Scaling laws for Rayleigh-Bénard convection between Navier-slip boundaries

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Abstract

We consider the two-dimensional Rayeigh-Bénard convection problem between Navier-slip fixed-temperature boundary conditions and present a new upper bound for the Nusselt number. The result, based on a localization principle for the Nusselt number and an interpolation bound, exploits the regularity of the flow. On one hand our method yields a shorter proof of the celebrated result in Whitehead & Doering (2011) in the case of free-slip boundary conditions. On the other hand, its combination with a new, refined estimate for the pressure gives a substantial improvement of the interpolation bounds in Drivas *et al.* (2022) for slippery boundaries. A rich description of the scaling behaviour arises from our result: depending on the magnitude of the Prandtl number and slip-length, our upper bounds indicate five possible scaling laws: $Nu \sim (L_s^{-1}Ra)^{\frac{1}{3}}$, $Nu \sim (L_s^{-\frac{2}{5}}Ra)^{\frac{5}{13}}$, $Nu \sim Ra^{\frac{5}{12}}$, $Nu \sim Pr^{-\frac{1}{6}}(L_s^{-\frac{4}{3}}Ra)^{\frac{1}{2}}$ and $Nu \sim Pr^{-\frac{1}{6}}(L_s^{-\frac{1}{3}}Ra)^{\frac{1}{2}}$.

1 Introduction

In this paper we consider a layer of fluid trapped between two parallel horizontal plates held at different temperatures. Specifically, in the rectangular domain $\Omega = [0,\Gamma] \times [0,1]$ the velocity $u = (u_1(\mathbf{x},t), u_2(\mathbf{x},t))$ and temperature $T = T(\mathbf{x},t)$ of the buoyancy-driven flow satisfy

$$\frac{1}{Pr}(\partial_t u + u\cdot\nabla u) - \Delta u + \nabla p = RaT\boldsymbol{e}_2 \tag{1}$$

$$\nabla\cdot u = 0 \tag{2}$$

$$\partial_t T + u\cdot\nabla T - \Delta T = 0\,, \tag{3}$$

with

$$\begin{aligned} T &= 0 \qquad \text{at } x_2 = 1 \\ T &= 1 \qquad \text{at } x_2 = 0\,. \end{aligned} \tag{4}$$

In (1), p denotes the pressure, $\boldsymbol{e}_2 = (0,1)^T$ the unit vector and $\mathbf{x} = (x_1, x_2)$ the spatial variable with $x_1 \in [0,\Gamma]$ and $x_2 \in [0,1]$. In the dimensionless equations of motion (1)–(3) (notice that the temperature gap and the height of the container are equal to one) only two non-dimensional numbers appear: the Rayleigh number $Ra = \frac{g\alpha\delta T h^3}{\kappa\nu}$ and the Prandtl number $Pr = \frac{\nu}{\kappa}$. In these definitions g is the gravitational constant, α the thermal expansion coefficient, ν the kinematic viscosity, κ the thermal diffusivity, h the height of the container and $\delta T = T_{\text{bottom}} - T_{\text{top}}$ the temperature gap. In the horizontal variable x_1 all variables are periodic. We study this problem assuming Navier-slip boundary conditions for the velocity

$$\begin{aligned} u_2 = 0, &\quad \partial_2 u_1 = -\tfrac{1}{L_s}u_1 \quad \text{at } x_2 = 1 \\ u_2 = 0, &\quad \partial_2 u_1 = \tfrac{1}{L_s}u_1 \qquad \text{at } x_2 = 0, \end{aligned} \tag{5}$$

where L_s is the constant slip-length. We are interested in quantifying the heat transport in the upward direction as measured by the non-dimensional Nusselt number

$$Nu = \left\langle \int_0^1 (u_2 T - \partial_2 T)\, dx_2 \right\rangle \tag{6}$$

where

$$\langle \cdot \rangle = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{\Gamma} \int_0^\Gamma (\cdot)\, dx_1\, ds.$$

This number, of utmost relevance in geophysics and industrial applications (Plumley & Julien, 2019), is predicted to obey a power law scaling of the type

$$Nu \sim Ra^\alpha Pr^\beta.$$

Although physical arguments suggest certain scaling exponents α and β (Malkus (1954); Kraichnan (1962); Spiegel (1971); Siggia (1994)) and where transitions between scalings could occur (Ahlers *et al.*, 2009), these theories need to be validated. While experiments are expensive and difficult (Ahlers, 2006), numerical studies are limited by the lack of computational power in reaching high-Rayleigh number regimes (Plumley & Julien, 2019). Even in two spatial dimensions there have been recent debates about the presence/absence of evidence of the "ultimate scaling" (scaling that holds in the regime of very large Rayleigh numbers) for the Nusselt number (Zhu *et al.*, 2018; Doering *et al.*, 2019; Zhu *et al.*, 2019; Doering, 2020). We use mathematical analysis in order to derive universal upper bounds for the Nusselt number, which serve as a rigorous indication for the scaling exponents holding in the turbulent regime $Ra \to \infty$. The properties of boundary layers and their thickness play a central role in the scaling laws for the Nusselt number in Rayleigh-Bénard convection (see Nobili (2023) and references therein). For this reason, it is interesting to study how heat transport properties change when varying the boundary conditions. In particular, we ask the following question: are there boundary conditions inhibiting or enhancing heat transport compared to the classical *no-slip* boundary conditions? While many theoretical studies focus on no-slip boundary conditions (Doering & Constantin, 1996, 2001; Doering *et al.*, 2006; Otto & Seis, 2011; Goluskin & Doering, 2016; Tobasco & Doering, 2017), other reasonable boundary conditions have been far less explored. In the early 2000, Ierley *et al.* (2006) considered the Rayleigh-Bénard convection problem at infinite Prandtl number and free-slip boundary conditions: combining the Busse's asymptotic expansion in multi-boundary-layers solutions (multi$-\alpha$ solutions) and the Constantine & Doering *background field method* approach (Doering & Constantin (1996, 1994)), the authors showed $Nu \lesssim Ra^{\frac{5}{12}}$, optimizing on all background profiles satisfying a spectral constraint. Inspired by this results and a numerical study in Otero's thesis (Otero, 2002) indicating the $Ra^{\frac{5}{12}}$-scaling, Doering & Whitehead rigorously proved

$$Nu \lesssim Ra^{\frac{5}{12}},$$

for the two dimensional, finite Prandtl model (Whitehead & Doering (2011)) and for the three dimensional, infinite Prandtl number model (Whitehead & Doering (2012)) by an elaborate application of the background field method. By a perturbation argument around Stokes equations, Wang & Whitehead (2013) proved

$$Nu \lesssim Ra^{\frac{5}{12}} + \mathrm{Gr}^2 Ra^{\frac{1}{4}}$$

in three-dimensions and small Grashof number ($\mathrm{Gr} = \frac{Ra}{Pr}$).

