $f:I\subset\mathbb{R}\to\mathbb{R},I\subset[0,\infty)$ *,be a differentiable function on I such that* $f\in AC^{n}L[a,b]$ *where* $a,b\in I$ *with* $0<a<t<x\leq b$. *If* $\alpha>0$ *and* $\alpha\notin\{1,2,3,...\}$, $n=[\alpha]+1$, $f^{(n)}>0$. *Then*

$$\int_{a}^{b}f^{(n)}(t)\,dt\leq \tag{2.1}$$

$$\frac{\Gamma(n-\alpha)\left[({}^{C}D_{a^+}^{\alpha}f)(x)+(-1)^{n}({}^{C}D_{b^-}^{\alpha}f)(x)\right]+\Gamma(\alpha-n+2)\left[({}^{C}D_{a^+}^{\alpha}f)(x)+(-1)^{\alpha}({}^{C}D_{b^-}^{\alpha}f)(x)\right]}{2}$$

Proof. First of all, since $(x-t)>0$ we can write the following inequality

$$(x-t)^{n-\alpha-1}+\frac{1}{(x-t)^{n-\alpha-1}}=(x-t)^{n-\alpha-1}+(x-t)^{\alpha-n+1}>2$$

Now If we multiply each side of the final inequality by $f^{(n)}>0$ and then integrate it over $[a,b]$ we have □

$$\begin{aligned}
2\int_a^b f^{(n)}(t)\,dt &< \int_a^b (x-t)^{n-\alpha-1}\ f^{(n)}(t)\,dt+\int_a^b (x-t)^{\alpha-n+1}f^{(n)}(t)\,dt\\
&= \int_a^x (x-t)^{n-\alpha-1}\ f^{(n)}(t)\,dt+\int_x^b (x-t)^{\alpha-n-1}\ f^{(n)}(t)\,dt\\
&\quad+\int_a^x (x-t)^{\alpha-n+1}\ f^{(n)}(t)\,dt+\int_x^b (x-t)^{\alpha-n+1}\ f^{(n)}(t)\,dt\\
&= \Gamma(n-\alpha)({}^{C}D_{a^+}^{\alpha}f)(x)+(-1)^{n}\Gamma(n-\alpha)({}^{C}D_{b^-}^{\alpha}f)(x)\\
&\quad+\Gamma(\alpha-n+2)({}^{C}D_{a^+}^{\alpha}f)(x)+(-1)^{\alpha}\Gamma(\alpha-n+2)({}^{C}D_{b^-}^{\alpha}f)(x)
\end{aligned}$$

Taking into account definition (1.2) we obtain inequality (2.1)

Theorem 2. *Let $\alpha>0$,and $\alpha\notin\{1,2,3,...\}$, $n=[\alpha]+1$, $f^{(n)}>0$. If $f:I\subset\mathbb{R}\to\mathbb{R}$, $I\subset[0,\infty)$,be a differentiable function on I such that $f\in AC^{n}L[a,b]$.*

where $a,b\in I$ with $0<t\leq a\leq x\leq b$. Then the following inequality holds :

$$\int_a^b\sqrt{|(x-t)|^{2(n-\alpha)}\,dt}\leq\Gamma(n-\alpha+1)\frac{\left[({}^{C}D_{a^+}^{\alpha}f)(x)+(-1)^{n}({}^{C}D_{b^-}^{\alpha}f)(x)\right]}{2}\tag{2.2}$$

Proof. According to relation between the Geometric and Arithmetic means we can write the basic inequality:

$$\begin{aligned}
\sqrt{|(x-t)|^{2(n-\alpha)}} &= \sqrt{|(x-t)|^{(n-\alpha)}|(t-x)|^{(n-\alpha)}}\\
&\leq \frac{1}{2}\left[|(x-t)|^{(n-\alpha)}+|(t-x)|^{(n-\alpha)}\right]\\
&\leq \frac{1}{2}\left[|(x-t)|^{(n-\alpha)}+|(t-x)|^{(n-\alpha)}\right]f^{(n)}(t)\\
&= \frac{1}{2}\left[|(x-t)|^{(n-\alpha)}f^{(n)}(t)+|(t-x)|^{(n-\alpha)}f^{(n)}(t)\right]
\end{aligned}$$

