COMPACT LINEAR COMBINATIONS OF COMPOSITION OPERATORS ON HARDY SPACES

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Abstract. Let φ_j, $j = 1, 2, \dots, N$, be holomorphic self-maps of the unit disk $\mathbb{D}$ of $\mathbb{C}$. We prove that the compactness of a linear combination of the composition operators $C_{\varphi_j} : f \mapsto f \circ \varphi_j$ on the Hardy space $H^p(\mathbb{D})$ does not depend on p for $0 < p < \infty$. This answers a conjecture of Choe et al. about the compact differences $C_{\varphi_1} - C_{\varphi_2}$ on $H^p(\mathbb{D})$, $0 < p < \infty$.

1. Introduction

Let $\mathcal{H}ol(\mathbb{D})$ denote the space of holomorphic functions in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and let $\mathcal{S}(\mathbb{D})$ denote the class of all holomorphic self-maps of $\mathbb{D}$. Given a $\varphi \in \mathcal{S}(D)$, the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in \mathcal{H}ol(\mathbb{D})$. Clearly, C_φ takes the space $\mathcal{H}ol(\mathbb{D})$ into itself.

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|^p_{H^p} := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty,$$

where m is the normalized Lebesgue measure on the unit circle $\mathbb{T} := \partial\mathbb{D}$.

Given a $\varphi \in \mathcal{S}$, a typical problem is to relate certain operator-theoretic properties of C_φ and function-theoretic properties of φ. Properties of C_φ on the Hardy spaces are of special interest. Recall that, by the classical Littlewood subordination principle, C_φ is bounded on $H^p(\mathbb{D})$, $p > 0$, for any symbol $\varphi \in \mathcal{S}$. We refer to standard references [7] and [15] for various aspects on the theory of composition operators on $H^p(\mathbb{D})$ and on classical function spaces in one and several complex variables.

More generally, one considers similar questions for linear combinations of C_{φ_j}, $\varphi_j \in \mathcal{S}$, $j = 1, 2, \dots, N$. In fact, the present paper is motivated by the following problem about differences of two composition operators:

Problem 1.1. Given a $p > 0$, characterize $\varphi, \psi \in \mathcal{S}$ such that $C_\varphi - C_\psi$ is a compact operator on $H^p(\mathbb{D})$.

2010 *Mathematics Subject Classification.* Primary 47B33; Secondary 32A35, 46B70, 46M35, 47B07.

Key words and phrases. Hardy space, compact difference of composition operators, real interpolation of Banach spaces.

The research on Sections 1 and 2 was supported by Russian Science Foundation (grant No. 24-11-00087), https://rscf.ru/project/24-11-00087/; the research on Sections 3 and 4 was supported by Russian Science Foundation (grant No. 23-11-00171), https://rscf.ru/project/23-11-00171/.

The above problem goes back to J. H. Shapiro and C. Sundberg [16]. For $p = 2$, Problem 1.1 was first solved in [4]. A more explicit solution of the problem for $p = 2$ was given in [3]. Moreover, it was shown in [11] that the solution of Problem 1.1 does not depend on p for $1 \le p < \infty$. Finally, B. R. Choe et al. [3] formulated the following conjecture.

Conjecture 1.1 ([3, Conjecture 4.3])**.** The compactness of $C_\varphi - C_\psi$ on $H^p(\mathbb{D})$ is independent of the parameter $p \in (0, \infty)$.

In the present paper, we obtain the following theorem.

Theorem 1.1. *Let $\varphi_j \in \mathcal{S}$, $j = 1, 2, \dots, N$, and let T denote a linear combination of the composition operators C_{φ_j}. Then the compactness of $T : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ does not depend on p for $0 < p < \infty$.*

Trivially, one has the following corollary, that is, Conjecture 1.1 holds.

Corollary 1.1. *Let $\varphi, \psi \in \mathcal{S}$. Then the compactness of $C_\varphi - C_\psi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ does not depend on p for $0 < p < \infty$.*

Organization of the paper. In Section 2, we first formulate Proposition 2.1, a one-sided compactness result for the real interpolation method of quasi-Banach Hardy spaces. We deduce Theorem 1.1 from Proposition 2.1 in Section 2. To obtain an explicit and complete proof of Proposition 2.1, in Section 3, we formulate and prove a more general version of a classical compactness result due to A. Persson [12], Theorem 3.1. The final Section 4 contains corollaries from Theorem 1.1 and comments.

2. Proof of Theorem 1.1

The arguments in the proof are based on the real interpolation method for Banach spaces, so we first formulate related basic facts.

Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. The interpolation space $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $0 < q \le \infty$, is defined by

$$(A_0, A_1)_{\theta,q} = \{f \in A_0 + A_1 : \|f\|_{\theta,q} < \infty\},$$

where

$$\|f\|_{\theta,q}^q = \int_0^\infty \left(t^{-\theta} K(t, f)\right)^q \, \frac{dt}{t}$$

and

$$\begin{aligned} K(t, f) &= K(t, f; A_0, A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\} \end{aligned}$$

is the K-functional. See, for example, [2] for further details.