In many physical situations the Navier-slip boundary conditions are used to describe the presence of slip at the solid–liquid interface (Uthe *et al.*, 2022; Neto *et al.*, 2005). We notice that, for any finite $L_s > 0$, these conditions imply vorticity production at the boundary. In the limit of infinite slip-length, the Navier-slip boundary conditions reduce to free-slip boundary conditions, while in the limit $L_s \to 0$ they converge to the no-slip boundary conditions.

Inspired by the seminal paper Whitehead & Doering (2011), Drivas, Nguyen and the second author of this paper considered the two-dimensional model with Navier-slip boundary conditions and rigorously proved the upper bound

$$Nu \lesssim Ra^{\frac{5}{12}} + L_s^{-2} Ra^{\frac{1}{2}} \tag{7}$$

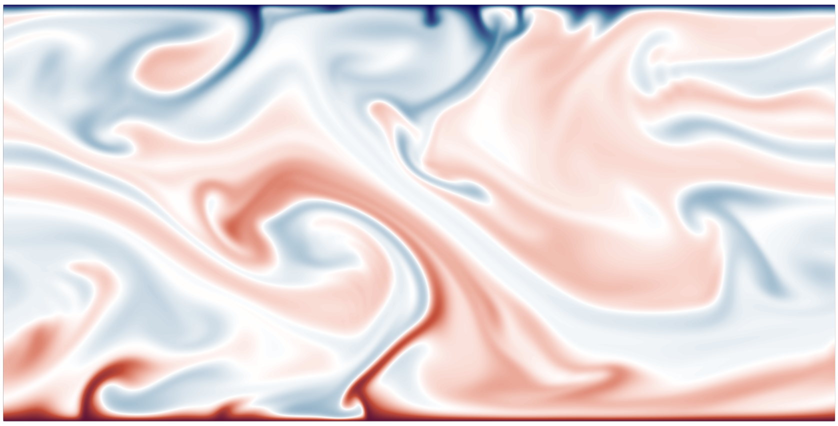

Figure 1: Snapshot of a simulation with $Pr = 1$, $L_s = 1$ and $Ra = 10^8$.

when $Pr \geq Ra^{\frac{3}{4}} L_s^{-1}$ and $L_s \gtrsim 1$ (Drivas *et al.*, 2022) . The authors refer to (7) as *interpolation bound*: assuming $L_s \sim Ra^{\alpha}$ with $\alpha \geq 0$ (the justification of this choice can be found in the appendix of Bleitner & Nobili (2024)), it can be easily deduced that

$$Nu \lesssim \begin{cases} Ra^{\frac{5}{12}} & \text{if } \alpha \geq \frac{1}{24} \\ Ra^{\frac{1}{2}-2\alpha} & \text{if } 0 \leq \alpha \leq \frac{1}{24} \, . \end{cases}$$

In Bleitner & Nobili (2024), the authors generalized the result in Drivas *et al.* (2022) to the case of rough walls and Navier-slip boundary conditions, proving interpolation bounds exhibiting explicit dependency on the spatially varying friction coefficient and curvature.

Given the model (1),(2),(3) with (4) and (5), the objective of this paper is twofold: on one hand we want to apply the so called "direct method" (as outlined in Otto & Seis (2011)) to derive upper bounds on the Nusselt number exhibiting a transparent relation with the thermal boundary layer's thickness. For discussions about different approaches to derive bounds on the Nusselt number we refer the readers to Chernyshenko (2022) and references therein. On the other hand, we aim at improving the bound in Drivas *et al.* (2022) by refining the estimates on the pressure. Our new result is stated in the following

Theorem 1. *Suppose $u_0 \in W^{1,4}$ and $0 \leq T_0 \leq 1$. If $L_s \geq 1$, then*

$$Nu \lesssim Ra^{\frac{5}{12}} + L_s^{-\frac{1}{6}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}} . \tag{8}$$

If $0 < L_s < 1$, then

$$Nu \lesssim L_s^{-\frac{1}{3}} Ra^{\frac{1}{3}} + L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}} + L_s^{-\frac{2}{13}} Ra^{\frac{5}{13}} + Ra^{\frac{5}{12}} . \tag{9}$$

We first remark that, for $L_s \geq 1$, this result improves the upper bound in (7): in fact if $L_s \sim Ra^{\alpha}$, with $\alpha \geq 0$, our new result yields

$$Nu \lesssim \begin{cases} Ra^{\frac{5}{12}} & \text{if } Pr \geq Ra^{\frac{1}{2}-\alpha} \\ Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}-\frac{\alpha}{6}} & \text{if } Pr \leq Ra^{\frac{1}{2}-\alpha} \, . \end{cases}$$

In particular, when $\alpha = 0$ we notice a crossover at $Pr \sim Ra^{\frac{1}{2}}$ between the $Ra^{\frac{1}{2}}$ and the $Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$ scaling regime. This is reminiscent of the upper bound in Choffrut *et al.* (2016) for no-slip boundary conditions. Our result also covers the case $0 < L_s < 1$ and, in this region, we can detect four scaling regimes depending on the magnitude of Prandtl number and $L_s < 1$. The dominating terms in (8) and (9) are summarized in Table 1. Observe that for $Pr \to \infty$, the term $L_s^{-\frac{1}{3}} Ra^{\frac{1}{3}}$ is dominating in the region $0 < L_s < Ra^{-\frac{2}{7}}$. On one hand, this seems to indicate the (expected) transition from