Now, If we integrate both sides of the first and last terms over $[a,b]$ we obtain □

$$\begin{aligned}
\int_a^b \sqrt{|(x-t)|^{2(n-\alpha)}}dt &\leq \frac{1}{2}\left[\int_a^x |(x-t)|^{(n-\alpha)} f^{(n)}(t)\,dt + \int_x^b |(t-x)|^{(n-\alpha)} f^{(n)}(t)\,dt\right] \\
&= \frac{1}{2}\left[\int_a^x |(x-t)|^{(n-\alpha)} f^{(n)}(t)\,dt + \int_x^b |(x-t)|^{(n-\alpha)} f^{(n)}(t)\,dt\right] \\
&\quad +\frac{1}{2}\left[\int_a^x |(t-x)|^{(n-\alpha)} f^{(n)}(t)\,dt + \int_x^b |(t-x)|^{(n-\alpha)} f^{(n)}(t)\,dt\right] \\
&= \frac{1}{2}\left[\int_a^x (x-t)^{n-\alpha} f^{(n)}(t)\,dt - \int_b^x (x-t)^{n-\alpha} f^{(n)}(t)\,dt\right] \\
&\quad +\frac{1}{2}\left[(t-x)^{n-\alpha} f^{(n)}(t)\,dt - \int_b^x |(t-x)|^{(n-\alpha)} f^{(n)}(t)\,dt\right] \\
&= \Gamma(n-\alpha+1)\frac{\left[({}^C D^{\alpha}_{a^+} f)(x) + (-1)^n ({}^C D^{\alpha}_{b^-} f)(x)\right]}{2}
\end{aligned}$$

This completes the proof of inequality (2.2)

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}, I \subset [0,\infty)$, be a differentiable function on I such that $f^{(n+1)} \in AC^n L[a,b]$. If and $\left|f^{(n+1)}\right|$ is $m-$convex on $[x,y]$ for $t \in [0,1]$, then for all $\alpha > 0$,and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1,\ m \in (0,1]$ we have*

$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{n}(b)}{2} - \frac{\Gamma(\alpha-n+1)}{2(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{a^+} f)(b) + (-1)^n ({}^C D^{\alpha}_{b^-} f)(a)\right]\right| \\
&\leq \frac{b-a}{2}\left(\frac{1}{2(n-\alpha+1)} + \frac{1}{2(n-\alpha+2)}\right)\left(\left|f^{(n+1)}(a)\right| + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right)
\end{aligned} \tag{2.3}$$

Proof. We know from our elementary knowledge that for $\alpha \in [0,1]$ and $\forall t_1, t_2 \in [0,1]$, $\left|t_1^{n-\alpha} - t_2^{n-\alpha}\right| \leq |t_1 - t_2|^{n-\alpha}$. □

let be

$$K = \frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{a^+} f)(b) + ({}^C D^{\alpha}_{b^-} f)(a)\right]$$

In Lemma 1, using the properties of the modulus as well as the fact that $\left|f^{(n+1)}\right|$ is $m-$convex on $[a,b]$, we can write the relation below.

