

The inversion number of dijoins and blow-up digraphs

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Abstract

For an oriented graph D , the *inversion* of $X \subseteq V(D)$ in D is the digraph obtained from D by reversing the direction of all arcs with both ends in X . The *inversion number* of D , denoted by $\text{inv}(D)$, is the minimum number of inversions needed to transform D into an acyclic digraph. In this paper, we first show that $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D) + 1$ for any oriented graph D with even inversion number $\text{inv}(D)$, where the dijoin $\vec{C}_3 \Rightarrow D$ is the oriented graph obtained from the disjoint union of \vec{C}_3 and D by adding all arcs from \vec{C}_3 to D . Thus we disprove the conjecture of Aubian et al. [2] and the conjecture of Alon et al. [1]. We also study the blow-up graph which is an oriented graph obtained from a tournament by replacing all vertices into oriented graphs. We construct a tournament T with order n and $\text{inv}(T) = \frac{n}{3} + 1$ using blow-up graphs.

Keywords: inversion number; tournament; oriented graph; dijoin; blow-up graph.

1 Introduction

In this paper we only consider digraphs without loops, parallel edges and 2-cycles. In the following content, we will use digraph and oriented graph interchangeably. We denote by $V(D)$ and $A(D)$ the vertex set and the arc set of a digraph D , respectively. The subdigraph of D induced by a subset $X \subseteq V(D)$ is denoted by $D \langle X \rangle$. The *inversion* of X in D is the digraph obtained from D by reversing all arcs in $A(D \langle X \rangle)$, which we denote by $\text{Inv}(D; X)$. If $(X_i)_{i \in I}$ is a family of subsets of $V(D)$, then $\text{Inv}(D; (X_i)_{i \in I})$ is the digraph obtained from D by inverting X_i one after another. Note that the ordering of the inversions doesn't matter. We say $(X_i)_{i \in I}$ is a *decycling family* if $\text{Inv}(D; (X_i)_{i \in I})$ is acyclic. Furthermore, if $|I| = k$, we say $(X_i)_{i \in I}$ is a *k-decycling family*. The *inversion number* of D is the minimum cardinality

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of a decycling family, which we denote by $\text{inv}(D)$. For a positive integer n , denote $\text{inv}(n) = \max\{\text{inv}(D) \mid D \text{ oriented graph of order } n\}$.

The inversion problem was first introduced by Belkhechine et al. ([4] [5]). They mainly concerned about $\text{inv}(n)$. Recently, Bang-Jensen et al. [3], Alon et al. [1] and Aubian et al. [2] have obtained many results regarding all aspects of the problem.

It is natural to consider the inversion number under some graph operations. For digraphs L and R , the *dijoin* $L \Rightarrow R$ from L to R is the digraph composed by disjoint union of L and R , with an edge uv for all $u \in V(L)$ and $v \in V(R)$. Bang-Jensen et al. [3] showed that, for two strongly connected oriented graphs L and R such that $\text{inv}(L), \text{inv}(R) \geq 2$, $\text{inv}(L \Rightarrow R) \geq 4$. They also gave the following conjecture.

Conjecture 1.1 (Bang-Jensen et al. [3]) *For oriented graphs L and R , we have $\text{inv}(L \Rightarrow R) = \text{inv}(L) + \text{inv}(R)$.*

This is called “dijoin conjecture”, which is an important cornerstone of inversion number problem. Conjecture 1.1 is trivial when one of $\text{inv}(L)$ and $\text{inv}(R)$ is zero. Bang-Jensen et al. [3] showed it is true when $\text{inv}(L) + \text{inv}(R) \leq 3$, and Alon et al. [1] showed it is true when $\text{inv}(L) = \text{inv}(R) = 2$. However, Alon et al. [1] and Aubian et al. [2] independently found Conjecture 1.1 is not correct in general.

Note that the dijoin operation is not a symmetric operation, which means $L \Rightarrow R$ and $R \Rightarrow L$ are different in general. However, Conjecture 1.1 is symmetric in some sense, since the graphs obtained by inverting the whole vertex set give the corresponding examples in the other direction. Actually, we have $\text{inv}(D) = \text{inv}(\text{Inv}(D; V(D)))$.

To disprove Conjecture 1.1, Aubian et al. [2] gave an explicit construction.

Theorem 1.2 (Aubian et al. [2]) *For every odd integer $k \geq 3$, there is a tournament T_k with $\text{inv}(T_k) = k$ such that $\text{inv}(T_k \Rightarrow R) \leq k + \text{inv}(R) - 1$ for every oriented graph R with $\text{inv}(R) \geq 1$.*

They also conjectured that the same statement also holds for every even integer $k \geq 4$.

Conjecture 1.3 (Aubian et al. [2]) *For any $k \geq 3$, there is a tournament T_k with $\text{inv}(T_k) = k$ such that $\text{inv}(T_k \Rightarrow R) < k + \text{inv}(R)$ for all R with $\text{inv}(R) \geq 1$.*

Independently and almost simultaneously, Alon et al. [1] exhibited a tournament R with 9 vertices such that $\text{inv}(R) = \text{inv}(\vec{C}_3 \Rightarrow R) = 3$, where \vec{C}_3 is the directed cycle on three vertices. They also give the following conjecture, which is similar with Conjecture 1.3.

Conjecture 1.4 (Alon et al. [1]) *For all $l, r \in \mathbb{N}$ with $l \geq 3$ or $r \geq 3$, there exists oriented graphs L and R with $\text{inv}(L) = l$ and $\text{inv}(R) = r$, but $\text{inv}(L \Rightarrow R) < l + r$.*

In this paper, we prove the following theorem and then disprove both Conjecture 1.3 and Conjecture 1.4. It is quite surprising and tells us that Conjecture 1.1 is still correct in some cases.

Theorem 1.5 *Let $k \geq 2$ be an even integer. For all oriented graphs L and R with $\text{inv}(L) = 1$ and $\text{inv}(R) = k$, we have $\text{inv}(L \Rightarrow R) = \text{inv}(R \Rightarrow L) = \text{inv}(L) + \text{inv}(R) = 1 + k$.*

In conclusion, here are the current results for “dijoin conjecture”. For oriented graphs L and R , if $\text{inv}(L) = l$ and $\text{inv}(R) = r$, we have $\text{inv}(L \Rightarrow R) = l + r$ for

$$(l, r) \in \{(0, 0), (0, k), (k, 0), (1, 1), (1, 2k), (2k, 1), (2, 2)\},$$

where k is a positive integer. Also, there are counterexamples with $\text{inv}(L \Rightarrow R) < l + r$ when either l or r is an odd number at least 3 and $lr \neq 0$. The remaining cases are still open, which are the cases when l and r are both positive even numbers except for $l = r = 2$.

On the other hand, a natural generalization of dijoin operation is so-called k -join, which is a graph obtained by dijoin k digraphs one after another. Additionally, here is a further generalization.

Definition 1.6 (blow-up graph) *Let H be a digraph with $|V(H)| = n$. Label the vertices in H as v_1, v_2, \dots, v_n . For pairwise vertex disjoint digraphs D_1, \dots, D_n , define the blow-up graph $H[D_1, D_2, \dots, D_n]$ is the digraph whose vertex set and arc set are*

$$V(H[D_1, D_2, \dots, D_n]) = \bigcup_{i=1}^n V(D_i),$$

$$A(H[D_1, D_2, \dots, D_n]) = \bigcup_{v_i v_j \in A(H)} \{xy | x \in V(D_i), y \in V(D_j)\} \cup \bigcup_{k=1}^n A(D_k).$$

Moreover, if $D_i = D$ are all the same, we simply denote $H[D_1, D_2, \dots, D_n]$ by $H[D]_n$. If $V(H) = \{x, y\}$ and $A(H) = \{xy\}$, it gives dijoin $H[D_1, D_2] = D_1 \Rightarrow D_2$. Also, k -join comes from $H = TT_k$, which is a tournament with order k and arc set $A(TT_k) = \{v_i v_j | i < j\}$, then we denote $TT_k[D_1, D_2, \dots, D_k]$ by $[D_1, D_2, \dots, D_k]$ and denote $TT_k[D]_k$ by $[D]_k$.

