DERIVED FUNCTORS AND HILBERT POLYNOMIALS OVER HYPERSURFACE RINGS

TONY J. PUTHENPURAKAL

Abstract. Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. We show that there is a non-negative integer r_I (depending only on I) such that if M is any non-free maximal Cohen-Macaulay (= MCM) A-module the function $n \to \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$ (which is of polynomial type) has degree r_I. Analogous results hold for Hilbert polynomials associated to Ext-functors. Surprisingly a key ingredient is the classification of thick subcategories of the stable category of MCM A-modules (obtained by Takahashi, see [8, 6.6]).

1. introduction

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. If N is an A-module of finite length then $\ell(N)$ denotes its length. Let M be a maximal Cohen-Macaulay (= MCM) A-module. The function $t_I(M, n) = \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$ is of polynomial type, see [9, Corollary 4] (also see [5, Proposition 17]). Let $t_I^M(z) \in \mathbb{Q}[z]$ be such that $t_I^M(n) = t_I(M, n)$ for all $n \gg 0$. It is easily shown that $\deg t_I^M(z) \leq d-1$. In [5, Theorem 18] we proved that $\deg t_{\mathfrak{m}}^M(z) = d-1$ for any non-free MCM A-module. It was also shown that if I is a parameter ideal then $t_I(M, n) = 0$ for all $n \geq 0$, see [5, Remark 20]. In general it is a difficult question to determine the degree of $t_I^M(z)$ and the answer is known only for a few classes of ideals and modules, see [3, 3.5] for some examples. The fact that $\deg t_{\mathfrak{m}}^M(z) = d-1$ for non-free MCM's has an important consequence in the study of associated graded modules (with respect to $\mathfrak{m}$) of MCM A-modules, see [6].

In this paper we prove few surprising results. Recall A is said to be a hypersurface ring if its completion $\widehat{A} = Q/(f)$ where $(Q, \mathfrak{n})$ is a regular local ring and $f \in \mathfrak{n}^2$ is non-zero. We show

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer r_I (depending only on I) such that if M is any non-free maximal MCM A-module then $\deg t_I^M(z) = r_I$.*

1.2. For the Ext functors we prove an analogous result. It is known that if M is a finitely generated A-module the function $n \to \ell(\operatorname{Ext}_A^1(M, A/I^{n+1}))$ is of polynomial type say of degree s_I^M, see [9, Corollary 4]. We prove

Theorem 1.3. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer s_I (depending only on I) such that if M is any non-free maximal MCM A-module then $s_I^M = s_I$.*

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It is also known that if M is a finitely generated A-module the function $n \to \ell(\operatorname{Ext}^{d+1}_A(A/I^{n+1}, M))$ is of polynomial type say of degree e^M_I, see [9, Theorem 5]. Let $\operatorname{Spec}^0(A) = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$. We prove

Theorem 1.4. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer e_I (depending only on I) such that if M is any non-free maximal MCM A-module free on* $\operatorname{Spec}^0(A)$ *then $e^M_I = e_I$.*

See 4.4 on why in Theorem 1.4 we need to restrict to the case of MCM modules free on $\operatorname{Spec}^0(A)$ while in Theorems 1.1 and 1.3 we do not have such restriction.

Technique used to prove the result: We first note that the function $t_I(M, n)$ is a function on $\underline{\mathrm{CM}}(A)$ the stable category of MCM A-modules. We also note that $\underline{\mathrm{CM}}(A)$ is a triangulated category [1, 4.4.1]. Let $\underline{\mathrm{CM}}_0(A)$ be the thick subctegory of MCM A-modules which are free on the punctured spectrum $\operatorname{Spec}^0(A)$ of A. The crucial ingredient in our proofs is that $\underline{\mathrm{CM}}_0(A)$ has *no* proper thick subcategories, see [8, 6.6]. We first prove Theorem 1.1 for non-free MCM modules in $\underline{\mathrm{CM}}_0(A)$ and then prove for all non-free MCM A-modules by using an induction on $\dim \underline{\operatorname{Hom}}_A(M, M)$. The techniques to prove Theorems 1.3 and 1.4 are similar.