$$\begin{aligned}
|K| &\leq \frac{b-a}{2}\int_0^1 \left|(1-t)^{n-\alpha}-t^{n-\alpha}\right| \left|f^{(n+1)}(ta+(1-t)b)\right| dt \\
&\leq \frac{b-a}{2}\int_0^1 |1-2t|^{n-\alpha}\left|f^{(n+1)}(ta+(1-t)b)\right| dt \\
&= \frac{b-a}{2}\int_0^1 |1-2t|^{n-\alpha}\left|f^{(n+1)}\left(ta+m(1-t)\frac{b}{m}\right)\right| dt \\
&\leq \frac{b-a}{2}\left\{\begin{array}{c} \left|f^{(n+1)}(a)\right|\left(\int_0^{\frac{1}{2}} t(1-t)^{n-\alpha}dt+\int_{\frac{1}{2}}^1 t(2t-1)^{n-\alpha}dt\right)+ \\ +m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\left(\int_0^{\frac{1}{2}}(1-t)(1-2t)^{n-\alpha}dt+\int_{\frac{1}{2}}^1(1-t)(2t-1)^{n-\alpha}dt\right)\end{array}\right\}
\end{aligned}$$

Calculate the integrals in parentheses and multiply by their coefficients,we obtain inequality (2.3) .

Corollary 1. *If* $\alpha = n \in \{1,2,3,...\}$ *and usual derivative* $f^{(n)}(a)$ *of order* n *exists, then Caputo fractional derivatives* $({}^C D^{\alpha}_{a^+} f)(a)$ *coincides with* $f^{(n)}(a)$ *whereas* $({}^C D^{\alpha}_{b^-} f)(b)$ *coincides* $f^{(n)}(b)$ *to a constant multipler* $(-1)^n$ *. Thus if we choose and* $\alpha = 0$ *In* (2.3) *with* $m = 1$ *we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[f(b)+(-1)^n f(a)\right]\right| \leq \frac{b-a}{4}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$

Theorem 4. *Let* $f:[a,b]\to(-\infty,\infty)$ *be a differentiable mapping on* $a<b$*, If* $\alpha>0$ *and* $\alpha\notin\{1,2,3,...\}$*,* $n=[\alpha]+1$*,* $q>1$*,* $p=\frac{q}{q-1}$ *and* $f^{(n+1)}\in L[a,b]$ *and* $\left|f^{(n+1)}\right|^q$ *is* $m-$*convex,* $m\in(0,1]$

Then the following inequality holds:

$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{a^+} f)(b)+(-1)^n({}^C D^{\alpha}_{b^-} f)(a)\right]\right| \\
&\leq \frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{1}{(p(n-\alpha)+1)^{\frac{1}{p}}}\right)\left(\left|f^{(n+1)}(a)\right|^q+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned} \tag{2.4}$$

Proof. Let the left side of Lemma 1 be K. □

Since $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$, $\left|t_t^{n-\alpha}-t_2^{n-\alpha}\right|\leq|t_1-t_2|^{n-\alpha}$ we can write the following inequality with properties of modulus:

$$|K|\leq\frac{b-a}{2}\int_0^1|1-2t|^{n-\alpha}\left|f^{(n+1)}(ta+(1-t)b)\right|dt$$

By applying Hölder's inequality to the right hand side of the above inequality with properties of modulus and after If we use $\left| f^{(n+1)} \right|^q$ is $m-$convex, we have

$$\begin{aligned}
|K| &\leq \frac{b-a}{2}\int_0^1 |1-2t|^{n-\alpha}\left|f^{(n+1)}\left(ta+(1-t)b\right)\right|dt \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^{p(n-\alpha)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}\left(ta+m(1-t)\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{(p(n-\alpha)+1)^{\frac{1}{p}}}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n+1)}(a)\right|^q+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{1}{(p(n-\alpha)+1)^{\frac{1}{p}}}\right)^{\frac{1}{p}}\left(\left|f^{(n+1)}(a)\right|^q+m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

This completes the proof of inequality(2.4) .Here it can be easily checked that

$$\begin{aligned}
\left(\int_0^1 |1-2t|^{p(n-\alpha)}dt\right)^{\frac{1}{p}} &= \frac{1}{(p(n-\alpha)+1)^{\frac{1}{p}}}, \\
\left|f^{(n+1)}(a)\right|^q\int_0^1 tdt &= \frac{\left|f^{(n+1)}(a)\right|^q}{2}, \\
m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q\int_0^1(1-t)\,dt &= m\frac{\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|^q}{2}
\end{aligned}$$

