SHARP QUASI-INVARIANCE THRESHOLD FOR THE CUBIC SZEGŐ EQUATION

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ABSTRACT. We consider the 1-dimensional cubic Szegő equation with data distributed according to the Gaussian measure with inverse covariance operator $(1 - \partial_x^2)^s$, where $s > \frac{1}{2}$. We show that, for $s > 1$, this measure is quasi-invariant under the flow of the equation, while for $s < 1$, $s \neq \frac{3}{4}$, the transported measure and the initial Gaussian measure are mutually singular for almost every time. This is the first observation of a transition from quasi-invariance to singularity in the context of the transport of Gaussian measures under the flow of Hamiltonian PDEs.

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1. INTRODUCTION

1.1. **Main result.** In this work, we consider the Cauchy problem for the cubic Szegő equation on the 1-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$,

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0) = u_0 = \Pi(u_0), \end{cases} \tag{1.1}$$

where Π is the Szegő projection onto non-negative Fourier frequencies:

$$\Pi\Big(\sum_{n \in \mathbb{Z}} \widehat{u}(n)e^{inx}\Big) = \sum_{n \geq 0} \widehat{u}(n)e^{inx},$$

and the condition $u_0 = \Pi(u_0)$ simply means that we restrict our attention to initial data which has Fourier support on nonnegative frequencies. Our goal is to study certain statistical properties of the evolution of (1.1), when its initial data u_0 is randomly distributed, according to a particular family of Gaussian measures μ_s.

More precisely, we concern ourselves with the following question: denoting with $\Phi_t(u_0)$ the solution of (1.1) at time t, does it hold that

$$(\Phi_t)_\# \mu_s \ll \mu_s?$$

When this phenomenon happens, we say that the measure is *quasi-invariant* under the flow of (1.1).

The study of quasi-invariance for the flow of Hamiltonian PDEs has been initiated by Tzvetkov in [32]. In particular, in this pioneering work, Tzvetkov showed that in the particular case of the regularized long wave (BBM) equation, and of Gaussian probability measures of the form

$$\mu_s \sim \exp\Big(-\|u\|_{H^s}^2\Big)du,\,^1 \tag{1.2}$$

quasi-invariance holds in regime that is much wider than what one would expect from the classical results of Cameron-Martin [7] and Ramer [30]. Throughout the years, the technology to show quasi-invariance has been greatly developed, especially in the context of dispersive PDEs. In particular, we signal quasi-invariance results for the BBM and Benjamin-Ono equations [15, 16, 32], KdV type equations [29], wave equations [20, 27, 31], and Schrödinger equations [9, 12–15, 23–26, 28].

In the results above, the dispersive nature of the equation plays a significant role in the proof, and indeed, if one considers the ODE

$$i\partial_t u = |u|^2 u,$$

it is possible to show that quasi-invariance fails in a dramatic fashion, and $(\Phi_t)_\# \mu_s \not\ll \mu_s$, see [24].

However, the relationship between the dispersive nature of the equation and quasi-invariance is not fully understood. In particular, when the regularity of the initial data (distributed according to the measure μ_s) becomes low enough, it is unclear if quasi-invariance is preserved or not. On the one hand, from the study of *invariant* measures, one could expect that the evolution eventually becomes singular with respect to the underlying

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[1] For a rigorous definition of these measures in the setting of this paper, see Section 2.

Gaussian measure.[2] On the other hand, no such example is available in the literature, and only positive results are known.

Here lies our interest into (1.1). This equation was introduced by Gérard-Grellier in [17], as a toy model for non-dispersive Hamiltonian dynamics. More precisely, this equation informally serves as a toy model for the "0-dispersion" limit of fractional NLS/ half-wave equation

$$i\partial_t u - (-\Delta)^{\frac{1}{2}} u = |u|^2 u,$$

and similar dispersionless models. Indeed, equation (1.1) shares many features with such models. To begin with, it is formally a Hamiltionian equation on the phase space $L^2_+ = \Pi(L^2(\mathbb{T}))$,[3] with Hamiltonian given by

$$E(u) = \|u\|^4_{L^4(\mathbb{T})}.$$

Moreover, it has a number of conserved quantities, including the L^2-norm and the $\dot{H}^{\frac{1}{2}}$ norm of the solutions, i.e.

$$M(u) = \|u\|^2_{L^2(\mathbb{T})}, \quad P(u) = \|u\|^2_{\dot{H}^{\frac{1}{2}}} =: \int_{\mathbb{T}} u(-\Delta)^{\frac{1}{2}}\overline{u}.$$

In [17], Gérard and Grellier showed that this equation admits a Lax pair, which in particular implies that the equation is completely integrable and the existence of infinitely many conservation laws. Exploiting these conservation laws, in [18], Gérard and Koch showed that equation (1.1) is globally well-posed in the space $\text{BMO}_+ = \Pi(L^\infty(\mathbb{T}))$.

From a statistical point of view, the conservation of M and P above suggest that the white-noise measure μ_0 and (a variant of) the measure $\mu_{\frac{1}{2}}$ should be invariant (and hence quasi-invariant) for the flow of (1.1). While these measures are concentrated on distributions of too low regularity to be able to show strong invariance statements, a weak form of invariance for $\mu_{\frac{1}{2}}$ has been proven in [6]. For $s > \frac{1}{2}$, one has that $\mu_s(\text{BMO}_+) = 1$, and so the flow map Φ_t and the push-forward measure $(\Phi_t)_\# \mu_s$ are well-defined for every $t \in \mathbb{R}$. In view of the conservation laws for the Szegő equation and of the analogous results for dispersive equations, one might conjecture that $(\Phi_t)_\# \mu_s \ll \mu_s$.

On the other hand, as discussed, equation (1.1) enjoys no dispersion nor multilinear smoothing, even in the probabilistic setting [22]. Consequently, one could expect its behaviour to be similar to the one of the ODE

$$i\partial_t u = |u|^2 u,$$

for which we recall that no measure μ_s (for $s > \frac{1}{2}$) is quasi-invariant, see [24].

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[2]Famously, the Φ^4_3 measure of quantum field theory

$$\Phi^4_3 \sim \exp\left(-\frac{1}{4}\int u^4 - \frac{1}{2}\int u^2 - \frac{1}{2}\int |\nabla u|^2\right)$$

is singular with respect to the underlying Gaussian measure, and it is (formally) an invariant measure for the Schrödinger equation on $\mathbb{T}^3$

$$i\partial_t u - \Delta u - u^3 = 0,$$

and in [5], it has been shown to be invariant for the wave equation on $\mathbb{T}^3$ (up to renormalisation)

$$\partial_t^2 u - \Delta u - u^3 = 0.$$

[3]Hence our choice $u_0 = \Pi(u_0)$.

The main goal of this paper is to provide a clear picture of quasi-invariance and lack thereof for equation (1.1), and develop a better understanding of when one should expect quasi-invariance to hold/fail (respectively). In particular, our main result is the following.

Theorem 1.1. *Let $s > \frac{1}{2}$, and let Φ_t be the flow map of equation (1.1) on the space BMO_+. We have the following.*

(i) *If $s > 1$, then the measure μ_s is quasi-invariant. More precisely, for every $t \in \mathbb{R}$, we have that $(\Phi_t)_{\#}\mu_s \ll \mu_s$.*

(ii) *If $\frac{1}{2} < s < 1$ and $s \neq \frac{3}{4}$, then the transported measure $(\Phi_t)_{\#}\mu_s$ is singular with respect to μ_s for almost every time $t \in \mathbb{R}$. More precisely, there exists a countable set $\mathscr{N}_s \subset \mathbb{R}$, such that for every $t \in \mathbb{R} \setminus \mathscr{N}_s$, $(\Phi_t)_{\#}\mu_s \perp \mu_s$.*

To the authors' knowledge, this is the first time that such a transition from quasi-invariance to singularity has been observed.

1.2. **A heuristic for quasi-invariance and a strategy for singularity.** In order to understand the numerology of our result in Theorem 1.1, consider (as a toy model) the dynamical system on $\mathbb{R}^d$

$$\begin{cases} \dot{y} = b(y), \\ y(0) = x, \end{cases} \tag{1.3}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field. For the purpose of this subsection, denote by $\Phi_t(x)$ the solution of (1.3). For every probability measure ν_0 on $\mathbb{R}^d$, the transported measure $\nu_t := (\Phi_t(x))_{\#}\nu_0$ will satisfy the Liouville's equation

$$\partial_t \nu_t = -\operatorname{div}(b\,\nu_t). \tag{1.4}$$

When ν_0 is a probability measure of the form

$$\nu_0 = \frac{1}{Z}\exp(-E(y))dy$$

for some smooth function $E : \mathbb{R}^d \to \mathbb{R}$ with $\exp(-E) \in L^1(\mathbb{R}^d)$, then we can solve (1.4) for

$$f_t := \log\frac{d\nu_t}{dx} - \log\frac{d\nu_0}{dx}$$

explicitly with the method of characteristics, and obtain that

$$f_t = E(x) - E(\Phi_{-t}(x)) + \int_0^t \operatorname{div}(b)\big(\Phi_{-t'}(x)\big)dt'.$$

For convenience, define

$$Q(x) = \frac{d}{dt}E(y(t))\Big|_{t=0} + \operatorname{div}(b), \tag{1.5}$$

from which we obtain

$$\nu_t = \exp\Big(\int_0^t Q(\Phi_{-t'}(x))dx\Big)\nu_0. \tag{1.6}$$

One can now apply (formally) this formula to equation (1.1) and $\nu_0 = \mu_s$ in (1.2), for which $E(u) = \|u\|_{H^s}^2$, obtaining that

$$(\Phi_t)_{\#}\mu_s(u_0) = \exp\Big(\int_0^t Q\big(\Phi_{-t'}(u_0)\big)dt'\Big)\mu_s(u_0),$$

where

$$Q(u) := 2\,\mathrm{Im}\left(\int \Pi(|u|^2 u)\overline{\langle\nabla\rangle^{2s}u}\right)$$

$$= \frac{i}{2}\sum_{n_1-n_2+n_3-n_4=0,n_j\geq 0}\left(\langle n_1\rangle^{2s}-\langle n_2\rangle^{2s}+\langle n_3\rangle^{2s}-\langle n_4\rangle^{2s}\right)\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)},$$

(1.7)

and the last equality is obtained by exploiting Plancherel and the symmetry between the Fourier coefficients of u and $\overline{u}$. Then, a natural benchmark to test quasi-invariance is the following: denoting by $S(t)$ the propagator for the *linear* flow for (1.1), is it true that

$$\mathbb{E}_{\mu_s}\left|\int_0^t Q\big(S(-t')(u_0)\big)dt'\right|^2 < \infty?$$

In the case of equation (1.1), one has that $S(t)$ is the identity, and this condition translates to

$$\mathbb{E}_{\mu_s}\big|Q(u_0)\big|^2 < \infty.$$

Exploiting the cancellations of the symbol $\left(\langle n_1\rangle^{2s}-\langle n_2\rangle^{2s}+\langle n_3\rangle^{2s}-\langle n_4\rangle^{2s}\right)$, one can check that for $s > \frac{1}{2}$, this is the case if and only if $s > 1$, which is exactly the regime in which we prove quasi-invariance in Theorem 1.1. Actually, the finiteness of this quantity is what allows us to perform the proof of quasi-invariance in this regime. We essentially rely on the strategy of [28] adapted to the Szegő equation, together with some recent technology (namely, the variational formula (3.7)) in order to estimate

$$\int \exp(TQ(u_0))d\mu_s(u_0).$$

In this paper, we re-introduce the strategy of [28] starting from a different point of view, that relies on the formula (1.6) in a more crucial way, see the proof of Proposition 3.6.[4] While in the particular case of equation (1.1), the results we obtain are completely equivalent, we believe that the argument presented in this paper is far more flexible, and indeed it was necessary to exploit some of this flexibility in the work by Forlano and the second author [13].

We now move to discussing the case $s < 1$, where instead we have $\mathbb{E}_{\mu_s}\big|Q(u_0)\big|^2 = \infty$. As it turns out, the heuristic above not only suggests that singularity should hold, but also a strategy for its proof. Indeed, if two probability measures ν, ν' are mutually singular, there exists a set E such that $\nu(E) = 1$, $\nu'(E^c) = 1$. Therefore, if we formally write

$$f = \frac{d\nu'}{d\nu},$$

we must have that

$$\int_E f\,d\nu = \nu'(E) = 0,$$

which implies that $f = 0$ ν-almost surely. By flipping the roles of ν, ν', we also get

$$\frac{1}{f} = 0 \ \nu'-\text{a.s.} \ \Leftrightarrow f = +\infty \ \nu'-\text{a.s.}$$

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[4]A similar argument appeared also in [1].

In the case of the Szegő equation, since (formally) $\operatorname{div}(b) = 0$, we can rewrite (1.6) as

$$(\Phi_t)_{\#}\mu_s(u_0) = \exp\left(\|u_0\|_{H^s}^2 - \|\Phi_{-t}(u_0)\|_{H^s}^2\right)\mu_0(u_0).$$

Therefore, the conditions above can be rewritten as

$$\|u_0\|_{H^s}^2 - \|\Phi_{-t}(u_0)\|_{H^s}^2 = -\infty \ \ \mu_s - \text{a.s.} \ ,$$

and

$$\|u_0\|_{H^s}^2 - \|\Phi_{-t}(u_0)\|_{H^s}^2 = +\infty \ \ (\Phi_t)_{\#}\mu_s - \text{a.s.} \ \Leftrightarrow \|\Phi_t(u_0)\|_{H^s}^2 - \|u_0\|_{H^s}^2 = +\infty \ \ \mu_s - \text{a.s.}$$

In other words, singularity should correspond exactly to the fact that for a generic u_0 distributed according to μ_s, then $t = 0$ is a (local) *minimum* for the (infinite) quantity $\|\Phi_t(u_0)\|_{H^s}^2$. This suggests the following strategy for showing singularity.

(1) Pick a well-defined approximation $R_N(u_0)$ of $\|u_0\|_{H^s}^2$.

(2) Show that for some $c > 0$, $\alpha > 0$,

$$\frac{d^2}{dt^2}R_N(\Phi_t(u_0))\bigg|_{t=0} = cN^\alpha(1 + o(1))$$

for μ_s-a.e. u_0, and that

$$\frac{d}{dt}R_N(\Phi_t(u_0))\bigg|_{t=0} = o(N^\alpha).$$

(3) Exploiting the local theory for equation (1.1), show that for $t \ll 1$,

$$R_N(\Phi_t(u_0)) - t\frac{d}{dt}R_N(\Phi_t(u_0))\bigg|_{t=0} - \frac{t^2}{2}\frac{d^2}{dt^2}R_N(\Phi_t(u_0))\bigg|_{t=0} \ll t^2N^\alpha,$$

which formally corresponds to the fact that $t = 0$ is a local minimum for

$$\lim_{N\to\infty} N^{-\alpha}R_N(\Phi_t(u_0)).$$

(4) Since functions on $\mathbb{R}$ can have at most countably many local minimum points, deduce that for up to countably many times, $(\Phi_t)_{\#}\mu_s \perp \mu_s$.

This list of steps is essentially the strategy that we perform in Section 4 in order to show Theorem 1.1, (ii).

More precisely, we first show an abstract formulation of (4) in Proposition 4.1.

In Section 4.2, we then consider an approximation R_N of the square of the H^1 norm,[5] and proceed to estimating

$$\left|\frac{d}{dt}R_N(\Phi_t(u_0))\bigg|_{t=0}\right| \lesssim N^{2-2s}$$

$$\frac{d^2}{dt^2}R_N(\Phi_t(u_0))\bigg|_{t=0} \sim N^{4-4s}$$

This is where our restrictions on s, namely $s \neq 1, \frac{3}{4}$, originate from. Indeed, we have that

- For $s = 1$, both $\frac{d}{dt}R_N(\Phi_t(u_0))\big|_{t=0}$ and $\frac{d}{dt^2}R_N(\Phi_t(u_0))\big|_{t=0}$ are $O(1)$, and similarly the error terms coming from (3) are also of the same size,

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[5]Choosing the H^1 norm instead of the H^s norm makes no difference in the subsequent steps, but some of the formulas become slightly easier.

- When $s = \frac{3}{4}$, the quantity

$$\frac{d^2}{dt^2} R_N(\Phi_t(u_0))\bigg|_{t=0}$$

transitions from being positive when $\frac{3}{4} < s < 1$, to being *negative* when $\frac{1}{2} < s < \frac{3}{4}$. When $s = \frac{3}{4}$, the sign of this quantity is unclear, but in any case it is of smaller order than N^{4-4s}, which is the minimum of what the rest of the argument can handle.

We then proceed to estimate the error of the term

$$R_N(\Phi_t(u_0)) - t\,\frac{d}{dt} R_N(\Phi_t(u_0))\bigg|_{t=0} - \frac{t^2}{2}\,\frac{d^2}{dt^2} R_N(\Phi_t(u_0))\bigg|_{t=0}.$$

This requires a very precise decomposition of the solution as $u(t) = X(t) + Y(t)$, where $X(t)$ is a semi-explicit, pseudo-Gaussian term of regularity $s - \frac{1}{2}$ (the same as a typical data sampled according to μ_s), and $Y(t)$ has regularity $2s - 1$. We perform this in Section 4.3. This decomposition is closely related to the ones used by Bringmann in [4] and Deng, Nahmod and Yue in [10, 11]. However, we point out that the main properties that we need to exploit are completely determinstic, and are not improved by the random structure of the initial data u_0.

Finally, in Section 4.4, we use this decomposition in order to show a series of (sharp) error bounds. We note that the only term that does not have a satisfactory deterministic bound is (4.29), for which a random estimate is necessary.

Remark 1.2. It might be possible that, in the case $s = 1$, performing sharper estimates and choosing a better approximation R_N of $\|u_0\|_{H^1}^2$, would allows us to conclude singularity also in this case. Similarly, it might be possible to show that for $s = \frac{3}{4}$,

$$\frac{d^4}{dt^4} R_N(\Phi_t(u_0))\bigg|_{t=0} \sim cN^{4-4s}$$

for some $c \neq 0$. However, this would require substantial more work and ideas in Section 4.4 and 4.2 respectively, and at this stage, it is unclear if such results are true.

Remark 1.3. As a sanity check, we can also test the heuristic above for the ODE

$$i\partial_t u = |u|^2 u,$$

for which we know that quasi-invariance fails for every $s > \frac{1}{2}$, see [24]. In this case, the condition above for quasi-invariance translates into

$$\mathbb{E}_{\mu_s}\bigg|\sum_{n_1 - n_2 + n_3 - n_4 = 0} \big(\langle n_1\rangle^{2s} - \langle n_2\rangle^{2s} + \langle n_3\rangle^{2s} - \langle n_4\rangle^{2s}\big)\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}\bigg|^2 < \infty,$$

where the only difference with respect to (1.1) is that we dropped the condition $n_j \geq 0$ from the sum. For this sum, we have that

$$\mathbb{E}_{\mu_s} \left| \sum_{n_1 - n_2 + n_3 - n_4 = 0} \left(\langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n_4 \rangle^{2s} \right) \widehat{u}(n_1) \overline{\widehat{u}(n_2)} \widehat{u}(n_3) \overline{\widehat{u}(n_4)} \right|^2$$

$$\sim \sum_{n_1 - n_2 + n_3 - n_4 = 0} \frac{\left(\langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n_4 \rangle^{2s} \right)^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s} \langle n_4 \rangle^{2s}}$$

$$\gtrsim \sum_{n_1 = -n_3, n_2 = n_4 = 0} 1$$

$$= \infty.$$

Therefore, according to our heuristic, we do not expect quasi-invariance to hold for any $s > \frac{1}{2}$, which is indeed the case.

1.3. **Some reductions.** In practice, while working with the Szegő equation, we will need to keep track of the regularity of solutions emanating from an initial data sampled according to μ_s. Consequently, we will not be able to work in the full space BMO_+, where the global flow for (1.1) is defined, but we will rather have to consider some space $X_s \subseteq \mathrm{BMO}_+$.

Recall from [17] that (1.1) is globally well-posed in the space H_+^σ for every $\sigma > \frac{1}{2}$. In view of Lemma 2.3, we have that for μ_s-a.e. initial data u_0, we have $u_0 \in H_+^\sigma$ for every $\sigma < s - \frac{1}{2}$. Therefore, when $s > 1$, one recovers a good global well-posedness statement by referring to well known results.

When $s \leq 1$, the situation is a bit more complicated. Indeed, while global well-posedness in L_+^2 has recently been shown in [21], it is still unclear if the equation (1.1) is well-posed in H_+^σ for any $0 < \sigma < \frac{1}{2}$. It is indeed possible (in principle) that for a general initial data in $H^\sigma \cap \mathrm{BMO}_+$, the solution will belong to a space $H_+^{\sigma(t)}$ for some $\sigma(t)$ which is strictly decreasing in time, see [18]. On the other hand, for the purpose of Section 4, we need to keep track of the optimal regularity of the solution. The considerations above lead to the following choice for X_s.

$$X_s = \begin{cases} H_+^\sigma \text{ for any } \frac{1}{2} < \sigma < s - \frac{1}{2}, & \text{when } s > 1, \\ B_{p,\infty}^{s-\frac{1}{2},+} \text{ for any } p > \min\left(100, \frac{1}{s-\frac{1}{2}}\right), & \text{when } \frac{1}{2} < s \leq 1, \end{cases} \quad (1.8)$$

where $B_{p,\infty}^{s-\frac{1}{2},+}$ is a Besov space for functions supported on non-negative frequencies, defined in Section 2. In view of Lemma 2.3, this space is well-adapted to the Gaussian measure μ_s. Moreover, it satisfies the following.

Proposition 1.4. *Let* $s > \frac{1}{2}$, *then the Banach space* $X_s \subset \mathrm{BMO}_+$ *defined above satisfies* $\mu_s(X_s) = 1$ *and the equation* (1.1) *is locally well posed in* X_s.

Unfortunately, the restriction to the space X_s makes it unclear if a generic $X_s \subset \mathrm{BMO}_+$ solution $\Phi_t(u_0)$ actually belongs to X_s for infinite time. What is worse, is that in principle it is possible that the set of times in which the solution belongs to X_s, i.e.

$$\mathcal{G}(u_0) := \{t \in \mathbb{R} : \Phi_t(u_0) \in X_s\}$$

can be *disconnected*.

To deal with these (potential) issues, we need to introduce some notation. For $t \in \mathbb{R}$, we denote by $\mathrm{WP}(t)$ the set of "good" initial data for which the solution to (1.1) exists in X_s for every time τ between 0 and t, or more precisely

$$\mathrm{WP}(t) := \{u_0 \in X_s : \text{the equation (1.1) admits a solution } u \in C([0,t], X_s) \text{ with } u(0) = u_0\}$$

when $t \geq 0$, and

$$\mathrm{WP}(t) := \{u_0 \in X_s : \text{the equation (1.1) admits a solution } u \in C([t,0], X_s) \text{ with } u(0) = u_0\}$$

when $t \leq 0$. By the local well posedness in the space X_s, and the global well posedness in the space $\mathrm{BMO}_+ \supseteq X_s$, we must have that for $u_0 \in \mathrm{WP}(t)$, the (local) solution in X_s must coincide with $\Phi_t(u_0)$. Moreover, by Proposition 1.4, we obtain that

$$\Phi_t : X_s \supseteq \mathrm{WP}(t) \to X_s$$

is a continuous map, and that $\mathrm{WP}(t) \subseteq X_s$ is an open set for every $t \in \mathbb{R}$.

Then, the main results of Sections 3, 4 are the following (respectively).

Proposition 1.5. *Let $s > 1$. Then for every $t \in \mathbb{R}$, we have that $\mathrm{WP}(t) = X_s$. Moreover, we have that*

$$(\Phi_t)_\# \mu_s \ll \mu_s$$

for every $t \in \mathbb{R}$.

Proposition 1.6. *Let $\frac{1}{2} < s < 1$, with $s \neq \frac{3}{4}$. Then there exists a countable set $\mathcal{N} \subseteq \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus \mathcal{N}$ with $\mu_s(\mathrm{WP}(t)) > 0$,*

$$(\Phi_t)_\# (\mathbb{1}_{\mathrm{WP}(t)} \mu_s) \perp \mu_s.$$

It is clear that Proposition 1.5 immediately implies Theorem 1.1, (i). The situation in the singular case is more unclear, but it can be proven via a soft argument that Proposition 1.6 automatically implies Theorem 1.1, (ii). We will show this at the end of Section 4.

2. PRELIMINARIES

2.1. **Notation.** To fix notation, we will always interpret N, M and N_j as dyadic integers (and sums over such as just sums over dyadic integers), and given a family $N_1, ..., N_k$ of dyadic integers, we denote $N^{(1)} \geq N^{(2)} \geq ... \geq N^{(k)}$ to be a decreasing arrangement of the N_j. We take as notation $M \ll N$ to mean $M < 2^{-20}N$, $M \gtrsim N$ to mean $M \geq 2^{-20}N$, $M \sim N$ to mean $M \gtrsim N \gtrsim M$, and $N \approx M$ to mean $M/4 \leq N \leq 4M$. This (unconventional) choice of symbols will help with probabilistic decoupling of different frequency scales.

When A, B are not dyadic numbers, we will instead use the notation $A \lesssim B$ to denote that there exists a constant $C \in (0, +\infty)$ such that $A \leq CB$.

We also take the convention $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with normalized integral

$$\int_\mathbb{T} f(x)dx := \frac{1}{2\pi} \int_0^{2\pi} f(x)dx,$$

and we define the Fourier transform of a function f to be

$$\widehat{f}(n) := \mathcal{F}[f](n) := \int_\mathbb{T} f(x)e^{-inx}dx,$$

where $n \in \mathbb{Z}$.