Here is an overview of the contents of this paper. In section two we discuss a few preliminaries that we need. In section three we prove Theorems 1.1, 1.3, 1.4 when M is free on the punctured spectrum of A. Finally in section four we prove Theorems 1.1 and 1.3.

2. Preliminaries

In this section we discuss a few preliminary results that we need. We use [4] for notation on triangulated categories. However we will assume that if $\mathcal{C}$ is a triangulated category then $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set for any objects X, Y of $\mathcal{C}$.

2.1. Let $\mathcal{C}$ be a skeletally small triangulated category with shift operator Σ and let $\mathbb{I}(\mathcal{C})$ be the set of isomorphism classes of objects in $\mathcal{C}$. By a *weak triangle function* on $\mathcal{C}$ we mean a function $\xi \colon \mathbb{I}(\mathcal{C}) \to \mathbb{Z}$ such that

(1) $\xi(X) \geq 0$ for all $X \in \mathcal{C}$.
(2) $\xi(0) = 0$.
(3) $\xi(X \oplus Y) = \xi(X) + \xi(Y)$ for all $X, Y \in \mathcal{C}$.
(4) $\xi(\Sigma X) = \xi(X)$ for all $X \in \mathcal{C}$.
(5) If $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{C}$ then $\xi(Z) \leq \xi(X) + \xi(Y)$.

2.2. Set

$$\ker \xi = \{X \mid \xi(X) = 0\}.$$

The following result (essentially an observation) is a crucial ingredient in our proof of Theorem 1.1.

Lemma 2.3. *(see* [7, 2,3] *) (with hypotheses as above)* $\ker \xi$ *is a thick subcategory of $\mathcal{C}$.*

2.4. Let $(A, \mathfrak{m})$ be a hypersurface ring and let I be an $\mathfrak{m}$-primary ideal in A. Let M be a MCM A-module. Set for $n \geq 0$

$$t_I(M,n) = \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$$
$$s_I(M,n) = \ell(\operatorname{Ext}_A^1(M, A/I^{n+1})))$$
$$e_I(M,n) = \ell(\operatorname{Ext}_A^{d+1}(A/I^{n+1}, M))).$$

Let $\Omega_A^i(M)$ denote the i^{th}-syzygy of M. We prove

Lemma 2.5. *(with hypotheses as above)*

(1) *For all $n \geq 0$ the functions $t_I(-,n), s_I(-,n)$ and $e_I(-,n)$ are functions on $\underline{\mathrm{CM}}(A)$*
(2) *For all $n \geq 0$ we have $t_I(M,n) = t_I(\Omega_A^1(M), n), s_I(M,n) = s_I(\Omega_A^1(M), n)$ and $e_I(M,n) = e_I(\Omega_A^1(M), n)$.*

Proof. (1) Let $E = M \oplus F = N \oplus G$ where F, G are free A-modules. Then by definition $t_I(E,n) = t_I(M,n) = t_I(N,n)$. Thus $t_I(-,n)$ is a function on $\underline{\mathrm{CM}}(A)$.

The proof for assertions on $s_I(-,n)$ and $e_I(-,n)$ are similar.

(2) We may assume that M has no free summands. Set $N = \Omega_1^A(M)$. Let $0 \to N \to F \to M \to 0$ be the minimal presentation of M with $F = A^r$. Then note as A is a hypersurface ring and M is MCM without free summands we get that a minimal presentation of N is as follows $0 \to M \to G \to N \to 0$ where $G = A^r$. By using the first exact sequence we get

$$0 \to \operatorname{Tor}_1^A(M, A/I^{n+1}) \to N/I^{n+1}N \to F/I^{n+1}F \to M/I^{n+1} \to 0.$$

So we have

$$t_I(M,n) = \ell(N/I^{n+1}N) + \ell(M/I^{n+1}M) - r\ell(A/I^{n+1}A).$$

Using the second exact sequence we find that $t_I(M,n) = t_I(N,n)$. The result follows.