Assumptions		Bound
$Ra^{\frac{11}{7}} \le Pr$	$Ra^{-\frac{5}{24}} \le L_s$	$Nu \lesssim Ra^{\frac{5}{12}}$
	$Ra^{-\frac{2}{7}} \le L_s \le Ra^{-\frac{5}{24}}$	$Nu \lesssim L_s^{-\frac{2}{13}} Ra^{\frac{5}{13}}$
	$Pr^{-\frac{1}{2}} Ra^{\frac{1}{2}} \le L_s \le Ra^{-\frac{2}{7}}$	$Nu \lesssim L_s^{-\frac{1}{3}} Ra^{\frac{1}{3}}$
	$L_s \le Pr^{-\frac{1}{2}} Ra^{\frac{1}{2}}$	$Nu \lesssim L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$
$Ra^{\frac{4}{3}} \le Pr \le Ra^{\frac{11}{7}}$	$Ra^{-\frac{5}{24}} \le L_s$	$Nu \lesssim Ra^{\frac{5}{12}}$
	$Pr^{-\frac{13}{40}} Ra^{\frac{9}{40}} \le L_s \le Ra^{-\frac{5}{24}}$	$Nu \lesssim L_s^{-\frac{2}{13}} Ra^{\frac{5}{13}}$
	$L_s \le Pr^{-\frac{13}{40}} Ra^{\frac{9}{40}}$	$Nu \lesssim L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$
$Ra^{\frac{1}{2}} \le Pr \le Ra^{\frac{4}{3}}$	$Pr^{-\frac{1}{4}} Ra^{\frac{1}{8}} \le L_s$	$Nu \lesssim Ra^{\frac{5}{12}}$
	$L_s \le Pr^{-\frac{1}{4}} Ra^{\frac{1}{8}}$	$Nu \lesssim L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$
$Pr \le Ra^{\frac{1}{2}}$	$Pr^{-1} Ra^{\frac{1}{2}} \le L_s$	$Nu \lesssim Ra^{\frac{5}{12}}$
	$1 \le L_s \le Pr^{-1} Ra^{\frac{1}{2}}$	$Nu \lesssim L_s^{-\frac{1}{6}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$
	$L_s \le 1$	$Nu \lesssim L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}}$

Table 1: Overview of the results in Theorem 1. The coloring corresponds the cases $L_s \le 1$ and $1 \le L_s$. In all the other (uncolored) cases, L_s may be smaller or bigger than one.

Navier-slip to no-slip boundary conditions in the bounds. In order to contextualize this remark, we recall that for no-slip boundary conditions, the upper bound $Nu \lesssim Ra^{\frac{1}{3}}$ was proven at infinite Prandtl number (Otto & Seis (2011)) and for $Pr \gtrsim Ra^{\frac{1}{3}}$ (Choffrut *et al.* (2016)). On the other hand we stress that our bounding method breaks down in the limit $L_s \to 0$ and, consequently, all but the last scaling prefactors in (9) blow up.

Differently from the result in Whitehead & Doering (2011), the proof of our theorem does not rely on the background field method but rather exploits the regularity properties of the flow through a localization principle. In fact the Nusselt number can be localized in the vertical variable

$$Nu = \frac{1}{\delta}\left\langle \int_0^\delta (u_2 T - \partial_2 T)\, dx_2 \right\rangle \le \frac{1}{\delta}\left\langle \int_0^\delta u_2 T\, dx_2 \right\rangle + \frac{1}{\delta}. \tag{10}$$

Here we used the boundary conditions for the temperature and the maximum principle $\sup_x |T(x,t)| \le 1$ for all t. Notice that this localization principle comes from the fact that the (long time and horizontal average of the) heat flux is the same for each $x_2 \in (0,1)$ and this can be deduced by merely using the non-penetration boundary conditions $u_2 = 0$ at $x_2 = 0, x_2 = 1$ (see the proof of Lemma 1 below).

The second crucial point in our proof is the following interpolation inequality

$$\frac{1}{\delta}\left\langle \int_0^\delta u_2 T\, dx_2 \right\rangle \le \frac{1}{2}\left\langle \int_0^1 |\partial_2 T|^2\, dx_2 \right\rangle + C\delta^3 \left\langle \int_0^1 |\omega|^2\, dx_2 \right\rangle^{\frac{1}{2}} \left\langle \int_0^1 |\partial_1 \omega|^2\, dx_2 \right\rangle^{\frac{1}{2}}, \tag{11}$$

which was first derived (in a slightly different form) in Drivas *et al.* (2022) and it is proved in Lemma 8 below. We again observe that this bound holds only relying on the assumption $u_2 = 0$ at the boundaries $x_2 = 0, x_2 = 1$.

In Lemma 5 we prove the new pressure estimate

$$\|p\|_{H^1} \le C\left(\frac{1}{L_s}\|\partial_2 u\|_{L^2} + \frac{1}{Pr}\|\omega\|_{L^2}\|\omega\|_{L^r} + Ra\|T\|_{L^2}\right), \tag{12}$$

and notice that the improvement (compared to the pressure estimate in Proposition 2.7 in Drivas *et al.* (2022)) lies in the first term on the right-hand side, stemming from the new trace-type estimate

$$\left| \int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\, dx_1 \right| \le 3\|p\|_{H^1}\|\partial_2 u\|_{L^2},$$

used to control the boundary terms in the H^1 pressure identity (28). We observe that the new pressure estimate enables us to treat the case of small slip-length, i.e. $0 < L_s \leq 1$. This small slip-length regime was not treatable in Drivas *et al.* (2022) due to the presence of the term $\frac{1}{L_s}\|\partial_1\omega\|_{L^2}$ on the right-hand side of (12), which, in turn, needed to be balanced by $\|\nabla\omega\|_{L^2}^2$ in order to ensure the positivity of the spectral constraint (see the proof of Proposition 3.4 in Drivas *et al.* (2022)).

In order to illustrate the proof of Theorem 1, we re-derive the seminal result in Whitehead & Doering (2011) in the case of free-slip boundary conditions, which corresponds to the case $L_s = \infty$ in (5). This new proof of the $Ra^{\frac{5}{12}}$-scaling is based on the combination of the localization principle together with the interpolation bound (11). First, we insert in (11) the bounds coming from the energy and enstrophy balances

$$\left\langle \int_0^1 |\omega|^2\, dx_2 \right\rangle = \left\langle \int_0^1 |\nabla u|^2\, dx_2 \right\rangle \leq NuRa,$$
$$\left\langle \int_0^1 |\partial_1\omega|^2\, dx_2 \right\rangle \leq \left\langle \int_0^1 |\nabla\omega|^2\, dx_2 \right\rangle \leq NuRa^{\frac{3}{2}},$$

and the identity

$$\left\langle \int_0^1 |\nabla T|^2\, dx_2 \right\rangle = Nu\,. \tag{13}$$

Then, using the localization (10), we obtain

$$Nu \leq \frac{1}{2}Nu + C\delta^3 (NuRa)^{\frac{1}{2}}(Ra^{\frac{3}{2}}Nu)^{\frac{1}{2}} + \frac{1}{\delta}$$

which yields

$$\frac{1}{2}Nu \leq C\delta^3 NuRa^{\frac{5}{4}} + \frac{1}{\delta}\,.$$

Optimizing in δ

$$\delta \sim \frac{1}{Ra^{\frac{5}{16}}Nu^{\frac{1}{4}}},$$

we deduce

$$Nu \lesssim Ra^{\frac{5}{12}}\,.$$

The paper is divided in two sections: in Section 2 we prove all the a-priori estimates that we will need to prove the main theorem in Section 3. The crucial localization and interpolation lemmas are proven in Lemma 1 and 8 respectively. The improvement of the upper bounds on the Nusselt number stems from the new pressure estimates in Lemma 5.