2.2. **Function spaces.** Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define the L^2-based Sobolev space $H^s(\mathbb{T})$ by the norm:

$$\|f\|_{H^s} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell_n^2},$$

and the space restricted to non-negative frequencies

$$H_+^s = \Pi(H^s(\mathbb{T})).$$

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\varphi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. We set $\phi_1(\xi) = \varphi(|\xi|)$ and

$$\phi_N(\xi) = \varphi\big(\tfrac{\xi}{2^j}\big) - \varphi\big(\tfrac{\xi}{2^{j-1}}\big) \tag{2.1}$$

where $N = 2^j$, and $j \in \mathbb{N}$. Then, for $N = 2^j$ for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the Littlewood-Paley projector P_N as the Fourier multiplier operator with a symbol ϕ_N. Note that we have

$$\sum_{N \in 2^{\mathbb{N}_0}} \phi_N(\xi) = 1$$

for each $\xi \in \mathbb{R}$. Thus, we have

$$f = \sum_{N \in 2^{\mathbb{N}_0}} P_N f.$$

We will also denote by $\phi_{\ll N}, \phi_{\lesssim N}, \phi_{\sim N}, \phi_{\approx N}, \phi_{\gtrsim N}, \phi_{\gg N}$ the multipliers

$$\phi_{\ll N} = \sum_{M \ll N} \phi_M, \qquad \phi_{\lesssim N} = \sum_{M \lesssim N} \phi_M, \qquad \phi_{\sim N} = \sum_{M \sim N} \phi_M,$$

$$\phi_{\approx N} = \sum_{M \approx N} \phi_M, \qquad \phi_{\gtrsim N} = \sum_{M \gtrsim N} \phi_M, \qquad \phi_{\gg N} = \sum_{M \gg N} \phi_M,$$

and $P_{\ll N}, P_{\lesssim N}, P_{\sim N}, P_{\approx N}, P_{\gtrsim N}, P_{\gg N}$ will denote the corresponding Fourier multiplier operators.

Next, we recall the basic properties of the Besov spaces $B_{p,q}^s(\mathbb{T})$ defined by the norm:

$$\|u\|_{B_{p,q}^\sigma} = \Big\| N^\sigma \|P_N u\|_{L_x^p} \Big\|_{\ell_N^q(2^{\mathbb{N}_0})}.$$

Essentially, σ denotes the regularity of a function $u \in B_{p,q}^\sigma$, and p the integrability of such a function, with the extra parameter q allowing for some extra flexibility at a "logarithmic" scale.

We shall also denote

$$B_{p,q}^{\sigma,+} := \Pi\Big(B_{p,q}^\sigma(\mathbb{T})\Big),$$

i.e. the Besov space on non-negative frequencies. Note that, from the boundedness of the Hilbert transform of $L^p(\mathbb{T})$ for $1 < p < \infty$, we also have that $B_{p,q}^{\sigma,+} \subset B_{p,q}^\sigma$ whenever $p \neq 1, \infty$.

We recall the basic estimates in Besov spaces. See [2,20] for example, for (i) – (iii), while (iv) is a simple consequence of the well-known boundedness of the Hilbert transform on L^p for $p \in (1, \infty)$.

Lemma 2.1. *The following estimates hold.*

(i) *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$. Then, we have*

$$\|u\|_{B_{p_1,q_1}^{s_1}} \lesssim \|u\|_{B_{p_2,q_2}^{s_2}} \qquad \text{for } s_1 \leq s_2, \ p_1 \leq p_2, \text{ and } q_1 \geq q_2,$$

$$\|u\|_{B_{p_1,q_1}^{s_1}} \lesssim \|u\|_{B_{p_1,\infty}^{s_2}} \qquad \text{for } s_1 < s_2, \tag{2.2}$$

$$\|u\|_{B_{p_1,\infty}^{0}} \lesssim \|u\|_{L^{p_1}} \lesssim \|u\|_{B_{p_1,1}^{0}}.$$

(ii) *Let $1 \leq p_2 \leq p_1 \leq \infty$, $q \in [1, \infty]$, and $s_2 \geq s_1 + \left(\frac{1}{p_2} - \frac{1}{p_1}\right)$. Then, we have*

$$\|u\|_{B_{p_1,q}^{s_1}} \lesssim \|u\|_{B_{p_2,q}^{s_2}}. \tag{2.3}$$

(iii) *Let $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$. Then, for every $\sigma > 0$, we have*

$$\|uv\|_{B_{p,q}^{\sigma}} \lesssim \|u\|_{B_{p_1,q}^{\sigma}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|v\|_{B_{p_4,q}^{\sigma}}. \tag{2.4}$$

(iv) *For $p \in (1, \infty)$,*

$$\|\Pi(u)\|_{L^p} \lesssim \|u\|_{L^p}. \tag{2.5}$$

In particular, in view of (2.4), (2.3) and (2.2) we have that whenever $p > \frac{1}{\sigma}$, the space $B_{p,q}^{\sigma}$ has the algebra property, i.e.

$$\|fg\|_{B_{p,q}^{\sigma}} \lesssim \|f\|_{B_{p,q}^{\sigma}} \|g\|_{B_{p,q}^{\sigma}}. \tag{2.6}$$

Combining with (2.5), we have for $\max(1, \frac{1}{\sigma}) < p < \infty$, the space $B_{p,q}^{\sigma,+}$ has the following algebra property:

$$\|\Pi(fg)\|_{B_{p,q}^{\sigma,+}} \lesssim \|f\|_{B_{p,q}^{\sigma,+}} \|g\|_{B_{p,q}^{\sigma,+}}. \tag{2.7}$$

Next we recall the Coifman-Meyer multiplier theorem, which we shall use extensively. See [8, Theorem 3.3].

Proposition 2.2. *Let $R \geq 1$ and let $m : \mathbb{R}^d \to \mathbb{C}$ be smooth away from the origin such that*

$$\|m\|_{CM,d} := \sup_{|\alpha| \leq d(d+3)} \sup_{x \in \mathbb{R}^d} \|x\|^{|\alpha|} |\partial^{\alpha} m(x)| \leq R < \infty.$$

Consider the multilinear map given by

$$\mathcal{F}[T_m(f_1, ..., f_d)](n) = \sum_{\substack{n_1, ..., n_d \in \mathbb{Z} \\ n_1 + ... n_d = n}} m(n_1, ..., n_d) \prod_{j=1}^{d} \widehat{f_j}(n_j).$$

Then for all $p \in [1, \infty)$ and $p_1, ..., p_d \in (1, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_d}$, there is a $C(R, p, p_i) < \infty$ such that

$$\|T_m(f_1, ..., f_d)\|_{L^p} \leq C(R, p, p_i) \prod_{j=1}^{d} \|f_j\|_{L^{p_j}}.$$

for all $f_1, ..., f_d$.

We refer to $\|m\|_{CM,d}$ as the Coifman-Meyer norm of m. A key feature of this result is for any family of smooth multipliers $\{m_N\}_N$ with uniformly bounded Coifman-Meyer norms, we may choose a uniform operator bound $C(R, p, p_i)$ for the family $\{T_{m_N}\}_N$.

2.3. **On the Gaussian measures μ_s.** We now define the Gaussian measures μ_s that we are going to consider throughout the paper, conditioned to be concentrated on spaces of functions which only have nonnegative frequencies.

To this goal, for $s > \frac{1}{2}$ we consider random initial data of the form

$$u_0(x) = \sum_{n \geq 0} \frac{g_n}{\langle n \rangle^s} e^{inx}, \tag{2.8}$$

where $\{g_n\}_{n \geq 0}$ are independent standard complex Gaussian random variables[6], and $\langle n \rangle = (1 + n^2)^{1/2}$. We denote

$$\mu_s := \text{Law}(u_0),$$

which is the unique centred Gaussian measure on L_+^2 with covariance operator $(1 - \Delta)^{-s}$ (restricted to non-negative frequencies). This measure may be written formally as

$$d\mu_s = \frac{1}{Z_s} e^{-\|u\|_{H_+^s}^2} \Pi(u) du,$$

where du is the formal Lebesgue measure on $L^2(\mathbb{T})$. We recall for $\sigma < s - \frac{1}{2}$,

$$\mu_s \left(H^\sigma \setminus H^{s-\frac{1}{2}} \right) = 1,$$

and so one may view u_0 as random initial data of regularity $s - \frac{1}{2}-$ (in Sobolev norm).

We now verify that μ_s is concentrated on a Besov space with critical regularity $s - \frac{1}{2}$.

Lemma 2.3. *Let $s > \frac{1}{2}$, and let $p \in [1, \infty)$. Then $\mu_s(B_{p,\infty}^{s-\frac{1}{2},+}) = 1$.*

Proof. We shall show the result for $p = 2k$ for $k \in \mathbb{N}$, and the general case follows by interpolation. We note that

$$\|u\|_{B_{p,\infty}^{s-\frac{1}{2},+}} = \sup_{N \in 2^{\mathbb{N}_0}} N^{s-\frac{1}{2}} \|P_N u\|_{L^p},$$

and so we define the random variable X_N for $N \in 2^{\mathbb{N}_0}$ as

$$X_N = N^{2k\left(s-\frac{1}{2}\right)} \|P_N u\|_{L^{2k}}^{2k},$$

where $u = u_0$ is given by (2.8). We see that

$$X_N = N^{k(2s-1)} \int_{\mathbb{T}} \left| \sum_{n \in \mathbb{N}_0} \frac{\phi_N(n) g_n}{\langle n \rangle^s} e^{inx} \right|^{2k} dx$$

$$= N^{k(2s-1)} \int_{\mathbb{T}} \sum_{\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k} \prod_{j=1}^{k} \frac{\phi_N(n_j) \phi_N(m_j) g_{n_j} \overline{g_{m_j}}}{\langle n_j \rangle^s \langle m_j \rangle^s} e^{i(n_j - m_j)x} dx$$

$$= N^{k(2s-1)} \sum_{\substack{\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k \\ n_1 + \ldots + n_k = m_1 + \ldots + m_k}} C_{\mathbf{n}, \mathbf{m}} \prod_{j=1}^{k} g_{n_j} \overline{g_{m_j}},$$

where $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{m} = (m_1, \ldots, m_k)$, and

$$C_{\mathbf{n}, \mathbf{m}} = \prod_{j=1}^{k} \frac{\phi_N(n_j) \phi_N(m_j)}{\langle n_j \rangle^s \langle m_j \rangle^s}.$$

---

[6]i.e. $\text{Re}(g_n) \sim \text{Im}(g_n) \sim N(0, \frac{1}{2})$, and $\text{Re}(g_n), \text{Im}(g_n)$ are independent.

We note that

$$C_{\mathbf{n},\mathbf{m}} = 0, \text{ for } (\mathbf{n},\mathbf{m}) \notin [N/2, 2N]^{2k},$$

$$C_{\mathbf{n},\mathbf{m}} \lesssim \frac{1}{N^{2ks}} \text{ for } (\mathbf{n},\mathbf{m}) \in [N/2, 2N]^{2k}.$$

Now, for $j = 0, ..., k$, let E_j denote the set of $(\mathbf{n},\mathbf{m})$ such that $\mathbf{n}$ and $\mathbf{m}$ have exactly j entries in common, i.e.

$$E_j = \left\{ (\mathbf{n},\mathbf{m}) \in [N/2, 2N]^{2k} : \max_{\sigma \in S_k} \left[\#\{l \text{ s.t. } n_l = m_{\sigma(l)}\} \right] = j \right\},$$

where S_k denotes the set of permutations of $\{1, \ldots, k\}$. We define

$$Y_{N,j} = N^{k(2s-1)} \sum_{\substack{(\mathbf{n},\mathbf{m})\in E_j \\ n_1+...+n_k=m_1+...+m_k}} C_{\mathbf{n},\mathbf{m}} \prod_{j=1}^{k} g_{n_j} \overline{g_{m_j}}.$$

Since $[N/2, 2N]^{2k} = \bigsqcup_{j=0}^{k} E_j$, we have that

$$X_N = \sum_{j=0}^{k} Y_{N,j}. \tag{2.9}$$

Since the $\{g_j\}$ are independent standard complex Gaussians, we observe that

$$\mathbb{E}\Big[\prod_{j=1}^{k} g_{n_j} \overline{g_{m_j}} \Big] = 0, \text{ whenever } (\mathbf{n},\mathbf{m}) \notin E_k. \tag{2.10}$$

Therefore, for $0 \le j < k$,

$$\mathbb{E}[Y_{N,j}] = 0.$$

For convenience of notation, denote

$$\Delta = \left\{ (\mathbf{n},\mathbf{m}) \in [N/2, 2N]^{2k} : n_1 + ... + n_k = m_1 + ... + m_k \right\}.$$

For $0 \le j < k$, we have that

$$\mathbb{E}[Y_{N,j}^2] = N^{2k(2s-1)} \sum_{\substack{(\mathbf{n},\mathbf{m})\in E_j\cap\Delta \\ (\mathbf{l},\mathbf{p})\in E_j\cap\Delta}} C_{\mathbf{n},\mathbf{m}} C_{\mathbf{l},\mathbf{p}} \mathbb{E}\Big[\prod_{j=1}^{k} g_{n_j} g_{l_j} \overline{g_{m_j} g_{p_j}} \Big].$$

By similar considerations to the ones leading to (2.10), we have that

$$\mathbb{E}\Big[\prod_{j=1}^{k} g_{n_j} g_{l_j} \overline{g_{m_j} g_{p_j}} \Big] = 0 \text{ unless } (\mathbf{m},\mathbf{p}) = \sigma(\mathbf{n},\mathbf{l}) \text{ for some } \sigma \in S_{2k},$$

and by Hölder,

$$\mathbb{E}\Big[\prod_{j=1}^{k} g_{n_j} g_{l_j} \overline{g_{m_j} g_{p_j}} \Big] \le \mathbb{E}|g_{n_j}|^{4k} \lesssim_k 1.$$

For $\sigma \in S_{2k}$, denote

$$\sigma_1(\mathbf{n},\mathbf{l}) := (\sigma(\mathbf{n},\mathbf{l})_1, \ldots, \sigma(\mathbf{n},\mathbf{l})_k), \quad \sigma_2(\mathbf{n},\mathbf{l}) := (\sigma(\mathbf{n},\mathbf{l})_{k+1}, \ldots, \sigma(\mathbf{n},\mathbf{l})_{2k}).$$

We then obtain

$$\mathbb{E}[Y_{N,j}^2] = N^{2k(2s-1)} \sum_{\substack{(\mathbf{n},\mathbf{m})\in E_j\cap\Delta \\ (\mathbf{l},\mathbf{p})\in E_j\cap\Delta}} C_{\mathbf{n},\mathbf{m}}C_{\mathbf{l},\mathbf{p}}\mathbb{E}\Big[\prod_{j=1}^k g_{n_j}g_{l_j}\overline{g_{m_j}g_{p_j}}\Big]$$

$$\leq N^{2k(2s-1)} \sum_{\sigma\in S_{2k}} \sum_{\substack{(\mathbf{n},\mathbf{m})\in E_j\cap\Delta \\ (\mathbf{l},\mathbf{p})\in E_j\cap\Delta \\ (\mathbf{m},\mathbf{p})=\sigma(\mathbf{n},\mathbf{l})}} C_{\mathbf{n},\mathbf{m}}C_{\mathbf{l},\mathbf{p}}\Big|\mathbb{E}\Big[\prod_{j=1}^k g_{n_j}g_{l_j}\overline{g_{m_j}g_{p_j}}\Big]\Big|$$

$$\lesssim N^{-2k} \sum_{\sigma\in S_{2k}} \sum_{\substack{(\mathbf{n},\sigma_1(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta \\ (\mathbf{l},\sigma_2(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta}} 1$$

$$= N^{-2k}\Big|\{(\sigma,\mathbf{n},\mathbf{l}) \in S_{2k}\times[N/2,2N]^k\times[N/2,2N]^k :$$

$$(\mathbf{n},\sigma_1(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta, (\mathbf{l},\sigma_2(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta\}\Big|$$

We now bound the size of this set.

Firstly, we note that since $(\mathbf{n},\sigma_1(\mathbf{n},\mathbf{l}))\in E_j, (\mathbf{l},\sigma_2(\mathbf{n},\mathbf{l}))\in E_j$, there exist permutations $\tau_{\mathbf{n}},\tau_{\mathbf{l}},\tau_1,\tau_2\in S_k$ such that

$$\tau_{\mathbf{n}}(\mathbf{n})_1 = \tau_1(\sigma_1(\mathbf{n},\mathbf{l}))_1,\ldots,\tau_{\mathbf{n}}(\mathbf{n})_j = \tau_1(\sigma_1(\mathbf{n},\mathbf{l}))_j,$$
$$\tau_{\mathbf{l}}(\mathbf{l})_1 = \tau_2(\sigma_2(\mathbf{n},\mathbf{l}))_1,\ldots,\tau_{\mathbf{l}}(\mathbf{l})_j = \tau_2(\sigma_2(\mathbf{n},\mathbf{l}))_j,$$
$$\tau_{\mathbf{n}}(\mathbf{n})_{j+1} = \tau_2(\sigma_2(\mathbf{n},\mathbf{l}))_{j+1},\ldots,\tau_{\mathbf{n}}(\mathbf{n})_k = \tau_2(\sigma_2(\mathbf{n},\mathbf{l}))_k,$$
$$\tau_{\mathbf{l}}(\mathbf{l})_{j+1} = \tau_1(\sigma_1(\mathbf{n},\mathbf{l}))_{j+1},\ldots,\tau_{\mathbf{l}}(\mathbf{l})_k = \tau(\sigma_1(\mathbf{n},\mathbf{l}))_k.$$

Namely, the permutations $\tau_{\mathbf{n}},\tau_{\mathbf{l}},\tau_1,\tau_2\in S_k$ put the common terms between $\mathbf{n}$ and $\sigma_1(\mathbf{n},\mathbf{l})$ and between $\mathbf{m}$ and $\sigma_2(\mathbf{n},\mathbf{l})$ in corresponding positions $1,\ldots,j$, and the common terms between $\mathbf{n}$ and $\sigma_2(\mathbf{n},\mathbf{l})$ and between $\mathbf{m}$ and $\sigma_1(\mathbf{n},\mathbf{l})$ in corresponding positions $j+1,\ldots,k$.

Since $|S_k|^4 = (k!)^4 \lesssim_k 1$, we obtain that

$$\Big|\{(\sigma,\mathbf{n},\mathbf{l})\in S_{2k}\times[N/2,2N]^k\times[N/2,2N]^k : (\mathbf{n},\sigma_1(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta, (\mathbf{l},\sigma_2(\mathbf{n},\mathbf{l}))\in E_j\cap\Delta\}\Big|$$

$$\lesssim_k \Big|\{\mathbf{n},\mathbf{l}\in[N/2,2N]^k\times[N/2,2N]^k : n_{j+1}+\cdots+n_k = l_{j+1}+\cdots+l_k\}\Big|$$

$$\lesssim_k N^{2k-1}.$$

Therefore, we deduce that

$$\mathbb{E}[Y_{N,j}^2] \lesssim N^{-2k}N^{2k-1} = \frac{1}{N}.$$

Summing over $N\in 2^{\mathbb{N}_0}$, we obtain that for every $\varepsilon > 0$, and every $j < k$,

$$|Y_{N,j}| \leq C_\varepsilon N^{-(\frac{1}{2}-\varepsilon)} \tag{2.11}$$

holds μ-a.s., for a (random) constant $C_\varepsilon < \infty$.

It remains to consider the $j = k$ case. Noting that E_k is the set of $(\mathbf{n}, \mathbf{m}) \in [N/2, 2N]^{2k}$ such that $\mathbf{m}$ is a permutation of $\mathbf{n}$, we see that

$$
\begin{aligned}
Y_{N,k} &= k! N^{k(2s-1)} \sum_{\mathbf{n} \in \mathbb{Z}^k} C_{\mathbf{n},\mathbf{n}} \prod_{j=1}^{k} |g_{n_j}|^2 \\
&\lesssim N^{-k} \sum_{\mathbf{n} \in [N/2,2N]^k} \prod_{j=1}^{k} |g_{n_j}|^2 \\
&= \left(N^{-1} \sum_{n \in [N/2,2N]} |g_n|^2 \right)^k =: (\widetilde{Y}_N)^k.
\end{aligned}
$$

We have that

$$
\mathbb{E}[\widetilde{Y}_N] = \mathbb{E}\Big[N^{-1} \sum_{n \in [N/2,2N]} |g_n|^2 \Big] \lesssim 1,
$$

and, by independence of the Gaussians g_n, g_m for $n \neq m$,

$$
\begin{aligned}
\mathbb{E}\Big[|\widetilde{Y}_N - \mathbb{E}[\widetilde{Y}_N]|^2 \Big] &= N^{-2} \mathbb{E}\Big[\Big| \sum_{n \in [N/2,2N]} |g_n|^2 - 1 \Big|^2 \Big] \\
&= N^{-2} \mathbb{E}\Big[\sum_{n \in [N/2,2N]} \big(|g_n|^2 - 1 \big)^2 \Big] \\
&\lesssim N^{-1}.
\end{aligned}
$$

Therefore, we obtain that μ_s-a.s.,

$$
\lim_{N \to \infty} \widetilde{Y}_N - \mathbb{E}[\widetilde{Y}_N] = 0,
$$

and so for some deterministic $C = C(k) > 0$, we have that

$$
\limsup_{N \to \infty} Y_{N,k} \leq \limsup_{N \to \infty} (\widetilde{Y}_N)^k = \limsup_{N \to \infty} (\mathbb{E}[\widetilde{Y}_N])^k \leq C.
$$

Therefore, together with (2.9) and (2.11), we obtain that μ_s-a.s.,

$$
\sup_N X_N < \infty,
$$

which implies that $u_0 \in B_{p,\infty}^{s-\frac{1}{2},+}$ μ_s-a.s. $\qquad\square$

2.4. **Local well posedness.** Recalling our choice of X_s:

$$
X_s = \begin{cases} H_+^\sigma \text{ for any } \frac{1}{2} < \sigma < s - \frac{1}{2}, & \text{when } s > 1, \\ B_{p,\infty}^{s-\frac{1}{2},+} \text{ for any } p > \min\left(100, \frac{1}{s-\frac{1}{2}}\right), & \text{when } \frac{1}{2} < s \leq 1. \end{cases} \tag{2.12}
$$

We have the following local well-posedness statement.

Proposition 2.4. *Let $s > \frac{1}{2}$. Then the equation (1.1) is locally well-posed in X_s. More precisely, for every $u_0 \in X_s$, there exists $T_* = T_*(\|u_0\|_{X_s}) > 0$ such that the (1.1) admits a unique solution u belonging to the space $u \in C([0, T_*], X_s)$. Moreover, this solution satisfies*

$$
u \in C^\infty([0, T_*], X_s).
$$

Proof. The proof is a standard application of Banach fixed-point theorem, where we only need the algebra property, i.e. the boundedness of the maps

$$H^\sigma \times H^\sigma \ni (f,g) \mapsto fg \in H^s \quad B_{p,\infty}^{s-\frac{1}{2}} \times B_{p,\infty}^{s-\frac{1}{2}} \ni (f,g) \mapsto fg \in B_{p,\infty}^{s-\frac{1}{2}}$$

respectively (for our choice of parameters), together with boundedness of

$$\Pi : H^\sigma \to H_+^\sigma, \quad \Pi : B_{p,\infty}^{s-\frac{1}{2}} \to B_{p,\infty}^{s-\frac{1}{2},+}.$$

Note that the fact that the solution is going to be infinitely smooth in time follows from the (formal) identity

$$i^k \partial_t^k u = \Pi(|u|^2 \Pi(|u|^2 \dots \Pi(|u^2|u) \dots)),$$

and the fact that the right-hand-side of this expression is bounded in X_s due to the algebra property again. $\qquad\square$

3. Quasi-invariance: $s > 1$

For $N \in \mathbb{N}$, we introduce the sharp Fourier truncation π_N on L_+^2 by

$$\pi_N \left(\sum_{n \geq 0} \widehat{u}(n) \right) = \sum_{0 \leq n < N} \widehat{u}(n),$$

and the truncated flow

$$\begin{cases} i\partial_t u_N = \pi_N \left(\Pi \left(|u_N|^2 u_N \right) \right), \\ u(0) = u_0 \in \pi_N(L_+^2). \end{cases} \tag{3.1}$$

We recall from [17] that on the space $\pi_N(L_+^2) \cong \mathbb{R}^{2N}$ endowed with the symplectic form

$$\omega(u,v) = 4\mathrm{Im}(u,v)_{L^2},$$

this truncated flow is the equation for a Hamiltonian equation associated with the Hamiltonian

$$E_N(u) = \|\pi_N(u)\|_{L^4}^4.$$

The Hamiltonian vector field $X_{E_N}(u) = \pi_N \left(\Pi \left(|u|^2 u \right) \right)$ on $\pi_N(L_+^2)$ is smooth, and by the conservation of the Hamiltonian, we can define a global flow map $\Phi_{t,N}$ for (3.1) on this space. Moreover, using standard arguments, one can easily show the following approximation statement.