The proof for assertions on $s_I(-,n)$ and $e_I(-,n)$ are similar. □

3. $\underline{\mathrm{CM}}_0(A)$

In this section we give proofs of Theorem 1.1, 1.3 and 1.4 when M is free on $\operatorname{Spec}^0(A)$.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer r_I (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg t_I^M(z) = r_I$.*

Proof. We first note that for any MCM M we have $\deg t_I^M(z) \leq d-1$, see [9, Corollary 4]. We set the degree of the zero polynomial to be -1. Set

$$r = \max\{\deg t_I^M(z) \mid M \in \underline{\mathrm{CM}}_0(A)\}.$$

If $r = -1$ then we have nothing to prove. So assume $r \geq 0$. For $M \in \underline{\mathrm{CM}}_0(A)$ define

$$\xi_I(M) = \lim_{n\to\infty} \frac{r!}{n^r} t_I(M,n).$$

We note that $\xi_I(M) \geq 0$ and is zero precisely when $\deg t_I(M,z) < r$.
Claim: $\xi_I(-)$ is a weak triangle function on $\underline{\mathrm{CM}}_0(A)$, see 2.1.
Assume the claim for the time being. Then $\ker \xi$ is a thick subcategory of $\underline{\mathrm{CM}}_0(A)$. Also if $\deg t_i^L(z) = r$ then $L \notin \ker \xi$. So $\ker \xi \neq \underline{\mathrm{CM}}_0(A)$. As $\underline{\mathrm{CM}}_0(A)$ has no proper

thick subcategories, see [8, 6.6], it follows that $\ker \xi = 0$. Therefore $\deg t_I^M(z) = r$ for all $M \neq 0$ in $\underline{\mathrm{CM}}_0(A)$.

It remains to show ξ_I is a weak triangle function on $\underline{\mathrm{CM}}_0(A)$. The first three conditions are trivial to satisfy. By 2.5(2) it follows that $\xi_I(\Omega_A^{-1}(M)) = \xi_I(M)$. Let $L \to M \to N \to \Omega^{-1}(L)$ is a triangle in $\underline{\mathrm{CM}}_0(A)$ then note that we have a short exact sequence of A-modules

$$0 \to M \to N \oplus F \to \Omega^{-1}(L) \to 0, \quad \text{where } F \text{ is free.}$$

Therefore we have an inequality

$$t_I(N, n) \leq t_I(M, n) + t_I(\Omega^{-1}(L), n).$$

The result follows. □

The following two results can be proved similarly as in 3.1. We have to use that $\deg s_I^M(z) \leq d - 1$ (see [9, Corollary 4]) and that $\deg e_I^M(z) \leq d$ (see [9, Corollary 7]).

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer s_I (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg s_I^M(z) = r_I$.*

Theorem 3.3. *(= Theorem 1.4) Let $(A, \mathfrak{m})$ be a hypersurface local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. Then there is a non-negative integer e_I (depending only on I) such that if $M \in \underline{\mathrm{CM}}_0(A)$ is non-zero then $\deg e_I^M(z) = r_I$.*

4. Proofs of Theorem 1.1 and 1.3

In this section we give proofs of Theorem 1.1 and 1.3. We need a few preliminaries.

4.1. Let M be any finitely generated A-module. Set $L_i(M) = \bigoplus_{n \geq 0} \mathrm{Tor}_i^A(M, A/I^{n+1})$ for $i \geq 0$. Let $\mathcal{R} = A[It]$ be the Rees algebra of I. We have an exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(1) \to A[t](1) \to L_0(A) \to 0.$$

Tensoring with M yields an inclusion $0 \to L_1(M) \subseteq \mathcal{R}(1) \otimes M$ and isomorphisms $L_i(M) \cong \mathrm{Tor}_{i-1}^A(\mathcal{R}(1), M)$ for $i \geq 2$. It follows that $L_i(M)$ are finitely generated $\mathcal{R}$-module for all $i \geq 1$. We note that if $\Omega_2^A(M) \cong M$ then we have $L_i(M) \cong L_{i+2}(M)$ for all $i \geq 1$.