2 Identities and a-priori bounds

In this section we derive a-priori bounds for the Rayleigh-Bénard convection problem with Navier-slip boundary conditions. These will be used in the proof of Theorem 1 in the next section. We anticipate that all identities and estimates can be adapted to the case of free-slip boundary condition, by setting $L_s = \infty$.

Recall that the temperature equation enjoys a maximum principle: if $0 \leq T_0(\mathbf{x}) \leq 1$ then

$$0 \leq T(\mathbf{x}, t) \leq 1 \text{ for all } \mathbf{x}, t. \tag{14}$$

Lemma 1 (Localization of the Nusselt number)**.** *The Nusselt number Nu defined in (6) is independent of x_2, that is*

$$Nu = \langle u_2 T - \partial_2 T\rangle \qquad \textit{for all } x_2 \in [0,1]. \tag{15}$$

In particular, for any $\delta \in (0,1)$ it holds

$$Nu = \frac{1}{\delta}\left\langle \int_0^\delta (u_2 T - \partial_2 T)\, dx_2 \right\rangle \tag{16}$$

Proof. Taking the long-time and horizontal average of the equation for T we have

$$0 = \langle \partial_t T \rangle + \langle \nabla \cdot (uT - \nabla T) \rangle = \partial_2 \langle u_2 T - \partial_2 T \rangle .$$

Hence $\langle u_2 T - \partial_2 T\rangle$ is constant in x_2, proving (15). The identity (16) is a direct consequence of (15). □

Notice that from the boundary condition for T at $x_2 = 0$ and the maximum principle (14) it follows

$$-\int_0^\delta \partial_2 T \, dx_2 = -T(x_1, \delta) + 1 \le 1 \, .$$

As a consequence we obtain

$$Nu \le \frac{1}{\delta} \left\langle \int_0^\delta u_2 T \, dx_2 \right\rangle + \frac{1}{\delta} \, . \tag{17}$$

Thanks to (15), we can now derive another useful representation of the Nusselt number

Lemma 2 (Representation of the Nusselt number)**.** *The Nusselt number, defined in* (6)*, has the following alternative representation*

$$Nu = \left\langle \int_0^1 |\nabla T|^2 \, dx_2 \right\rangle \, . \tag{18}$$

Proof. Testing the temperature equation with T and integrating by parts we obtain

$$\frac{1}{2} \frac{d}{dt} \|T\|_{L^2}^2 = -\|\nabla T\|_{L^2}^2 - \int_0^\Gamma \partial_2 T|_{x_2=0} \, dx_1 ,$$

where we used the incompressibility condition (2), the non-penetration condition $u \cdot e_2 = u_2 = 0$ at $x_2 = \{0, 1\}$ and the boundary conditions (4) for T. The statement follows from taking the long-time averages, observing that $\limsup_{t\to\infty} \int_0^t \frac{d}{ds} \frac{1}{\Gamma} \int_0^\Gamma \int_0^1 |T|^2 \, dx_2 \, dx_1 \, ds = 0$ thanks to the maximum principle for T (14), and using $Nu = \langle u_2 T - \partial_2 T \rangle|_{x_2=0}$ by (15). □

Lemma 3 (Energy)**.** *Suppose $u_0 \in L^2$, then there exists a constant $C = C(\Gamma) > 0$ such that*

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + C \max\{1, L_s\} Ra \tag{19}$$

for all times $t \in [0, \infty)$ and

$$\left\langle \int_0^1 |\nabla u|^2 \, dx_2 \right\rangle \le Nu Ra. \tag{20}$$

Proof. We test the velocity equation with u, integrate by parts, use the boundary conditions (5) for u and the incompressibility condition (2) to find

$$\frac{1}{2Pr} \frac{d}{dt} \int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \frac{1}{L_s} \int_0^\Gamma (u_1^2|_{x_2=0} + u_1^2|_{x_2=1}) \, dx_1 = Ra \int_\Omega T u_2 \, dx. \tag{21}$$

By (5) one has $|u| = |u_1|$ on the boundaries and therefore the fundamental theorem of calculus implies the existence of a constant $C(\Gamma) > 0$ such that

$$\begin{aligned} \|u\|_{L^2}^2 \le \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 &\le C \left(\|\partial_2 u_1\|_{L^2}^2 + \int_0^\Gamma (u_1^2|_{x_2=0} + u_1^2|_{x_2=1}) \, dx_1 + \|\partial_2 u_2\|_{L^2}^2 \right) \\ &\le C \left(\|\nabla u\|_{L^2}^2 + \int_0^\Gamma (u_1^2|_{x_2=0} + u_1^2|_{x_2=1}) \, dx_1 \right), \end{aligned}$$

which applied to (21) yields

$$\frac{1}{2Pr}\frac{d}{dt}\int_\Omega |u|^2\,dx + \frac{1}{C}\min\{1, L_s^{-1}\}\|u\|_{L^2}^2 \le Ra\int_\Omega Tu_2\,dx \le \frac{1}{4\epsilon}\Gamma Ra^2 + \epsilon\|u_2\|_{L^2}^2,$$

where we used Hölder's inequality, Young's inequality and $\|T\|_{L^\infty} \le 1$ because of the maximum principle (14). Setting $\epsilon = \frac{1}{2C}\min\{1, L_s^{-1}\}$ implies

$$\frac{1}{Pr}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{1}{C}\min\{1, L_s^{-1}\}\|u\|_{L^2}^2 \le C\max\{1, L_s\}\Gamma Ra^2$$

and Grönwall's inequality now yields (19). Taking the long-time average of (21), using (19), one has

$$\left\langle \int_0^1 |\nabla u|^2\,dx_2 \right\rangle + \frac{1}{L_s}\left\langle (u_1^2|_{x_2=0} + u_1^2|_{x_2=1}) \right\rangle = Ra\left\langle \int_0^1 Tu_2\,dx_2 \right\rangle.$$

The claim follows by observing that, due to the boundary conditions for T, we have

$$Nu = \left\langle \int_0^1 Tu_2\,dx_2 \right\rangle + 1.$$

□

In order to bound the second derivatives of u we will exploit the equation for the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$:

$$\begin{aligned} Pr^{-1}(\partial_t\omega + u\cdot\nabla\omega) - \Delta\omega &= Ra\partial_1 T && \text{in } \Omega\\ \omega &= \tfrac{1}{L_s}u_1 && \text{at } x_2 = 1\\ \omega &= -\tfrac{1}{L_s}u_1 && \text{at } x_2 = 0\,. \end{aligned}$$