Our key technical tool from the area of real interpolation of quasi-Banach spaces is the following one-sided compactness assertion.

Proposition 2.1. *Assume that $T : H^{p_j} \to H^{p_j}$, $0 < p_j < \infty$, $j = 0, 1$, is a bounded linear operator such that $T : H^{p_1} \to H^{p_1}$ is compact. Then*

$$T : (H^{p_0}, H^{p_1})_{\theta,q} \to (H^{p_0}, H^{p_1})_{\theta,q}$$

is a compact operator for all admissible θ and q.

Proposition 2.1 follows from Theorem 3.1 below.

Proof of Theorem 1.1. Assume that $p_0 \in (0,\infty)$ and $C_\varphi : H^{p_0}(\mathbb{D}) \to H^{p_0}(\mathbb{D})$ is a compact operator. We need to verify that $C_\varphi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is also compact for all $p \in (0,\infty)$. In fact, we may assume that $p > p_0$.

Fix p and p_1 such that $p_1 > p > p_0$. Define $\theta \in (0,1)$ by the following identity:

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{2.1}$$

It is well known that (2.1) implies $(H^{p_0}, H^{p_1})_{\theta,p} = H^p(\mathbb{D})$. This result essentially goes back to R. Salem and A. Zygmund [14] (see [2, Chapter 1, Exercise 14]); for a modern treatment and appropriate references see [9]. Next, recall that $C_\varphi : H^{p_1}(\mathbb{D}) \to H^{p_1}(\mathbb{D})$ is a bounded operator by the Littlewood subordination principle. Thus, applying Proposition 2.1 for p_0, p_1 and $q = p$, we conclude that $C_\varphi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is compact, as required. □

3. Compactness and interpolation

3.1. **A compactness theorem.**

The following is a direct generalization of A. Persson's classical result [12]. For the sake of clarity we write it out in detail.

Theorem 3.1. *Let (E_0, E_1) and (F_0, F_1) be two compatible couples of quasi-Banach spaces, and suppose that E and F are quasi-Banach interpolation spaces for these couples, meaning that for any linear operator $U : E_0 \to F_0$, $U : E_1 \to F_1$ it follows that $U : E \to F$. Suppose also that the second couple satisfies the following approximation condition: there exists a sequence of linear operators $P_j : F_0, F_1 \to F_0 \cap F_1$, such that*

$$P_j f \to f \text{ in } F_0 \text{ for all } f \in F_0 \text{ and } \sup_j \|P_j\|_{F_1\to F_1} < \infty, \tag{H}$$

and the intermediate spaces satisfy the following conditions:

$$\lim_{\varepsilon\to 0} \sup \{\|U\|_{E\to F} : \|U\|_{E_0\to F_0} \leqslant \varepsilon, \|U\|_{E_1\to F_1} \leqslant 1\} = 0, \tag{C_0}$$

$$\lim_{t\to\infty} \quad t^{-1} \sup_{f\in B_E} K(t, f; E_0, E_1) = 0, \tag{C_1}$$

where B_E denotes the (closed) unit ball of a space E, and $K(t, f; E_0, E_1)$ is the K-functional. Then any linear operator T that is compact as $T : E_0 \to F_0$ and bounded as $T : E_1 \to F_1$ is also compact as $T : E \to F$.

Proof. We first establish the particular case $F_0 = F_1 = F$ due to J. L. Lions and J. Peetre (see [2, Theorem 3.8.1]). Let $a_j \in B_E$ and $\varepsilon > 0$. By the assumption (C_1), for large enough t there exist decompositions $a_j = a_{0,j} + a_{1,j}$ satisfying $\|a_{0,j}\|_{E_0} \leqslant t\varepsilon$ and $\|a_{1,j}\|_{E_1} \leqslant \varepsilon$. By passing to a subsequence, we may assume that

$$\|Ta_{0,j} - Ta_{0,j'}\|_F \leqslant \varepsilon$$

for large enough j and j'. On the other hand,

$$\|Ta_{1,j} - Ta_{1,j'}\|_F \leqslant 2\varepsilon\|T\|_{E_1\to F}$$

Adding these two estimates together yields

$$\|Ta_j - Ta_{j'}\|_F \leqslant c(1 + 2\|T\|_{E_1\to F})\varepsilon$$

for large enough j and j', where c is a constant in the triangle inequality for F. A convergent subsequence of Ta_j is then obtained by an application of the diagonal process.