To prove Theorem 1.5, we start with the case when $L = \vec{C}_3$.

Theorem 1.7 *Let $k \geq 2$ be an even integer and D be an oriented graph with $\text{inv}(D) = k$. Then $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(\vec{C}_3) + \text{inv}(D) = 1 + k$.*

The main technique to prove Theorem 1.7 is to investigate the rank of some related matrices over \mathbb{F}_2 . From the same idea, we have some other interesting results. We provide an equivalent condition when $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D) = k$, and prove the following theorems.

Theorem 1.8 *Let D be oriented graph. Then $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D \Rightarrow \vec{C}_3)$.*

Theorem 1.9 $\text{inv}([\vec{C}_3, \vec{C}_3, D]) = \text{inv}(\vec{C}_3 \Rightarrow D) + 1$ for every oriented graph D .

We also have the following generalized result as a corollary. Let $[m] = \{1, 2, \dots, m\}$.

Corollary 1.10 Let $k \geq 2$ and D_1, D_2, \dots, D_k be oriented graphs. Assume that there is $j \in [k]$ such that $\text{inv}(D_j) \geq 1$ and $\text{inv}(D_i) = 1$ for all $i \in [k] \setminus \{j\}$. Then

$$\text{inv}([D_1, D_2, \dots, D_k]) = \begin{cases} \sum_{i=1}^k \text{inv}(D_i) - 1 & \text{if } \text{inv}(\vec{C}_3 \Rightarrow D_j) = \text{inv}(D_j), \\ \sum_{i=1}^k \text{inv}(D_i) & \text{otherwise.} \end{cases}$$

Theorem 1.5 is actually a special case of this corollary.

Proof of Theorem 1.5. Since $\text{inv}(R) = k$ is even, we have $\text{inv}(\vec{C}_3 \Rightarrow R) = \text{inv}(R) + 1$ by Theorem 1.7. From Corollary 1.10, we are done. \square

Theorem 1.8 seems very reasonable but it is non-trivial. In fact, we believe that the direction of the dijoin operation does not affect the inversion number in general.

Conjecture 1.11 For every oriented graphs L and R , $\text{inv}(L \Rightarrow R) = \text{inv}(R \Rightarrow L)$.

Basing on Theorems 1.9, we also give the following conjecture.

Conjecture 1.12 For tournaments T_1, T_2 with $\text{inv}(T_1) \geq 2$, we have $\text{inv}(T_1 \Rightarrow T_2) > \text{inv}(T_2)$.

There are lots of results about $\text{inv}(n)$. Belkhechine et al. [5] first proved that $\frac{n-1}{2} - \log n \leq \text{inv}(n) \leq n - 3$ for all integer $n \geq 4$.

Bang-Jensen et al. [3] found that the constant term in the upper bound can be improved very slightly. Alon et al. [1] greatly improved the bounds of $\text{inv}(n)$, showed that $\text{inv}(n) = (1 + o(1))n$. As the proof of it relies on probabilistic methods, there is no explicit construction for a digraph with large inversion number close to $\text{inv}(n)$. The largest one found is $[\vec{C}_3]_{\frac{n}{3}}$ whose inversion number is $\frac{n}{3}$.

Let Q_n be the tournament obtained from the transitive tournament by reversing the arcs of its unique directed hamiltonian path (v_1, v_2, \dots, v_n) . Belkhechine et al. [5] conjectured that $\text{inv}(Q_n) = \lfloor \frac{n-1}{2} \rfloor$. It means that Q_n is possibly a specific example with much larger inversion number. It is obvious that $\text{inv}(Q_n) \leq \lfloor \frac{n-1}{2} \rfloor$ since we can give a $\lfloor \frac{n-1}{2} \rfloor$ -decycling family $X_i = \{v_{2i}, v_{2i+1}\}$ for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

Our another main result of the paper is to give a construction with inversion number slightly more than $\frac{n}{3}$ using blow-up graphs.

Theorem 1.13 Let T be a tournament of order n and $\text{inv}(T) = 1$. Then $\text{inv}(T[D_1, \dots, D_n]) = n + 1$ for every oriented graphs $\{D_i\}_{1 \leq i \leq n}$ with $\text{inv}(D_i) = 1$. Moreover, $\text{inv}(T[\vec{C}_3]_k) = k + 1$.

We can prove that for oriented graphs D_i with $\text{inv}(D_i) = 1$ ($1 \leq i \leq n$), the inequality

$$n \leq \text{inv}(T[D_1, D_2, \dots, D_n]) \leq n + \text{inv}(T)$$

holds. Then for $\text{inv}(T) \in \{0, 1\}$, we have $\text{inv}(T[D_1, D_2, \dots, D_n]) = n + \text{inv}(T)$ by Theorem 1.13. If $\text{inv}(T) \notin \{0, 1\}$, we have the following result.

Theorem 1.14 *Let $k \geq 3$ be an odd integer. Then there exists a tournament T_k with $\text{inv}(T_k) = k$ such that*

$$\text{inv}(T_k[D_1, D_2, \dots, D_n]) \leq n + k - 1$$

for every oriented graphs $\{D_i\}_{1 \leq i \leq n}$ with $\text{inv}(D_i) = 1$. Here $n = |V(T_k)|$ is the order of T_k .

We have the following conjecture for even case.

Conjecture 1.15 *Let $k \geq 2$ be an even integer, T be a tournament with $\text{inv}(T) = k$ and $|V(T)| = n$. For oriented graphs D_1, D_2, \dots, D_n with $\text{inv}(D_i) = 1$, we have*

$$\text{inv}(T[D_1, D_2, \dots, D_n]) = n + k.$$

We detail some of the notation and observations to be used in this paper. If $xy \in A(D)$, we say x dominates y , denoted by $x \rightarrow y$. If every vertex of $A \subseteq V(D)$ dominates every vertex of $B \subseteq V(D)$, then we say A dominates B , denoted by $A \rightarrow B$.

Given a tournament T and the result tournament T' of T after inverting. We introduce a sign ' $<$ ' to express the ordering of vertices in T' , that is $x < y$ if $xy \in A(T')$. Then $x < y < z$ means $xy, xz, yz \in A(T')$. Let $A, B \subseteq V(T)$. Denote $A < B$ (resp. $A \leq y$) if $x < y$ for every $x \in A$ and $y \in B$ (resp. $x < y$ for every $x \in A$ with $x \neq y$). Note that if the result tournament T' is acyclic, then ' $<$ ' is transitive and gives a total order on $V(T')$.

It is common to use vectors in \mathbb{F}_2 to investigate the inversion problem of graphs. For a digraph D , a decycling family $(X_i)_{1 \leq i \leq k}$ and a vertex $v \in V(D)$, we define the *characteristic vector* $\mathbf{v} \in \mathbb{F}_2^k$ of v in $(X_i)_{1 \leq i \leq k}$ where the i -th element of \mathbf{v} is 1 if and only if $v \in X_i$. For $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^k$, we write $\mathbf{u} \cdot \mathbf{v}$ to be the scalar product of \mathbf{u} and \mathbf{v} over \mathbb{F}_2^k . We say a collection $\{\mathbf{u}_i\}_{i \in I}$ is *orthonormal* if $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ and $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $i \neq j$. For $uv \in A(D)$, it is obvious that $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $uv \in A(\text{Inv}(D; (X_i)_{i \in I}))$.