We first observe that

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \qquad \|\nabla u\|_{L^p} \le C\|\omega\|_{L^p}. \tag{22}$$

While the identity in L^2 follows from a direct computation, the inequality in L^p follows by elliptic regularity: in fact let ψ be the the stream function for u, i.e. $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$. Then ψ is a solution of

$$\begin{aligned} \Delta\psi &= \omega && \text{in } \Omega\\ \psi &= c(t) && \text{at } x_2 = 1\\ \psi &= 0 && \text{at } x_2 = 0 \end{aligned}$$

and $\tilde\psi = \psi - x_2 c(t)$ solves

$$\begin{aligned} \Delta\tilde\psi &= \omega && \text{in } \Omega\\ \psi &= 0 && \text{at } x_2 \in \{0,1\}. \end{aligned}$$

One has

$$\begin{aligned} \|\nabla u\|_{L^p}^p &= \|\nabla u_1\|_{L^p}^p + \|\nabla u_2\|_{L^p}^p = \|-\nabla\partial_2\psi\|_{L^p}^p + \|\nabla\partial_1\psi\|_{L^p}^p\\ &\le \|\nabla^2\psi\|_{L^p}^p + \|\nabla^2\psi\|_{L^p}^p = \|\nabla^2\tilde\psi\|_{L^p}^p + \|\nabla^2\tilde\psi\|_{L^p}^p \le C\|\omega\|_{L^p}^p, \end{aligned}$$

where we used the Calderon-Zygmund estimate in the last inequality.

Lemma 4 (Enstrophy)**.** *Suppose $u_0 \in W^{1,4}$ and $L_s > 0$. Then there exists a constant $C = C(\Gamma) > 0$ such that for all $t > 0$*

$$\|\omega(t)\|_{L^4} \le C\max\left\{1, L_s^{-3}\right\}(\|u_0\|_{W^{1,4}} + Ra) \tag{23}$$

and

$$\left\langle \int_0^1 |\nabla\omega|^2\,dx_2 \right\rangle \le \frac{1}{L_s}\left|\left\langle p\partial_1 u_1|_{x_2=1}\right\rangle + \left\langle p\partial_1 u_1|_{x_2=0}\right\rangle\right| + NuRa^{\frac{3}{2}}. \tag{24}$$

Proof. For the proof of (23) we refer the reader to (Drivas *et al.*, 2022, Lemma 2.11), where the case of $L_s \geq 1$ is covered. The same argument also yields the bound in case of $L_s < 1$. Testing the equation with ω and integrating by parts we obtain

$$\frac{1}{2Pr}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \frac{1}{L_s}\|u_1\|_{L^2(x_2=0)}^2 + \frac{1}{L_s}\|u_1\|_{L^2(x_2=1)}^2\right) + \|\nabla\omega\|_{L^2}^2\,dx$$
$$+\frac{1}{L_s}\left[\int_0^\Gamma \partial_1 u_1 p|_{x_2=0}\,dx_1 + \int_0^\Gamma \partial_1 u_1 p|_{x_2=1}\right]dx_1 = Ra\int_\Omega \omega\partial_1 T\,dx\,, \quad (25)$$

where we used $-\partial_2\omega = \Delta u_1 = \frac{1}{Pr}(\partial_t u_1 + (u\cdot\nabla)u_1) - \partial_1 p$ and the fact that $\omega(u\cdot\nabla)u_1 = \pm\frac{1}{L_s}u_1^2\partial_1 u_1$ at $x_2 = \{0,1\}$ vanishes after integration (in x_1) due to periodicity. We integrate (25) in time and notice that, thanks to (22) and the trace estimate, the first bracket on the left-hand side is bounded by the H^1-norm of $\mathbf{u}$:

$$\|\omega\|_{L^2}^2 + \frac{1}{L_s}\|u_1\|_{L^2(x_2=0)}^2 + \frac{1}{L_s}\|u_1\|_{L^2(x_2=1)}^2 \leq C(L_s)\|\mathbf{u}\|_{H^1}^2.$$

Note also that due to Hölder's inequality, (23) additionally yields

$$\|\omega(t)\|_{L^2} \leq C\max\left\{1, L_s^{-3}\right\}(\|u_0\|_{W^{1,4}} + Ra),$$

which, combined with (19) and (22), implies that $\|\mathbf{u}(t)\|_{H^1}$ is universally bounded in time. Therefore claim (24) follows from taking the space and long-time average of (25), using the fact that the long-time average of the first term in (25) vanishes due to the argument above, and observing

$$\left\langle\int_\Omega \omega\partial_1 T\,dx_2\right\rangle \leq \left\langle\int_\Omega |\omega|^2 dx_2\right\rangle^{\frac{1}{2}}\left\langle\int_\Omega |\nabla T|^2 dx_2\right\rangle^{\frac{1}{2}} \leq (NuRa)^{\frac{1}{2}}Nu^{\frac{1}{2}},$$

where we used (22), (20) and (13). □

Notice that the pressure term appears at the boundary in (24) and, for this reason, we need to control its H^1−norm. Taking the divergence of (1), it is easy to see that the pressure p satisfies

$$\begin{aligned} \Delta p &= -\tfrac{1}{Pr}\nabla u^{\mathrm{T}} : \nabla u + Ra\partial_2 T && \text{in } \Omega \\ \partial_2 p &= \tfrac{1}{L_s}\partial_1 u_1 && \text{at } x_2 = 1 \\ -\partial_2 p &= \tfrac{1}{L_s}\partial_1 u_1 - Ra && \text{at } x_2 = 0, \end{aligned} \quad (26)$$

where the boundary conditions are derived by tracing the second component of (1) on the boundary.