Corollary 2. *If we write corollary 1 for the Theorem* (2.4) *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[f(b)+(-1)^n f(a)\right]\right| \leq \frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}$$

On the other hand , let $a_1=\left|f''(a)\right|^q$, $b_1=\left|f''(b)\right|^q$. *Here* $0<\frac{q-1}{q}<1$, *for* $q>1$. *Using the fact that* $\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s+b_k^s$, *for* $(0\leq s<1)$,

$a_1,\ a_2,\ a_3,...a_n \geq 0,\ b_1,\ b_2,\ b_3,...b_n \geq 0$ and Considering that

$$\lim_{p\to\infty}\frac{1}{(p+1)^{\frac{1}{p}}}=1 \text{ and } \lim_{q\to\infty}\frac{1}{2^{1+\frac{1}{q}}}=\frac{1}{2}$$

we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[f(b)+(-1)^n f(a)\right]\right| \leq \frac{b-a}{2}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$

Remark 1. *Note that the right side of Corollary 1 is a better upper bound than the right side of Corollary 2.*

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R},\ I \subset [0,\infty)$, be a differentiable function on I such that $f^{(n+1)} \in L[a,b]$ with $a \leq x < y \leq b,\ t \in [0,1]$ .If $f^{(n+1)}$ is $m-$convex on $[x,y]$. The for all $\alpha > 0, m \in (0,1]$*

(2.5)
$$\frac{1}{y-x} f^{(n)}(y) - \frac{(-1)^n \Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}} \left(C_{D^{\alpha}_{y^-}} f\right)(x) \leq f(x) \frac{n-\alpha}{n-\alpha+2} \beta(2, n-\alpha) + mf\left(\frac{y}{m}\right) \frac{1}{n-\alpha+1}.$$

Proof. From lemma 2, we have

$$\begin{aligned}
\frac{1}{y-x} f^{(n)}(y) - \frac{(-1)^n \Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}} \left(C_{D^{\alpha}_{y^-}} f\right)(x) &= \int_0^1 (1-t)^{n-\alpha} f^{(n+1)}\left(tx + m(1-t)\frac{y}{m}\right) dt. \\
&\leq f(x) \int_0^1 t(1-t)^{n-\alpha} dt + mf\left(\frac{y}{m}\right) \int_0^1 (1-t)^{2(n-\alpha)} dt \\
&= f(x) \beta(2, n-\alpha+1) + mf\left(\frac{y}{m}\right) \frac{1}{n-\alpha+1} \\
&= f(x) \frac{n-\alpha}{n-\alpha+2} \beta(2, n-\alpha) + mf\left(\frac{y}{m}\right) \frac{1}{n-\alpha+1}
\end{aligned}$$

which gives the required inequality (2.5). Here we used the property of the known function β.

$$\beta(2, n-\alpha+1) = \frac{n-\alpha}{n-\alpha+2} \beta(2, n-\alpha).$$

Corollary 3. *If we choose $x = a,\ y = b$ and $m = 1$ in (2.5) we have the following inequality*

$$\frac{1}{b-a} f^{(n)}(b) - \frac{(-1)^n \Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}} \left(C_{D^{\alpha}_{b^-}} f\right)(a) \leq f(a) \frac{n-\alpha}{n-\alpha+2} \beta(2, n-\alpha) + f(b) \frac{1}{n-\alpha+1}$$

□

Theorem 6. *$\alpha > 0$, let $f : I \subset \mathbb{R} \to \mathbb{R},\ I \subset [0,\infty)$, be a differentiable function on I such that $f^{(n+1)} \in L[a,b]$ with $a \leq x < y \leq b,\ t \in [0,1]$ .If $\left|f^{(n+1)}\right|^q$ is $m-$convex on $[x,y]$, $q > 1$, $p = \frac{q}{q-1}, m \in (0,1]$*