When we study the inversion number, we mainly focus on tournaments because of the following observations.

Observation 1.16 (Bang-Jensen et al. [3]) *If D' is a subdigraph of D , then $\text{inv}(D') \leq \text{inv}(D)$. Actually, if $(X_i)_{i \in I}$ is a decycling family of D , then $(X_i \cap V(D'))_{i \in I}$ is a decycling family of D' .*

Observation 1.17 (Bang-Jensen et al. [3]) *If $(X_i)_{i \in I}$ is a decycling family of an oriented graph D , then D can be extended to a tournament T such that $\text{inv}(D) = \text{inv}(T)$ and $(X_i)_{i \in I}$ is still a decycling family of T .*

Observation 1.17 shows that we only need to consider tournaments rather than oriented graphs in lots of questions, and it preserves the decycling family of the original graph.

After finishing the first version of this paper, we notice the independent work by Behague et al. [6]. They are also working on the inversion number of dijoins and disprove a different conjecture of Aubian et al. [2]. To our knowledge, their results do not imply any of our results in this paper.

The rest of this paper is organized as follows. In Section 2 we give proofs of Theorems 1.7 to 1.9. In Section 3 we prove Theorem 1.13 and study the inversion number of the blow-up graphs.

2 Proofs of Theorems 1.7 to 1.9

In this section we first prove Theorem 1.7.

2.1 Proof of Theorem 1.7

The proof of Theorem 1.7 relies on the ranks of matrices. We prove that the rank of characteristic vectors of any decycling family of an oriented graph D is no less than $\text{inv}(D)$ if $\text{inv}(D)$ is even. We first have the following lemma.

Lemma 2.1 *For any odd integer n and every symmetric matrix $M \in M_n(\mathbb{F}_2)$, there is $U \in M_n(\mathbb{F}_2)$ such that $U^t U = M$.*

Proof. By induction on n . It is trivial when $n = 1$. Let $M = (m_{ij})_{1 \leq i, j \leq n}$ be a symmetric matrix with $n \geq 3$. We complete the proof by considering two cases.

Case 1. There exists i such that $m_{ii} = 1$. By interchanging columns and corresponding rows we assume $i = 1$.

Case 1.1. There exists $j \neq 1$ such that the submatrix $A = \begin{pmatrix} m_{11} & m_{1j} \\ m_{j1} & m_{jj} \end{pmatrix}$ is non-singular. Also assume $j = 2$.

In this subcase, $M = \begin{pmatrix} A & B \\ B^t & M_0 \end{pmatrix}$. Note that

$$\begin{pmatrix} I_2 & 0 \\ -B^t A^{-1} & I_{n-2} \end{pmatrix} \begin{pmatrix} A & B \\ B^t & M_0 \end{pmatrix} \begin{pmatrix} I_2 & -A^{-1} B \\ 0 & I_{n-2} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & M_0 - B^t A^{-1} B \end{pmatrix}.$$

Since $m_{11} = 1$ and A is non-singular, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, there is $U_1 \in M_2(\mathbb{F}_2)$ such that $U_1^t U_1 = A$. By induction, there is $U_2 \in M_{n-2}(\mathbb{F}_2)$ such that $U_2^t U_2 = M_0 - B^t A^{-1} B$. Thus the result holds.

Case 1.2. For every $j > 1$, $A_j = \begin{pmatrix} m_{11} & m_{1j} \\ m_{j1} & m_{jj} \end{pmatrix}$ is singular.

In this subcase, $A_j = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ since M is symmetric. By elementary column transformations and corresponding row transformations we can change all $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus we can assume

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_{23} & \cdots & m_{2n} \\ 0 & m_{32} & 0 & \cdots & m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & m_{n2} & m_{n3} & \cdots & 0 \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & m_{23} & \cdots & m_{2n} \\ 0 & m_{32} & 0 & \cdots & m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & m_{n2} & m_{n3} & \cdots & 0 \end{pmatrix}.$$

There is $V \in M_n(\mathbb{F}_2)$ such that $V^t V = C$ by **Case 1.1**. Assume $V = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $U = (\mathbf{1}, \alpha_1 + \alpha_2, \alpha_3, \dots, \alpha_n)$, where $\mathbf{1}$ is the all-ones vector. Then $U \in M_n(\mathbb{F}_2)$. Note that $\mathbf{1} \cdot \mathbf{1} = 1$, $\mathbf{1} \cdot \alpha_i = \alpha_i \cdot \alpha_i = 0$ for $i \geq 3$, $\mathbf{1} \cdot \alpha_i = \alpha_i \cdot \alpha_i = 1$ for $i \leq 2$ and $\alpha_1 \cdot \alpha_i = 0$ for $i \geq 2$. Hence $U^t U = M$.

Case 2. $m_{ii} = 0$ holds for every $i \in [n]$.

Let

$$C_0 = M + \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

There is $U_0 \in M_n(\mathbb{F}_2)$ such that $U_0^t U_0 = C_0$ by **Case 1**. Assume $U_0 = (\alpha'_1, \alpha'_2, \dots, \alpha'_n)$. Let $U = (\alpha'_1 + \mathbf{1}, \alpha'_2, \alpha'_3, \dots, \alpha'_n)$. Note that $\mathbf{1} \cdot \mathbf{1} = 1$ and $\mathbf{1} \cdot \alpha'_i = \alpha'_i \cdot \alpha'_i = 0$ for all $1 \leq i \leq n$. Hence $M = U^t U$ and we are done. \square

Lemma 2.2 *For every even integer k and a tournament T with $\text{inv}(T) = k$, $\text{rank}(\Lambda) \geq k$, where $\Lambda = \{\mathbf{u} | u \in V(T)\}$ is the set of characteristic vectors of a decycling family of T . Moreover, $\text{rank}(\Lambda) = k$ if the decycling family of T is a k -decycling family.*

Proof. Suppose for a contradiction that $\text{rank}(\Lambda) < k$. Then there are $k-1$ vectors $\{\alpha_1, \dots, \alpha_{k-1}\} \subset \Lambda$ such that \mathbf{u} is a linear combination of them for each $u \in V(T)$. Let $\mathbf{u} = \sum_{i=1}^{k-1} \lambda_{u,i} \alpha_i$ and $M \in M_{k-1}(\mathbb{F}_2)$ the Gram matrix of $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$. Then there exists $U \in M_{k-1}(\mathbb{F}_2)$ such that $M = U^t U$ by Lemma 2.1. Assume $U = (\beta_1, \beta_2, \dots, \beta_{k-1})$. Then $\beta_i \in \mathbb{F}_2^{k-1}$ and $\beta_i \cdot \beta_j = \alpha_i \cdot \alpha_j$ for any $i, j \in \{1, \dots, k-1\}$. We have a new $(k-1)$ -decycling family given by

the characteristic vector

$$\mathbf{u} = \sum_{i=1}^{k-1} \lambda_{u,i} \beta_i$$

for each $u \in V(T)$, because the scalar products between vertices are fixed. A contradiction with $\text{inv}(T) = k$. \square

Remark 2.3 *The results of Lemma 2.1 only hold when k is even. If k is odd, then $k - 2$ is odd. By Lemma 2.1, there is $U \in M_{k-2}(\mathbb{F}_2)$ such that $U^t U = M$. Hence we can prove $\text{rank}(\Lambda) \geq k - 1$ by using the same method.*

To prove Theorem 1.7, we give a stronger conclusion as following.

Lemma 2.4 *Let D be an oriented graph. If $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D) = k$, then there exists a k -decycling family of D with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(D)\}$ such that $\text{rank}(\Lambda) = k - 1$.*

Proof. Note that $\text{inv}(D) = k$. By Observation 1.17, Lemma 2.2 and Remark 2.3, we can assume $\text{rank}(\Lambda) = k$ for every k -decycling family of D .