Lemma 5 (Pressure bound)**.** *Let $r > 2$. Then there exists a constant $C = C(r,\Gamma) > 0$ such that*

$$\|p\|_{H^1} \leq C\left(\frac{1}{L_s}\|\partial_2 u\|_{L^2} + \frac{1}{Pr}\|\omega\|_{L^2}\|\omega\|_{L^r} + Ra\|T\|_{L^2}\right). \quad (27)$$

Proof. The proof is a consequence of the following two claims:

$$\|\nabla p\|_{L^2}^2 = \frac{1}{L_s}\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\,dx_1 + \frac{1}{Pr}\int_\Omega p\nabla u^{\mathrm{T}} : \nabla u\,dx$$
$$+ Ra\int_\Omega \partial_2 p T\,dx, \quad (28)$$

$$\left|\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\,dx_1\right| \leq 3\|p\|_{H^1}\|\partial_2 u\|_{L^2}. \quad (29)$$

In fact, applying the Poincaré-Wirtinger inequality[1] and combining (28) and (29), we obtain

$$
\begin{aligned}
\|p\|_{H^1}^2 &\lesssim \frac{1}{L_s}\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\,dx_1 + \frac{1}{Pr}\int_\Omega p\nabla u^T : \nabla u\,dx \\
&\quad + Ra\int_\Omega \partial_2 pT\,dx \\
&\lesssim L_s^{-1}\|p\|_{H^1}\|\partial_2 u\|_{L^2} + Pr^{-1}\|p\|_{L^q}\|\nabla u\|_{L^2}\|\nabla u\|_{L^r} + Ra\|p\|_{H^1}\|T\|_{L^2} \\
&\lesssim L_s^{-1}\|p\|_{H^1}\|\partial_2 u\|_{L^2} + Pr^{-1}\|p\|_{H^1}\|\nabla u\|_{L^2}\|\nabla u\|_{L^r} + Ra\|p\|_{H^1}\|T\|_{L^2},
\end{aligned}
$$

where q and r are related by $\frac{1}{p}+\frac{1}{r}=\frac{1}{2}$. Here, besides the trace estimate, we used the fact that $\|p\|_{L^q} \le C\|p\|_{H^1}$ for any $q\in(2,\infty)$ by Sobolev embedding in bounded domains. It is left to prove the claims.

Argument for (28). Integrating by parts and using (26)

$$
\begin{aligned}
\|\nabla p\|_{L^2}^2 &= \int_0^\Gamma p\partial_2 p|_{x_2=1}\,dx_1 - \int_0^\Gamma p\partial_2 p|_{x_2=0}\,dx_1 - \int_\Omega p\Delta p\,dx \\
&= \frac{1}{L_s}\int_0^\Gamma p(\partial_1 u_1|_{x_2=1} + \partial_1 u_1|_{x_2=0})\,dx_1 - Ra\int_0^\Gamma p|_{x_2=0}\,dx_1 \\
&\quad + \frac{1}{Pr}\int_\Omega p\nabla u^{\mathrm{T}} : \nabla u\,dx - Ra\int_\Omega \partial_2 Tp\,dx \\
&= \frac{1}{L_s}\int_0^\Gamma p(\partial_1 u_1|_{x_2=1} + \partial_1 u_1|_{x_2=0})\,dx_1 + \frac{1}{Pr}\int_\Omega p\nabla u^{\mathrm{T}} : \nabla u\,dx + Ra\int_\Omega T\partial_2 p\,dx,
\end{aligned}
$$

where in the last identity we used $-Ra\int_\Omega \partial_2 Tp - Ra\int_0^\Gamma p|_{x_2=0}\,dx_1 = Ra\int_\Omega T\partial_2 p\,dx$ thanks to the boundary conditions for T.

Argument for (29). Since u is divergence free we have

$$
0 = \int_\Omega p(-1+2x_2)\partial_2(\nabla\cdot u)\,dx = \int_\Omega p(-1+2x_2)\nabla\cdot(\partial_2 u),
$$

and integration by parts yields

$$
\begin{aligned}
\int_\Omega p(-1+2x_2)\nabla\cdot(\partial_2 u)\,dx &= -\int_\Omega \nabla p\cdot(-1+2x_2)\partial_2 u\,dx - 2\int_\Omega p\partial_2 u_2\,dx \\
&\quad - \left(\int_0^\Gamma p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0}\right)dx_1,
\end{aligned}
$$

where we used $\partial_1 u_1 = -\partial_2 u_2$ by incompressibility. Combining the two identities we obtain

$$
\begin{aligned}
\left|\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\,dx_1\right| &\le 2\|p\|_{L^2}\|\partial_2 u_2\|_{L^2} + \|\partial_2 u\|_{L^2}\|\nabla p\|_{L^2} \\
&\le 3\|p\|_{H^1}\|\partial_2 u\|_{L^2}\,.
\end{aligned}
$$

□

We conclude this section with bounds on second derivatives of the velocity field. First we relate the L^2-norm of $\nabla^2 u$ to the L^2-norm of $\nabla\omega$.

[1] p can be assumed to have zero mean, since $p - \langle p\rangle$ satisfies the equations (26)

Lemma 6.

$$\|\nabla^2 u\|_{L^2} \le \|\nabla\omega\|_{L^2}\,. \tag{30}$$

Proof. First we show that $\|\nabla^2 u\|_{L^2} \le \|\Delta u\|_{L^2}$: integrating by parts twice yields

$$\begin{aligned}
\|\nabla^2 u\|^2_{L^2(\Omega)} &= \int_\Omega \partial_i\partial_j u_k \partial_i\partial_j u_k \, dx \\
&= \int_\Omega \partial_i^2 u_k \partial_j^2 u_k \, dx - \int_{\partial\Omega} \partial_i^2 u_k \partial_j u_k n_j \, dS + \int_{\partial\Omega} \partial_i\partial_j u_k \partial_j u_k n_i \, dS \\
&= \|\Delta u\|^2_{L^2(\Omega)} - \int_0^\Gamma \partial_1^2 u_k \partial_2 u_k n_2 \, dx_1 + \int_0^\Gamma \partial_2\partial_1 u_k \partial_1 u_k n_2 \, dx_1\,,
\end{aligned}$$

where we used periodicity in the horizontal direction and the fact that the terms with $i = j$ cancel. Note that, due to (5), the boundary terms have a sign:

$$\begin{aligned}
&-\int_0^\Gamma \partial_1^2 u_k \partial_2 u_k n_2 \, dx_1 + \int_0^\Gamma \partial_2\partial_1 u_k \partial_1 u_k n_2 \, dx_1 \\
&\quad= -\int_0^\Gamma \partial_1^2 u_1 \partial_2 u_1 n_2 \, dx_1 + \int_0^\Gamma \partial_2\partial_1 u_1 \partial_1 u_1 n_2 \, dx_1 \\
&\quad= \frac{1}{L_s}\int_0^\Gamma \partial_1^2 u_1 u_1 \, dx_1 - \int_0^\Gamma (\partial_1 u_1)^2 \, dx_1 = -\frac{2}{L_s}\int_0^\Gamma (\partial_1 u_1)^2 \, dx_1 \le 0\,,
\end{aligned}$$

where in the last identity we used the periodicity in the horizontal direction. This proves the first claim.