4.2. We also need the following notion. Let $M \in \underline{\mathrm{CM}}(A)$. Let

$$\underline{\mathrm{Supp}}(M) = \{P \mid M_P \text{ is not free } A_P - \text{module}\}.$$

It is readily verified that $\underline{\mathrm{Supp}}(M) = V(\underline{\mathrm{Hom}}(M, M))$.

Proof of Theorem 1.1. By Theorem 3.1 we have that there exists r_I such that for any non-free MCM module $E \in \underline{\mathrm{CM}}_0(A)$ we have $\deg t_I^E(z) = r_I$.

Claim: For any non-free MCM A-module M we have $\deg t_I^M(z) = r_I$.
We prove this assertion by induction on $\dim \underline{\mathrm{Supp}}(M)$. If $\dim \underline{\mathrm{Supp}}(M) = 0$ then M is free on $\mathrm{Spec}^0(A)$. In this case we have nothing to show.

Now assume $\dim \underline{\mathrm{Supp}}(M) > 0$. As $L_1(M)_n, L_2(M)_n$ have finite length for all n and as $L_1(M), L_2(M)$ are finitely generated $\mathcal{R}$-modules it follows that there exists

l such that $\mathfrak{m}^l L_i(M)_n = 0$ for all n and for $i = 1, 2$. As M has period two it follows that $\mathfrak{m}^l L_i(M)_n = 0$ for all $i \geq 1$ and all $n \geq 0$.

Let

$$x \in \mathfrak{m}^l \setminus \bigcup_{\substack{P \supseteq \operatorname{ann} \underline{\operatorname{Hom}}(M,M) \\ P \text{ minimal}}} P.$$

Let $M \xrightarrow{x} M \to N \to \Omega^{-1}(A)$ be a triangle in $\underline{\mathrm{CM}}(A)$. It is readily verified that support of $\underline{\operatorname{Hom}}(N,N)$ is contained in the intersection of support of $\underline{\operatorname{Hom}}(M,M)$ and $A/(x)$. So $\dim \underline{\operatorname{Supp}}(N) \leq \dim \underline{\operatorname{Supp}}(M) - 1$. It is also not difficult to prove that N is not free A-module. By induction hypotheses $\deg t_I^N(z) = r_I$. By the structure of triangles in $\underline{\mathrm{CM}}(A)$, see [1, 4.4.1], we have an exact sequence $0 \to G \to N \to M/xM \to 0$ with G-free. It follows that $L_3(N) = L_3(M/xM)$. We also have an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. As $x \in \operatorname{ann} L_i(M)$ it follows that we have an exact sequence

$$0 \to L_3(M) \to L_3(M/xM) \to L_2(M) \to 0.$$

As the Hilbert function of $L_3(M)$ and $L_2(M)$ are identical, 2.5(2) we get that $2t_I^M(z) = t_I^N(z)$. It follows that $\deg t_I^M(z) = r_I$. By induction the result follows. $\square$

4.3. To prove Theorem 1.3 we need a few preliminaries. Let M be a finitely generated Cohen-Macaulay$A$-module of dimension r. Let
$E^i(M) = \bigoplus_{n \geq 0} \operatorname{Ext}_A^i(M, A/I^{n+1})$. The exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(1) \to A[t](1) \to L_0(A) \to 0,$$

induces an isomorphism $E^i(M) \cong \operatorname{Ext}_A^{i+1}(M, \mathcal{R}(1))$ for all $i > d - r$. In particular $E_A^i(M)$ are finitely generated $\mathcal{R}$-modules for all $i > d-r$. We note that if $\Omega_2^A(M) \cong M$ then we have $E^i(M) \cong E^{i+2}(M)$ for all $i \geq 1$. The proof of Theorem 1.3 is mostly similar to the proof of Theorem 1.1. So we mostly sketch the proof.

Sketch of a proof of Theorem 1.3. By Theorem 3.2 we have that there exists r_I such that for any non-free MCM module $L \in \underline{\mathrm{CM}}_0(A)$ we have $\deg s_I^L(z) = s_I$.