Proposition 3.1. *Let $\sigma > \frac{1}{2}$, and let $u_0 \in H^\sigma$. Let $T \geq 0$. Then for every $\sigma' < \sigma$,*

$$\lim_{N \to \infty} \|\Phi_{t,N}(u_0) - \Phi_t(u_0)\|_{C([-T,T],H^{\sigma'})} = 0,$$

and for every $R > 0$, convergence is uniform on the H^σ ball

$$B_R := \{u_0 \in H^\sigma : \|u_0\|_{H^\sigma} \leq R\}.$$

We next define the truncated measure

$$\mu_{s,N} := (\pi_N)_\# \mu_s,$$

which is a Gaussian measure on $\pi_N(L_+^2)$, given by

$$d\mu_{s,N} = \frac{1}{Z_{s,N}} e^{-\|\pi_N(u)\|_{H_+^s}^2} du\,d\overline{u},$$

where $du\overline{du}$ is the natural Lebesgue measure on $\pi_N(L_+^2) \cong \mathbb{C}^N \cong \mathbb{R}^{2N}$. By Liouville's theorem, the Lebesgue measure is invariant under the flow map, and so we see that for all $t \in \mathbb{R}$:

$$d(\Phi_{t,N})_\# \mu_{s,N} = \exp\left(-\|\Phi_{-t,N}(u)\|_{H_+^s}^2 + \|u\|_{H_+^s}^2\right) d\mu_{s,N},$$

and so we define $f_{t,N}$ to be the density in the expression above, i.e.

$$f_{t,N} := \frac{d\Phi_{t,N}\# \mu_{s,N}}{d\mu_{s,N}} = \exp\left(-\|\Phi_{-t,N}(u)\|_{H_+^s}^2 + \|u\|_{H_+^s}^2\right). \tag{3.2}$$

By the fundamental theorem of calculus, we can write for $u \in \pi_N(L_+^2)$:

$$-\|\Phi_{-t,N}(u)\|_{H_+^s}^2 + \|u\|_{H_+^s}^2 = -\int_0^{-t} Q_{\pi_N}(\Phi_{\tau,N}(u)) d\tau, \tag{3.3}$$

where (with an abuse of notation) we denote

$$Q_{\pi_N}(u) = Q_{\pi_N}(u, u, u, u),$$

and Q_{π_N} is the multi-linear map

$$Q_{\pi_N}(u_1, u_2, u_3, u_4) := \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ 0 \le n_i < N}} \frac{i}{2} \Psi_s(\mathbf{n}) \widehat{u_1}(n_1) \overline{\widehat{u_2}(n_2)} \widehat{u_3}(n_3) \overline{\widehat{u_4}(n_4)},$$

with Ψ_s being the multiplier

$$\Psi_s(\mathbf{n}) = \langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n_4 \rangle^{2s}.$$

We have the following bound on Ψ_s.

Lemma 3.2. *Let $n_j \in \mathbb{N}_0$ with $n_j \sim N_j$ and $n_1 - n_2 + n_3 - n_4 = 0$. We have*

$$|\Psi_s(\mathbf{n})| \lesssim (N^{(1)})^{2s-1} N^{(3)}. \tag{3.4}$$

Proof. Without loss of generality, we can assume $N_1 \sim N_2 \sim N^{(1)}$. Indeed, if otherwise we had (say) $N_1 \sim N_3 \sim N^{(1)} \gg N_2, N_4$, then the condition $n_j \ge 0$ prevents us from having $n_1 - n_2 + n_3 - n_4 = 0$. Therefore, from the mean value theorem, we obtain that

$$\begin{aligned}
|\Psi_s(\mathbf{n})| &\le \left|\langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s}\right| + \langle n_3 \rangle^{2s} + \langle n_4 \rangle^{2s} \\
&\lesssim (N^{(1)})^{2s-1} |n_1 - n_2| + (N^{(3)})^{2s} + (N^{(4)})^{2s} \\
&= (N^{(1)})^{2s-1} |n_3 - n_4| + (N^{(3)})^{2s} + (N^{(4)})^{2s} \\
&\lesssim (N^{(1)})^{2s-1} N^{(3)}.
\end{aligned}$$

$\square$

Remark 3.3. Recalling the calculations of Remark 1.3 for the ODE setting, we see the diverging term of $Q(u)$ is given by the interaction of $|n_1|, |n_3| \gg |n_2|, |n_4|$. By restricting to non-negative frequencies no such interaction can exist, from which we deduce this improved bound on Ψ_s which we use to prove integrability of Q.

We wish to show (local) exponential integrability (with respect to $\mu_{s,N}$) of Q_{π_N}, uniformly in N. To do this we first need a deterministic multi-linear bound on Q_{π_N}.

Lemma 3.4. *For all $p_1, p_2, p_3, p_4 \in (1, \infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1$, there is a constant $C(p_i, s) > 0$, such that for all $N \in \mathbb{N}$ and N_i dyadic,*

$$|Q_{\pi_N}(P_{N_1} u_1, P_{N_2} u_2, P_{N_3} u_3, P_{N_4} u_4)| \leq C(p_i, s) \left(N^{(1)}\right)^{2s-1} N^{(3)} \prod_{j=1}^4 \|P_{N_j} u_j\|_{L^{p_j}}, \quad (3.5)$$

for all $u_j \in \pi_N(L_+^2)$.

Proof. Without loss of generality it suffices to consider the case $N_1 = N^{(1)}$ and $N_1 \approx N_2$. For $M \geq 1$ dyadic, we let

$$\Psi_{M,s}(\underline{n}) = \left(\phi_{\lesssim M}(n_1)\langle n_1\rangle^{2s} - \phi_{\lesssim M}(n_2)\langle n_2\rangle^{2s} + \phi_{\lesssim M}(n_3)\langle n_3\rangle^{2s} - \phi_{\lesssim M}(n_4)\langle n_4\rangle^{2s}\right) \prod_{j=1}^4 \phi_{\lesssim M}(n_j),$$

and we write

$$\Psi_{M,s}(\underline{n}) = (n_4 - n_3)(n_2 - n_3) m_M(n_1, n_2, n_3, n_4).$$

By the double mean-value theorem, m_M corresponds to the multiplier

$$m_M(n_1, n_2, n_3, n_4) = \prod_{j=1}^4 \phi_{\lesssim M}(n_j) \int_0^1 \int_0^1 (\langle \cdot \rangle^{2s} \phi_M)''(n_3 + s(n_2 - n_3) + t(n_4 - n_3)) \, ds \, dt.$$

We easily see that the Coifman-Meyer norm of m_M is bounded by M^{2s-2}. Denote by T_{m_M} the quadrilinear operator given by the multiplier m_M, i.e.

$$T_{m_M}(u_1, u_2, u_3, u_4) = \sum_{n_1 - n_2 + n_3 - n_4 = 0} m_M(n_1, n_2, n_3, n_4) \widehat{u_1(n_1)} \overline{\widehat{u_2(n_2)}} \widehat{u_3(n_3)} \overline{\widehat{u_4(n_4)}}.$$

We calculate

$$Q_{\pi_N}(P_{N_1} u_1, P_{N_2} u_2, P_{N_3} u_3, P_{N_4} u_4)$$

$$= \sum_{M \lesssim N_1} \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ 0 \leq n_i < N}} \frac{i}{2} \Psi_{M,s}(\underline{n}) \widehat{P_{N_1} u_1}(n_1) \overline{\widehat{P_{N_2} u_2}(n_2)} \widehat{P_{N_3} u_3}(n_3) \overline{\widehat{P_{N_4} u_4}(n_4)}$$

$$= \sum_{M \lesssim N_1} \frac{i}{2} T_{m_M}(P_{N_1} u_1, i\partial_x P_{N_2} u_2, P_{N_3} u_3, i\partial_x P_{N_4} u_4) + \text{ similar terms,}$$

where each of the similar terms is obtained by writing each of the terms of $(n_4 - n_3)(n_2 - n_3) = n_2 n_4 - n_2 n_3 - n_3 n_4 + n_3^2$ as derivatives. Applying the Coifman-Meyer multiplier theorem on m_M and using Bernstein's inequality yields (with a constant $C(p_i, s)$ that changes line-by-line):

$$|Q_{\pi_N}(P_{N_1} u_1, P_{N_2} u_2, P_{N_3} u_3, P_{N_4} u_4)| \leq C(p_i, s) \sum_{M \lesssim N_1} M^{2s-2} N_1 N^{(3)} \prod_{j=1}^4 \|P_{N_j} u_j\|_{L^{p_j}}$$

$$\leq C(p_i, s) N_1^{2s-1} N^{(3)} \prod_{j=1}^4 \|P_{N_j} u_j\|_{L^{p_j}}.$$

$\square$

From now on, we fix $s > 1$, and $\frac{1}{2} < \sigma < s - \frac{1}{2}$ as in the definition (1.8) of X_s, with the understanding that σ is sufficiently close to $s - \frac{1}{2}$. Also, we let

$$B_R := \{u_0 \in H^\sigma : \|u_0\|_{H^\sigma} \leq R\}.$$

Proposition 3.5. *There exist constants $A(s, \sigma), B(s, \sigma), C(s, \sigma) > 0$ such that for every $M > 0$, $R \geq 1$, and $N \in \mathbb{N}$ we have*

$$\int \exp\left(M|Q_{\pi_N}(u)|\right) \mathbb{1}_{B_R(0)}(u) d\mu_{s,N}(u) \leq \exp(CM \max\{1, M^A\}(1 + R)^B).$$

Proof. We recall from [13, Lemma 2.4] the simplified Boué-Dupuis variational formula, which in this context reads

$$\log \int \exp\left(M|Q_{\pi_N}(u)|\right) \mathbb{1}_{B_R(0)}(u) d\mu_{s,N}(u)$$

$$\leq \mathbb{E}\left[\sup_{V \in H^s} M|Q_{\pi_N}(u_0 + V)| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) - \frac{1}{2}\|V\|_{H^s}^2\right], \tag{3.6}$$

where u_0 is a random variable distributed according to μ_s. By symmetry we may expand

$$|Q_{\pi_N}(u_0 + V)|$$

$$\leq \sum_{\substack{N_1, N_2, N_3, N_4 \lesssim N \\ N_2 = N^{(1)}, N_4 \sim N^{(4)}, N_1 \sim N_2}} C_0 |Q_{\pi_N}(P_{N_1}(u_0 + V), P_{N_2}(u_0 + V), P_{N_3}(u_0 + V), P_{N_4}(u_0 + V))|$$

for some combinatorial constant C_0. For ease of notation, we write

$$Q_{\underline{N}}(f_1, f_2, f_3, f_4) = Q_{\pi_N}(P_{N_1}(f_1), P_{N_2}(f_2), P_{N_3}(f_3), P_{N_4}(f_4)),$$

and again denote $Q_{\underline{N}}(f) = Q_{\underline{N}}(f, f, f, f)$.

Fix $\varepsilon > 0$ to be chosen later, then pick $\delta > 0$ so that

$$\sum_{\substack{N_1, N_2, N_3, N_4 \lesssim N \\ N_2 \sim N^{(1)}, N_4 = N^{(4)}, N_1 \sim N_2}} 6\delta N_2^{-\varepsilon} < 1.$$

For some universal constant $C_0 > 0$, from (3.6) we then have that

$$\log \int \exp\left(M|Q_{\pi_N}(u)|\right) \mathbb{1}_{B_R(0)}(u) d\mu_{s,N}(u)$$

$$\leq \sum_{\substack{N_1, N_2, N_3, N_4 \lesssim N \\ N_2 \sim N^{(1)}, N_4 = N^{(4)}, N_1 \sim N_2}} \mathbb{E}\left[\sup_{V \in H^s} C_0 M|Q_{\underline{N}}(u_0 + V)| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) - \frac{6\delta N_2^{-\varepsilon}}{2}\|V\|_{H^s}^2\right].$$

$$\tag{3.7}$$

We now expand $Q_{\underline{N}}(u_0 + V)$:

$$Q_{\underline{N}}(u_0 + V, u_0 + V, u_0 + V, u_0 + V)$$

$$= Q_{\underline{N}}(u_0, u_0, u_0, u_0 + V) \tag{I}$$

$$+ Q_{\underline{N}}(u_0, u_0, V, u_0 + V) + Q_{\underline{N}}(u_0, V, u_0, u_0 + V) + Q_{\underline{N}}(V, u_0, u_0, u_0 + V) \tag{II}$$

$$+ Q_{\underline{N}}(V, V, u_0 + V, u_0 + V) \tag{III}$$

$$+ Q_{\underline{N}}(V, u_0, V, u_0 + V) + Q_{\underline{N}}(u_0, V, V, u_0 + V). \tag{IV}$$

We now bound each term.

(I): We express (I) as a random kernel acting on the low-frequency function $P_{N_4}(u_0 + V)$, namely

$$Q_{\underline{N}}(u_0, u_0, u_0, u_0 + V) = \sum_{0 \le n_4 < N} K_{\underline{N}}(n_4) \overline{\left(P_{N_4}(u_0 + V)\right)\widehat{}(n_4)},$$

where

$$K_{\underline{N}}(n_4) = \sum_{\substack{n_1 - n_2 + n_3 = n_4 \\ 0 \le n_1, n_2, n_3 < N}} \frac{i \Psi_s(\underline{\mathbf{n}}) \phi_{N_1}(n_1) \phi_{N_2}(n_2) \phi_{N_3}(n_3) g_{n_1} \overline{g_{n_2}} g_{n_3}}{2 \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s} \phi_{\sim N_4}(n_4).$$

We see that

$$|Q_{\underline{N}}(u_0, u_0, u_0, u_0 + V)| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V))$$
$$\le \|K_{\underline{N}}(n_4)\|_{\ell_{n_4}^2} \|P_{N_4} \pi_N(u_0 + V)\|_{L^2} \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V))$$
$$\le R N_4^{-\sigma} \|K_{\underline{N}}(n_4)\|_{\ell_{n_4}^2},$$

which is a quantity independent of V. Now using (3.4), we have

$$\mathbb{E}\left[\sup_{V \in H^s} C_0 M |Q_{\underline{N}}(u_0, u_0, u_0, u_0 + V)| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) \right]$$

$$\lesssim MR \Big(N_4^{-2\sigma} \mathbb{E}\Big[\sum_{n_4} |K_{\underline{N}}(n_4)|^2 \Big] \Big)^{\frac{1}{2}}$$

$$\lesssim MR \Bigg(N_4^{-2\sigma} \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_i \sim N_i \\ 0 \le n_i < N}} \frac{|\Psi_s(\underline{\mathbf{n}})|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Bigg)^{\frac{1}{2}}$$

$$\lesssim MR \left(N_4^{-2\sigma} N_2 N_3 N_4 N_2^{4s-2} N_3^2 N_2^{-4s} N_3^{-2s} \right)^{\frac{1}{2}},$$

so

$$\mathbb{E}\left[\sup_{V \in H^s} |(\mathrm{I})| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) \right] \lesssim MR \left(N_2^{-1} N_3^{3-2s} N_4^{1-2\sigma} \right)^{\frac{1}{2}}, \tag{3.8}$$

which is summable in N_i.

(II): We again use a random kernel representation. Note that each of the terms in (II) can be written as

$$\sum_{0 \le n_3, n_4 < N} K_{\underline{N}}(n_3, n_4) \langle n_3 \rangle^s \langle n_4 \rangle^\sigma \widehat{P_{N_{\alpha(3)}}(V)}(n_3) \overline{\widehat{P_{N_4}(u_0 + V)}(n_4)},$$

where

$$K_{\underline{N}}(n_3, n_4) = \sum_{\substack{n_{\alpha(1)} - n_{\alpha(2)} + n_{\alpha(3)} - n_4 = 0 \\ 0 \le n_1, n_2 < N}} \frac{i \Psi_s(\underline{\mathbf{n}}) \phi_{N_{\alpha(1)}}(n_1) \phi_{N_{\alpha(2)}}(n_2) \widetilde{g_{n_1}} \widetilde{g_{n_2}}}{2 \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s \langle n_4 \rangle^\sigma} \phi_{\sim N_{\alpha(3)}}(n_3) \phi_{\sim N_4}(n_4),$$

α is a permutation of $\{1, 2, 3\}$, and $\widetilde{g}_{n_1}, \widetilde{g}_{n_2}$ are normal Gaussian random variables that satisfy $\mathbb{E}[\widetilde{g}_{n_1} \widetilde{g}_{n_2}] = 0$ unless $n_1 = n_2$ and $\alpha(2) = 2$, in which case $\widetilde{g}_{n_1} = \overline{\widetilde{g}_{n_1}}$. Note that in

this case, we must also have $\Psi_s(\mathbf{n}) = 0$. Therefore,

$$K_{\underline{N}}(n_3, n_4) = \sum_{\substack{n_{\alpha(1)}-n_{\alpha(2)}+n_{\alpha(3)}-n_4=0 \\ 0 \leq n_1, n_2 < N \\ \mathbb{E}[\widetilde{g}_{n_1}\widetilde{g}_{n_2}] \neq 0}} \frac{i\Psi_s(\underline{\mathbf{n}})\phi_{N_{\alpha(1)}}(n_1)\phi_{N_{\alpha(2)}}(n_2)\widetilde{g_{n_1}}\,\widetilde{g_{n_2}}}{2\langle n_1\rangle^s\langle n_2\rangle^s\langle n_3\rangle^s\langle n_4\rangle^\sigma}\phi_{\sim N_{\alpha(3)}}(n_3)\phi_{\sim N_4}(n_4).$$

We denote by $\|K\|_{\mathrm{op}}$ the optimal constant C such that the inequality

$$\sum_{0 \leq n_3, n_4 < N} K(n_3, n_4)a(n_3)b(n_4) \leq C\|a\|_{l^2}\|b\|_{l^2}$$

holds. Then it follows that

$$|(\text{II})|\mathbb{1}_{\mathrm{B_R}(0)}(\pi_{\mathrm{N}}(\mathrm{u}_0 + \mathrm{V})) \lesssim \max_\alpha \|K_{\underline{N}}\|_{\mathrm{op}}\|V\|_{H^s}\|\pi_N(u_0+V)\|_{H^\sigma}\mathbb{1}_{B_R(0)}(\pi_N(u_0+V))$$
$$\lesssim R\max_\alpha\|K_{\underline{N}}\|_{\mathrm{op}}\|V\|_{H^s}.$$

Taking Hilbert-Schmidt norms, and using (3.4) again, we see that

$$\mathbb{E}[\|K_{\underline{N}}\|_{\mathrm{op}}^2]$$
$$\lesssim \mathbb{E}[\|K_{\underline{N}}\|_{l^2(n_3,n_4)}^2]$$
$$\leq \sum_{\substack{n_{\alpha(1)}-n_{\alpha(2)}+n_{\alpha(3)}-n_4=0 \\ n_1 \neq n_2}} \frac{|\Psi_s(\mathbf{n})|^2\phi_{N_{\alpha(1)}}(n_1)^2\phi_{N_{\alpha(2)}}^2(n_2)}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}\langle n_4\rangle^{2\sigma}}\phi_{\sim N_{\alpha(3)}}(n_3)^2\phi_{\sim N_4}(n_4)^2$$
$$\lesssim N_2^{-1}N_3^{3-2s}N_4^{1-2\sigma}.$$

Therefore, for some universal constant $C > 0$, by Young's inequality, we have

$$\mathbb{E}\left[\sup_{V \in H^s} C_0M|(\text{II})|\mathbb{1}_{\mathrm{B_R}(0)}(\pi_{\mathrm{N}}(\mathrm{u}_0 + \mathrm{V})) - \frac{\delta\mathrm{N}_2^{-\varepsilon}}{2}\|\mathrm{V}\|_{\mathrm{H^s}}^2\right]$$
$$\leq \mathbb{E}\left[\sup_{V \in H^s} CMR\|K_{\underline{N}}\|_{\mathrm{op}}\|V\|_{H^s} - \frac{\delta N_2^{-\varepsilon}}{2}\|V\|_{H^s}^2\right] \tag{3.9}$$
$$\leq \frac{2C^2M^2R^2N_2^\varepsilon}{\delta}\mathbb{E}[\|K_{\underline{N}}\|_{\mathrm{op}}^2]$$
$$\leq \delta^{-1}C^2M^2R^2N_2^{\varepsilon-1}N_3^{3-2s}N_4^{1-2\sigma},$$

which is summable in N_i (for ε small enough).

$\underline{(\text{III})}$: To bound this term, we use (3.5) and Bernstein's inequality, and obtain (on the set $\pi_N(u_0 + V) \in B_R(0)$):

$$|(\text{III})| \lesssim N_2^{2s-1}N_3\|P_{N_1}\pi_N V\|_{L^2}\|P_{N_2}\pi_N V\|_{L^2}\|P_{N_3}\pi_N(u_0+V)\|_{L^\infty}\|P_{N_4}\pi_N(u_0+V)\|_{L^\infty}$$
$$\lesssim N_2^{-1}N_3^{\frac{3}{2}-\sigma}\|P_{\lesssim N_2}\pi_N V\|_{H^s}^2\|\pi_N(u_0+V)\|_{H^\sigma}^2$$
$$\lesssim R^2N_2^{-\min(1,\sigma-\frac{1}{2})}\|P_{\lesssim N_2}\pi_N V\|_{H^s}^2.$$

Therefore, for some universal constant C, we have

$$|(\text{III})| \leq \mathrm{CR}^2\mathrm{N}_2^{-\min(1,\sigma-\frac{1}{2})}\|\mathrm{P}_{\lesssim\mathrm{N}_2}\pi_{\mathrm{N}}\mathrm{V}\|_{\mathrm{H^s}}^2.$$

Let $N_0 = N_0(R, M, \delta, \sigma, \varepsilon)$ be such that for every $N' \geq N_0$,

$$CR^2(N')^{-\min(1, \sigma - \frac{1}{2})} \leq \frac{\delta}{2M}(N')^{-\varepsilon}.$$

Note that this exists as long as $\varepsilon < \min(1, \sigma - \frac{1}{2})$, and for some $b_1, b_2 > 0$, we can choose N_0 such that

$$N_0 \lesssim M^{b_1} R^{b_2}.$$

Then, for a constant $C' = C'(\delta, \sigma, \varepsilon)$ that can change line to line, distinguishing the cases $N_2 < N_0$ and $N_2 \geq N_0$, we obtain that for some $a_1, a_2 > 0$,

$$C_0 M |(\text{III})| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) \leq N_2^{-\varepsilon} C' M^{a_1} R^{a_2} \|\pi_N V\|_{H^\sigma}^2 + \frac{\delta}{2} N_2^{-\varepsilon} \|V\|_{H^s}^2$$

$$\leq N_2^{-\varepsilon} M^{a_1} R^{a_2} C'(R + \|u_0\|_{H^\sigma})^2 + \frac{\delta}{2} N_2^{-\varepsilon} \|V\|_{H^s}^2.$$

Therefore, by taking expectations, we get for some universal constant $C > 0$,

$$\mathbb{E}\left[\sup_{V \in H^s} C_0 M |(\text{III})| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) - \frac{\delta N_2^{-\varepsilon}}{2} \|V\|_{H^s}^2\right] \leq CM^A(1 + R^A)N_2^{-\varepsilon}, \quad (3.10)$$

which is summable in N_i.

(IV): We use once again (3.5), and obtain (on the set $\pi_N(u_0 + V) \in B_R(0)$):

$$|(\text{IV})| \lesssim N_2^{2s-1} N_3 N_2^{-s} N_3^{-s} \|P_{\lesssim N_2} \pi_N V\|_{H^s}^2 \|P_{\approx N_2} u_0\|_{L^\infty} \|P_{N_4} \pi_N(u_0 + V)\|_{L^\infty}$$

$$\lesssim R N_2^{s-1} N_3^{1-s} N_2^{\frac{3}{4}-s} \|P_{\lesssim N_2} \pi_N V\|_{H^s}^2 \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}$$

$$\lesssim R N_2^{-\frac{1}{4}} \|P_{\lesssim N_2} \pi_N V\|_{H^s}^2 \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}.$$

Proceeding as for (III), we pick $N_0 = N_0(R, \delta, u_0, M, \varepsilon)$ such that for all $N' \geq N_0$

$$CR(N')^{-\frac{1}{4}} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}} \leq \frac{\delta}{2M}(N')^{-\varepsilon},$$

where C is the implicit constant in the inequality above. Moreover we may pick

$$N_0 \sim M^{a_1} R^{a_2} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}^{a_3} \delta^{-a_4},$$

for some $a_i > 0$ depending only on ε. Therefore, for $0 < \theta < 1$ such that

$$\theta s + (1 - \theta)\sigma \geq s - \frac{1}{8},$$

and for constants $C = C(s, \sigma, \theta, \delta) > 0$ and $A_1 = A_1(\varepsilon, \theta) > 0$, $A_2 = A_2(\varepsilon, \theta) > 0$ (that can change line to line), distinguishing the cases $N_2 \geq N_0$ and $N_2 < N_0$,

$$C_0 M |(\text{IV})| \mathbb{1}_{B_R(0)}(\pi_N(u_0 + V)) \leq N_2^{-\varepsilon} C M^{A_1} R^{A_2} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}^{A_2} \|P_{\lesssim N_2} \pi_N V\|_{H^{s-\frac{1}{8}}}^2 + \frac{\delta}{2} N_2^{-\varepsilon} \|V\|_{H^s}^2$$

$$\leq N_2^{-\varepsilon} C M^{A_1} R^{A_2} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}^{A_2} \|V\|_{H^s}^{2\theta} \|\pi_N V\|_{H^\sigma}^{2(1-\theta)} + \frac{\delta}{2} N_2^{-\varepsilon} \|V\|_{H^s}^2$$

$$\leq N_2^{-\varepsilon} C M^{A_1} R^{A_2} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}^{A_2} \|\pi_N V\|_{H^\sigma}^2 + \delta N_2^{-\varepsilon} \|V\|_{H^s}^2$$

$$\leq N_2^{-\varepsilon} C M^{A_1} R^{A_2} \|u_0\|_{B_{4,\infty}^{s-\frac{1}{2},+}}^{A_2} (R + \|u_0\|_{H^\sigma})^2 + \delta N_2^{-\varepsilon} \|V\|_{H^s}^2.$$

In particular, we have that for a (different) constant $C > 0$,

$$\mathbb{E}\left[\sup_{V \in H^s} C_0 M |(\mathrm{IV})| \mathbb{1}_{\mathrm{B}_\mathrm{R}(0)}(\pi_\mathrm{N}(u_0 + \mathrm{V})) - \delta \mathrm{N}_2^{-\varepsilon} \|\mathrm{V}\|_{\mathrm{H}^s}^2\right] \leq \mathrm{N}_2^{-\varepsilon'} C M^{A_1} R^{A_2}, \qquad (3.11)$$

which is summable in N_i. Finally, from (3.7), (3.8), (3.9), (3.10), (3.11), we get that, for some constant $K > 0$, and some $A, B > 0$,

$$\log \int \exp\left(M|Q_{\pi_N}(u)|\right) \mathbb{1}_{B_R(0)}(u) d\mu_{s,N}(u) \leq KM \max\{1, M^A\}(1+R)^B.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We now introduce the sets $E_{N,R,t} \subset \pi_N(H^\sigma)$ for $R \geq 1$ and $t \in \mathbb{R}$

$$E_{N,R,t} = \bigcap_{\tau \in I_t} \Phi_{\tau,N}\left(\pi_N(B_R(0))\right) = \left\{u \in \pi_N(H^\sigma) \,\bigg|\, \sup_{\tau \in I_t} \|\Phi_{-\tau,N}(u)\|_{H^\sigma} \leq R\right\}, \qquad (3.12)$$

where $I_t = [0, t]$ for $t \geq 0$ and $I_t = [t, 0]$ for $t < 0$. These sets will allow us to obtain a-priori estimates on the L^p norm of $f_{t,N}\mathbb{1}_{E_{N,R,t}}$, in a manner not too dissimilar from the arguments in [1, 13].