Now, a direct computation yields $\Delta u = \nabla^\perp \omega$, and from this follows $\|\Delta u\|_{L^2(\Omega)} = \|\nabla\omega\|_{L^2(\Omega)}$. We conclude that

$$\|\nabla^2 u\|_{L^2(\Omega)} \le \|\Delta u\|_{L^2(\Omega)} = \|\nabla\omega\|_{L^2(\Omega)}$$

yielding (30). □

Next, we relate $\|\nabla^2 u\|_{L^2}$ to Nu, Ra and L_s, via an upper bound for $\|\nabla\omega\|_{L^2}$. This is done by combining the enstrophy balance, the pressure bound (27) and the boundary integral estimate (29).

Lemma 7 (Hessian bound). *Let $L_s > 0$ and $u_0 \in W^{1,4}$. Then there exists a constant $C = C(\Gamma) > 0$ such that*

$$\langle\|\nabla^2 u\|^2_{L^2}\rangle \le C\Big(L_s^{-2} + \frac{\max\{1, L_s^{-3}\}(\|u_0\|_{W^{1,4}} + Ra)}{Pr L_s} + L_s^{-1} Nu^{-\frac12} Ra^{\frac12} + Ra^{\frac12}\Big) Nu Ra. \tag{31}$$

Proof. From (30) and (24) we have

$$\left\langle \int_0^1 |\nabla^2 u|^2 \, dx \right\rangle \le L_s^{-1}\left(\langle p\partial_1 u_1|_{x_2=1}\rangle + \langle p\partial_1 u_1|_{x_2=0}\rangle\right) + Nu Ra^{\frac32}.$$

Using (29) and (27) we obtain

$$\begin{aligned}
\left\langle \int_0^1 |\nabla^2 u|^2 \, dx \right\rangle &\le L_s^{-1}\langle\|p\|_{H^1}\|\partial_2 u\|_{L^2}\rangle + Nu Ra^{\frac32} \\
&\le L_s^{-2}\langle\|\partial_2 u\|^2_{L^2}\rangle + L_s^{-1} Pr^{-1}\langle\|\omega\|_{L^2}\|\omega\|_{L^4}\|\partial_2 u\|_{L^2}\rangle \\
&\quad + L_s^{-1} Ra\langle\|T\|_{L^2}\|\partial_2 u\|_{L^2}\rangle + Nu Ra^{\frac32}.
\end{aligned}$$

The identity in (22) and pointwise (in time) vorticity bound (23) yield

$$\begin{aligned}
\left\langle \int_0^1 |\nabla^2 u|^2 \, dx_2 \right\rangle &\le L_s^{-2}\langle\|\partial_2 u\|^2_{L^2}\rangle + \frac{C\max\{1, L_s^{-3}\}}{Pr L_s}\langle\|\nabla u\|_{L^2}(\|u_0\|_{W^{1,4}} + Ra)\|\partial_2 u\|_{L^2}\rangle \\
&\quad + L_s^{-1} Ra\langle\|T\|_{L^2}\|\partial_2 u\|_{L^2}\rangle + Nu Ra^{\frac32}.
\end{aligned}$$

Finally using $\langle \|\partial_2 u\|^2_{L^2}\rangle \le \langle \|\nabla u\|^2_{L^2}\rangle$, the upper bound in (20) and the maximum principle for the temperature (14) we deduce

$$\left\langle \int_0^1 |\nabla^2 u|^2 \, dx_2 \right\rangle \le L_s^{-2} NuRa + \frac{C \max\{1, L_s^{-3}\}}{Pr L_s} (\|u_0\|_{W^{1,4}} + Ra) NuRa$$
$$+ L_s^{-1} \Gamma^{\frac{1}{2}} Nu^{\frac{1}{2}} Ra^{\frac{3}{2}} + NuRa^{\frac{3}{2}} .$$

□

3 Proof of Theorem 1

A crucial ingredient of the proof of Theorem 1 is the following interpolation bound, relating the integral of the product $u_2 T$ with $\|\nabla T\|_{L^2}$, $\|\nabla u\|_{L^2}$ and $\|\nabla^2 u\|_{L^2}$. Notice that, in turn, these quantities are estimated in terms of Nusselt, Rayleigh and Prandtl numbers in (18), (20) and (31).

Lemma 8. *The interpolation bound*

$$\frac{1}{\delta} \left\langle \int_0^\delta u_2 T \, dx_2 \right\rangle \le \frac{1}{2} \left\langle \int_0^1 |\partial_2 T|^2 \, dx_2 \right\rangle + C\delta^3 \left\langle \int_0^1 |\nabla u|^2 \, dx_2 \right\rangle^{\frac{1}{2}} \left\langle \int_0^1 |\nabla^2 u|^2 \, dx_2 \right\rangle^{\frac{1}{2}} \tag{32}$$

holds for some constant $C > 0$.

Proof. First notice that, due to incompressibility $\partial_2 u_2 = -\partial_1 u_1$ and horizontal periodicity

$$\partial_2 \frac{1}{\Gamma} \int_0^\Gamma u_2(x_1, x_2) \, dx_1 = -\frac{1}{\Gamma} \int_0^\Gamma \partial_1 u_1(x_1, x_2) \, dx_1 = 0,$$

implying

$$\frac{1}{\Gamma} \int_0^\Gamma u_2(x_1, x_2) \, dx_1 = 0 \tag{33}$$

thanks to the boundary conditions $u_2 = 0$ at $x_2 = \{0, 1\}$. Let $\theta(x_1, x_2) := T(x_1, x_2) - 1$, then $\theta(x_1, x_2) = 0$ at $x_2 = 0$ and by (33)

$$\frac{1}{\delta} \left\langle \int_0^\delta u_2 T \, dx_2 \right\rangle = \frac{1}{\delta} \left\langle \int_0^\delta u_2 \theta \, dx_2 \right\rangle .$$

By the fundamental theorem of calculus and the homogeneous Dirichlet boundary conditions for θ and u_2 we have

$$|\theta(x_1, x_2)| \le x_2^{\frac{1}{2}} \|\partial_2 \theta\|_{L^2(0,1)}$$
$$|u_2(x_1, x_2)| \le x_2 \|\partial_2 u_2\|_{L^\infty(0,1)} .$$

Furthermore, since $\int_0^1 \partial_2 u_2(x_1, x_2) \, dx_2 = 0$, there exists $\xi = \xi(x_1) \in (0, 1)$ such that $\partial_2 u_2(x_1, \xi) = 0$. Then

$$|\partial_2 u_2(x_1, x_2)|^2 = \left| \int_\xi^{x_2} \partial_2 (\partial_2 u_2(x_1, z))^2 \, dz \right| = \left| 2 \int_\xi^{x_2} \partial_2 u_2 \partial_2^2 u_2 \, dz \right| ,$$

which implies

$$|u_2(x_1, x_2)| \le x_2 \|\partial_2 u_2(x_1, \cdot)\|^{\frac{1}{2}}_{L^2(0,1)} \|\partial_2^2 u_2(x_1, \cdot)\|^{\frac{1}{2}}_{L^2(0,1)} .$$