Assume $V(\vec{C}_3) = \{u, v, w\}$. Fix a k -decycling family $(X_i)_{1 \leq i \leq k}$ of $\vec{C}_3 \Rightarrow D$. We extend D to a tournament T such that $\vec{C}_3 \Rightarrow T$ has the same k -decycling family and $\text{inv}(T) = \text{inv}(D) = k$ by Observation 1.17. After the inversions, assume the new ordering of $V(\vec{C}_3 \Rightarrow T)$ is

$$P < u < Q < v < R < w < S,$$

where $\{P, Q, R, S\}$ is a partition of $V(T)$ with possibly empty set. Comparing with the old ordering of $V(\vec{C}_3 \Rightarrow T)$, that is

$$\{u, v, w\} \rightarrow P \cup Q \cup R \cup S,$$

we can give the scalar products between $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{F}_2^k$ and $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in \mathbb{F}_2^k$ as following, where $p \in P, q \in Q, r \in R, s \in S$.

$$\begin{aligned} \mathbf{p} \cdot \mathbf{u} &= 1, & \mathbf{q} \cdot \mathbf{u} &= 0, & \mathbf{r} \cdot \mathbf{u} &= 0, & \mathbf{s} \cdot \mathbf{u} &= 0, \\ \mathbf{p} \cdot \mathbf{v} &= 1, & \mathbf{q} \cdot \mathbf{v} &= 1, & \mathbf{r} \cdot \mathbf{v} &= 0, & \mathbf{s} \cdot \mathbf{v} &= 0, \\ \mathbf{p} \cdot \mathbf{w} &= 1, & \mathbf{q} \cdot \mathbf{w} &= 1, & \mathbf{r} \cdot \mathbf{w} &= 1, & \mathbf{s} \cdot \mathbf{w} &= 0. \end{aligned} \tag{1}$$

Before the inversions, three arcs of \vec{C}_3 are either $u \rightarrow v, v \rightarrow w, w \rightarrow u$ or $u \rightarrow w, w \rightarrow v, v \rightarrow u$. Then we have either $\mathbf{u} \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 1$, or $\mathbf{u} \cdot \mathbf{v} = 1, \mathbf{v} \cdot \mathbf{w} = 1$ and $\mathbf{u} \cdot \mathbf{w} = 0$. In both cases, we have $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = 1$ and $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = 1$. Thus $\mathbf{u}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{v}$ are nonzero vectors.

Since $\text{inv}(T) = k$, $\Lambda = \{\mathbf{p} | p \in P\} \cup \{\mathbf{q} | q \in Q\} \cup \{\mathbf{r} | r \in R\} \cup \{\mathbf{s} | s \in S\}$ is the set of characteristic vectors of the k -decycling family $(X_i \cap V(T))_{1 \leq i \leq k}$, of T . By assumption,

$\text{rank}(\Lambda) = k$. If $P = \emptyset$ (resp. $Q = \emptyset$ or $R = \emptyset$), then $\mathbf{u} \perp \Lambda$ (resp. $(\mathbf{u} + \mathbf{v}) \perp \Lambda$ or $(\mathbf{v} + \mathbf{w}) \perp \Lambda$), a contradiction with $\text{rank}(\Lambda) = k$. Hence we have P, Q and R all are non-empty sets. Note that the digraph $\text{Inv}(T; (X_i \cap V(T))_{i \in I})$ is acyclic and the ordering of $V(T)$ in $\text{Inv}(T; (X_i \cap V(T))_{i \in I})$ is $P < Q < R < S$. Particularly, $\text{Inv}(T \langle Y \rangle; (X_i \cap Y)_{i \in I})$ is acyclic for any $Y \in \{P, Q, R, S\}$.

Now we choose $p_0 \in P, q_0 \in Q, r_0 \in R$. From (1), it is not difficult to show that $\{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}$ and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linear independent. Let $\Phi = \{\mathbf{x} \in \mathbb{F}_2^k \mid \mathbf{x} \perp \mathbf{u}, \mathbf{x} \perp \mathbf{v}, \mathbf{x} \perp \mathbf{w}\}$. Then $\text{rank}(\Phi) = k - 3$. For any $p \in P$, we have $\mathbf{p} + \mathbf{p}_0 \in \Phi$ which implies $\{\mathbf{p} \mid p \in P\} \subset \mathbf{p}_0 + \Phi$. So for any $p \in P$, we can set $\mathbf{p} = \mathbf{p}_0 + \phi_p$, where $\phi_p \in \Phi$. Similarly, for any $q \in Q, r \in R$, we set $\mathbf{q} = \mathbf{q}_0 + \phi_q, \mathbf{r} = \mathbf{r}_0 + \phi_r$, where $\phi_q, \phi_r \in \Phi$. Note that any vector is a linear combination of $\{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}$ and a vector in Φ . Assume

$$\mathbf{u} + \mathbf{v} = \lambda_1 \mathbf{p}_0 + \lambda_2 \mathbf{q}_0 + \lambda_3 \mathbf{r}_0 + \phi.$$

Since $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = 1, \mathbf{u} + \mathbf{v} \notin \Phi$. Then $|\{\lambda_1, \lambda_2, \lambda_3\} \cap (\mathbb{F}_2 \setminus \{0\})| \geq 1$. We consider the following three cases.

Case 1. $\lambda_1 = 1$.

In this case, $\tau := \mathbf{p}_0 + \mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{q}_0, \mathbf{r}_0$ and a vector in Φ . Let

$$\Lambda' = \{\mathbf{p} + \mathbf{u} + \mathbf{v} \mid p \in P\} \cup \{\mathbf{q} \mid q \in Q\} \cup \{\mathbf{r} \mid r \in R\} \cup \{\mathbf{s} \mid s \in S\}.$$

Then Λ' is the set of characteristic vectors of a k -family, say $(X'_i)_{1 \leq i \leq k}$, of T . Obviously, $\text{rank}(\Lambda') = k - 1$. So we just need to show that $(X'_i)_{1 \leq i \leq k}$ is a k -decycling family of T , that is $\text{Inv}(T; (X'_i)_{i \in I})$ is acyclic. By comparing Λ and Λ' , we have $\text{Inv}(T \langle Y \rangle; (X'_i \cap Y)_{i \in I})$ is acyclic for any $Y \in \{Q, R, S\}$ and $Q < R < S$ in $\text{Inv}(T; (X'_i)_{i \in I})$.

Since $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{r} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{s} = 0$, we have $(\mathbf{p} + \mathbf{u} + \mathbf{v}) \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}$ for all $p \in P$ and $x \in R \cup S$ which implies $P < R < S$ in $\text{Inv}(T; (X'_i)_{i \in I})$. Since $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{q} = 1$, we have $(\mathbf{p} + \mathbf{u} + \mathbf{v}) \cdot \mathbf{q} = \mathbf{p} \cdot \mathbf{q} + 1$ for all $p \in P$ and $q \in Q$ which implies $Q < P$ in $\text{Inv}(T; (X'_i)_{i \in I})$. For any $p_1, p_2 \in P$, we have

$$(\mathbf{p}_1 + \mathbf{u} + \mathbf{v}) \cdot (\mathbf{p}_2 + \mathbf{u} + \mathbf{v}) = (\mathbf{p}_1 \cdot \mathbf{p}_2) + (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}).$$

Note that $\mathbf{u} + \mathbf{v}$ is fixed, so the arcs in $\text{Inv}(T \langle P \rangle; (X'_i \cap P)_{i \in I})$ are either all inverted or not which implies $\text{Inv}(T \langle P \rangle; (X'_i \cap P)_{i \in I})$ is acyclic. As a conclusion, $\text{Inv}(T \langle Y \rangle; (X'_i \cap Y)_{i \in I})$ is acyclic for all $Y \in \{P, Q, R, S\}$ and $Q < P < R < S$ in $\text{Inv}(T; (X'_i)_{i \in I})$. Thus $(X'_i)_{1 \leq i \leq k}$ is a k -decycling family of T .