Then

$$\left| \frac{1}{y-x} f^{(n)}(y) - \frac{(-1)^n \Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}} \left(C_{D^{\alpha}_{y^-}} f\right)(x) \right|$$

(2.6)
$$\leq \left(\frac{1}{(n-\alpha+1)^{\frac{1}{p}}}\right) \left(\left|f^{(n+1)}\right|^q (x) \beta(2, n-\alpha+1) + m \left|f^{(n+1)}\right|^q \left(\frac{y}{m}\right) \frac{1}{2(n-\alpha)+1}\right)^{\frac{1}{q}}$$

Proof. Firstly, from lemma 2 and with properties of modulus and $m-$convex of the function $\left|f^{(n+1)}\right|^q$, secondly If we use power mean inequality ;

$$\left|\frac{1}{y-x}f^{(n)}(y)-\frac{(-1)^{n}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha+1}}\left(C_{D^{\alpha}_{y^{-}}}f\right)(x)\right|\leq\int_{0}^{1}\left|(1-t)^{n-\alpha}f^{(n+1)}(tx+(1-t)y)\right|dt.$$

$$=\int_{0}^{1}(1-t)^{n-\alpha}\left|f^{(n+1)}\left(tx+m(1-t)\frac{y}{m}\right)\right|dt$$

$$=\left(\int_{0}^{1}(1-t)^{n-\alpha}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}(1-t)^{n--\alpha}\left|f^{(n+1)}\left(tx+m(1-t)\frac{y}{m}\right)\right|^{\frac{1}{q}}dt\right)^{\frac{1}{q}}$$

□

$$\leq\left(\frac{1}{(n-\alpha+1)^{\frac{1}{p}}}\right)\cdot\left(\left|f^{(n+1)}\right|^{q}(x)\int_{0}^{1}t(1-t)^{n--\alpha}dt+m\left|f^{(n+1)}\right|^{q}\left(\tfrac{y}{m}\right)\int_{0}^{1}(1-t)^{2(n-\alpha)}dt\right)^{\frac{1}{q}}$$

$$=\left(\frac{1}{(n-\alpha+1)^{\frac{1}{p}}}\right)\left(\left|f^{(n+1)}\right|^{q}(x)\beta(2,n-\alpha+1)+m\left|f^{(n+1)}\right|^{q}\left(\frac{y}{m}\right)\frac{1}{2(n-\alpha)+1}\right)^{\frac{1}{q}}$$

which gives the desired inequality(2.6). Here we used

$$\beta(2,n-\alpha+1)=\int_{0}^{1}t(1-t)^{n-\alpha}dt\quad\text{and}\quad\int_{0}^{1}(1-t)^{2(n-\alpha)}dt=\frac{1}{2(n-\alpha)+1}$$

Corollary 4. *If we choose $x=a,\ y=b$ and $m=1$ in* (2.6)

$$\left|\frac{1}{b-a}f^{(n)}(b)-\frac{(-1)^{n}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}\left(C_{D^{\alpha}_{y^{-}}}f\right)(a)\right|$$

$$\leq\left(\frac{1}{(n-\alpha+1)^{\frac{1}{p}}}\right)\left(\left|f^{(n+1)}\right|^{q}(a)\beta(2,n-\alpha+1)+\left|f^{(n+1)}\right|^{q}(b)\frac{1}{2(n-\alpha)+1}\right)^{\frac{1}{q}}$$

The result in corollary (3) is more general than the result in corollary (2).

3. CONCLUSION

Where it is known that a subset of the set of real numbers has an infinite number of upper bounds. But, the smallest upper bound of the same set is unique. In terms of optimization theory, the aim is to capture the supremum of the upper bounds. Inequalities involving both right-sided and left-sided FC derivatives of non-integer order offer new estimations for integral inequalities under convex functions. Considering (1.1) researchers working in this field can write the above theorems once for Liouville derivatives.

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