Combining these estimates we obtain

$$
\begin{aligned}
&\frac{1}{\delta}\left\langle \int_0^\delta u_2\theta\, dx_2 \right\rangle \\
&\quad \le \frac{1}{\delta}\left\langle \int_0^\delta x_2^{\frac{3}{2}} \|\partial_2 u_2(x_1,\cdot)\|_{L^2(0,1)} \|\partial_2^2 u_2(x_1,\cdot)\|_{L^2(0,1)} \|\partial_2\theta(x_1,\cdot)\|^2_{L^2(0,1)}\, dx_2 \right\rangle \\
&\quad \le \delta^{\frac{3}{2}} \left\langle \|\partial_2 u_2(x_1,\cdot)\|_{L^2(0,1)} \|\partial_2^2 u_2(x_1,\cdot)\|_{L^2(0,1)} \right\rangle^{\frac{1}{2}} \left\langle \|\partial_2\theta(x_1,\cdot)\|^2_{L^2(0,1)} \right\rangle^{\frac{1}{2}} \\
&\quad \le C\delta^3 \left\langle \|\partial_2 u_2(x_1,\cdot)\|^2_{L^2(0,1)} \right\rangle^{\frac{1}{2}} \left\langle \|\partial_2^2 u_2(x_1,\cdot)\|^2_{L^2(0,1)} \right\rangle^{\frac{1}{2}} + \frac{1}{2}\left\langle \|\partial_2\theta(x_1,\cdot)\|^2_{L^2(0,1)} \right\rangle.
\end{aligned}
$$

□

Proof of Theorem 1. Using the localization of the Nusselt number (17) and (32) we have

$$
\begin{aligned}
Nu &\le \frac{1}{\delta}\left\langle \int_0^\delta u_2 T\, dx_2 \right\rangle + \frac{1}{\delta} \\
&\le \frac{1}{2}\left\langle \int_0^1 |\nabla T|^2\, dx_2 \right\rangle + C\delta^3 \left\langle \int_0^1 |\nabla u|^2\, dx_2 \right\rangle^{\frac{1}{2}} \left\langle \int_0^1 |\nabla^2 u|^2\, dx_2 \right\rangle^{\frac{1}{2}} + \frac{1}{\delta}
\end{aligned}
$$

yielding

$$
\frac{1}{2}Nu \le C\delta^3 \left\langle \int_0^\delta |\nabla u|^2\, dx_2 \right\rangle^{\frac{1}{2}} \left\langle \int_0^1 |\nabla^2 u|^2\, dx_2 \right\rangle^{\frac{1}{2}} + \frac{1}{\delta},
$$

where we used (18). Finally we insert the bounds (20) and (31) in the last inequality and, if Ra is large enough such that $\|u_0\|_{W^{1,4}} \lesssim Ra$ then, up to redefining constants, we find

$$
Nu \le C\delta^3\left(L_s^{-1} Nu Ra + \max\{1, L_s^{-\frac{3}{2}}\} L_s^{-\frac{1}{2}} Pr^{-\frac{1}{2}} Nu Ra^{\frac{3}{2}} + L_s^{-\frac{1}{2}} Nu^{\frac{3}{4}} Ra^{\frac{5}{4}} + Nu Ra^{\frac{5}{4}}\right) + \frac{2}{\delta}. \tag{34}
$$

We first cover the case $L_s \ge 1$. Then, as $L_s, Nu \ge 1$, the first and third terms on the right-hand side of (34) are dominated by $Nu Ra^{\frac{5}{4}}$, hence

$$
Nu \le C\delta^3 Nu \left(L_s^{-\frac{1}{2}} Pr^{-\frac{1}{2}} Ra^{\frac{3}{2}} + Ra^{\frac{5}{4}}\right) + 2\delta^{-1}
$$

and optimizing by setting $\delta = Nu^{-\frac{1}{4}}\left(L_s^{-\frac{1}{2}} Pr^{-\frac{1}{2}} Ra^{\frac{3}{2}} + Ra^{\frac{5}{4}}\right)^{-\frac{1}{4}}$ implies

$$
Nu \lesssim L_s^{-\frac{1}{6}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}} + Ra^{\frac{5}{12}}.
$$

If instead $L_s < 1$, then (34) is given by

$$
Nu \le C\delta^3\left(L_s^{-1} Nu Ra + L_s^{-2} Pr^{-\frac{1}{2}} Nu Ra^{\frac{3}{2}} + L_s^{-\frac{1}{2}} Nu^{\frac{3}{4}} Ra^{\frac{5}{4}} + Nu Ra^{\frac{5}{4}}\right) + 2\delta^{-1}.
$$

Optimizing by setting

$$
\delta = \begin{cases} Nu^{-\frac{1}{4}}\left(L_s^{-1} Ra + L_s^{-2} Pr^{-\frac{1}{2}} Ra^{\frac{3}{2}} + Ra^{\frac{5}{4}}\right)^{-\frac{1}{4}} & \text{if } L_s^{-\frac{1}{2}} Ra^{-\frac{1}{4}} + L_s^{-\frac{3}{2}} Pr^{-\frac{1}{2}} Ra^{\frac{1}{4}} + L_s^{\frac{1}{2}} \ge Nu^{-\frac{1}{4}} \\ L_s^{\frac{1}{8}} Nu^{-\frac{3}{16}} Ra^{-\frac{5}{16}} & \text{if } L_s^{-\frac{1}{2}} Ra^{-\frac{1}{4}} + L_s^{-\frac{3}{2}} Pr^{-\frac{1}{2}} Ra^{\frac{1}{4}} + L_s^{\frac{1}{2}} \le Nu^{-\frac{1}{4}} \end{cases}
$$

yields

$$
Nu \lesssim L_s^{-\frac{1}{3}} Ra^{\frac{1}{3}} + L_s^{-\frac{2}{3}} Pr^{-\frac{1}{6}} Ra^{\frac{1}{2}} + L_s^{-\frac{2}{13}} Ra^{\frac{5}{13}} + Ra^{\frac{5}{12}}.
$$

□

Aknowledgements

The authors acknowledge the support by the Deutsche Forschungsgemeinschaft (DFG) within the Research Training Group GRK 2583 ”Modeling, Simulation and Optimization of Fluid Dynamic Applications”.

Declaration of Interests

The authors report no conflict of interest.

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