Case 2. $\lambda_2 = 1$.

In this case, $\tau := \mathbf{q}_0 + \mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{p}_0, \mathbf{r}_0$ and a vector in Φ . Let

$$\Lambda' = \{\mathbf{p} \mid p \in P\} \cup \{\mathbf{q} + \mathbf{u} + \mathbf{v} \mid q \in Q\} \cup \{\mathbf{r} \mid r \in R\} \cup \{\mathbf{s} \mid s \in S\}.$$

Then Λ' is the set of characteristic vectors of a k -family, say $(X'_i)_{1 \leq i \leq k}$, of T . Obviously, $\text{rank}(\Lambda') = k - 1$. Since $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{p} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{r} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{s} = 0$, by the same argument

as that of Case 1, we have $\text{Inv}(T \langle Y \rangle; (X'_i \cap Y)_{i \in I})$ is acyclic for all $Y \in \{P, Q, R, S\}$ and $P < Q < R < S$ in $\text{Inv}(T; (X'_i)_{i \in I})$. Thus $(X'_i)_{1 \leq i \leq k}$ is a k -decycling family of T .

Case 3. $\lambda_3 = 1$.

In this case, $\tau := \mathbf{r}_0 + \mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{p}_0, \mathbf{q}_0$ and a vector in Φ . Let

$$\Lambda' = \{\mathbf{p}|p \in P\} \cup \{\mathbf{q}|q \in Q\} \cup \{\mathbf{r} + \mathbf{u} + \mathbf{v}|r \in R\} \cup \{\mathbf{s}|s \in S\}.$$

Then Λ' is the set of characteristic vectors of a k -family, say $(X'_i)_{1 \leq i \leq k}$, of T . Obviously, $\text{rank}(\Lambda') = k - 1$. Note that $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{p} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{s} = 0$ and $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{q} = 1$. By the same argument as that of Case 1, we have $\text{Inv}(T \langle Y \rangle; (X'_i \cap Y)_{i \in I})$ is acyclic for all $Y \in \{P, Q, R, S\}$ and $P < R < Q < S$ in $\text{Inv}(T; (X'_i)_{i \in I})$. Thus $(X'_i)_{1 \leq i \leq k}$ is a k -decycling family of T . \square

Now we are going to prove Theorem 1.7.

Proof of Theorem 1.7. By Observation 1.16, $\text{inv}(\vec{C}_3 \Rightarrow D) \geq k$. As we can invert \vec{C}_3 and D respectively to transform $\vec{C}_3 \Rightarrow D$ into an acyclic digraph, we have $\text{inv}(\vec{C}_3 \Rightarrow D) \leq k + 1$. If $\text{inv}(\vec{C}_3 \Rightarrow D) = k$, then there exists a k -decycling family of D with the set of characteristic vectors $\Lambda = \{\mathbf{u}|u \in V(D)\}$ such that $\text{rank}(\Lambda) = k - 1$ by Lemma 2.4, which contradicts with Lemma 2.2. Hence we have $\text{inv}(\vec{C}_3 \Rightarrow D) = k + 1$ as required. \square

2.2 Proofs of Theorems 1.8 to 1.9

We observe that in the proof of Theorem 1.7, we only use the property that the rank of the characteristic vectors and the inversion number are the same. Hence we consider applying the conclusion to general cases.

Lemma 2.5 *For every even integer n and a symmetric matrix $M \in M_n(\mathbb{F}_2)$, there is $U \in M_n(\mathbb{F}_2)$ such that $M = U^t U$ if and only if M has a non-zero diagonal entry or M is a singular matrix.*

Proof. We prove by induction on n . It is trivial when $n = 2$. Assume $M = (m_{ij})_{1 \leq i, j \leq n}$ with $n \geq 4$. We consider three cases.

Case 1. There exists $i \in [n]$ such that $m_{ii} = 1$. By interchanging columns and corresponding rows we assume $i = 1$. Let $M = \begin{pmatrix} 1 & B \\ B^t & M_0 \end{pmatrix}$. Note that

$$\begin{pmatrix} 1 & 0 \\ -B^t A^{-1} & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ B^t & M_0 \end{pmatrix} \begin{pmatrix} 1 & -A^{-1} B \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & M_0 - B^t B \end{pmatrix}.$$

Since $M_0 - B^t B \in M_{n-1}(\mathbb{F}_2)$, by Lemma 2.1, $M_0 - B^t B$ has a decomposition which implies the result holds.

Case 2. $m_{ii} = 0$ for every $i \in [n]$ and M is a non-singular symmetric matrix. Suppose there is $U \in M_n(\mathbb{F}_2)$ such that $U^t U = M$. Assume $U = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Since $\mathbf{1} \cdot \alpha_j = \alpha_j \cdot \alpha_j =$

$m_{jj} = 0$, for any $j \in [n]$, the vector α_j has an even number of 1s. So $\text{rank}(U) < n$ which implies $\text{rank}(M) \leq \text{rank}(U) < n$, a contradiction.

Case 3. $m_{ii} = 0$ for every $i \in [n]$, and M is a singular symmetric matrix. Then we can change the first column and first row to all zeros by elementary column transformations and corresponding row transformations. Thus we can assume $M = \begin{pmatrix} 0 & 0 \\ 0 & M' \end{pmatrix}$. Since $M' \in M_{n-1}(\mathbb{F}_2)$, by Lemma 2.1, M' has a decomposition which implies the result holds. \square

Lemma 2.6 *Let D be an oriented graph. If $\text{inv}(D) = k \geq 3$, then the following propositions are equivalent.*

- (1) $\text{inv}(\vec{C}_3 \Rightarrow D) = k$.
- (2) k is odd and there exists a k -decycling family of D with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(D)\}$ such that $\Lambda \subset \{x \in \mathbb{F}_2^k | x \perp \mathbf{1}\}$.

Proof. By Observation 1.17, we assume D is a tournament.

(1) \Rightarrow (2). Obviously k is odd by Theorem 1.7. By Lemma 2.4, there exists a k -decycling family of D with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(D)\}$ such that $\text{rank}(\Lambda) = k - 1$.

Under the fixed decycling family, we claim $\mathbf{z} \cdot \mathbf{z} = 0$ for each $z \in V(D)$, then the conclusion can be deduced since $\mathbf{1} \cdot \mathbf{z} = \mathbf{z} \cdot \mathbf{z} = 0$. Suppose there is $z_1 \in V(D)$ such that $\mathbf{z}_1 \cdot \mathbf{z}_1 = 1$. Then we can choose z_2, z_3, \dots, z_{k-1} such that $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{k-1}$ forms a base of Λ . So for each $z \in V(D)$, we have $\mathbf{z} = \sum_{i=1}^{k-1} \lambda_{z,i} \mathbf{z}_i$, where $\lambda_{z,i} \in \mathbb{F}_2$.

Let $M \in M_{k-1}(\mathbb{F}_2)$ be the Gram matrix of $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{k-1}\}$. Since $\mathbf{z}_1 \cdot \mathbf{z}_1 = 1$, there exists $U \in M_{k-1}(\mathbb{F}_2)$ such that $M = U^t U$ by Lemma 2.5. Let $U = (\mathbf{z}'_1, \mathbf{z}'_2, \dots, \mathbf{z}'_{k-1})$. Then $\mathbf{z}'_i \cdot \mathbf{z}'_j = \mathbf{z}_i \cdot \mathbf{z}_j$. Now we have a new $(k-1)$ -decycling family given by the characteristic vector

$$\mathbf{z} = \sum_{i=1}^{k-1} \lambda_{z,i} \mathbf{z}'_i$$

for each $z \in V(D)$, because the scalar products between vertices are fixed. Note that $\mathbf{z}'_i \in \mathbb{F}_2^{k-1}$ and this contradicts with $\text{inv}(D) = k$.