Proposition 3.6. *For $s > 1$ and $\frac{1}{2} < \sigma < s - \frac{1}{2}$ with $s - \frac{1}{2} - \sigma$ sufficiently small, there are constants $0 < A, B, C < \infty$ such that for all $t \in \mathbb{R}$, $N \in \mathbb{N}$, $R \geq 1$, and $p \in (1, \infty)$, we have*

$$\|f_{t,N}\mathbb{1}_{E_{N,R,t}}\|_{L^p(H^\sigma, d\mu_{s,N})} \leq \left\|\exp\left(|tQ_{\pi_N}(u)|\right)\mathbb{1}_{B_R(0)}(u)\right\|_{L^p(H^\sigma, d\mu_{s,N})} \leq \exp(C|1 + pt|^A R^B). \tag{3.13}$$

Proof. From (3.2), (3.3), and by Jensen's inequality together with convexity of the exponential, we have

$$\|f_{t,N}\mathbb{1}_{E_{N,R,t}}\|_{L^p(H^\sigma, d\mu_{s,N})}^p$$

$$= \int (f_{t,N}(u))^{p-1} \mathbb{1}_{E_{N,R,t}}(u) f_{t,N}(u) d\mu_{s,N}(u)$$

$$= \int (f_{t,N}(\Phi_{t,N}(u)))^{p-1} \mathbb{1}_{E_{N,R,t}}(\Phi_{t,N}(u)) d\mu_{s,N}(u)$$

$$= \int \exp\left(-(p-1)\int_0^{-t} Q_{\pi_N}(\Phi_{t+\tau,N}(u))d\tau\right) \mathbb{1}_{E_{N,R,t}}(\Phi_{t,N}(u)) d\mu_{s,N}(u)$$

$$= \int \exp\left(\frac{t}{|t|}(p-1)\int_{I_t} Q_{\pi_N}(\Phi_{\tau,N}(u))d\tau\right) \mathbb{1}_{E_{N,R,t}}(\Phi_{t,N}(u)) d\mu_{s,N}(u)$$

$$\leq \int \frac{1}{|t|}\int_{I_t} \exp\left(t(p-1)Q_{\pi_N}(\Phi_{\tau,N}(u))\right) \mathbb{1}_{E_{N,R,t}}(\Phi_{t,N}(u)) d\tau d\mu_{s,N}(u)$$

$$= \int \frac{1}{|t|}\int_{I_t} \exp\left(t(p-1)Q_{\pi_N}(u)\right) \mathbb{1}_{E_{N,R,t}}(\Phi_{N,t-\tau}(u)) f_{\tau,N}(u) d\tau d\mu_{s,N}(u).$$

For $\tau \in I_t$ we note

$$\Phi_{N,t-\tau}(u) \in E_{N,R,t} \implies \sup_{r \in I_{t-\tau}} \|\Phi_{N,r}(u)\|_{H^\sigma} \leq R \implies u \in E_{N,R,\tau}.$$

Combining this with the fact $E_{N,R,\tau} \subset B_R(0)$, we see

$$
\|f_{t,N}\mathbb{1}_{E_{N,R,t}}\|^p_{L^p(H^\sigma, d\mu_{s,N})}
$$
$$
\leq \int \frac{1}{|t|} \int_{I_t} \exp\left[t(p-1)Q_{\pi_N}(u)\right] \mathbb{1}_{B_R(0)}(u)\mathbb{1}_{E_{N,R,\tau}}(u)f_{\tau,N}(u)d\tau d\mu_{s,N}(u)
$$
$$
\leq \left\|\exp\left[t(p-1)|Q_{\pi_N}(u)|\right]\mathbb{1}_{B_R(0)}(u)\right\|_{L^{p'}(H^\sigma, d\mu_{s,N})} \sup_{\tau \in I_t}\|f_{\tau,N}\mathbb{1}_{E_{N,R,\tau}}\|_{L^p(H^\sigma, d\mu_{s,N})}.
$$

Taking the supremum over t in some interval I_T and using $p'(p-1) = p$ yields the first inequality in (3.13). The second inequality then follows from Proposition 3.5. $\square$

Remark 3.7. If one were able to obtain a similar result with a $H^{\frac{1}{2}}$-cutoff instead of a H^σ-cutoff, due to conservation of the $H^{\frac{1}{2}}$ norm, one could then apply Bourgain's (quasi-)invariant measure argument ([3], see [13, Theorem 6.1] for a precise statement), and obtain polynomial-in-time bounds on solutions for μ_s-a.e. initial data, due to the conservation of the $\dot{H}^{\frac{1}{2}}$ norm. However, these estimates appear out of reach. We point out that in [19], a dense sets of turbulent solutions for (1.1) has been constructed, but it is unclear if such solutions are typical or not. It would be interesting to either show good bounds on the growth of Sobolev norms for a.e. initial data distributed according to μ_s, or in alternative to exhibit a set of turbulent solutions with positive probability under such a Gaussian measure.

With this uniform L^p bound on the truncated densities, we are ready to prove Proposition 1.5.

Proof of Proposition 1.5. We first fix $\frac{1}{2} < \sigma < s - \frac{1}{2}$ satisfying the conditions of Proposition 3.6, and we recall that $X_s = H^\sigma$ where we have global well-posedness. By Proposition 3.1, we have that for all $u \in H^\sigma$ and $T > 0$, for every $\sigma' < \sigma$,

$$
\Phi_{t,N}(\pi_N u) \to \Phi_t(u) \text{ in } C([-T,T], H^{\sigma'}) \text{ as } N \to \infty,
$$

and this convergence is uniform on $B_R(0)$ for every $R \geq 0$. We now fix $\frac{1}{2} < \sigma' < \sigma$ that also satisfies the hypoteses of Proposition 3.6. By global well-posedness for (1.1) (see [17]) and this uniform convergence, we have that for any $R \geq 1$ and $t \in \mathbb{R}$, there exists $C(R,t) > 0$ such that for all N big enough (depending on R, t),

$$
\Phi_{t,N}\left(\pi_N(B_R(0))\right) \subset E'_{N,C(R,t),t},
$$

where the latter set corresponds to (3.12) with σ' replacing σ.

From the estimate (3.13), for any $t \in \mathbb{R}$ and $R \geq 1$, there exists $f_{t,R} \in L^2(H^\sigma, d\mu_s)$ such that (up to a subsequence) $f_{t,N}\mathbb{1}_{E_{N,C(R,t),t}} \circ \pi_N \to f_{t,R}$ weakly in $L^2(H^\sigma, d\mu_s)$. With this, we consider any non-negative, continuous and bounded function $F : H^{\sigma'} \to [0,\infty)$. By

dominated convergence, we have

$$\int F(u_0)d(\Phi_t)_{\#}(\mathbb{1}_{B_R(0)}\mu_s)(u_0)$$

$$= \int F(\Phi_t(u_0))d(\mathbb{1}_{B_R(0)}\mu_s)(u_0)$$

$$= \lim_{N\to\infty} \int F(\Phi_{t,N}(\pi_N u_0))\mathbb{1}_{B_R(0)}(u_0)d\mu_s(u_0)$$

$$\leq \limsup_{N\to\infty} \int F(\Phi_{t,N}(\pi_N u_0))\mathbb{1}_{\pi_N(B_R(0))}(\pi_N u_0)d\mu_s(u_0)$$

$$= \limsup_{N\to\infty} \int F(\pi_N u_0)\mathbb{1}_{\Phi_{t,N}(\pi_N(B_R(0)))}(\pi_N u_0)f_{t,N}(\pi_N u_0)d\mu_s(u_0)$$

$$\leq \limsup_{N\to\infty} \int F(\pi_N u_0)\mathbb{1}_{E_{N,C(R,t),t}}(\pi_N u_0)f_{t,N}(\pi_N u_0)d\mu_s(u_0).$$

We note by the dominated convergence theorem and the uniform L^2 bound on $f_{t,N}\mathbb{1}_{E_{N,C(R,t),t}} \circ \pi_N$:

$$\lim_{N\to\infty} \int |F(\pi_N u_0) - F(u_0)|\mathbb{1}_{\Phi_{t,N}\pi_N(B_R(0))}(\pi_N u_0)f_{t,N}(\pi_N u_0)d\mu_s(u_0) = 0,$$

and so

$$\int F(u_0)d(\Phi_t)_{\#}(\mathbb{1}_{B_R(0)}\mu_s)(u_0) \leq \limsup_{N\to\infty} \int F(u_0)\mathbb{1}_{E_{N,C(R,t),t}}(\pi_N u_0)f_{t,N}(\pi_N u_0)d\mu_s(u_0)$$

$$= \int F(u_0)f_{t,R}(u_0)d\mu_s(u_0).$$

Therefore $(\Phi_t)_{\#}(\mathbb{1}_{B_R(0)}\mu_s) \ll \mu_s$ for any $R \geq 1$. Since R is arbitrary, we deduce that $(\Phi_t)_{\#}\mu_s \ll \mu_s$ as well. $\qquad\square$

4. SINGULARITY: $s < 1$

4.1. **An abstract singularity result.** We start this section by showing a condition that guarantees singularity for up to countably many times. Most of the remaining of the paper will be dedicated to showing that this condition is indeed satisfied by the flow of (1.1) whenever $\frac{1}{2} < s < 1$ and $s \neq \frac{3}{4}$.

Proposition 4.1. *Let $g : X_s \times X_s \to \mathbb{R} \cup \{-\infty, \infty\}$ be a measurable function that satisfies*

$$g(x,y) > 0 \Rightarrow g(y,x) < 0.$$

Suppose that there exists $\tau = \tau(u_0) \geq 0$ such that $u_0 \in \mathrm{WP}(\tau) \cap \mathrm{WP}(-\tau)$, and for every $0 < |t| < \tau$,

$$g(\Phi_t(u_0), u_0) > 0,$$

and $\tau > 0$ for μ_s-a.e. u_0.[7] Then there exists a countable set $\mathscr{N} \subseteq \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus \mathscr{N}$,

$$(\Phi_t)_{\#}(\mathbb{1}_{\mathrm{WP}(t)}\mu_s) \perp \mu_s.$$

---

[7]Note that $\tau(u_0) = 0$ always (vacuously) satisfies the previous properties, so one just needs to check the existence of such a map on a set of full measure for μ_s.

Proof. First of all, we note that without loss of generality, we can assume that the function τ is measurable. Indeed, if it is not, we can simply redefine τ to be

$$\tau(u_0) = \sup\left\{\frac{1}{n} : n \in \mathbb{N}, u_0 \in \mathrm{WP}\left(\frac{1}{n}\right), \{t : |t| \leq \frac{1}{n}, g(\Phi_t(u_0), u_0) > 0\} = \left[-\frac{1}{n}, \frac{1}{n}\right]\right\},$$

where we define $\tau(u_0) = 0$ if the family on which we take the sup is empty. Note that this is measurable since $\mathrm{WP}(t)$ is an open set for every $t \in \mathbb{R}$, and the map $(t, u_0) \mapsto g(\Phi_t(u_0), u_0)$ is measurable due to continuity of the map $(t, u_0) \mapsto \Phi_t(u_0)$ (for $u_0 \in \mathrm{WP}(n^{-1})$).

Suppose by contradiction that the set

$$\mathcal{N} := \mathbb{R} \setminus \left(\{t : \mu_s(\mathrm{WP}(t)) > 0, (\Phi_t)_{\#}(\mathbb{1}_{\mathrm{WP}(t)}\mu_s) \perp \mu_s\} \cup \{t : \mu_s(\mathrm{WP}(t)) = 0\}\right)$$

is uncountable. Since $\tau > 0$ μ_s-a.s., for every $t \in \mathcal{N}$, there must exist $m_1, m_2 \in \mathbb{N}$ such that

$$(\Phi_t)_{\#}(\mathbb{1}_{\mathrm{WP}(t)}\mu_s)(\{\tau > 1/m_1\}) > 1/m_2.$$

Since $\mathcal{N}$ is uncountable, we deduce that there exist $\tau_0, \varepsilon_0 > 0$ such that the set

$$\mathcal{N}_{\tau_0, \varepsilon_0} = \{t : (\Phi_t)_{\#}(\mathbb{1}_{\mathrm{WP}(t)}\mu_s)(\{\tau > 2\tau_0\}) > \varepsilon_0\}$$

is uncountable as well. Since $\mathcal{N}_{\tau_0, \varepsilon_0} \subseteq \mathbb{R}$ is uncountable, it must have at least one accumulation point, i.e. there exist distinct $t_1, t_2, \ldots, t_n, \ldots$ such that $t_j \in \mathcal{N}_{\tau_0, \varepsilon_0}$ for every $j \in \mathbb{N}$, and $t_* \in \mathbb{R}$ such that

$$\lim_{j \to \infty} t_j = t_*.$$

Therefore, up to extracting a subsequence, we can assume that $|t_j - t_*| < \tau_0$ for every $j \in \mathbb{N}$. Now consider the sets

$$E_j := \Phi_{t_j}^{-1}(\{u_0 : \tau(u_0) > 2\tau_0\} \cap \mathrm{WP}(-t_j))$$

Recalling that $t_j \in \mathcal{N}_{\tau_0, \varepsilon_0}$ and that by definition $\Phi_{t_j}^{-1}(\mathrm{WP}(-t_j)) = \mathrm{WP}(t_j)$, we obtain that

$$\mu_s(E_j) = (\Phi_{t_j})_{\#}\mu_s(\{\tau > 2\tau_0\} \cap \mathrm{WP}(-t_j)) = (\Phi_{t_j})_{\#}(\mathbb{1}_{\mathrm{WP}(t_j)}\mu_s)(\{\tau > 2\tau_0\}) > \varepsilon_0.$$

Therefore, there must exist j_1, j_2 such that $E_{j_1} \cap E_{j_2} \neq \emptyset$, otherwise we would have

$$1 \geq \mu_s\left(\bigcup_j E_j\right) = \sum_j \mu_s(E_j) \geq \sum_j \varepsilon_0 = \infty.$$

Let $u_0 \in E_{j_1} \cap E_{j_2}$. By definition of E_j, we have that $\tau(\Phi_{t_{j_k}}(u_0)) > 2\tau_0$ for $k = 1, 2$. Therefore, by definition of τ, since $|t_{j_1} - t_{j_2}| \leq |t_{j_1} - t_*| + |t_* - t_{j_2}| < 2\tau_0$, we have that

$$g(\Phi_{t_{j_1}}(u_0), \Phi_{t_{j_2}}(u_0)) = g(\Phi_{t_{j_1} - t_{j_2}}(\Phi_{t_{j_2}}(u_0)), \Phi_{t_{j_2}}(u_0)) > 0.$$

Therefore, by hypothesis on g, we have that $g(\Phi_{t_{j_2}}(u_0), \Phi_{t_{j_1}}(u_0)) < 0$. However, proceeding similarly,

$$g(\Phi_{t_{j_2}}(u_0), \Phi_{t_{j_1}}(u_0)) = g(\Phi_{t_{j_2} - t_{j_1}}(\Phi_{t_{j_1}}(u_0)), \Phi_{t_{j_1}}(u_0)) > 0,$$

which is a contradiction.   □

Our main goal for this section will be showing the following.

Proposition 4.2. *Let $\frac{1}{2} < s < 1$ with $s \neq \frac{3}{4}$, and for $x, y \in X_s$, let*

$$g(x, y) := \liminf_{N \to \infty} \frac{\|P_N x\|^2_{\dot{H}^1} - \|P_N y\|^2_{\dot{H}^1}}{(4s - 3)N^{4-4s}}. \tag{4.1}$$

Then g satisfies the hypotheses of Proposition 4.1. More precisely, for every $x, y \in X_s$, we have that $g(x, y) > 0$ implies that $g(y, x) < 0$, and for μ_s-a.e. u_0, there exists $\tau(u_0) > 0$ such that for all $0 < |t| < \tau(u_0)$ we have

$$\liminf_{N \to \infty} \frac{\|P_N \Phi_t(u_0)\|^2_{\dot{H}^1} - \|P_N u_0\|^2_{\dot{H}^1}}{(4s - 3)N^{4-4s}} > 0.$$

We discuss briefly the main ideas that go into Proposition 4.2. It is not unreasonable to expect that, for small times t, we have

$$\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1} - \|P_N u_0\|^2_{\dot{H}^1} \approx t \frac{d}{dt}\|P_N \Phi_t(u_0)\|^2_{\dot{H}^1}\big|_{t=0} + \frac{t^2}{2}\frac{d^2}{dt^2}\|P_N u_0\|^2_{\dot{H}^1}\big|_{t=0} + O(t^3). \tag{4.2}$$

We then exploit the equation (1.1) to compute

$$\frac{d}{dt}\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1}\big|_{t=0}, \quad \frac{d^2}{dt^2}\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1}\big|_{t=0},$$

as polynomial objects in the frequencies of u_0. By taking expectations, we obtain that

$$\mathbb{E}\Big[\frac{d}{dt}\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1}\big|_{t=0}\Big] = 0,$$

$$\mathbb{E}\Big[\frac{d^2}{dt^2}\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1}\big|_{t=0}\Big] \sim (4s - 3)N^{4-4s}.$$

This suggests that for small times, $\|P_N \Phi_t(u_0))\|^2_{\dot{H}^1} - \|P_N u_0\|^2_{\dot{H}^1}$ has a definite sign as $N \to \infty$ (depending on the sign of $4s - 3$), which in turn leads us to the choice of g in (4.1). In particular, Theorem 4.2 follows once we show that this heuristic is correct, and once we bound the various error terms that will appear in the $O(t^3)$ term with the correct power on N. It actually turns out that the expansion in (4.2) is not fully correct, and we will need to allow for extra error terms of the form $o(N^{4-4s})$ (that disappear in the limit in the definition of (4.1)).

Note that estimating the remainder in (4.2) deterministically, i.e. using only the fact that the solution map $t \mapsto \Phi_t(u_0)$ is smooth in time with values in X_s, we are only able bound the error term as $O(N^{2s+1}t^3) \gg N^{4-4s}$. Therefore, in order to conclude, we will need to show appropriate multilinear random estimates for the solution of (1.1) at time t.

4.2. Time derivatives of the norm of the solution. Define the random objects

$$F_N(u_0) = N^{4s-4}\frac{d}{dt}\|P_N u(t)\|^2_{\dot{H}^1}\Big|_{t=0}, \tag{4.3}$$

$$G_N(u_0) = N^{4s-4}\frac{d^2}{dt^2}\|P_N u(t)\|^2_{\dot{H}^1}\Big|_{t=0}. \tag{4.4}$$

Let $u(t)$ be a (local) solution of (1.1) on X_s. By a direct computation, we have that

$$\frac{d}{dt}\|P_N u(t)\|^2_{\dot{H}^1} = Q_N(u, u, u, u)(t),$$

where Q_N is the quadrilinear map given by

$$Q_N(u_1, u_2, u_3, u_4) = \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_i \geq 0}} \frac{i}{2} \Psi_N(\mathbf{n}) \widehat{u_1}(n_1) \overline{\widehat{u_2}(n_2)} \widehat{u_3}(n_3) \overline{\widehat{u_4}(n_4)}, \qquad (4.5)$$

where we define the multiplier $\Psi_N(\mathbf{n})$ for $\mathbf{n} = (n_1, n_2, n_3, n_4)$ as

$$\Psi_N(\mathbf{n}) = \prod_{i=1}^{4} \mathbb{1}_{\mathbb{N}_0}(n_i) \Big(n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2 + n_3^2 \phi_N(n_3)^2 - n_4^2 \phi_N(n_4)^2 \Big). \qquad (4.6)$$

We note that (by construction) Ψ_N is anti-symmetric, so if $n_1 - n_2 + n_3 - n_4 = 0$ with $\{n_1, n_3\} \cap \{n_2, n_4\} \neq \emptyset$, then $\Psi_N(\mathbf{n}) = 0$. We start by showing a simple bound on Ψ_N.

Lemma 4.3. *For $n_i \in \mathbb{N}_0$ with $n_1 - n_2 + n_3 - n_4 = 0$ and $n_i \sim N_i$, we have*

$$|\Psi_N(\mathbf{n})| \lesssim \mathbb{1}_{N^{(1)} \gtrsim N} N \min\{N, N^{(3)}\}.$$

Proof. It is clear that if $N^{(1)} \ll N$, we then have $\Psi_N(\mathbf{n} = 0)$, so it is enough to show the bound. Similarly, from $n^2 \Phi_N(n) \lesssim N^2$, the bound

$$|\Psi_N(\mathbf{n})| \lesssim N^2$$

is immediate, so we focus on showing that $|\Psi_N(\mathbf{n})| \lesssim N N^{(3)}$. Since $n_i \geq 0$, in order to have $n_1 - n_2 + n_3 - n_4 = 0$, we cannot have that the biggest two frequencies are $\{n_1, n_3\}$ or $\{n_2, n_4\}$. Therefore, up to swapping n_1 with n_3 and n_2 with n_4, we have that

$$\min(n_1, n_2) \geq \max(n_3, n_4).$$

Up to further swapping n_1 with n_2 and n_3 with n_4 (which does not change $|\Psi_N(\mathbf{n})|$, we can put ourselves in the case

$$n_1 \geq n_2 \geq n_4 \geq n_3.$$

By the mean value theorem, noting that $\|\phi_N'\|_{L^\infty} \leq \frac{1}{N}$, we obtain that

$$\begin{aligned}
|\Psi_N(\mathbf{n})| &\leq |n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2| + |n_4^2 \phi_N(n_4)^2 - n_3^2 \phi_N(n_3)^2| \\
&\lesssim N|n_1 - n_2| + N|n_4 - n_3| \\
&= 2N|n_4 - n_3| \\
&\lesssim N_4 N \\
&= N^{(3)} N,
\end{aligned}$$

and so the bound follows. $\qquad \square$

We are now ready to estimate F_N and G_N in (4.3), (4.4).

Proposition 4.4. *For $\frac{1}{2} < s < 1$, we have that*

$$\mathbb{E}[F_N(u_0)] = 0, \quad \mathbb{E}[|F_N(u_0)|^2] \lesssim N^{2s-2}.$$

In particular, we have that

$$\lim_{N \to \infty} F_N(u_0) = 0 \ \mu_s\text{-almost surely,}$$

Proof. For N dyadic we have

$$N^{4-4s}F_N(u_0) = Q_N(u_0, u_0, u_0, u_0) = \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_i \geq 0}} \frac{i}{2}\Psi_N(\mathbf{n})\widehat{u_0}(n_1)\overline{\widehat{u_0}(n_2)}\widehat{u_0}(n_3)\overline{\widehat{u_0}(n_4)}$$

(4.7)

Note that F_N is antisymmetric under conjugation, so we have that

$$\overline{F_N(u_0)} = -F_N(u_0).$$

However, since $\text{Law}(\widehat{u_0}(n)) = \text{Law}(\overline{\widehat{u_0}(n)})$, we obtain

$$\mathbb{E}[F_N(u_0)] = -\mathbb{E}[\overline{F_N(u_0)}] = -\mathbb{E}[F_N(u_0)],$$

so we must have that

$$\mathbb{E}[F_N(u_0)] = 0.$$

We now move to computing the variance of $F_N(u_0)$. Recalling that whenever $n_1 = n_2$ or $n_1 = n_4$ we have $\Psi_N(\mathbf{n}) = 0$, and by Lemma 4.3, we get that

$$\begin{aligned}
\mathbb{E}\Big[|F_N(u_0)|^2\Big] &= N^{8s-8} \sum_{n_1-n_2+n_3-n_4=0} |\Psi_N(\mathbf{n})|^2 \prod_{j=1}^{4} \frac{1}{\langle n_j\rangle^{2s}} \\
&\lesssim N^{8s-8} \sum_{N_1,N_2,N_3,N_4} \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_j \sim N_j}} \frac{N^2\min\{N^2,(N^{(3)})^2\}}{N_1^{2s}N_2^{2s}N_3^{2s}N_4^{2s}} \\
&\lesssim N^{8s-8} \sum_{N_1,N_2,N_3,N_4} \frac{N^2\min\{N^2,(N^{(3)})^2\}N^{(2)}N^{(3)}N^{(4)}}{N_1^{2s}N_2^{2s}N_3^{2s}N_4^{2s}} \\
&\lesssim N^{8s-8}N^{6-6s} = N^{2s-2}.
\end{aligned}$$

In particular, this implies that $\mathbb{E}\Big[\sum_N |F_N(u_0)|^2\Big] < \infty$, so we must have

$$\lim_{N\to\infty} F_N(u_0) = 0$$

μ_s-almost surely. □

Proposition 4.5. *For $\frac{1}{2} < s < 1$, define*

$$I_s := (4s-2)\Big(\int_0^\infty (x\phi(x))^2 x^{1-4s}dx\Big)\Big(\int_0^1\int_0^1\int_0^1 y^{2-2s}(1-y+(\sigma+\tau)y)^{4s-4}d\sigma d\tau dy\Big).$$

(4.8)

Then $I_s > 0$, and we have

$$\lim_{N\to\infty} G_N(u_0) = 8(4s-3)I_s\|u_0\|_{L^2(\mathbb{T})}^2$$

(4.9)

for μ_s-almost every u_0.