Claim: For any non-free MCM A-module M we have $\deg s_I^M(z) = s_I$.
We prove this assertion by induction on $\dim \underline{\operatorname{Supp}}(M)$. If $\dim \underline{\operatorname{Supp}}(M) = 0$ then M is free on $\operatorname{Spec}^0(A)$. In this case we have nothing to show.

Now assume $\dim \underline{\operatorname{Supp}}(M) > 0$. As $E^1(M)_n, E^2(M)_n$ have finite length for all n and as $E^1(M), E^2(M)$ are finitely generated $\mathcal{R}$-modules it follows that there exists l such that $\mathfrak{m}^l E^i(M)_n = 0$ for all n and for $i = 1, 2$. As M has period two it follows that $\mathfrak{m}^l E^i(M)_n = 0$ for all $i \geq 1$ and all $n \geq 0$. Let

$$x \in \mathfrak{m}^l \setminus \bigcup_{\substack{P \supseteq \operatorname{ann} \underline{\operatorname{Hom}}(M,M) \\ P \text{ minimal}}} P.$$

Let $M \xrightarrow{x} M \to N \to \Omega^{-1}(A)$ be a triangle in $\underline{\mathrm{CM}}(A)$. As before we have $\dim \underline{\operatorname{Supp}}(N) \leq \dim \underline{\operatorname{Supp}}(M) - 1$ and N is not free. By induction hypotheses $\deg s_I^N(z) = r_I$. By the structure of triangles in $\underline{\mathrm{CM}}(A)$, see [1, 4.4.1], we have an exact sequence $0 \to G \to N \to M/xM \to 0$ with G-free. It follows that $E^3(N) = E^3(M/xM)$. We also have an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. As $x \in \operatorname{ann} L_i(M)$ it follows that we have an exact sequence

$$0 \to E^2(M) \to E^3(M/xM) \to E^3(M) \to 0.$$

As the Hilbert function of $E^3(M)$ and $E^2(M)$ are identical, 2.5(2) we get that $2s_I^M(z) = s_I^N(z)$. It follows that $\deg s_I^M(z) = s_I$. By induction the result follows. $\square$

Remark 4.4. Consider $U^i(M) = \bigoplus_{n \geq 0} \operatorname{Ext}_A^i(A/I^{n+1}, M)$. Then for $i \geq d+1$ it is possible to give a natural $\mathcal{R}$-module structure on $U^i(M)$. However with this structure $U^i(M)$ is NOT finitely generated (note if $xt \in \mathcal{R}_1$ then $x_1 t U^i(M)_n \subseteq U^i(M)_{n-1}$). Thus it is not possible to extend the result in 3.3 to all MCM modules.

References

[1] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings*, Mathematical Surveys and Monographs, 262. American Mathematical Society, Providence, RI, 2021.

[2] W. Bruns and J. Herzog, Cohen-Macaulay Rings, revised edition, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, 1998.

[3] D. Katz and E. Theodorescu, *On the degree of Hilbert polynomials associated to the torsion functor*, Proc. Amer. Math. Soc, 135, (2007), 3073–3082.

[4] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001.

[5] T. J. Puthenpurakal, *Hilbert-coefficients of a Cohen-Macaulay module*, J. Algebra 264 (2003), no. 1, 82–97.

[6] ———, *A sub-functor for Ext and Cohen-Macaulay associated graded modules with bounded multiplicity*, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2567-–2589.

[7] ———, *Derived functors and Hilbert polynomials over regular local rings*, arXiv:2312.16982.

[8] R. Takahashi, *Classifying thick subcategories of the stable category of Cohen-Macaulay modules*, Adv. Math. 225 (2010), no. 4, 2076–2116.

[9] E. Theodorescu, *Derived functors and Hilbert polynomials*, Math. Proc. Camb. Phil. Soc. 132 (2002), 75–88.

Department of Mathematics, IIT Bombay, Powai, Mumbai 400 076
Email address: tputhen@math.iitb.ac.in