(2) \Rightarrow (1). Obviously, $\text{inv}(\vec{C}_3 \Rightarrow D) \geq \text{inv}(D) = k$. Let $V(\vec{C}_3) = \{u, v, w\}$ and define $\mathbf{u} = 0, \mathbf{v} = \mathbf{w} = \mathbf{1} \in \mathbb{F}_2^k$. It is easy to check that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \cup \Lambda$ is the set of characteristic vectors of a k -decycling family of $\vec{C}_3 \Rightarrow D$ which implies $\text{inv}(\vec{C}_3 \Rightarrow D) \leq k$. \square

Proof of Theorem 1.8. Let D^- be the digraph obtained by reversing all arcs of D . Then D and D^- share the same decycling family and $\text{inv}(D) = \text{inv}(D^-)$.

Assume $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D) = k$. By Lemma 2.6 and the fact that D and D^- share the same decycling family, we have k is odd and there exists a k -decycling family of D^- with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(D^-)\}$ such that $\Lambda \subset \{x \in \mathbb{F}_2^k | x \perp \mathbf{1}\}$. By Lemma 2.6

again, $\text{inv}(\vec{C}_3 \Rightarrow D^-) = \text{inv}(D^-)$. Since $\text{inv}(\vec{C}_3 \Rightarrow D^-) = \text{inv}(D \Rightarrow \vec{C}_3)$ and $\text{inv}(D) = \text{inv}(D^-)$, we have $\text{inv}(D \Rightarrow \vec{C}_3) = \text{inv}(\vec{C}_3 \Rightarrow D)$.

Conversely, assume $\text{inv}(D \Rightarrow \vec{C}_3) = \text{inv}(\vec{C}_3 \Rightarrow D^-) = \text{inv}(D^-)$. By the same argument as above, we have $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D \Rightarrow \vec{C}_3)$.

Since $\text{inv}(\vec{C}_3 \Rightarrow D), \text{inv}(D \Rightarrow \vec{C}_3) \in \{\text{inv}(D), \text{inv}(D) + 1\}$, we have $\text{inv}(\vec{C}_3 \Rightarrow D) = \text{inv}(D \Rightarrow \vec{C}_3)$. \square

Lemma 2.7 *Let D be an oriented graph with $\text{inv}(D) = 1$. Then $\text{inv}(\vec{C}_3 \Rightarrow H) = \text{inv}(D \Rightarrow H)$ and $\text{inv}(H \Rightarrow \vec{C}_3) = \text{inv}(H \Rightarrow D)$ for every oriented graph H .*

Proof. We just show that $\text{inv}(\vec{C}_3 \Rightarrow H) = \text{inv}(D \Rightarrow H)$ holds for every oriented graph H . Actually, we have the conclusion in the other direction if we reverse all arcs. Obviously $\text{inv}(\vec{C}_3 \Rightarrow H), \text{inv}(D \Rightarrow H) \in \{\text{inv}(H), \text{inv}(H) + 1\}$.

If $\text{inv}(\vec{C}_3 \Rightarrow H) = \text{inv}(H) + 1$, the result holds since $\vec{C}_3 \Rightarrow H$ is a subdigraph of $D \Rightarrow H$.

If $\text{inv}(\vec{C}_3 \Rightarrow H) = \text{inv}(H) = k$, then k is odd by Theorem 1.7. By Lemma 2.4, there exists a k -decycling family of H with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(H)\}$ such that $\text{rank}(\Lambda) = k - 1$. Then we obtain a $(k + 1)$ -decycling family of $D \Rightarrow H$ with the set of characteristic vectors $\Lambda' = \{\mathbf{z} | z \in V(D \Rightarrow H)\}$ such that $\text{rank}(\Lambda') \leq k$ if we invert D and H respectively. Obviously $\text{inv}(D \Rightarrow H) = k$ or $k + 1$. If $\text{inv}(D \Rightarrow H) = k + 1$, $\text{rank}(\Lambda') \geq k + 1$ by Lemma 2.2, a contradiction.

Then $\text{inv}(D \Rightarrow H) = k = \text{inv}(\vec{C}_3 \Rightarrow H)$ which proves the result. \square

Now we are going to prove Theorem 1.9.

Proof of Theorem 1.9. By contradiction. Assume D is a tournament by Observation 1.17. Suppose $\text{inv}([\vec{C}_3, \vec{C}_3, D]) = \text{inv}(\vec{C}_3 \Rightarrow D) = k$. Then k is odd by Theorem 1.7. By Lemma 2.6, there exists a k -decycling family $(X_i)_{1 \leq i \leq k}$ of $\vec{C}_3 \Rightarrow D$ with the set of characteristic vectors $\Lambda = \{\mathbf{z} | z \in V(\vec{C}_3 \Rightarrow D)\}$ such that $\Lambda \subset F$ where $F = \{x \in \mathbb{F}_2^k | x \perp \mathbf{1}\}$. Then $\text{rank}(F) = k - 1$. Let $\Gamma = \{\mathbf{x} \in \Lambda | x \in V(D)\}$ be the set of the characteristics vectors of the k -decycling family $(X_i \cap V(D))_{1 \leq i \leq k}$ of D . Since $\text{inv}(D)$ is either k or $k - 1$, $\text{rank}(\Gamma) \geq k - 1$ by Remark 2.3 and Lemma 2.2. By Remark 2.3 and $\Lambda \subset F$, $\text{rank}(\Lambda) = k - 1$. From $\Gamma \subset \Lambda$, we have $\text{rank}(\Gamma) = k - 1$. Assume $V(\vec{C}_3) = \{u, v, w\}$. Under the fixed decycling family, assume the new ordering of $V(\vec{C}_3 \Rightarrow D)$ is

$$P < u < Q < v < R < w < S,$$

where P, Q, R, S form a partition of $V(D)$ with possibly empty set. Then $\Gamma = \{\mathbf{p} | p \in P\} \cup \{\mathbf{q} | q \in Q\} \cup \{\mathbf{r} | r \in R\} \cup \{\mathbf{s} | s \in S\}$. Comparing with the original tournament

$$\{u, v, w\} \rightarrow P \cup Q \cup R \cup S,$$

we can give the scalar products between $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{F}_2^k$ and $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in \mathbb{F}_2^k$, where $p \in P, q \in$

$Q, r \in R, s \in S$. That is

$$\begin{aligned} \mathbf{p} \cdot \mathbf{u} &= 1, & \mathbf{q} \cdot \mathbf{u} &= 0, & \mathbf{r} \cdot \mathbf{u} &= 0, & \mathbf{s} \cdot \mathbf{u} &= 0, \\ \mathbf{p} \cdot \mathbf{v} &= 1, & \mathbf{q} \cdot \mathbf{v} &= 1, & \mathbf{r} \cdot \mathbf{v} &= 0, & \mathbf{s} \cdot \mathbf{v} &= 0, \\ \mathbf{p} \cdot \mathbf{w} &= 1, & \mathbf{q} \cdot \mathbf{w} &= 1, & \mathbf{r} \cdot \mathbf{w} &= 1, & \mathbf{s} \cdot \mathbf{w} &= 0. \end{aligned} \tag{2}$$

By the same argument as the proof in Lemma 2.4, we have $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = 1$ and $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = 1$ which implies that $\mathbf{u}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{v}$ are nonzero vectors.