Proof. We use the symmetry of F_N to compute

$$\frac{d^2}{dt^2}\|P_N u(t)\|_{\dot{H}^1}^2 = 2\operatorname{Re}\Big(\sum_{n_1-n_2+n_3-n_4=0}\Psi_N(\mathbf{n})\widehat{i\partial_t u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}\Big)$$

$$= 2\sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0\\n_1-n_2+n_3\geq 0\\n_i\geq 0}}\Psi_N(n_4-n_5+n_6,n_4,n_5,n_6)\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}\widehat{u}(n_5)\overline{\widehat{u}(n_6)}.$$

Note that the the coefficients Ψ_N are real, and the real part of

$$\prod_{j=1,3,5}\widehat{u}(n_j)\prod_{j=2,4,6}\overline{\widehat{u}(n_j)}$$

is invariant under permutations in the set

$$\sigma\in S := \big\{\sigma\in S^6 : \sigma(\{1,3,5\})=\{1,3,5\}\text{ or }\sigma(\{1,3,5\})=\{2,4,6\}\big\}\cong (S^3\times S^3)\rtimes\mathbb{Z}_2.$$

Therefore, we obtain that

$$\frac{d^2}{dt^2}\|P_N u(t)\|_{\dot{H}^1}^2 = 2\sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0\\n_1-n_2+n_3\geq 0\\n_i\geq 0}}f_N(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}\widehat{u}(n_5)\overline{\widehat{u}(n_6)},$$

where f_N is the symmetrisation of Ψ_N under the action of S, i.e.

$$f_N(n_1,n_2,n_3,n_4,n_4,n_6) = \frac{1}{72}\sum_{\sigma\in S}\Psi_N(n_{\sigma(1)}-n_{\sigma(2)}+n_{\sigma(3)},n_{\sigma(4)},n_{\sigma(5)},n_{\sigma(6)}).\qquad(4.10)$$

We now show a bound on f_N analogous to the one in Lemma 4.3, i.e.

$$|f_N(\mathbf{n})| \lesssim \mathbb{1}_{N^{(1)}\gtrsim N}N\min(N,N^{(3)}),\qquad(4.11)$$

under the condition $n_1-n_2+n_3-n_4+n_5-n_6=0$. To see this, by symmetry it suffices to bound $\Psi_N(n_{\sigma(4)}-n_{\sigma(5)}+n_{\sigma(6)},n_{\sigma(4)},n_{\sigma(5)},n_{\sigma(6)})$ by this quantity. This bound follows by Lemma 4.3, in combination with the observation that we cannot have three of the values n_4,n_5,n_6 and $n_4-n_5+n_6$ being bigger than $100N^{(3)}$. Indeed, by definition of $N^{(j)}$, we must have $\min(n_4,n_5,n_6)\lesssim N^{(3)}$, and if $n_1-n_2+n_3=n_4-n_5+n_6\geq 100N^{(3)}$, then $\max(n_1,n_2,n_3)\geq 6N^{(3)}$, and so at most one of n_4,n_5,n_6 can be bigger that $6N^{(3)}$ as well.

We see that

$$G_N = 2N^{4s-4}\sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0\\n_i\geq 0}}f_N(\mathbf{n})\prod_{j=1}^6\langle n_j\rangle^{-s}g_{n_1}\overline{g_{n_2}}g_{n_3}\overline{g_{n_4}}g_{n_5}\overline{g_{n_6}},$$

which we decompose based on the number of pairings:

$$G_N = G_{N,0}+G_{N,1}+G_{N,3},$$

where

$$G_{N,0} = 2N^{4s-4} \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0 \\ n_i \geq 0 \\ \{n_1,n_3,n_5\}\cap\{n_2,n_4,n_6\}=\emptyset}} f_N(\mathbf{n})\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}\widehat{u}(n_5)\overline{\widehat{u}(n_6)},$$

$$G_{N,1} = 18N^{4s-4} \sum_{\substack{n_1-n_2+n_3-n_4=0 \\ n_i \geq 0 \\ \{n_1,n_3\}\cap\{n_2,n_4\}=\emptyset \\ n_5=n_6}} f_N(\mathbf{n})\widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)\overline{\widehat{u}(n_4)}|\widehat{u}(n_5)|^2,$$

$$G_{N,3} = 12N^{4s-4} \sum_{n_1,n_2,n_3\geq 0} f_N(n_1,n_1,n_2,n_2,n_3,n_3)\prod_{j=1}^{3}|\widehat{u}(n_j)|^2.$$

The fact that $G_N = G_{N,0} + G_{N,1} + G_{N,3}$ can be easily verified using the symmetry of f_N under the action of S. For the 0-pairing term, by independence of different frequencies we have that $\mathbb{E}[G_{N,0}(u_0)] = 0$, and we use (4.11) to bound its variance:

$$\mathbb{E}[|G_{N,0}(u_0)|^2] \lesssim N^{8s-8} \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0 \\ n_i \geq 0}} f_N(\mathbf{n})^2\prod_{j=1}^{6}\langle n_j\rangle^{-2s}$$

$$\lesssim N^{8s-8} \sum_{\substack{N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 \geq N_6 \\ N_1 \gtrsim N, N_1 \sim N_2}} N^2\min(N^2,N_3^2)N_1^{-2s}\prod_{j=2}^{6}N_j^{1-2s}$$

$$\lesssim N^{8s-8} \sum_{\substack{N_1 \geq N_2 \geq N_3 \\ N_3 \gtrsim N}} N^4 N_1^{-2s}N_2^{1-2s}N_3^{1-2s}$$

$$+ N^{8s-8} \sum_{\substack{N_1 \geq N_2 \geq N_3 \\ N_1 \gtrsim N \gg N_3}} N^2 N_1^{-2s}N_2^{1-2s}N_3^{3-2s}$$

$$\lesssim N^{2s-2} + N^{6s-6}$$

$$\lesssim N^{2s-2},$$

which is summable in N. Hence we obtain that

$$\lim_{N\to\infty} G_{N,0}(u_0) = 0$$

for μ_s-a.e. u_0.

We now move to estimating $G_{N,1}(u_0)$. We further decompose it as

$$G_{N,1}(u_0) = G_{N,1,0}(u_0) + G_{N,1,2}(u_0),$$

defined respectively by[8]

$$G_{N,1,0} = 18N^{4s-4} \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0 \\ n_i \geq 0 \\ \{n_1,n_3\} \cap \{n_2,n_4\} = \emptyset \\ n_5 = n_6}} f_N(\mathbf{n}) \prod_{j=1}^{6} \langle n_j \rangle^{-s} g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_{n_4}},$$

$$G_{N,1,2} = 18N^{4s-4} \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0 \\ n_i \geq 0 \\ \{n_1,n_3\} \cap \{n_2,n_4\} = \emptyset \\ n_5 = n_6}} f_N(\mathbf{n}) \prod_{j=1}^{6} \langle n_j \rangle^{-s} g_{n_1} \overline{g_{n_2}} g_{n_3} \overline{g_{n_4}} (|g_{n_5}|^2 - 1),$$

where we fixed the representation

$$u_0 = \sum_{n \geq 0} \langle n \rangle^{-s} g_n e^{inx}.$$

Similarly to the situation for $G_{N,0}$, we have that $\mathbb{E}[G_{N,1,2}(u_0)] = 0$ and

$$\mathbb{E}[|G_{N,1,2}(u_0)|^2] \lesssim N^{8s-8} \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0 \\ n_i \geq 0, n_5 = n_6}} f_N(\mathbf{n})^2 \prod_{j=1}^{6} \langle n_j \rangle^{-2s}$$

$$\lesssim N^{2s-2},$$

so $\lim_{N \to \infty} G_{N,0}(u_0) = 0$ for μ_s-a.e. u_0. We now move to estimating $G_{N,1,0}$. It is again easy to check that $\mathbb{E}[G_{N,1,0}] = 0$. Using again (4.11), note that for $n_1 \sim N_1, \ldots, n_4 \sim N_4$ with $n_1 - n_2 + n_3 - n_4 = 0$, letting $N^{(1)} \geq \ldots \geq N^{(4)}$ be a decreasing rearrangement of $N_1 \ldots N_4$, recalling that this implies $N^{(1)} \sim N^{(2)}$, we get that

$$\sum_{m \geq 0} |f_N(n_1, n_2, n_3, n_4, m, m)| \langle m \rangle^{-2s}$$

$$\lesssim \sum_{m \lesssim N^{(3)}, N^{(1)} \gtrsim N} N \min(N, N^{(3)}) \langle m \rangle^{-2s} + \sum_{N^{(2)} \gg m \gg N^{(3)}, N^{(1)} \gtrsim N} N \min(N, \langle m \rangle) \langle m \rangle^{-2s}$$

$$+ \sum_{m \gtrsim N^{(2)} \gtrsim N, \max(\langle m \rangle, N^{(1)}) \gtrsim N} N \min(N, N^{(2)}) \langle m \rangle^{-2s}$$

$$\sim N \min(N, N^{(3)}) \mathbb{1}_{N^{(1)} \gtrsim N} + N^{3-2s} \mathbb{1}_{N^{(1)} \gtrsim N} + N \min(N, N^{(1)})(\max(N, N^{(1)}))^{1-2s}.$$

---

[8]The attentive reader would notice that $G_{N,1,0}$ and $G_{N,1,2}$ are the projection to homogenous Wiener chaos of order 0 and order 2 (respectively), hence the choice of notation.

Therefore,

$$
\mathbb{E}[|G_{N,1,0}(u_0)|^2]
$$

$$
\lesssim N^{8s-8} \sum_{\substack{n_1,n_2,n_3,n_4,n_5,n_6 \geq 0 \\ n_1-n_2+n_3-n_4=0}} |f_N(n_1,n_2,n_3,n_4,n_5,n_5)| \cdot |f_N(n_1,n_2,n_3,n_4,n_6,n_6)| \prod_{j=1}^{6} \langle n_j \rangle^{-2s}
$$

$$
= N^{8s-8} \sum_{\substack{n_1,n_2,n_3,n_4 \geq 0 \\ n_1-n_2+n_3-n_4=0}} \prod_{j=1}^{4} \langle n_j \rangle^{-2s} \Big(\sum_{m \geq 0} |f_N(n_1,n_2,n_3,n_4,m,m)| \langle m \rangle^{-2s} \Big)^2
$$

$$
\lesssim N^{8s-8} \sum_{N_1 \sim N_2 \geq N_3 \geq N_4} \big(N^2 \min(N,N_3)^2 \mathbb{1}_{N_1 \gtrsim N} + N^{6-4s} \mathbb{1}_{N^{(1)} \gtrsim N}
$$

$$
+ N^2 \min(N,N_1)^2 \max(N,N_1)^{2-4s} \big) N_1^{-1} \prod_{j=1}^{4} N_j^{1-2s}
$$

$$
\lesssim N^{2s-2} + N^{-1} + N^{8s-8} \sum_{N_1} N^2 \min(N,N_1)^2 \max(N,N_1)^{2-4s} N_1^{1-4s}
$$

$$
\lesssim N^{2s-2} + N^{-1} N^{4s-4} \max(N^{3-4s},1),
$$

which is again summable in N. Therefore, we get that $\lim_{N \to \infty} G_{N,0,2}(u_0) = 0$ for μ_s-a.e. u_0, and so

$$
\lim_{N \to \infty} (G_N(u_0) - G_{N,3}(u_0)) = 0
$$

for μ_s-a.e. u_0.

In order to evaluate the behaviour of $G_{N,3}$ in the limit $N \to \infty$, it is convenient to work with the non-symmetric form Ψ_N instead of f_N, and so we write

$$
G_{N,3}(u) = 2N^{4s-4} \sum_{n_1,n_2,n_3 \geq 0} (2 - \mathbb{1}_{\{n_1=n_3\}}) \Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) \prod_{j=1}^{3} |\widehat{u}(n_j)|^2.
$$

Indeed, note that all the pairing in (4.10) that have $n_{\sigma(5)} = n_{\sigma(4)}$ or $n_{\sigma(5)} = n_{\sigma(6)}$ result in $\Psi_N = 0$, and so one must have that $n_{\sigma(5)} = n_{\sigma(2)}$, which becomes $n_5 = n_2$ after we undo the symmetrisation procedure.

First of all, we note that in the case $n_1 = n_3$, the third highest frequency between $2n_1 - n_2, n_1, n_2, n_1$ has size $\lesssim N_1$. Therefore,

$$
N^{4s-4} \Big| \sum_{n_1,n_2 \geq 0} \Psi_N(2n_1 - n_2, n_1, n_2, n_1) |\widehat{u}_0(n_1)|^4 |\widehat{u}_0(n_2)|^2 \Big|
$$

$$
\lesssim N^{4s-4} \sum_{n_1,n_2 \geq 0, \max(\langle n_1 \rangle, \langle n_2 \rangle) \gtrsim N} N \min(N, \langle n_1 \rangle) \frac{|g_{n_1}|^4}{\langle n_1 \rangle^{4s}} \frac{|g_{n_2}|^2}{\langle n_2 \rangle^{2s}}
$$

By taking expectations, we obtain

$$N^{4s-4}\mathbb{E}\Big|\sum_{n_1,n_2\geq 0}\Psi_N(2n_1-n_2,n_1,n_2,n_1)|\widehat{u_0}(n_1)|^4|\widehat{u_0}(n_2)|^2\Big|$$

$$\lesssim N^{4s-4}\sum_{n_1,n_2\geq 0,\max(\langle n_1\rangle,\langle n_2\rangle)\gtrsim N}N\min(N,\langle n_1\rangle)\frac{1}{\langle n_1\rangle^{4s}}\frac{1}{\langle n_2\rangle^{2s}}$$

$$\lesssim N^{4s-4}\sum_{N_1,N_2:\max(N_1,N_2)\gtrsim N}N\min(N,N_1)N_1^{1-4s}N_2^{1-2s}$$

$$\lesssim N^{4s-4}\sum_{M\gtrsim N}N^{3-4s}M^{1-2s}+N^2M^{1-4s}$$

$$\lesssim N^{-1},$$

which is again summable. When $n_2=n_3$, we simply get that $\Psi_N(n_1-n_2+n_3,n_1,n_2,n_3)=0$. Therefore, for μ_s-a.e. u_0, we deduce that

$$G_{N,3}(u_0)=4N^{4s-4}\sum_{n_1,n_2,n_3\geq 0:\ n_j\ \text{are distinct}}\Psi_N(n_1-n_2+n_3,n_1,n_2,n_3)\prod_{j=1}^{3}|\widehat{u}(n_j)|^2+o(1).$$

Therefore, we can anti-symmetrise further and order the n_j. We obtain

$$G_N+o(1)$$
$$=8N^{4s-4}\sum_{n_1>n_2>n_3\geq 0}\frac{\Psi_N(n_1-n_2+n_3,n_1,n_2,n_3)+\Psi_N(n_1-n_3+n_2,n_1,n_3,n_2)}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|g_{n_1}|^2|g_{n_2}|^2|g_{n_3}|^2.$$

Note that here we neglected the term with $\Psi_N(n_2-n_1+n_3,n_2,n_1,n_3)$, because in that case we must either have that $n_3\gtrsim N$, and so

$$\mathbb{E}\Big|8N^{4s-4}\sum_{n_1>n_2>n_3\gtrsim N}\frac{\Psi_N(n_2-n_1+n_3,n_2,n_1,n_3)}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|g_{n_1}|^2|g_{n_2}|^2|g_{n_3}|^2\Big|$$

$$\lesssim N^{4s-4}\sum_{N_1,N_2,N_3\gtrsim N}N^2(N_1N_2N_3)^{1-2s}$$

$$\lesssim N^{4s-4}N^{2+3-6s}\lesssim N^{1-2s},$$

which is summable in N. Otherwise, we have $n_3\ll N$, so in order to have $\Psi_N\neq 0$ we must have $n_2<n_1\leq n_2+n_3$ and $\Psi_N(n_2-n_1+n_3,n_2,n_1,n_3)=\phi_N(n_1)^2n_1^2-\phi_N(n_2)^2n_2^2$, in which case

$$\mathbb{E}\Big|8N^{4s-4}\sum_{n_1>n_2\gg n_3\geq 0}\frac{\Psi_N(n_2-n_1+n_3,n_2,n_1,n_3)}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}}|g_{n_1}|^2|g_{n_2}|^2|g_{n_3}|^2\Big|$$

$$\lesssim N^{4s-4}\sum_{N_1,N_2\approx N\gg N_3}NN_3N_1^{1-2s}(N_2N_3)^{1-2s}$$

$$\lesssim N^{1-2s},$$

which is again summable in N. We now apply a further decomposition to the expression of $G_{N,3}$. We write

$$G_{N,3,\leq 2} = 8N^{4s-4} \sum_{n_1 > n_2 > n_3 \geq 0} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} |g_{n_3}|^2,$$

$$G_{N,3,>2} = 8N^{4s-4} \sum_{n_1 > n_2 > n_3 \geq 0} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}}$$
$$\times \left(|g_{n_1}|^2 |g_{n_2}|^2 - 1 \right) |g_{n_3}|^2.$$

By the above, we have that

$$G_N = G_{N,3,\leq 2} + G_{N,3,>2} + o(1)$$

as $N \to \infty$. We now show that $G_{N,3,>2}$ is a further error term. Recall that

$$\mathbb{E}[(|g_{n_1}|^2 |g_{n_2}|^2 - 1)(|g_{n_1'}|^2 |g_{n_2'}|^2 - 1)] = 0$$

whenever $n_1' \neq n_1, n_2$ and $n_2' \neq n_1, n_2$. Therefore, from the inequality $ab \lesssim a^2 + b^2$, and Lemma 4.3, we obtain

$$\mathbb{E}[|G_{N,3,>2}(u_0)|^2]$$
$$\lesssim N^{8s-8} \sum_{n_1} \Big(\sum_{\substack{n_2, n_3: \\ 0 < n_3 < n_2 < n_1}} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)^2$$
$$+ N^{8s-8} \sum_{n_2} \Big(\sum_{\substack{n_1, n_3: \\ 0 < n_3 < n_2 < n_1}} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)^2$$
$$+ N^{8s-8} \sum_{n_1} \Big(\sum_{\substack{n_2, n_3: \\ 0 < n_3 < n_2 < n_1}} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)$$
$$\times \Big(\sum_{\substack{n_2, n_3: \\ 0 < n_3 < n_1 < n_2}} \frac{\Psi_N(n_2 - n_1 + n_3, n_2, n_1, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)$$
$$\lesssim N^{8s-8} \sum_{n_1} \Big(\sum_{\substack{n_2, n_3: \\ 0 < n_3 < n_2 < n_1}} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)^2$$
$$+ N^{8s-8} \sum_{n_2} \Big(\sum_{\substack{n_1, n_3: \\ 0 < n_3 < n_2 < n_1}} \frac{\Psi_N(n_1 - n_2 + n_3, n_1, n_2, n_3) + \Psi_N(n_1 - n_3 + n_2, n_1, n_3, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \Big)^2$$
$$\lesssim N^{8s-8} \sum_{N_1 \geq N_2 \geq N_3, N_1 \gtrsim N} \left(N_1^{1-4s} N_2^{1-4s} N_3^{2-4s} N^2 \min(N, N_2)^2 \right)(N_2 + N_1)$$
$$\lesssim N^{8s-8} \sum_{N_1 \geq N_2 \geq N_3, N_1 \gtrsim N} N_1^{2-4s} N_2^{1-4s} N_3^{2-4s} N^2 \min(N, N_2)^2$$
$$\lesssim N^{8s-8} \sum_{N_2} N_2^{1-4s} N^2 \max(N, N_2)^{2-4s} \min(N, N_2)^2$$
$$\lesssim N^{8s-8} N^{4-4s+\max(3-4s,0)} \lesssim N^{4s-4} + N^{-1},$$

which is again summable in N. This shows that

$$G_N = G_{N,3,\leq 2} + o(1)$$

as $N \to \infty$. Therefore, in order to conclude (4.9), we just need to show that $G_{N,3,\leq 2}(u_0) = 8(4s-3)I_s \|u_0\|_{L^2}^2 + o(1)$. For convenience, write

$$G_{N,3,\leq 2}(u_0) = 8 \sum_{n \geq 0} |\widehat{u_0}(n)|^2 A_N(n), \qquad (4.12)$$

i.e.

$$A_N(n) = \sum_{n_1 > n_2 > n} \left[N^{4s-4} \frac{\Psi_N(n_1 - n_2 + n, n_1, n_2, n) + \Psi_N(n_1 - n + n_2, n_1, n, n_2)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}} \right].$$

Note that $A_N(n)$ is a quantity independent from u_0. We first show that for every $n \geq 0$,

$$\lim_{N \to \infty} A_N(n) = (4s-3)I_s, \qquad (4.13)$$

and that $I_s > 0$. We start by proving that, for every $0 < \varepsilon \leq \varepsilon_0(s) \ll 1$, as $N \to \infty$ we have

$$A_N(n) = N^{4s-4} \sum_{N^{1+\varepsilon} \geq n_1 > n_2 \geq N^{1-\varepsilon}} \frac{\Psi_N(n_1 + n_2, n_1, 0, n_2) + \Psi_N(n_1 - n_2, n_1, n_2, 0)}{n_1^{2s} n_2^{2s}} + o_N(1). \qquad (4.14)$$

Indeed, recalling the definition of Ψ_N in (4.6), by the mean value theorem, we have that for $N \gg \langle n \rangle$,

$$\left| A_N(n) - N^{4s-4} \sum_{N^{1+\varepsilon} \geq n_1 > n_2 \geq N^{1-\varepsilon}} \frac{\Psi_N(n_1 + n_2, n_1, 0, n_2) + \Psi_N(n_1 - n_2, n_1, n_2, 0)}{n_1^{2s} n_2^{2s}} \right|$$

$$\lesssim N^{4s-4} \sum_{N^{1+\varepsilon} \geq n_1 > n_2 \geq N^{1-\varepsilon}} N \frac{\langle n \rangle}{n_1^{2s} n_2^{2s}}$$

$$+ N^{4s-4} \sum_{n_1 > N^{1+\varepsilon}, n_1 > n_2 > n} \frac{N^2 \mathbb{1}_{|n_2| \gtrsim N}}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}} + N^{4s-4} \sum_{n_2 < N^{1-\varepsilon}, n_1 > n_2 > n} \frac{N \langle n_2 \rangle \mathbb{1}_{|n_1| \gtrsim N}}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}}$$

$$\lesssim \langle n \rangle N^{4s-3} N^{(1-\varepsilon)(2-4s)} + N^{2s-1} N^{(1+\varepsilon)(1-2s)} + N^{2s-2} N^{(1-\varepsilon)(2-2s)} = o(1)$$

as $N \to \infty$, as long as $0 < \varepsilon \ll 1$. Now, let

$$h(x) = x^2 \phi(x)^2.$$

Recalling the definition of Ψ_N in (4.6), and by writing Ψ_N in terms of h, by (4.14) we have

$$A_N(n) = \frac{1}{N^2} \sum_{N^{1+\varepsilon} \geq n_1 > n_2 \geq N^{1-\varepsilon}} \frac{h\left(\frac{n_1}{N} + \frac{n_2}{N}\right) - 2h\left(\frac{n_1}{N}\right) + h\left(\frac{n_1}{N} - \frac{n_2}{N}\right)}{\left(\frac{n_1}{N}\right)^{2s} \left(\frac{n_3}{N}\right)^{2s}} + o_N(1),$$

which we can view as an (improper) Riemann sum, and so

$$\lim_{N \to \infty} A_N(n) = \int_{x \geq y \geq 0} \frac{h(x+y) - 2h(x) + h(x-y)}{x^{2s} y^{2s}} dx dy.$$

Both the existence of the integral and convergence of the Riemann sums can be justified by the fact h is smooth and compactly supported, and then using the double mean value theorem:

$$\frac{|h(x+y) - 2h(x) + h(x-y)|}{x^{2s}y^{2s}}$$
$$\lesssim \|h''\|_{L^\infty} x^{-2s} y^{2-2s} \mathbb{1}_{0 \leq y \leq x \leq 4} + \|h\|_{L^\infty} x^{-2s} y^{-2s} \mathbb{1}_{1 \leq y \leq x},$$

which is integrable. Together with the analogous estimates for h', one can easily show convergence of the Riemann sum.