Now, we verify the general case of Theorem 3.1. Since $T(B_{E_0})$ is relatively compact in F_0,

$$\lim_{j\to\infty} \|P_j T - T\|_{E_0\to F_0} = 0$$

by the assumption (H). By the assumption (C_0),

$$\lim_{j\to\infty} \|P_j T - T\|_{E\to F} = 0,$$

that is, T is approximated uniformly by $P_j T$, and it suffices to verify that $P_j T : E \to F$ is compact. Finally,

$$P_j T : E_0, E_1 \to F_0 \cap F_1 \subset F,$$

and $P_j T : E_0 \to F_0 \cap F_1$ is compact as a composition of a bounded and a compact operator. Thus its compactness is reduced to the particular case $F_0 = F_1 = F$. $\square$

3.2. **Applications.** Assumptions (C_0) and (C_1) are satisfied by all interpolation functors of type $\mathcal{C}_\theta$, $0 < \theta < 1$ (see [2, Section 3.5]), but they are also satisfied in many other applications of interest, such as couples of the quasinormed Orlicz spaces (L^{Φ_0}, L^{Φ_1}) and the corresponding intermediate Orlicz spaces

$$L^\Phi,\ L^{\Phi_0} \cap L^{\Phi_1} \subset L^\Phi \subset L^{\Phi_0} + L^{\Phi_1}$$

such that $L^\Phi \neq L^{\Phi_1}$. See, for example, [10]. However, the details in full generality may be somewhat complicated, so we leave it as a note in passing.

Assumption (H) is easily seen to be satisfied for the entire range of Hardy and weighted Bergman spaces (excluding $p = \infty$) if we take $P_j f(z) = f(r_j z)$ with some sequence $0 < r_j < 1$, $r_j \to 1$. Moreover, the same is true for the Hardy-type spaces for rearrangement invariant quasi-Banach lattices (see [1, Theorem 2.1]), which are also stable under the real interpolation by the results of [9]; for the BMO-regularity property of these lattices see, for example, [13, Proposition 2]. In particular, Theorem 3.1 is also applicable to the Hardy-Orlicz spaces H^Φ.

3.3. **Remarks on the compactness theorem.** Theorem 3.1 seems to be one of the earliest (taken from [12]) and indeed the easiest compactness result for couples of lattices that is perfectly applicable in many settings. It should be noted that there exists a large body of work on compactness and interpolation which is also relevant to the research; for instance, see [6, 5, 8]. In particular, the proof of the Hayakawa-Cwikel theorem [8, Theorem 1.1], establishing this result for the standard real interpolation functors, seems to work also for general quasi-Banach spaces. Thus, for these interpolation spaces the assumption (H) is clearly superfluous. This ([8, Theorem 1.1]) is indeed the result that one would use for general weighted Hardy spaces, where approximation might not work so easily. However, whether a compactness theorem holds true for the complex interpolation in the general case is still a notable open problem in the interpolation theory even in the standard Banach space setting.

4. Corollaries and comments

4.1. **A family of equivalent conditions.** We need some notation from [3]. As usual, let φ^* and ψ^* denote the boundary values of φ and ψ, respectively, on $\mathbb{T}$.

Given $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, $0 < p < \infty$ and $s \geq 0$, define

$$\lambda_{p,s}(a) = \lambda_{p,s}^{\varphi,\psi}(a) := \int_{(\varphi^*)^{-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\psi^*} \right|^p dm + \int_{(\psi^*)^{-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\varphi^*} \right|^p dm,$$

where

$$Q_s(a) := \{z \in \overline{\mathbb{D}} : |z - a| < 2^{s+1}(1 - |a|)\}$$

for $a \in \mathbb{D}$.

The following corollary extends [3, Corollary 4.1] from $1 \leq p < \infty$ to all p, $0 < p < \infty$.

Corollary 4.1. *Let $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$. Then the following assertions are equivalent:*

(a) *$C_\varphi - C_\psi$ is compact on $H^p(\mathbb{D})$;*
(b) *$\lim_{|a|\to 1-} \frac{\lambda_{p,s}(a)}{1-|a|} = 0$ for some (equivalently, for all) $s \geq 0$.*

Proof. As indicated above, by [3, Corollary 4.1], (a) $\Leftrightarrow$ (b) for $1 \leq p < \infty$. Next, by [3, Property (4.2)], assertion (b) does not depend on p for $0 < p < \infty$. Thus, application of Theorem 1.1 finishes the proof of the corollary. $\square$

4.2. **Symbols φ and ψ of bounded multiplicity.**

Corollary 4.2. *Let $0 < p < \infty$ and let $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ be of bounded multiplicity. Then $C_\varphi - C_\psi$ is compact on $H^p(\mathbb{D})$ if and only if*

$$\lim_{|z|\to 1-} \left[\frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right] \rho(\varphi(z), \psi(z)) = 0,$$

where ρ denotes the pseudohyperbolic distance.

Proof. For $p = 2$, the required equivalence holds true by [3, Theorem 4.5]. Thus, application of Theorem 1.1 finishes the proof of the corollary. $\square$

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