If $P = \emptyset$ (resp. $Q = \emptyset$ or $R = \emptyset$), then $\mathbf{u} \perp \Gamma$ (resp. $(\mathbf{u} + \mathbf{v}) \perp \Gamma$ or $(\mathbf{v} + \mathbf{w}) \perp \Gamma$). Note that \mathbf{u} (resp. $\mathbf{u} + \mathbf{v}$ or $\mathbf{v} + \mathbf{w}$) is linear independent with $\mathbf{1}$ and $\Gamma \subset F$. Thus we have $\text{rank}(\Gamma) \leq k - 2$, a contradiction with $\text{rank}(\Gamma) = k - 1$. Hence we have P, Q and R all are non-empty sets.

Now we can choose $p_0 \in P, q_0 \in Q, r_0 \in R$. From (2), it is not difficult to show that $\{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}$ and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linear independent. Let $\Phi = \{\mathbf{x} \in F \mid \mathbf{x} \perp \mathbf{u}, \mathbf{x} \perp \mathbf{v}, \mathbf{x} \perp \mathbf{w}\}$. Since $\text{rank}(\{\mathbf{1}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}) = 4$, $\text{rank}(\Phi) = k - 4$. By the same argument as the proof in Lemma 2.4, for any $p \in P$ (resp. $q \in Q$ or $r \in R$), there is $\phi_p \in \Phi$ (resp. $\phi_q \in \Phi$ or $\phi_r \in \Phi$) such that $\mathbf{p} = \mathbf{p}_0 + \phi_p$ (resp. $\mathbf{q} = \mathbf{q}_0 + \phi_q$ or $\mathbf{r} = \mathbf{r}_0 + \phi_r$). Note that any vector is a linear combination of $\{\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0\}$ and a vector in Φ . Assume

$$\mathbf{u} + \mathbf{v} = \lambda_1 \mathbf{p}_0 + \lambda_2 \mathbf{q}_0 + \lambda_3 \mathbf{r}_0 + \phi.$$

Since $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = 1$, $\mathbf{u} + \mathbf{v} \notin \Phi$. Then $|\{\lambda_1, \lambda_2, \lambda_3\} \cap (\mathbb{F}_2 \setminus \{0\})| \geq 1$. Let

$$\Gamma' = \begin{cases} \{\mathbf{p} + \mathbf{u} + \mathbf{v} \mid p \in P\} \cup \{\mathbf{q} \mid q \in Q\} \cup \{\mathbf{r} \mid r \in R\} \cup \{\mathbf{s} \mid s \in S\} & \text{if } \lambda_1 = 1, \\ \{\mathbf{p} \mid p \in P\} \cup \{\mathbf{q} + \mathbf{u} + \mathbf{v} \mid q \in Q\} \cup \{\mathbf{r} \mid r \in R\} \cup \{\mathbf{s} \mid s \in S\} & \text{if } \lambda_2 = 1, \\ \{\mathbf{p} \mid p \in P\} \cup \{\mathbf{q} \mid q \in Q\} \cup \{\mathbf{r} + \mathbf{u} + \mathbf{v} \mid r \in R\} \cup \{\mathbf{s} \mid s \in S\} & \text{if } \lambda_3 = 1. \end{cases}$$

By the same argument as the proof in Lemma 2.4, we have Γ' is the set of characteristic vectors of a k -decycling family of D . Obviously, $\text{rank}(\Gamma') = k - 2$, we have a contradiction because the rank of the characteristics vectors of any decycling family of D is at least $k - 1$. \square

Proof of Corollary 1.10. We prove it by induction on k . The situation when $k = 2$ is proved by Lemma 2.7. Assume $k \geq 3$. By Lemma 2.7 and Theorem 1.8, we can assume $j \neq 1$ and then $\text{inv}([D_1, D_2, \dots, D_k]) = \text{inv}([\vec{C}_3, D_2, \dots, D_k])$ by Lemma 2.7. If $j \neq k$, $\text{inv}([\vec{C}_3, D_2, \dots, D_k]) = \text{inv}([\vec{C}_3, D_2, \dots, D_{k-1}, \vec{C}_3])$ by Lemma 2.7. If $j = k$, $\text{inv}([\vec{C}_3, D_2, \dots, D_k]) = \text{inv}([D_2, \dots, D_k, \vec{C}_3])$ by Theorem 1.8. By Lemma 2.7, $\text{inv}([D_2, \dots, D_k, \vec{C}_3]) = \text{inv}([\vec{C}_3, D_3, \dots, D_k, \vec{C}_3])$.

Hence we can assume $\text{inv}([D_1, D_2, \dots, D_k]) = \text{inv}([\vec{C}_3, D_2, \dots, D_{k-1}, \vec{C}_3])$. By Theorems 1.8 and 1.9, for any oriented graph D , $\text{inv}([\vec{C}_3, \vec{C}_3, D]) = \text{inv}([\vec{C}_3, D, \vec{C}_3]) = \text{inv}([D, \vec{C}_3, \vec{C}_3]) = \text{inv}(\vec{C}_3 \Rightarrow D) + 1$. Then we obtain the result as required. \square

3 Blow-up graphs

The content in this section is inspired by Alon et al. [1]. They mainly concerned about the graph $D = TT_k[D_1, D_2, \dots, D_k]$. We generalize the base graph TT_k to any tournament H , since it creates a larger inversion number sometimes.

Proof of Theorem 1.13. By contradiction. Assume D_i is tournament for all $1 \leq i \leq n$. First $\text{inv}(T[D_1, D_2, \dots, D_n]) \leq n + 1$ as we can invert T to acyclic and then invert D_i one after another. Since $T[\vec{C}_3]_n$ is a subdigraph of $T[D_1, D_2, \dots, D_n]$, we only need to prove the situation when $D_i = \vec{C}_3$ by Observation 1.16. Let $\text{inv}(T[D_1, D_2, \dots, D_n]) = k$. For short, set $G = T[D_1, D_2, \dots, D_n]$. Then there exists a k -decycling family $(X_i)_{i \in I}$ of G with the set of characteristic vectors $\Lambda = \{\mathbf{u} \in \mathbb{F}_2^k | u \in V(G)\}$. Note that the digraph $\text{Inv}(G; (X_i)_{i \in I})$ is acyclic.

Recall $u < v$ (resp. $u \rightarrow v$) means $uv \in A(\text{Inv}(G; (X_i)_{i \in I}))$ (resp. $uv \in A(G)$).

Let $V(D_i) = \{u_i, v_i, w_i\}$, $1 \leq i \leq n$. Since $\text{Inv}(D_i; (X_j)_{j \in I})$ is acyclic, we can assume $u_i < v_i, u_i < w_i$ and $\mathbf{u}_i \cdot \mathbf{v}_i = 0, \mathbf{u}_i \cdot \mathbf{w}_i = 1$, which means $v_i \rightarrow w_i$ for $1 \leq i \leq n$.

Claim 3.1 $\{\mathbf{u}_i\}_{1 \leq i \leq n}$ are linearly independent.

Proof of Claim 3.1. It is equivalent to show that for any nonempty set $I \subseteq [n]$, $\sum_{i \in I} \mathbf{u}_i \neq 0$. Let $I \subseteq [n]$. Note that \mathbf{u}_i is a nonzero vector for all $1 \leq i \leq n$. Then we can assume $|I| \geq 2$. Choose $m \in I$ such that $u_{m'} < u_m$ for any $m' \in I \setminus \{m\}$. Since $u_m < \{v_m, w_m\}$, then $u_{m'} < \{v_m, w_m\}$ for any $m' \in I \setminus \{m\}$. Note that either $u_{m'} \rightarrow \{v_m, w_m\}$ or $\{v_m, w_m\} \rightarrow u_{m'}$, which means $\mathbf{u}_{m'} \cdot (\mathbf{v}_m + \mathbf{w}_m) = 0$. Since $\mathbf{u}_m \cdot (\mathbf{v}_m + \mathbf{w}_m) = 1$, we have $(\sum_{i \in I} \mathbf{u}_i) \cdot (\mathbf{v}_m + \mathbf{w}_m) = 1$. Thus $\sum_{i \in I} \mathbf{u}_i \neq 0$ as required. ■

Let $V(T) = \{x_1, x_2, \dots, x_n\}$, where x_i corresponds to D_i for $1 \leq i \leq n$.