We now manipulate this integral in order to show its relationship with I_s, and verify that $I_s > 0$. Using the fact h is supported away from the origin, by change of variable, we have

$$\lim_{\varepsilon \downarrow 0} \left(\int_{x \geq y \geq \varepsilon x} \frac{h(x+y) - 2h(x) + h(x-y)}{x^{2s}y^{2s}} dx dy \right)$$

$$= \lim_{\varepsilon \downarrow 0} \left(\int_{\substack{x \geq 0 \\ 1 \geq u \geq \varepsilon}} [h(x(1+u)) - 2h(x) + h(x(1-u))] x^{1-4s} u^{-2s} dx du \right)$$

$$= \lim_{\varepsilon \downarrow 0} \left(\int_{\substack{x \geq 0 \\ 1 \geq u \geq \varepsilon}} h(x(1+u)) x^{1-4s} u^{-2s} dx du - 2 \int_{\substack{x \geq 0 \\ 1 \geq u \geq \varepsilon}} h(x) x^{1-4s} u^{-2s} dx du \right.$$

$$\left. + \int_{\substack{x \geq 0 \\ 1 \geq u \geq \varepsilon}} h(x(1-u)) x^{1-4s} u^{-2s} dx du \right)$$

$$= \left(\int_0^\infty h(x) x^{1-4s} dx \right) \lim_{\varepsilon \downarrow 0} \left(\int_\varepsilon^1 u^{-2s} (1+u)^{4s-2} du - 2 \int_\varepsilon^1 u^{-2s} du + \int_\varepsilon^1 u^{-2s} (1-u)^{4s-2} du \right)$$

$$= \left(\int_0^\infty h(x) x^{1-4s} dx \right) \lim_{\varepsilon \downarrow 0} \left(\int_\varepsilon^1 u^{-2s} \left[(1+u)^{4s-2} - 2 + (1-u)^{4s-2} \right] du \right).$$

We note that the final integral converges since in a neighbourhood of 0,

$$u^{-2s}(1+u)^{4s-2} - 2 + (1-u)^{4s-2} \lesssim u^{2-2s},$$

which is integrable. Moreover, by using the fundamental theorem of calculus twice, we get that

$$\int_0^1 u^{-2s} \left[(1+u)^{4s-2} - 2 + (1-u)^{4s-2} \right] du$$

$$= (4s-3)(4s-2) \left(\int_0^1 \int_0^1 \int_0^1 u^{2-2s} (1 - u + (\sigma + \tau)u)^{4s-4} d\sigma d\tau du \right),$$

and so (4.13) follows. The fact that $I_s > 0$ follows from the fact that on the interval $[0,1]^3$, we have that $1 - u + (\sigma + \tau)u \geq 0$, and $h \geq 0$ as well.

In order to conclude the proof, we just need to show that we can swap limit and summation in (4.12). By Lemma 4.3, we have that

$$|A(n)| \lesssim N^{4s-4} \sum_{n_1 > n_2} \frac{N \min(N, n_2) \mathbb{1}_{n_1 \gtrsim N}}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}}$$
$$\lesssim N^{4s-4} N^{3-2s} N^{1-2s} \lesssim 1.$$

Therefore, for μ-a.e. u_0, by dominated convergence,

$$\lim_{N\to\infty} G_N(u_0) = \lim_{N\to\infty} G_{N,3,\leq 2}(u_0) = 8 \lim_{N\to\infty} \sum_{n\geq 0} |\widehat{u_0}(n)|^2 A_N(n) = 8(4s-3)I_s\|u_0\|_{L^2}^2.$$

$\square$

4.3. Paralinear decomposition of the flow.

We now need to justify the fact that (for short times) we can approximate $\|P_N\Phi_t(u_0)\|_{\dot{H}^1}^2$ by its second order expansion in time. In all of the following we let $\frac{1}{2} < s < 1$ and recall our choice of X_s as $X_s = B_{p,\infty}^{s-\frac{1}{2},+}$ with $p > \min\left\{100, \frac{1}{s-\frac{1}{2}}\right\}$. We also let

$$B_R := \left\{u_0 \in X_s \Big| \|u_0\|_{X_s} \leq R\right\}.$$

The first step to showing the validity of the expansion (4.2) is to linearise (1.1) at high-frequency. In particular, note the following (para-product) decomposition:

$$i\partial_t u = \Pi(|u|^2 u) = \sum_{N_1,N_2,N_3} \Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u)$$

$$= 2\sum_{N_1,N_2\ll N_3} \Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u) + 2\sum_{\substack{N_1\ll N_3 \\ N_2\sim N_3}} \Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u)$$

$$+ \sum_{\substack{N_1\sim N_3 \\ N_2\ll N_3}} \Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u) + \sum_{N_1,N_2\sim N_3} \Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u). \qquad (4.15)$$

This first term will serve as our linearisation around high-frequency. For ease of notation, we introduce the trilinear para-product operator describing this Low-Low-High interaction:

$$\Pi_{L,L,H}(u,v,w) := \sum_{N_1,N_2\ll N_3} \Pi(P_{N_1}u\overline{P_{N_2}v}P_{N_3}w).$$

Now, consider the system of equations

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) \\ i\partial_t X = 2\Pi_{L,L,H}(u,u,X), \\ Y = u - X, \\ (u,X,Y)|_{t=0} = (u_0,u_0,0). \end{cases} \qquad (4.16)$$

Namely, $u = X + Y$ represents the solution to (1.1), while X flows under the linearised Low-Low-High interaction. The main advantage of this decomposition is that the term X captures the main nonlinear obejct with regularity $s - \frac{1}{2}$, while Y will be a smoother remainder. In particular, we have the following.

Proposition 4.6. *For all $R \geq 1$, there exists $T_R > 0$ such that for all $u_0 \in B_R$, there exists a unique triple*

$$(u,X,Y) \in C^2\big([-T_R,T_R]; X_s \times X_s \times B_{p/2,\infty}^{2s-1,+}\big)$$

that satisfies (4.16) with

$$\|(u,X,Y)\|_{C^2\left([-T_R,T_R];B_{p,\infty}^{s-\frac{1}{2},+}\times B_{p,\infty}^{s-\frac{1}{2},+}\times B_{p/2,\infty}^{2s-1,+}\right)} \leq C(1+R^5) \qquad (4.17)$$

for some universal constant $C > 0$.

Proof. First, consider the following multilinear bound, for N dyadic:

$$\|P_N\big(\Pi_{L,L,H}(u,v,w)\big)\|_{L^q} \lesssim \sum_{N_1,N_2 \ll N_3} \|P_N\big(P_{N_1}u\overline{P_{N_2}v}P_{N_3}w\big)\|_{L^q}$$

$$\lesssim \sum_{\substack{N_1,N_2 \ll N_3 \\ N_3 \approx N}} \|P_{N_1}u\|_{L^\infty}\|P_{N_2}v\|_{L^\infty}\|P_{N_3}w\|_{L^q}$$

$$\lesssim \|u\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|v\|_{B_{p,\infty}^{s-\frac{1}{2}}} \sum_{N_3 \approx N} \|P_{N_3}w\|_{L^q},$$

where we just used Hölder's inequality and the embedding $B_{p,\infty}^{s-\frac{1}{2}} \hookrightarrow B_{\infty,1}^0$. In particular, we obtain that for every $\sigma \in \mathbb{R}, q \geq 1$,

$$\|\Pi_{L,L,H}(u,v,w)\|_{B_{q,\infty}^{\sigma,+}} \lesssim \|u\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|v\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|w\|_{B_{q,\infty}^{\sigma,+}}. \tag{4.18}$$

Moreover, by (4.15), for every $q > \frac{1}{2s-1}$, we have

$$\|P_N\big(\Pi(|u|^2u) - 2\Pi_{L,L,H}(u,u,u)\big)\|_{L^q}$$

$$= \Big\|P_N\Big(2\sum_{\substack{N_1 \ll N_3 \\ N_2 \sim N_3}}\Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u) + \sum_{\substack{N_1 \sim N_3 \\ N_2 \ll N_3}}\Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u) + \sum_{N_1,N_2 \sim N_3}\Pi(P_{N_1}u\overline{P_{N_2}u}P_{N_3}u)\Big)\Big\|$$

$$\lesssim \|u\|_{B_{2q,\infty}^{s-\frac{1}{2},+}}^2\Big(\sum_{\substack{N_1 \ll N_3 \\ N_2 \sim N_3 \gtrsim N}}\|P_{N_1}u\|_{L^\infty}(N_2N_3)^{\frac{1}{2}-s}$$

$$+ \sum_{\substack{N_1 \sim N_3 \gtrsim N \\ N_2 \ll N_3}}\|P_{N_2}u\|_{L^\infty}(N_1N_3)^{\frac{1}{2}-s} + \sum_{N_1 \sim N_2 \sim N_3 \gtrsim N}\|P_{N_1}u\|_{L^\infty}(N_2N_3)^{\frac{1}{2}-s}\Big)$$

$$\lesssim N^{1-2s}\|u\|_{B_{2q,\infty}^{s-\frac{1}{2},+}}^3,$$

where we used Hölder's inequality $\|fgh\|_{L^q} \leq \|f\|_{L^\infty}\|g\|_{L^{2q}}\|h\|_{L^{2q}}$, and the embedding $B_{2q,\infty}^{s-\frac{1}{2},+} \hookrightarrow B_{\infty,1}^0$. We obtain that

$$\|P_N\big(\Pi(|u|^2u) - 2\Pi_{L,L,H}(u,u,u)\big)\|_{B_{q,\infty}^{2s-1,+}} \lesssim \|u\|_{B_{2q,\infty}^{s-\frac{1}{2},+}}^3. \tag{4.19}$$

Now, rewrite the equations for X, Y in (4.16) as

$$i\partial_t X = 2\Pi_{L,L,H}(u,u,X),$$

$$i\partial_t Y = \Pi(|u|^2u) - 2\Pi_{L,L,H}(u,u,u) + 2\Pi_{L,L,H}(u,u,Y).$$

Then, by a standard Banach fixed point argument, from the estimates (4.18),(4.19), we obtain that there exists a unique couple $(X,Y) \in C([-T_R,T_R], B_{p,\infty}^{s-\frac{1}{2},+} \times B_{p/2,\infty}^{2s-1,+})$, while the existence of u follows from Proposition 2.4. The fact that $(X,Y) \in C^2([-T_R,T_R], B_{p,\infty}^{s-\frac{1}{2},+} \times B_{p/2,\infty}^{2s-1,+})$ follows from similar considerations to the ones in Proposition 2.4, together with (4.18) and the analogous of (4.19) for the difference

$$\partial_t^2\big(\Pi(|u|^2u) - 2\Pi_{L,L,H}(u,u,u)\big).$$

The reason of the power R^5 in the RHS of (4.17) is the fact that $\partial_t^2 u|_{t=0}$ is a quintic expression in R. We omit the details. $\square$

Recall that, in view of (4.2), Proposition 4.4 and Proposition 4.5, our goal is showing that, in an appropriate sense,

$$\|P_N \Phi_t(u_0))\|_{\dot{H}^1}^2 - \|P_N u_0\|_{\dot{H}^1}^2$$
$$= \frac{t^2}{2} \frac{d^2}{dt^2} \|P_N u_0\|_{\dot{H}^1}^2 \big|_{t=0} + R(t, u_0),$$

with

$$|R(t, u_0)| \le \varepsilon t^2 N^{4-4s} + o(N^{4-4s})$$

for t small enough (depending both on ε and u_0). Recalling the definition of Q_N in (4.5) and of F_N, G_N in (4.3), (4.4) respectively, such a statement reduces to showing that, for every fixed $\varepsilon > 0$, and for every t small enough (depending on u_0, ε), we have

$$\left| N^{4s-4} Q_N(u(t), u(t), u(t), u(t)) - F_N(u_0) - t G_N(u_0) \right| \le \varepsilon t.$$

In order to see this, one can write $u(t)$ as $u(t) = X(t) + Y(t)$, and expand the terms in the expression for Q_N by using quadrilinearity. The next lemma essentially states that the estimate above holds whenever the expansion of Q contains at least one term in Y that does not appear in the lowest frequency.

Lemma 4.7. *Let N_1, N_2, N_3, N_4 be dyadic numbers, and let $f_1, \ldots, f_4 \in L_+^4(\mathbb{T})$. We have that*

$$\left| Q_N(P_{N_1} f_1, P_{N_2} f_2, P_{N_3} f_3, P_{N_4} f_4) \right| \lesssim N \min(N, N^{(3)}) \prod_{j=1}^4 \|f_j\|_{L^4}, \qquad (4.20)$$

where we recall that $N^{(1)} \ge N^{(2)} \ge N^{(3)} \ge N^{(4)}$ is a reordering of $N_1, \ldots, N_4$.

Proof. Recall the formula (4.6) for the multiplier $\Psi_N(\mathbf{n})$ appearing in (4.5),

$$\Psi_N(\mathbf{n}) = n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2 + n_3^2 \phi_N(n_3)^2 - n_4^2 \phi_N(n_4)^2,$$

where we omitted the indicator of nonnegative frequencies due to the assumption $f_j \in L_+^4$.

Exploiting the symmetries of Ψ_N, we can assume without loss of generality that $N_1 \sim N_2 \gtrsim N$, and $N_3, N_4 \lesssim N_1$. Under the condition $n_1 - n_2 + n_3 - n_4 = 0$, and $n_j \sim N_j$, we can decompose Ψ_N as

$$\Psi_N(\mathbf{n}) = N(n_4 - n_3) \Big(\frac{n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2}{N(n_1 - n_2)} \prod_{j=1}^4 \phi_{\lesssim N}(n_j) \Big) + n_3^2 \phi_N(n_3)^2 - n_4^2 \phi_N(n_4)^2$$

$$=: N(n_4 - n_3) h_N(n_1, n_2, n_3, n_4) + n_3^2 \phi_N(n_3)^2 - n_4^2 \phi_N(n_4)^2.$$

It is fairly easy to check that $h_N(x_1, x_2, x_3, x_4) = h(\frac{x_1}{N}, \frac{x_2}{N}, \frac{x_3}{N}, \frac{x_4}{N})$, and h is a smooth, compactly supported function of x, y. Therefore, by Proposition 2.2, we obtain

$$\left|Q_N(P_{N_1}f_1, P_{N_2}f_2, P_{N_3}f_3, P_{N_4}f_4)\right|$$

$$\leq \left|\sum_{n_1-n_2+n_3-n_4=0} Nh_N(n_1, n_2, n_3, n_4)(n_4-n_3)\widehat{P_{N_1}f_1}(n_1)\overline{\widehat{P_{N_2}f_2}(n_2)}\widehat{P_{N_3}f_3}(n_3)\overline{\widehat{P_{N_4}f_4}(n_4)}\right|$$

$$+ \left|\sum_{n_1-n_2+n_3-n_4=0} n_3^2\phi_N(n_3)^2\widehat{P_{N_1}f_1}(n_1)\overline{\widehat{P_{N_2}f_2}(n_2)}\widehat{P_{N_3}f_3}(n_3)\overline{\widehat{P_{N_4}f_4}(n_4)}\right|$$

$$+ \left|\sum_{n_1-n_2+n_3-n_4=0} n_4^2\phi_N(n_4)^2\widehat{P_{N_1}f_1}(n_1)\overline{\widehat{P_{N_2}f_2}(n_2)}\widehat{P_{N_3}f_3}(n_3)\overline{\widehat{P_{N_4}f_4}(n_4)}\right|$$

$$\lesssim \left(N\|P_{N_1}f_1\|_{L^4}\|P_{N_2}f_2\|_{L^4}(\|(P_{\lesssim N}P_{N_3}f_3)'\|_{L^4}\|P_{N_4}f_4\|_{L^4} + \|P_{N_3}f_3\|_{L^4}\|(P_{\lesssim N}P_{N_4}f_4)'\|_{L^4})\right)$$

$$+ \|P_{N_1}f_1\|_{L^4}\|P_{N_2}f_2\|_{L^4}\|(P_N P_{N_3}f_3)''\|_{L^4}\|P_{N_4}f_4\|_{L^4}$$

$$+ \|P_{N_1}f_1\|_{L^4}\|P_{N_2}f_2\|_{L^4}\|P_{N_3}f_3\|_{L^4}\|(P_N P_{N_4}f_4)''\|_{L^4})$$

$$\lesssim \prod_{j=1}^{4}\|f_j\|_{L^4}\left(N(\min(N, N_3) + \min(N, N_4)) + \min(N, N_3)^2 + \min(N, N_4)^2\right)$$

$$\lesssim N\min(N, N^{(3)})\prod_{j=1}^{4}\|f_j\|_{L^4}.$$

$\square$

In view of the previous lemma and of Proposition 4.4, we are left with estimating the difference

$$\left|N^{4s-4}Q_N(X(t), X(t), X(t), u(t)) - tG_N(u_0)\right|$$

under the extra assumption that the terms in $u(t)$ in the expression above appear at the lowest frequency. However, one can check that for generic $f_1, \ldots, f_4 \in B_{p,\infty}^{s-\frac{1}{2},+}$, one cannot expect an estimate which is any better than

$$Q_N(f_1, f_2, f_3, f_4) \lesssim N^{\frac{7}{2}-3s}.$$

Therefore, in order to conclude, we need to exploit the random oscillations of $X(t)$ in order to show that the expression

$$Q_N(X(t), X(t), X(t), u(t))$$

has the correct growth in the parameter N (and that has the correct dependency in the parameter t). In order to achieve this, we will need some further reductions. As it turns out, it is actually convenient to treat $Y(t)$ together with further error terms. In particular, the goal of this subsection is to show the following.

Proposition 4.8. *For $u_0 \in X_s$ and N dyadic, denote*

$$u_{0,N} = \sum_{M \ll N} P_M u_0 = P_{\ll N}u_0.$$

Let $u_N(t) = \Phi_t(u_{0,N})$. Then there exists $T_0 = T_0(\|u_0\|_{X_s}) > 0$ such that for every N dyadic,

$$P_N u(t) = P_N\Pi(X_N)(t) + v_N(t),$$

where

$$X_N(t) = \exp\left(-2i\int_0^t |P_{\ll N}u_N|^2(\tau)d\tau\right)P_{\approx N}u_0, \tag{4.21}$$

and u, v_N satisfy:

$$\|u(t)\|_{C^2\left([-T_0,T_0];X_s\right)} \lesssim 1 + \|u_0\|_{X_s}^5,$$

$$\|v_N(t)\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim (1 + \|u_0\|_{X_s}^5)N^{1-2s}. \tag{4.22}$$

We defer the proof of this proposition to the end of this subsection, since we will need a number of preliminary lemmas.

Firstly, we have the following bound on the difference of solutions $u(t)$ and $u_N(t)$.

Lemma 4.9. *For every $u_0 \in X_s$, there exists $T_0 = T_0(\|u_0\|_{X_s}) > 0$ such that for every N dyadic,*

$$\|u(t) - u_N(t)\|_{C^2\left([-T_R,T_R];B_{p,1}^{0,+}\right)} \lesssim N^{\frac{1}{2}-s}\|u_0\|_{X_s}(1 + \|u_0\|_{X_s}^4),$$

where $p > \max(100, \frac{1}{s-\frac{1}{2}})$ is the same parameter that appears in the definition (1.8) of X_s.

Proof. Let $R = 1 + \|u_0\|_{X_s}$. Note that, by definition,

$$\|u_{0,N}\|_{X_s} \lesssim \|u_0\|_{X_s}.$$

Therefore, by Proposition 2.4 we may pick $C, \widetilde{T_R} > 0$ such that $\|u(t)\|_{C^1\left([-\widetilde{T_R},\widetilde{T_R}];X_s\right)} \leq CR^5$ and $\|u_N(t)\|_{C^1\left([-\widetilde{T_R},\widetilde{T_R}];X_s\right)} \leq CR^5$ for all N. We now show a product estimate on Besov spaces (for functions not necessarily supported on non-negative frequencies). For every $0 < \varepsilon \ll s - \frac{1}{2} - \frac{1}{p}$, we have that

$$\|fgh\|_{B_{p,1}^0} = \sum_M \|P_M(fgh)\|_{L^p}$$

$$\leq \sum_M \sum_{N_1,N_2,N_3} \|P_M(P_{N_1}f P_{N_2}g P_{N_3}h)\|_{L^p}$$

$$\lesssim \|g\|_{B_{\infty,\infty}^\varepsilon}\|h\|_{B_{\infty,\infty}^\varepsilon} \sum_M \left(\sum_{\substack{N_1,N_2,N_3 \\ N_1 \lesssim \max(N_2,N_3) \\ M \lesssim \max(N_2,N_3)}} + \sum_{\substack{N_1,N_2,N_3 \\ N_1 \gg \max(N_2,N_3) \\ N_1 \sim M}}\right)\|P_{N_1}f\|_{L^p} N_2^{-\varepsilon}N_3^{-\varepsilon}$$

$$\lesssim \|g\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|h\|_{B_{p,\infty}^{s-\frac{1}{2}}}\left(\sum_M M^{-\varepsilon}\|f\|_{B_{p,1}^0} + \sum_M \sum_{N_1 \sim M}\|P_{N_1}f\|_{L^p}\right)$$

$$\lesssim \|f\|_{B_{p,1}^0}\|g\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|h\|_{B_{p,\infty}^{s-\frac{1}{2}}}.$$

With this, we see for $|t| \leq \widetilde{T_R}$:

$$\|u(t) - u_N(t)\|_{B_{p,1}^0} \leq \|u_0 - u_{0,N}\|_{B_{p,1}^0} + \int_0^t \|\Pi(|u|^2u(\tau) - |u_N|^2u_N(\tau))\|_{B_{p,1}^0}d\tau$$

$$\lesssim RN^{\frac{1}{2}-s} + \int_0^t \|(u-u_N)(u+u_N)\overline{u}\|_{B_{p,1}^0} + \|\overline{(u-u_N)}u_N^2\|_{B_{p,1}^0}d\tau$$

$$\lesssim \|u_0\|_{X^s}N^{\frac{1}{2}-s} + R^5|t| \sup_{|\tau| \leq |t|}\|u(\tau) - u_N(\tau)\|_{B_{p,1}^0},$$

and so it follows that picking a $T_0 = T_0(R) > 0$ small enough, we obtain

$$\|u(t) - u_N(t)\|_{C^0\left([-T_0,T_0];B_{p,1}^{0,+}\right)} \lesssim N^{\frac{1}{2}-s}\|u_0\|_{X_s}.$$

The analogous C^2 bound follows by taking derivatives and a similar argument. $\square$

Lemma 4.10 (Commutator estimates). *For N dyadic, consider the following trilinear maps*

$$\mathcal{C}_N^1[f,g,h] = P_N\Pi(P_{\ll N}(f)\overline{P_{\ll N}(g)}h) - P_{\ll N}(f)\overline{P_{\ll N}(g)}P_N\Pi(h)$$
$$\mathcal{C}_N^2[f,g,h] = P_N\Pi\big(\Pi_{L,L,H}(f,g,h) - P_{\ll N}(f)\overline{P_{\ll N}(g)}h\big)$$

For $j = 1, 2$, we have the following estimate

$$\|\mathcal{C}_N^j[f,g,h]\|_{L^{\frac{p}{3}}} \lesssim N^{1-2s}\|f\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|g\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|h\|_{B_{p,\infty}^{s-\frac{1}{2}}},$$

which holds for all f, g, h not-necessarily supported on non-negative frequencies.