Claim 3.2 For any nonempty set $I \subseteq [n]$. Let $S = \{\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i | i \in I\}$. If $\text{rank}(S) = |I|$, then $T \langle \{x_i\}_{i \in I} \rangle$ is acyclic.

Proof of Claim 3.2. Let $R = \{u_i, v_i, w_i | i \in I\}$. We prove it by induction on $|I|$. Let $Z = \{v_i, w_i | i \in I\}$ and choose $z \in Z$ such that $z < y$ for any $y \in Z \setminus \{z\}$. Assume $z \in \{v_t, w_t\}$, where $t \in I$. Since $\text{rank}(S) = |I|$, there exists $J \subseteq I$ such that $\mathbf{z} = \sum_{j \in J} \mathbf{u}_j$ by Claim 3.1.

If $J = \emptyset$, then $\mathbf{z} = 0$ which implies all arcs incident with z are not inverted. Hence $z \rightarrow y$ for all $y \in Z \setminus \{z\}$. So $x_t \rightarrow x_i$ for each $i \in I \setminus \{t\}$ in T and $z = v_t$ by $v_t \rightarrow w_t$. Let $R' = R \setminus \{u_t, v_t, w_t\}$ and $S' = S \setminus \{\mathbf{u}_t, \mathbf{v}_t, \mathbf{w}_t\}$. Since $u_t \rightarrow R'$ and $u_t < R'$, we have $\mathbf{u}_t \perp S'$ which implies $\text{rank}(S') = |I| - 1$. By the inductive hypothesis, $T \langle \{x_i\}_{i \in I \setminus \{t\}} \rangle$ is acyclic. Hence $T \langle \{x_i\}_{i \in I} \rangle$ is acyclic as required.

Now we consider the case $J \neq \emptyset$. If $J = \{t\}$, then $\mathbf{z} = \mathbf{u}_t$. Recall $z \in \{v_t, w_t\}$. If $z = v_t$ (resp. $z = w_t$), then $\mathbf{v}_t = \mathbf{u}_t$ and $v_t < w_t$ (resp. $\mathbf{w}_t = \mathbf{u}_t$ and $w_t < v_t$) which implies $1 = \mathbf{u}_t \cdot \mathbf{w}_t = \mathbf{v}_t \cdot \mathbf{w}_t$ (resp. $0 = \mathbf{u}_t \cdot \mathbf{v}_t = \mathbf{w}_t \cdot \mathbf{v}_t$), a contradiction with $v_t \rightarrow w_t$. So we have $J \neq \{t\}$. Let $u_s \in \{u_j | j \in J\}$ such that $x < u_s$ for any $x \in \{u_j | j \in J\} \setminus \{u_s\}$. We consider two cases.

Case 1. $s \neq t$. Then $\{z\} \cup \{u_j | j \in J\} < \{v_s, w_s\}$. For any $y \in \{z\} \cup \{u_j | j \in J \setminus \{s\}\}$, we have either $y \rightarrow \{v_s, w_s\}$ or $\{v_s, w_s\} \rightarrow y$ which implies $\mathbf{y} \cdot (\mathbf{v}_s + \mathbf{w}_s) = 0$. Hence $(\mathbf{z} + \sum_{j \in J} \mathbf{u}_j) \cdot (\mathbf{v}_s + \mathbf{w}_s) = 1$ by $\mathbf{u}_s \cdot (\mathbf{v}_s + \mathbf{w}_s) = 1$, a contradiction with $\mathbf{z} = \sum_{j \in J} \mathbf{u}_j$.

Case 2. $s = t$. In this case, $u_s < z$ by assumption. Let $u_{s'} \in \{u_j | j \in J \setminus \{t\}\}$ such that $x < u_{s'}$ for any $x \in \{u_j | j \in J \setminus \{t, s'\}\}$. Then $z < \{v_{s'}, w_{s'}\}$ and $\{u_j | j \in J\} \leq u_s < z < \{v_{s'}, w_{s'}\}$.

For any $y \in \{z\} \cup \{u_j | j \in J \setminus \{s'\}\}$, we have either $y \rightarrow \{v_{s'}, w_{s'}\}$ or $\{v_{s'}, w_{s'}\} \rightarrow y$ which implies we have $\mathbf{y} \cdot (\mathbf{v}_{s'} + \mathbf{w}_{s'}) = 0$. Therefore $(\mathbf{z} + \sum_{j \in J} \mathbf{u}_j) \cdot (\mathbf{v}_{s'} + \mathbf{w}_{s'}) = 1$ by $\mathbf{u}_{s'} \cdot (\mathbf{v}_{s'} + \mathbf{w}_{s'}) = 1$, a contradiction with $\mathbf{z} = \sum_{j \in J} \mathbf{u}_j$. \blacksquare

Now we complete the proof of Theorem 1.13. By Claim 3.1, we have $\text{rank}(\Lambda) \geq n$ which implies $k \geq n$. If $k = n$, then $\text{rank}(\Lambda) = n$. By Claim 3.2, $\text{inv}(T) = 0$, our final contradiction. \square

Since we can invert T to acyclic and then invert D_i one after another, then for oriented graphs D_i with $\text{inv}(D_i) = 1$, the following inequality holds

$$n \leq \text{inv}(T[D_1, D_2, \dots, D_n]) \leq n + \text{inv}(T).$$

Here we prove that there exists T such that it cannot reach the upper bound.

Proof of Theorem 1.14. From Theorem 1.2, for each odd integer $k \geq 3$, there is a tournament D with $\text{inv}(D) = k$ such that $\text{inv}(D \Rightarrow \vec{C}_3) \leq k$. Since D is a subgraph of $D \Rightarrow \vec{C}_3$, we have $\text{inv}(D \Rightarrow \vec{C}_3) = k$. From Theorem 1.8, we know $\text{inv}(D) = \text{inv}(\vec{C}_3 \Rightarrow D) = k$. Let $T_k = u_1 \Rightarrow D$. It's not hard to find that $\text{inv}(T_k) = k$, since D is a subgraph of T_k and the decycling family of D gives a decycling family of T_k . Assume $V(T_k) = \{u_1, u_2, \dots, u_{n_k}\}$. We need to show $\text{inv}(T_k[D_1, D_2, \dots, D_{n_k}]) \leq n_k + k - 1$ when $\text{inv}(D_i) = 1$ for all i .

From Lemma 2.6, there is a k -decycling family $(X_i)_{1 \leq i \leq k}$ of D such that every characteristic vector is orthogonal to $\mathbf{1}$. It means $|\{i \in [k] | v \in X_i\}|$ is even for any $v \in V(D)$. The corresponding blow-up of X_i is $X'_i = \bigcup_{u_j \in X_i} V(D_j)$. Since $\text{inv}(D_j) = 1$, for each $j \in [n_k]$, there exists a $Y_j \subset V(D_j)$ such that $\text{Inv}(D_j; Y_j)$ is acyclic. Then it is not hard to check that there exists a $(n_k + k - 1)$ -decycling family as following:

$$\begin{aligned} Z_i &= Y_1 \cup X'_i & 1 \leq i \leq k, \\ Z_{k+j-1} &= Y_j & 2 \leq j \leq n_k. \end{aligned}$$

Hence $\text{inv}(T_k[D_1, D_2, \dots, D_{n_k}]) \leq n_k + k - 1$ and we are done. \square

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