Proof. We compute for $n \in \mathbb{Z}$

$$\left(\mathcal{C}_N^1[f,g,h]\right)^{\widehat{}}(n)$$

$$= \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ n_1-n_2+n_3=n}} [\mathbb{1}_{\mathbb{N}_0}(n)\phi_N(n) - \mathbb{1}_{\mathbb{N}_0}(n_3)\phi_N(n_3)]\widehat{P_{\ll N}f}(n_1)\overline{\widehat{P_{\ll N}g}(n_2)}\widehat{h}(n_3)$$

$$= \mathbb{1}_{\mathbb{N}_0}(n)\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ n_1-n_2+n_3=n}} [\phi_N(n) - \phi_N(n_3)]\widehat{P_{\ll N}f}(n_1)\overline{\widehat{P_{\ll N}g}(n_2)}\widehat{\Pi P_{\approx N}h}(n_3)$$

$$= \frac{\mathbb{1}_{\mathbb{N}_0}(n)}{N}\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ n_1-n_2+n_3=n}} (n_1-n_2)\int_0^1 \phi'\Big(\frac{n_3+\sigma(n_1-n_2)}{N}\Big)d\sigma\, \widehat{P_{\ll N}f}(n_1)\overline{\widehat{P_{\ll N}g}(n_2)}\widehat{\Pi P_{\approx N}h}(n_3)$$

$$= -i\frac{\mathbb{1}_{\mathbb{N}_0}(n)}{N}\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ n_1-n_2+n_3=n}} m_N(n_1,n_2,n_3)(\widehat{P_{\ll N}f})'(n_1)\overline{\widehat{P_{\ll N}g}(n_2)}\widehat{\Pi P_{\approx N}h}(n_3)$$

$$+ i\frac{\mathbb{1}_{\mathbb{N}_0}(n)}{N}\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ n_1-n_2+n_3=n}} m_N(n_1,n_2,n_3)\widehat{P_{\ll N}f}(n_1)\overline{(\widehat{P_{\ll N}g})'(n_2)}\widehat{\Pi P_{\approx N}h}(n_3),$$

where m_N is the Fourier multiplier given by

$$m_N(\xi_1,\xi_2,\xi_3) = \int_0^1 \phi'\Big(\frac{\xi_3+\sigma(\xi_1-\xi_2)}{N}\Big)d\sigma \prod_{j=1}^3 \Big(\sum_{M \leq 100N} \phi_M(\xi_j)\Big),$$

We note that the m_N are smooth and bounded on $\mathbb{R}^3$ (uniformly in N), and are supported on frequencies $|\xi_{max}| \lesssim N$. Using the fact $\Big(\sum_{M \leq 100N} \phi_M(x)\Big)$ is supported on $(-200N, 200N)$, we note that for all multi-indices α,

$$|\partial^\alpha m_N(\xi_1,\xi_2,\xi_3)| \lesssim N^{-|\alpha|}\mathbb{1}_{\{|\xi_{max}| \lesssim N\}} \lesssim \|(\xi_1,\xi_2,\xi_3)\|^{-\alpha},$$

and so the m_N are Coifmann-Meyer multipliers, with norms uniform in N. So by Proposition 2.2, we have that uniform in N,

$$\|\mathcal{C}_N^1[f,g,h]\|_{L^{\frac{p}{3}}} \lesssim \frac{1}{N}(\|P_{\ll N}f'\|_{L^p}\|P_{\ll N}g\|_{L^p} + \|P_{\ll N}f\|_{L^p}\|P_{\ll N}g'\|_{L^p})\|\Pi P_{\approx N}h\|_{L^p}$$

$$\lesssim N^{\frac{1}{2}-s}N^{-1}\sum_{N_1 \ll N} N_1 N_1^{\frac{1}{2}-s}\|f\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|g\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|h\|_{B_{p,\infty}^{s-\frac{1}{2}}}$$

$$\lesssim N^{1-2s}\|f\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|g\|_{B_{p,\infty}^{s-\frac{1}{2}}}\|h\|_{B_{p,\infty}^{s-\frac{1}{2}}}.$$

The estimate for $\mathcal{C}_N^2$ is simpler, noting that

$$\mathcal{C}_N^2(f,g,h) = P_N\Pi\left(\sum_{M\approx N}(P_{\ll M}(f)\overline{P_{\ll M}(g)}P_M h) - P_{\ll N}(f)\overline{P_{\ll N}(g)}h\right)$$

$$= P_N\Pi\left(\sum_{M\approx N}\left(P_{\ll M}(f)\overline{P_{\ll M}(g)} - P_{\ll N}(f)\overline{P_{\ll N}(g)}\right)P_M h\right)$$

$$= P_N\Pi\Bigg(\sum_{M\approx N}\Big((P_{\ll M}(f) - P_{\ll N}(f))\overline{P_{\ll M}(g)}$$

$$+ P_{\ll N}(f)\big(\overline{P_{\ll M}(g)} - \overline{P_{\ll N}(g)}\big)\Big)P_M h\Bigg),$$

from which the estimate follows. $\square$

Proof of Proposition 4.8. Let $R = 1 + \|u_0\|_{X_s}$. Recall the decomposition $u = X + Y$, where (u,X,Y) solve (4.16). By Proposition 4.6 and Lemma 4.9, we may pick a constant $C > 0$ and $T_0 = T_0(R) > 0$ such that

$$\|X(t)\|_{C^2\left([-T_R,T_R];X_s\right)} \le CR^5,$$

$$\|Y(t)\|_{C^2\left([-T_R,T_R];B_{p/2,\infty}^{2s-1,+}\right)} \le CR^5,$$

$$\|u_N(t)\|_{C^2\left([-T_R,T_R];X_s\right)} \le CR^5, \tag{4.23}$$

$$\|u(t) - u_N(t)\|_{C^2\left([-T_R,T_R];B_{p,1}^{0,+}\right)} \le CN^{\frac{1}{2}-s}R^5,$$

for all N dyadic. We also note that X_N as defined in (4.21) solves the equation

$$\begin{cases} i\partial_t X_N = 2|P_{\ll N}u_N|^2 X_N, \\ X_N(0) = P_{\approx N}u_0, \end{cases}$$

for which one can easily prove the analogous of Proposition 2.4. Therefore, up to making T_0 smaller, we also have

$$\|X_N(t)\|_{C^2\left([-T_0,T_0];X_s\right)} \le CR^5 \tag{4.24}$$

for all N dyadic. Then we decompose

$$v_N = P_N\Pi(X) - P_N\Pi(X_N) + P_N Y.$$

From (4.23), we get

$$\|P_N Y(t)\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^5 N^{1-2s}.$$

and so it suffices to bound $\widetilde{v_N} = P_N \Pi(X - X_N)$. We compute

$$\widetilde{v_N}$$
$$= -2i \int_0^t P_N \Pi(\Pi_{L,L,H}(u,u,X))(\tau) - P_N \Pi(|P_{\ll N} u_N|^2 X_N)(\tau) d\tau$$
$$= -2i \int_0^t |P_{\ll N} u_N|^2 \widetilde{v_N} d\tau$$
$$\quad - 2i \int_0^t P_N \Pi(|P_{\ll N} u_N|^2 (X - X_N)) - |P_{\ll N} u_N|^2 P_N \Pi(X - X_N) d\tau$$
$$\quad - 2i \int_0^t P_N \Pi(\Pi_{L,L,H}(u,u,X) - |P_{\ll N} u_N|^2 X) d\tau.$$
$$= -2i \int_0^t \left(|P_{\ll N} u_N|^2 \widetilde{v_N} d\tau + \mathcal{C}_N^1(u_N, u_N, X - X_N) \right.$$
$$\qquad \left. + P_N \Pi(\Pi_{L,L,H}(u,u,X) - |P_{\ll N} u_N|^2 X) \right) d\tau.$$

First note that, by (4.23), for $|t| \leq T_0$,

$$\left\| \int_0^t |P_{\ll N} u_N|^2 \widetilde{v_N} d\tau \right\|_{L^{\frac{p}{3}}} \lesssim R^{10} T_0 \|\widetilde{v_N}\|_{C^0\left([-T_0,T_0];L^{\frac{p}{3}}\right)}.$$

From the commutator estimate in Lemma 4.10, and (4.23), (4.24), we see that

$$\left\| \int_0^t \mathcal{C}_N^1(u_N, \overline{u_N}, X - X_N) d\tau \right\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^{15} T_0 N^{1-2s}.$$

Next we compute

$$P_N \Pi(\Pi_{L,L,H}(u,u,X) - |P_{\ll N} u_N|^2 X)$$
$$= P_N \Pi(\Pi_{L,L,H}(u,u,X) - \Pi_{L,L,H}(u_N, u_N, X))$$
$$\quad + P_N \Pi(\Pi_{L,L,H}(u_N, u_N, X) - |P_{\ll N} u_N|^2 X)$$
$$= P_N \Pi(\Pi_{L,L,H}(u,u,X) - \Pi_{L,L,H}(u_N, u_N, X)) + \mathcal{C}_N^2(u_N, u_N, X).$$

Again from Lemma 4.10, (4.23), and (4.24), we have

$$\left\| \int_0^t \mathcal{C}_N^2(u_N, u_N, X) d\tau \right\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^{15} T_0 N^{1-2s}$$

By (4.23), and Lemma 4.9, we have that

$$\|P_N \Pi(\Pi_{L,L,H}(u,u,X) - \Pi_{L,L,H}(u_N, u_N, X))\|_{C^1\left([-T_0,T_0];L^{\frac{p}{3}}\right)}$$

$$\lesssim \sum_{\substack{N_1, N_2 \ll N_3 \\ N_3 \sim N}} \|(P_{N_1}(u - u_N)\overline{P_{N_2}(u_N)} + \overline{P_{N_2}(u - u_N)}P_{N_1}u)P_{N_3}X\|_{C^1\left([-T_0,T_0];L^{\frac{p}{3}}\right)}$$

$$\lesssim R^{10}N^{\frac{1}{2}-s}\|u(t) - u_N(t)\|_{C^1\left([-T_0,T_0];B^0_{p,1}\right)}$$

$$\lesssim R^{10}N^{1-2s}.$$

Therefore

$$\left\|\int_0^t P_N \Pi(\Pi_{L,L,H}(u,u,X) - |P_{\ll N}u_N|^2 X)d\tau\right\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^{15}T_0 N^{1-2s}.$$

Putting all this together we can deduce that, up to making T_0 even smaller (but depending only on R),

$$\|\widetilde{v_N}\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^5 N^{1-2s},$$

and so

$$\|v_N\|_{C^2\left([-T_0,T_0];L^{\frac{p}{3}}\right)} \lesssim R^5 N^{1-2s}$$

as well. $\qquad\qquad\square$

4.4. **Estimates for Q_N.** We now verify that the decomposition $u(t) = P_N\Pi(X_N)(t) + v_N(t)$ given by Proposition 4.8 allows us to estimate $Q_N(u(t), u(t), u(t), u(t))$ appropriately. For ease of notation, let

$$\mathscr{E}_N(t) = \exp\left(-2i\int_0^t |P_{\ll N}u_N|^2(\tau)d\tau\right),$$

and

$$E_N(t) = P_{\ll N}(\mathscr{E}_N(t)).$$

We will also define, for M, M' dyadic,

$$X_M^+ = P_M\Pi(X_M) = P_M\Pi(\mathscr{E}_M(t)P_{\approx M}u_0),$$
$$X_{M,M'} = E_{M'}(t)P_M u_0 = P_{\ll M'}(\mathscr{E}_{M'}(t))P_M u_0.$$

We have the following estimates.

Lemma 4.11. *Let $M' \leq M$ dyadic, $u_0 \in X_s$, and let $T_0 = T_0(\|u_0\|_{X_s})$ be as in Proposition 4.6. Then it holds*

$$\|X_M^+ - X_{M,M'}\|_{C^2([-T_0,T_0],L^4)} \lesssim M^{\frac{1}{2}-s}(M')^{\frac{1}{2}-s}(1 + \|u_0\|_{X_s})^4\|u_0\|_{X_s}. \qquad (4.25)$$

Proof. We write

$$X_M^+ - X_{M,M'} = P_M\Pi(\mathscr{E}_M(t)P_{\approx M}u_0) - P_{\ll M'}(\mathscr{E}_{M'}(t))P_M u_0$$

$$= P_M\Pi((\mathscr{E}_M(t) - \mathscr{E}_{M'}(t))P_{\approx M}u_0) \qquad\qquad (I)$$

$$+ P_M\Pi((\mathscr{E}_{M'}(t) - P_{\ll M'}\mathscr{E}_{M'}(t))P_{\approx M}u_0) \qquad\qquad (II)$$

$$+ P_M(P_{\ll M'}\mathscr{E}_{M'}(t)P_{\approx M}u_0) - P_{\ll M'}\mathscr{E}_{M'}(t)P_M P_{\approx M}u_0, \qquad (III)$$

where in the last line we used that $\Pi(P_{\ll M}\mathscr{E}_{M'}(t)P_{\approx M}u_0) = P_{\ll M}\mathscr{E}_{M'}(t)P_{\approx M}u_0$ and that $P_M P_{\approx M} = P_M$.

To estimate (I), from Lemma 4.9, we deduce that

$$\|\mathscr{E}_M(t) - \mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],L^8)} \lesssim (M')^{\frac{1}{2}-s}(1 + \|u_0\|_{X_s})^4,$$

from which by Hölder's inequality,

$$\begin{aligned}
\|(\mathrm{I})\|_{C^2([-T_0,T_0],L^4)} &\lesssim \|\mathscr{E}_M(t) - \mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],L^8)}\|P_M u_0\|_{L^8} \\
&\lesssim M^{\frac{1}{2}-s}(M')^{\frac{1}{2}-s}(1 + \|u_0\|_{X_s})^4\|u_0\|_{X_s}.
\end{aligned}$$

For (II), note that $\mathscr{E}_{M'}(t)$ solves the equation

$$i\partial_t \mathscr{E}_{M'}(t) = 2|P_{\ll M'}u_{M'}|^2\mathscr{E}_{M'}(t),$$

from which we immediately deduce that

$$\begin{aligned}
\|\mathscr{E}_{M'}\|_{C^2([-T_0,T_0],X_s)} &\lesssim 1 + \|u_{M'}\|_{C^0([-T_0,T_0],X_s)}\|u_{M'}\|_{C^1([-T_0,T_0],X_s)} \\
&\lesssim (1 + \|u_0\|_{X_s}^4)
\end{aligned} \tag{4.26}$$

Therefore, we obtain that

$$\begin{aligned}
\|\mathscr{E}_{M'}(t) - P_{\ll M'}\mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],L^8)} &\lesssim \sum_{L \gtrsim M'} \|P_L\mathscr{E}_{M'}\|_{C^2([-T_0,T_0],L^8)} \\
&\lesssim \sum_{L \gtrsim M'} L^{\frac{1}{2}-s}\|\mathscr{E}_{M'}\|_{C^2([-T_0,T_0],X_s)} \\
&\lesssim (M')^{\frac{1}{2}-s}(1 + \|u_0\|_{X_s})^4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(\mathrm{II})\|_{C^2([-T_0,T_0],L^4)} &\lesssim \|\mathscr{E}_{M'}(t) - P_{\ll M}\mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],L^8)}\|P_M u_0\|_{L^8} \\
&\lesssim M^{\frac{1}{2}-s}(M')^{\frac{1}{2}-s}(1 + \|u_0\|_{X_s})^4\|u_0\|_{X_s}.
\end{aligned}$$

To estimate (III), consider the multiplier

$$m_M(n_1, n_2) = M\frac{(\phi_M(n_1 + n_2) - \phi_M(n_1))}{n_2}\phi_{\leq M}(n_1)\phi_{\leq M}(n_2).$$

It is easy to check that this is a Coifman-Meyer multiplier with $\|m_M\|_{CM,2} \lesssim 1$. In particular, we deduce that for every $f, g : \mathbb{T} \to \mathbb{C}$ supported on frequencies $|n| \lesssim M$, we have

$$\|P_M(fg) - fP_M(g)\|_{L^4} \lesssim M^{-1}\|f'\|_{L^8}\|g\|_{L^8}.$$

Therefore,

$$
\begin{aligned}
\|(\text{III})\|_{C^2([-T_0,T_0],L^4)} &\lesssim \sum_{L \ll M'} M^{-1}\|\partial_x P_L \mathscr{E}_{M'}\|_{C^2([-T_0,T_0],L^8)}\|P_{\approx M} u_0\|_{L^8} \\
&\lesssim \sum_{L \ll M'} \frac{L}{M}\|P_L \mathscr{E}_{M'}\|_{C^2([-T_0,T_0],L^8)}\|P_{\approx M} u_0\|_{L^8} \\
&\lesssim \sum_{L \ll M'} \frac{L}{M} L^{\frac{1}{2}-s} M^{\frac{1}{2}-s}\|\mathscr{E}_{M'}\|_{C^2([-T_0,T_0],X_s)}\|u_0\|_{X_s} \\
&\lesssim (M')^{\frac{3}{2}-s} M^{-\frac{1}{2}-s}(1+\|u_0\|_{X_s})^4\|u_0\|_{X_s} \\
&\lesssim M^{\frac{1}{2}-s}(M')^{\frac{1}{2}-s}(1+\|u_0\|_{X_s})^4\|u_0\|_{X_s}.
\end{aligned}
$$

$\square$

Lemma 4.12. *Let $M' \leq M$ dyadic, $u_0 \in X_s$, and let $T_0 = T_0(\|u_0\|_{X_s})$ be as in Proposition 4.6. Then it holds*

$$
\|\partial_x^2 P_N^2(X_{M,M'}) - E_{M'(t)}\partial_x^2 P_N^2 P_M u_0\|_{C^2([-T_0,T_0],L^4)} \lesssim N^{\frac{3}{2}-s}(M')^{\frac{3}{2}-s}(1+\|u_0\|_{X_s})^4\|u_0\|_{X_s}.
\tag{4.27}
$$

Proof. Recall that

$$
X_{M,M'} = E_{M'}(t)P_M u_0 = P_{\ll M'}(\mathscr{E}_{M'}(t))P_M u_0.
$$

Moreover, as in the previous proof, we have that (see (4.26))

$$
\|\mathscr{E}_{M'}\|_{C^2([-T_0,T_0],X_s)} \lesssim (1+\|u_0\|_{X_s}^4).
$$

For f, g smooth, it is easy to check that the bilinear map

$$
\partial_x^2 P_N^2((P_{\ll 4M}f)g) - P_{\ll 4M}f \cdot \partial_x^2 P_N^2(g)
$$

is given by the multiplier

$$
\begin{aligned}
c_M(n_1,n_2) &= (n_1^2\phi_N(n_1)^2 - (n_1+n_2)^2\phi_N(n_1+n_2)^2)\phi_{\ll 4M}(n_2) \\
&= Mn_2 \cdot \phi_{\ll 4M}(n_2)\frac{n_1^2(\phi_N(n_1)^2 - \phi_N(n_1+n_2)^2)}{Mn_2} \\
&\quad - (2n_1n_2 + n_2^2)\phi_N(n_1+n_2)^2\phi_{\ll 4M}(n_2).
\end{aligned}
$$

Via a scaling argument, it is easy to check that for $N \sim M$, we have

$$
\left\|\phi_{\ll 4M}(n_2)\frac{n_1^2(\phi_N(n_1)^2 - \phi_N(n_1+n_2)^2)}{Mn_2}\right\|_{CM,2} \lesssim 1.
$$

Therefore, by Coifman-Meyer's theorem, we obtain that for $M \sim N$,

$$
\begin{aligned}
&\|\partial_x^2 P_N^2((P_{\ll 4M}f)g) - P_{\ll 4M}f \cdot \partial_x^2 P_N^2(g)\|_{L^4} \\
&\lesssim M\|f'\|_{L^8}\|g\|_{L^8} + \|P_N^2((P_{\ll 4M}f')g')\|_{L^4} + \|P_N^2((P_{\ll 4M}f'')g)\|_{L^4} \\
&\lesssim N\|f'\|_{L^8}\|g\|_{L^8} + \|f'\|_{L^8}\|g'\|_{L^8} + \|f''\|_{L^8}\|g\|_{L^8}.
\end{aligned}
$$

We now apply this to $f = \partial_t^k P_L \mathscr{E}_{M'}(t)$ for $k \in \{0,1,2\}$ and $L \ll M'$, and $g = P_M u_0$. Note that when $M \not\approx N$, the LHS of (4.27) is equal to 0. Therefore, we can assume that $M \approx N$,

and obtain

$$
\begin{aligned}
&\|\partial_x^2 P_N^2 X_{M,M'} - E_{M'(t)}\partial_x^2 P_N^2 P_M u_0\|_{C^2([-T_0,T_0],L^4)} \\
&\lesssim \sum_{L \ll M'} (ML + L^2)\|P_L\mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],L^4)}\|P_M u_0\|_{L^4} \\
&\lesssim \sum_{L \ll M'} ML \cdot L^{\frac{1}{2}-s}M^{\frac{1}{2}-s}\|\mathscr{E}_{M'}(t)\|_{C^2([-T_0,T_0],X_s)}\|u_0\|_{X_s} \\
&\lesssim N^{\frac{3}{2}-s}(M')^{\frac{3}{2}-s}(1 + \|u_0\|_{X_s})^4\|u_0\|_{X_s}
\end{aligned}
$$

$\square$

The previous deterministic estimates allow us to isolate the "diverging" part of Q_N for general $u_0 \in X_s$.[9] For ease of notation we introduce the conjugation relation

$$
\mathfrak{c}_j(z) = \begin{cases} z, & \text{for } j \text{ even,} \\ \overline{z}, & \text{for } j \text{ odd.} \end{cases}
$$

Lemma 4.13 (Decoupling of Q_N)**.** *Let $u_0 \in X_s$, and let $T_0 = T_0(\|u_0\|_{X_s})$ be as in Proposition 4.6. For N_1, N_2, N_3 dyadic, define*

$$
Y_{N_1,N_2,N_3}(x)
$$
$$
:= \frac{i}{2}\sum_{n_1,n_2,n_3\in\mathbb{Z}}(n_1^2\phi_N(n_1)^2 - n_2^2\phi_N(n_2)^2 + n_3^2\phi_N(n_3)^2)\prod_{j=1}^3 \phi_{N_j}(n_j)\mathfrak{c}_j(\widehat{u_0}(n_j))e^{i(n_1-n_2+n_3)x}.
$$

$$(4.28)$$

Then, for all N dyadic, we have

$$
\begin{aligned}
&\left\|Q_N(u,u,u,u)(t)\right. \\
&\left. - 4\sum_{\substack{N_1,N_2,N_3\gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N\gg N_4 \\ N^{(3)}\lesssim N}}\int_{\mathbb{T}}|E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4}u_{\min(N,N^{(3)})}(t,x)}Y_{N_1,N_2,N_3}(x)\right\|_{C^2([-T_0,T_0])} \\
&\lesssim N^{4-4s}(1 + \|u_0\|_{X_s}^{20}),
\end{aligned}
$$

where we recall that $N^{(1)} \geq N^{(2)} \geq N^{(3)} \geq N^{(4)}$ is a reordering of $N_1,\ldots,N_4$.[10]

---

[9]As we will see soon, this is an improper name, since the "diverging" part of Q_N will actually be of smaller order as a power of N. Nevertheless, for general functions $u_0 \in X_s$, this is not the case, and the "diverging" part can actually be much bigger than the other terms.

[10]It is actually possible to improve the power in the RHS to $(1 + \|u_0\|_{X_s}^8)$. However, this is inessential to the argument of this paper.

Proof. By symmetry, we decompose

$$Q_N(u, u, u, u)$$

$$= 4 \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N \gg N_4}} Q_N(P_{N_1} u, P_{N_2} u, P_{N_3} u, P_{N_4} u)$$

$$+ \sum_{\substack{N_1, N_2, N_3, N_4 \\ \min(N, N^{(3)}) \lesssim N^{(4)}}} Q_N(P_{N_1} u, P_{N_2} u, P_{N_3} u, P_{N_4} u)$$

$$= 4 \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N^{(1)} \sim N^{(2)} \gtrsim N \gg N_4 \\ N^{(3)} \lesssim N}} \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t, x) \overline{P_{N_4} u_{\min(N, N^{(3)})}(t, x)} Y_{N_1, N_2, N_3}(x)$$

$$+ 4 \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N^{(1)} \sim N^{(2)} \gtrsim N \gg N_4 \\ N^{(3)} \lesssim N}} \left(Q_N(X_{N_1, N^{(3)}}, X_{N_2, N^{(3)}}, X_{N_3, N^{(3)}}, P_{N_4} u_{\min(N, N^{(3)})}) \right.$$
$$\left. - \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t, x) \overline{P_{N_4} u_{\min(N, N^{(3)})}(t, x)} Y_{N_1, N_2, N_3}(x) \right) \qquad \text{(I)}$$

$$+ 4 \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N \gg N_4}} \left(Q_N(X_{N_1}^+, X_{N_2}^+, X_{N_3}^+, P_{N_4} u_{\min(N, N^{(3)})}) \right.$$
$$\left. - Q_N(X_{N_1, N^{(3)}}, X_{N_2, N^{(3)}}, X_{N_3, N^{(3)}}, P_{N_4} u_{\min(N, N^{(3)})}) \right) \qquad \text{(II)}$$

$$+ 4 \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N \gg N_4}} \left(Q_N(P_{N_1} u, P_{N_2} u, P_{N_3} u, P_{N_4} u) \right.$$
$$\left. - Q_N(X_{N_1}^+, X_{N_2}^+, X_{N_3}^+, P_{N_4} u_{\min(N, N^{(3)})}) \right) \qquad \text{(III)}$$

$$+ \sum_{\substack{N_1, N_2, N_3, N_4 \\ \min(N, N^{(3)}) \lesssim N^{(4)}}} Q_N(P_{N_1} u, P_{N_2} u, P_{N_3} u, P_{N_4} u). \qquad \text{(IV)}$$

Let $T_0 = T_0(R)$ be as in Proposition 4.8. From (4.20) and (4.22), we have that

$$\|\text{IV}\|_{C^2([-T_0, T_0])}$$

$$\lesssim \sum_{\substack{N_1, N_2, N_3, N_4 \\ \min(N, N^{(3)}) \lesssim N^{(4)} \\ N^{(1)} \sim N^{(2)} \gtrsim N}} N \min(N, N^{(3)}) \prod_{j=1}^4 N_j^{\frac{1}{2}-s} \|u(t)\|_{C^2([-T_0, T_0], B_{4,\infty}^{s-\frac{1}{2},+})}^4$$

$$\lesssim N \sum_{\substack{N_1, N_2, N_3, N_4 \\ \min(N, N^{(3)}) \lesssim N^{(4)} \\ N^{(1)} \sim N^{(2)} \gtrsim N}} \min(N, N^{(3)}) (N^{(1)})^{\frac{1}{2}-s} (N^{(2)})^{\frac{1}{2}-s} (N^{(3)})^{\frac{1}{2}-s} (N^{(4)})^{\frac{1}{2}-s} (1 + \|u_0\|_{X_s}^{20})$$

$$\lesssim N \sum_{\substack{N^{(3)} \gtrsim N^{(4)} \\ \min(N, N^{(3)}) \lesssim N^{(4)}}} \min(N, N^{(3)}) \max(N, N^{(3)})^{1-2s} (N^{(3)})^{\frac{1}{2}-s} (N^{(4)})^{\frac{1}{2}-s} (1 + \|u_0\|_{X_s}^{20})$$

$$\lesssim \|u_0\|_{X_s}^4 \left(N^{2-2s} \sum_{N^{(3)} \sim N^{(4)} \lesssim N} (N^{(3)})^{\frac{3}{2}-s} (N^{(4)})^{\frac{1}{2}-s} + N^2 \sum_{N^{(3)} \gtrsim N^{(4)} \gtrsim N} (N^{(3)})^{\frac{3}{2}-3s} (N^{(4)})^{\frac{1}{2}-s} \right)$$

$$\lesssim (N^{2-2s} + N^{4-4s})(1 + \|u_0\|_{X_s}^{20}) \lesssim N^{4-4s}(1 + \|u_0\|_{X_s}^{20}).$$

In order to estimate (III), for $j = 1, 2, 3$, decompose $P_{N_j} u(t) = X_{N_j}^+ + v_{N_j}(t)$, as in Proposition 4.8. Recalling that

$$\left\| v_{N_j}(t) \right\|_{C^2([-T_0, T_0], L^{\frac{p}{3}})} \lesssim N^{1-2s} \|u_0\|_{X_s},$$

and that $\frac{p}{3} \geq 4$, by (4.20) and Lemma 4.9 we obtain

$$\|(\text{III})\|_{C^2([-T_0, T_0])}$$
$$\lesssim \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N^{(1)} \sim N^{(2)} \gtrsim N \gg N_4}} N \min(N, N^{(3)})^{\frac{3}{2}-s} (N^{(1)})^{\frac{1}{2}-s} (N^{(2)})^{\frac{1}{2}-s} (N^{(3)})^{\frac{1}{2}-s} (N^{(4)})^{\frac{1}{2}-s} (1 + \|u_0\|_{X_s}^{20})$$
$$\lesssim \sum_{N^{(1)} \sim N^{(2)} \gtrsim N} (N^{(1)})^{\frac{1}{2}-s} (N^{(2)})^{\frac{1}{2}-s} N^{3-2s} (1 + \|u_0\|_{X_s}^{20})$$
$$\lesssim N^{4-4s} (1 + \|u_0\|_{X_s}^{20}).$$

For estimating (II), we proceed similarly, using (4.25) in lieu of the estimate on v_N. Using (4.20) once again, we obtain

$$\|(\text{II})\|_{C^2([-T_0, T_0])}$$
$$= 4 \Bigg\| \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N \gg N_4}} \Big(Q_N(X_{N_1}^+, X_{N_2}^+, X_{N_3}^+, P_{N_4} u_{\min(N, N^{(3)})})$$
$$\qquad\qquad - Q_N(P_{\approx N_1} X_{N_1, N^{(3)}}, P_{\approx N_2} X_{N_2, N^{(3)}}, P_{\approx N_3} X_{N_3, N^{(3)}}, P_{N_4} u_{\min(N, N^{(3)})}) \Big) \Bigg\|_{C_t^2}$$
$$\lesssim \sum_{\substack{N_1, N_2, N_3 \gg N_4 \\ N^{(1)} \sim N^{(2)} \gtrsim N \gg N_4}} N \min(N, N^{(3)}) (N^{(1)})^{\frac{1}{2}-s} (N^{(2)})^{\frac{1}{2}-s} (N^{(3)})^{1-2s} (N^{(4)})^{\frac{1}{2}-s} (1 + \|u_0\|_{X_s}^{20})$$
$$\lesssim N^{4-4s} (1 + \|u_0\|_{X_s}^{20}).$$

Finally, we move to the estimate of (I). Recall the definition of Ψ_N in (4.6), which we now view as inducing a differential operator in spatial representation. Then, by definition of $X_{M, M'}$, we have that

$$Q_N(X_{N_1, N^{(3)}}, X_{N_2, N^{(3)}}, X_{N_3, N^{(3)}}, P_{N_4} u_{\min(N, N^{(3)})})$$
$$\quad - \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t, x) \overline{P_{N_4} u_{\min(N, N^{(3)})}(t, x)} Y_{N_1, N_2, N_3}(x) dx$$
$$= \frac{i}{2} \int_{\mathbb{T}} \overline{P_{N_4} u_{\min(N, N^{(3)})}} \sum_{j=1}^{3} (-1)^j \Big(\mathfrak{c}_j \big(\partial_x^2 P_N^2 X_{N_j, N^{(3)}} \big) \prod_{k \in \{1,2,3\} \setminus \{j\}} \mathfrak{c}_k \big(X_{N_k, N^{(3)}} \big)$$
$$\qquad\qquad - \mathfrak{c}_j \big(E_{N^{(3)}}(t) \partial_x^2 P_N^2 P_{N_j} u_0 \big) \prod_{k \in \{1,2,3\} \setminus \{j\}} \mathfrak{c}_k \big(E_{N^{(3)}}(t) P_{N_k} u_0 \big) \Big) dx$$
$$= \frac{i}{2} \int_{\mathbb{T}} \overline{P_{N_4} u_{\min(N, N^{(3)})}} \sum_{j=1}^{3} (-1)^j \mathfrak{c}_j \big(\partial_x^2 P_N^2 X_{N_j, N^{(3)}} - E_{N^{(3)}}(t) \partial_x^2 P_N^2 P_{N_j} u_0 \big)$$
$$\qquad\qquad\qquad\qquad\qquad \times \prod_{k \in \{1,2,3\} \setminus \{j\}} \mathfrak{c}_k \big(X_{N_k, N^{(3)}} \big) dx.$$

By (4.27), we obtain that

$$
\begin{aligned}
&\Big\| Q_N(X_{N_1,N^{(3)}}, X_{N_2,N^{(3)}}, X_{N_3,N^{(3)}}, P_{N_4} u_{\min(N,N^{(3)})}) \\
&\quad - \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x)dx \Big\|_{C^2([-T_0,T_0])} \\
&\lesssim \|P_{N_4} u_{\min(N,N^{(3)})}\|_{C^2([-T_0,T_0],L^4)} \|E_{N^{(3)}}\|_{C^2([-T_0,T_0],L^\infty)}^2 \\
&\qquad \times \sum_{j=1}^{3} \|\partial_x^2 P_N X_{N_j,N^{(3)}} - E_{N^{(3)}}(t)\partial_x^2 P_N P_{N_j} u_0\|_{C^2([-T_0,T_0],L^4)} \prod_{k\in\{1,2,3\}\setminus\{j\}} \|P_{N_k} u_0\|_{L^4} \\
&\lesssim (1 + \|u_0\|_{X_s}^{20})(N^{(3)})^{\frac{3}{2}-s} N^{\frac{3}{2}-s}(N^{(2)}N^{(3)})^{\frac{1}{2}-s} N_4^{\frac{1}{2}-s} \\
&\lesssim N^{\frac{3}{2}-s}(N^{(1)})^{\frac{1}{2}-s}(N^{(3)})^{2-2s} N_4^{\frac{1}{2}-s}.
\end{aligned}
$$

Therefore,

$$
|(\mathrm{I})| \lesssim \sum_{\substack{N_1,N_2,N_3 \gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N \gg N_4 \\ N^{(3)}\lesssim N}} N^{\frac{3}{2}-s}(N^{(1)})^{\frac{1}{2}-s}(N^{(3)})^{2-2s} N_4^{\frac{1}{2}-s}
$$

$$
\lesssim N^{4-4s}.
$$

$\square$

In view of this estimate, we are left with bounding the expression.

$$
\sum_{\substack{N_1,N_2,N_3 \gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N \gg N_4 \\ N^{(3)}\lesssim N}} \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x)dx.
$$

In principle, if the various functions in the expression above are general elements of the Besov space $B_{p,\infty}^{s-\frac{1}{2},+}$, the best estimate that one could prove is

$$
\sum_{\substack{N_1,N_2,N_3 \gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N \gg N_4 \\ N^{(3)}\lesssim N}} \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x)dx \lesssim N^{\frac{7}{2}-3s},
$$

which is not sufficient for our goals. However, exploiting the (random) structure of u_0, we can prove the following.

Lemma 4.14 (Random estimate for Q_N). *Fix $R > 0$, and let $T_0(R)$ be as in Proposition 4.8. For every $0 \le |t| \le T_0(R)$, and for every $k \in \{0,1,2\}$, we have that*

$$
\mathbb{E}\left[\left| \frac{d^k}{dt^k}\Big(\sum_{\substack{N_1,N_2,N_3 \gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N \gg N_4 \\ N^{(3)}\lesssim N}} \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x)dx \Big) \right| \mathbb{1}_{B_R}(u_0) \right]
$$

$$
\lesssim (1 + R^{20})N^{3-3s}.
$$

(4.29)

Proof. Note that, by definition, the function Y_{N_1,N_2,N_3} does not depend on time. Therefore, when we take derivatives, we obtain that

$$\frac{d^k}{dt^k} \Big(\int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x) \overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x) dx \Big)$$

$$= \int_{\mathbb{T}} f_1 f_2 f_3 f_4 Y_{N_1,N_2,N_3} dx$$

where $f_1, \ldots f_4$ are functions of $P_{\ll N^{(3)}} u_0$ which satisfy

$$\|f_j\|_{C^2([-T_0,T_0],X_s)} \lesssim 1 + R^5$$

as long as $\|P_{\ll N^{(3)}} u_0\|_{X_s} \lesssim R$, and have Fourier support in $\{n : |n| \ll 2N^{(3)}\}$. In particular, we have that

$$\int_{\mathbb{T}} f_1 f_2 f_3 f_4 Y_{N_1,N_2,N_3} dx = \int_{\mathbb{T}} f_1 f_2 f_3 f_4 P_{\ll 16N^{(3)}} Y_{N_1,N_2,N_3} dx,$$

and that Y_{N_1,N_2,N_3} is independent from $(f_1, \ldots, f_4)$, since Y is a function of $P_{N_1} u_0, P_{N_2} u_0, P_{N_3} u_0$, which are independent from $P_{\ll N^{(3)}} u_0$.

For $|k| \ll 16N^{(3)}$, we now estimate $\mathbb{E}|(Y_{N_1,N_2,N_3})\widehat{}(k)|^2$. By (4.28), we have

$$\mathbb{E}|(Y_{N_1,N_2,N_3})\widehat{}(k)|^2$$

$$= \mathbb{E}\Big| \sum_{n_1-n_2+n_3=k} (n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2 + n_3^2 \phi_N(n_3)^2) \prod_{j=1}^3 \phi_{N_j}(n_j) \mathfrak{c}_j(\widehat{u_0}(n_j)) \Big|^2$$

$$\lesssim \sum_{\substack{n_1-n_2+n_3=k \\ n_2 \neq n_1, n_3}} \frac{(n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2 + n_3^2 \phi_N(n_3)^2)^2 \prod_{j=1}^3 \phi_{N_j}(n_j)^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}}$$

$$\lesssim N^2 \min(N, N^{(3)})^2 (N_1 N_2 N_3)^{-2s} N^{(1)} N^{(3)}$$

$$\lesssim N^2 \min(N, N^{(3)})^2 (N^{(3)})^{1-2s} (N^{(1)})^{1-4s}.$$

Here we used that under the condition $|n_1 - n_2 + n_3| = |k| \ll N^{(3)}$, we have the bound

$$|n_1^2 \phi_N(n_1)^2 - n_2^2 \phi_N(n_2)^2 + n_3^2 \phi_N(n_3)^2| \lesssim N \min(N, N^{(3)}),$$

which follows from Lemma 4.3, where we put $n_4 = k$.[11]

Moreover, for $k \neq h$ we have that

$$\mathbb{E}\Big[(Y_{N_1,N_2,N_3})\widehat{}(k) \overline{(Y_{N_1,N_2,N_3})\widehat{}(h)} \Big] = 0.$$

Indeed, it is easy to check that $k \neq h$ implies that we cannot pair the frequencies $\widehat{u_0(n_j)}, \widehat{u_0(n_j')}$ so that $\{n_1, n_3\} = \{n_1', n_3'\}$, $n_2 = n_2'$ and $n_1 - n_2 + n_3 = k$, $n_1' - n_2' + n_3' = h$, and so the expectation must vanish.

---

[11]In principle, Lemma 4.3 does not hold when $n_4 < 0$, while it is certainly possible for us to have $k < 0$. However, one can check that in the the proof of Lemma 4.3, the lowest frequency is allowed to be negative without any modification in the argument.

Therefore, by Plancherel, recalling that Y_{N_1,N_2,N_3} is independent from $P_{\ll 2N^{(3)}} u_0$ and $(f_j)_{j=1,\dots,4}$ are functions of $P_{\ll N^{(3)}} u_0$, for a universal constant $C > 0$, we have

$$\mathbb{E}\Big[\Big|\Big(\int_{\mathbb{T}} f_1 f_2 f_3 f_4 Y_{N_1,N_2,N_3} dx\Big)\mathbb{1}_{B_R}(u_0)\Big|^2\Big]$$

$$\lesssim \mathbb{E}\Big[\Big|\Big(\int_{\mathbb{T}} f_1 f_2 f_3 f_4 Y_{N_1,N_2,N_3} dx\Big)\mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big|^2\Big]$$

$$\lesssim \mathbb{E}\Big[\Big|\sum_{|k|\ll 16N^{(3)}} \Big(\prod_{j=1}^4 f_j\Big)^{\widehat{}}(k)\overline{(Y_{N_1,N_2,N_3})^{\widehat{}}(k)}\Big|^2 \mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big]$$

$$= \sum_{|k|\ll 16N^{(3)}} \mathbb{E}\Big[\Big|\Big(\prod_{j=1}^4 f_j\Big)^{\widehat{}}(k)\Big|^2 \mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big]\mathbb{E}|(Y_{N_1,N_2,N_3})^{\widehat{}}(k)|^2$$

$$\lesssim \sum_{|k|\ll 16N^{(3)}} \mathbb{E}\Big[\Big|\Big(\prod_{j=1}^4 f_j\Big)^{\widehat{}}(k)\Big|^2 \mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big] N^2 \min(N, N^{(3)})^2 (N^{(3)})^{1-2s}(N^{(1)})^{1-4s}$$

$$= \mathbb{E}\Big[\Big\|\prod_{j=1}^4 f_j\Big\|_{L^2}^2 \mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big] N^2 \min(N, N^{(3)})^2 (N^{(3)})^{1-2s}(N^{(1)})^{1-4s}.$$

Therefore,

$$\mathbb{E}\Big[\Big|\frac{d^k}{dt^k}\Big(\sum_{\substack{N_1,N_2,N_3\gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N\gg N_4 \\ N^{(3)}\lesssim N}} \int_{\mathbb{T}} |E_{N^{(3)}}|^2 E_{N^{(3)}}(t,x)\overline{P_{N_4} u_{\min(N,N^{(3)})}(t,x)} Y_{N_1,N_2,N_3}(x)dx\Big)\Big|\mathbb{1}_{B_R}(u_0)\Big]$$

$$\lesssim \sum_{\substack{N_1,N_2,N_3\gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N\gg N_4 \\ N^{(3)}\lesssim N}} \mathbb{E}\Big[\Big\|\,|E_{N^{(3)}}|^2 E_{N^{(3)}}\overline{P_{N_4} u_{\min(N,N^{(3)})}}\Big\|_{C^2([-T_0(R),T_0(R),L^2)} \mathbb{1}_{B_{CR}}(P_{\ll N^{(3)}} u_0)\Big]$$

$$\times N \min(N, N^{(3)})(N^{(3)})^{\frac{1}{2}-s}(N^{(1)})^{\frac{1}{2}-2s}$$

$$\lesssim (1+R^{20}) \sum_{\substack{N_1,N_2,N_3\gg N_4 \\ N^{(1)}\sim N^{(2)}\gtrsim N\gg N_4 \\ N^{(3)}\lesssim N}} N_4^{\frac{1}{2}-s} N \min(N, N^{(3)})(N^{(3)})^{\frac{1}{2}-s}(N^{(1)})^{\frac{1}{2}-2s}$$

$$\lesssim (1+R^{20}) N^{3-3s}.$$

$\square$

4.5. **Singularity of the evolved measure.** We are now ready to prove Proposition 4.2 and Proposition 1.6.

Proof of Proposition 4.2. We first show that $g(x, y) > 0$ implies that $g(y, x) < 0$. Indeed, by (4.1), we have that

$$
\begin{aligned}
g(y, x) &= \liminf_{N \to \infty} \frac{\|P_N y\|_{\dot{H}^1}^2 - \|P_N x\|_{\dot{H}^1}^2}{(4s - 3)N^{4-4s}} \\
&\leq \limsup_{N \to \infty} \frac{\|P_N y\|_{\dot{H}^1}^2 - \|P_N x\|_{\dot{H}^1}^2}{(4s - 3)N^{4-4s}} \\
&= \limsup_{N \to \infty} -\left(\frac{\|P_N x\|_{\dot{H}^1}^2 - \|P_N y\|_{\dot{H}^1}^2}{(4s - 3)N^{4-4s}} \right) \\
&= -\liminf_{N \to \infty} \frac{\|P_N x\|_{\dot{H}^1}^2 - \|P_N y\|_{\dot{H}^1}^2}{(4s - 3)N^{4-4s}} \\
&= -g(x, y) < 0.
\end{aligned}
$$

We now move to showing the existence of $\tau(u_0) > 0$ for μ_s-a.e. $u_0 \in X_s$. We simply pick

$$
\tau(u_0) = \min(T_0(\|u_0\|_{X_s}), \delta(\|u_0\|_{X_s}, \|u_0\|_{L^2}^2)),
$$

where T_0 is as in Proposition 4.6, and $\delta \ll 1$ is to be determined later in the proof. For ease of notation, let

$$
h_N(t) = \frac{\|P_N(\Phi_t(u_0))\|_{\dot{H}^1}^2 - \|P_N u_0\|_{\dot{H}^1}^2}{N^{4-4s}}.
$$

By Proposition 2.4, we deduce that $h_N(t) \in C^3([-T_0, T_0])$. Therefore, by Taylor's theorem with integral remainder, we have that

$$
\begin{aligned}
h_N(t) &= h_N(0) + t h'_N(0) + \frac{t^2}{2} h''_N(0) + \int_0^t h'''_N(\sigma) \frac{\sigma^3}{6} d\sigma \\
&= t F_N(u_0) + \frac{t^2}{2} G_N(u_0) + N^{4s-4} \int_0^t \left(\frac{d^2}{d\sigma^2} Q_N(u(\sigma), u(\sigma), u(\sigma), u(\sigma)) \right) \frac{\sigma^3}{6} d\sigma,
\end{aligned} \tag{4.30}
$$

where F_N, G_N and Q_N are defined in (4.3), (4.4) and (4.5) respectively. By Lemma 4.13 and Lemma 4.14, we can write

$$
N^{4s-4} \left(\frac{d^2}{d\sigma^2} Q_N(u(\sigma), u(\sigma), u(\sigma), u(\sigma)) \right) = a_1(\sigma) + a_2(\sigma),
$$

with

$$
\|a_1(\sigma)\|_{L^\infty([-T_0, T_0])} \lesssim 1 + \|u_0\|_{X_s}^{20}, \quad \mathbb{E}|a_2(\sigma) \mathbb{1}_{\{u_0 \in B_R\}}| \lesssim N^{s-1}(1 + R^{20}).
$$

In particular, since N^{s-1} is summable, we deduce that

$$
\lim_{N \to \infty} \int_{-T_0(u_0)}^{T_0(u_0)} |a_2(\sigma)| \frac{\sigma^3}{6} d\sigma = 0
$$

for μ_s-a.e. $u_0 \in X_s$, and for every $|t| < \tau(u_0)$,

$$
\int_{-|t|}^{|t|} |a_1(\sigma)| \frac{\sigma^3}{6} d\sigma \lesssim t^3 (1 + \|u_0\|_{X_s})^{20} \lesssim t^2 \delta (1 + \|u_0\|_{X_s})^{20}.
$$

Therefore, we can pick $\delta \ll |4s - 3|(1 + \|u_0\|_{X_s})^{-20}\|u_0\|_{L^2}^2$, so we can guarantee that

$$
\int_{-|t|}^{|t|} |a_1(\sigma)| \frac{\sigma^3}{6} d\sigma \leq |4s - 3| I_s \|u_0\|_{L^2}^2 t^2,
$$

where I_s is defined in (4.8). Making this choice, we obtain

$$\limsup_{N\to\infty}\Big|\int_0^t h_N'''(\sigma)\frac{\sigma^3}{6}d\sigma\Big| \le |4s-3|I_s\|u_0\|_{L^2}^2 t^2.$$

Therefore, from (4.30), Proposition 4.4, and (4.9), for every $0 < |t| \le \tau(u_0)$ we have that

$$
\begin{aligned}
g(\Phi_t(u_0), u_0) &= \liminf_{N\to\infty}\frac{h_N(t)}{4s-3}\\
&= \lim_{N\to\infty}\frac{F_N(u_0)}{4s-3}t + \lim_{N\to\infty}\frac{G_N(u_0)}{4s-3}\cdot\frac{t^2}{2} + \frac{1}{4s-3}\liminf_{N\to\infty}\int_0^t h_N'''(\sigma)\frac{\sigma^3}{6}d\sigma\\
&\ge 8I_s\|u_0\|_{L^2}^2 t^2 - I_s\|u_0\|_{L^2}^2 t^2\\
&\ge I_s\|u_0\|_{L^2}^2 t^2 > 0.
\end{aligned}
$$

$\square$

Proof of Proposition 1.6. This is an immediate corollary of Proposition 4.2 and Proposition 4.1. $\square$

We now complete the paper by showing that Proposition 1.6 implies Theorem 1.1, (ii).

Proof of Theorem 1.1, (ii). Fix $\delta > 0$, and define

$$\mathscr{N}_\delta := \Big\{t \in \mathbb{R} : (\Phi_t)_\#\mu_s \not\ll \mathbb{1}_{\mathrm{WP}(\delta)\cap\mathrm{WP}(-\delta)}\mu_s\Big\}.$$

We first show that $\mathscr{N}_\delta$ is countable for every $\delta > 0$. If $\mu_s(\mathrm{WP}(\delta)\cap\mathrm{WP}(-\delta)) = 0$, then by definition $\mathscr{N}_\delta = \emptyset$, so without loss of generality, we can assume that $\mu_s(\mathrm{WP}(\delta)\cap\mathrm{WP}(-\delta)) > 0$. For $t \in \mathscr{N}_\delta$, pick a representative for the Radon-Nykodim derivative $\frac{d(\Phi_t)_\#\mu_s}{d\mu_s}$ of (the absolutely continuous part of) $(\Phi_t)_\#\mu_s$ with respect to μ_s, such that the singular part of $(\Phi_t)_\#\mu_s$ with respect to μ_s is concentrated on the set

$$\Big\{\frac{d(\Phi_t)_\#\mu_s}{d\mu_s} = \infty\Big\}.$$

Define the set

$$E_t := \Phi_{-t}\Big(\Big\{0 < \frac{d(\Phi_t)_\#\mu_s}{d\mu_s}\mathbb{1}_{\mathrm{WP}(\delta)\cap\mathrm{WP}(-\delta)} < \infty\Big\}\Big).$$

Now fix an interval $I \subseteq \mathbb{R}$ with $|I| \le \delta$, and define

$$1 \ge a_{\delta,I} := \sup_{\substack{\mathscr{T}\subseteq\mathscr{N}_\delta\cap I,\\ \mathscr{T}\text{ countable}}}\mu_s\Big(\bigcup_{t\in\mathscr{T}} E_t\Big).$$

It is fairly easy to check that the sup in the definition of $a_{\delta,I}$ is a maximum, i.e. there exists a countable set $\mathscr{T}_{\delta,I} \subseteq \mathscr{N}_\delta \cap I$ such that

$$a_{\delta,I} = \mu_s\Big(\bigcup_{t\in\mathscr{T}_{\delta,I}} E_t\Big).$$

Call $E_I := \bigcup_{t \in \mathscr{T}_{\delta,I}} E_t$. From the definition of $a_{\delta,I}$, we deduce that, for every $t \in (\mathscr{N}_\delta \cap I) \setminus \mathscr{T}_{\delta,I}$,

$$
\begin{aligned}
0 &= \mu_s(E_I^c \cap E_t) \\
&= \mu_s\Big(E_I^c \cap \Phi_{-t}\big(\{0 < \frac{d(\Phi_t)_\# \mu_s}{d\mu_s} \mathbb{1}_{\mathrm{WP}(\delta) \cap \mathrm{WP}(-\delta)} < \infty\}\big)\Big) \\
&= (\Phi_t)_\#(\mathbb{1}_{E_I^c} \mu_s)\Big(\{0 < \frac{d(\Phi_t)_\# \mu_s}{d\mu_s} \mathbb{1}_{\mathrm{WP}(\delta) \cap \mathrm{WP}(-\delta)} < \infty\}\Big),
\end{aligned}
$$

and so for every $t \in (\mathscr{N}_\delta \cap I) \setminus \mathscr{T}_{\delta,I}$, $(\Phi_t)_\#(\mathbb{1}_{E_I^c} \mu_s) \perp \mathbb{1}_{\mathrm{WP}(\delta) \cap \mathrm{WP}(-\delta)} \mu_s$. From the definition $\mathscr{N}_\delta$, we deduce that

$$
\begin{aligned}
(\mathscr{N}_\delta \cap I) \setminus \mathscr{T}_{\delta,I} &\subseteq \{\tau \in I : (\Phi_\tau)_\#(\mathbb{1}_{E_I} \mu_s) \not\perp \mu_s\} \\
&= \bigcup_{t \in \mathscr{T}_{\delta,I}} \{\tau \in I : (\Phi_\tau)_\#(\mathbb{1}_{E_t} \mu_s) \not\perp \mu_s\}.
\end{aligned}
\tag{4.31}
$$

Now let $t \in \mathscr{T}_{\delta,I}$, and $\tau \in I$. By definition of E_t, we have that

$$
\Phi_t(E_t) \subseteq \mathrm{WP}(\delta) \cap \mathrm{WP}(-\delta) \subseteq \mathrm{WP}(\tau - t),
$$

and

$$
(\Phi_t)_\#(\mathbb{1}_{E_t} \mu_s) \ll \mu_s.
$$

Therefore, by Proposition 1.6, we deduce that

$$
\begin{aligned}
\{\tau \in I : (\Phi_\tau)_\#(\mathbb{1}_{E_t} \mu_s) \not\perp \mu_s\} &= \{\tau \in I : (\Phi_{\tau-t})_\#((\Phi_t)_\#(\mathbb{1}_{E_t} \mu_s)) \not\perp \mu_s\} \\
&= \{\tau \in I : (\Phi_{\tau-t})_\#(\mathbb{1}_{\mathrm{WP}(\tau-t)}(\Phi_t)_\#(\mathbb{1}_{E_t} \mu_s)) \not\perp \mu_s\} \\
&\subseteq \{\tau \in I : (\Phi_{\tau-t})_\#(\mathbb{1}_{\mathrm{WP}(\tau-t)} \mu_s) \not\perp \mu_s\}
\end{aligned}
$$

is countable. By (4.31), recalling that the set $\mathscr{T}_{\delta,I}$ is countable, we deduce that $\mathscr{N}_\delta \cap I$ is countable as well. Since I was an arbitrary interval with $|I| \le \delta$, taking unions over a countable family of I, we obtain that $\mathscr{N}_\delta$ is countable.

Now, define

$$
\mathscr{N} := \Big\{t \in \mathbb{R} : (\Phi_t)_\# \mu_s \not\perp \mu_s\Big\}.
$$

By Proposition 4.6, we have that $\mu_s = \lim_{n \to \infty} \mathbb{1}_{\mathrm{WP}(\frac{1}{n}) \cap \mathrm{WP}(-\frac{1}{n})} \mu_s$. Therefore,

$$
\mathscr{N} = \bigcup_{n \in \mathbb{N}} \mathscr{N}_{\frac{1}{n}}
$$

is a countable union of countable sets, so it must be countable as well. $\qquad\square$

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