The symmetric (2+1)-dimensional Lotka–Volterra equation with self-consistent sources

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Abstract

The symmetric $(2+1)$-dimensional Lotka–Volterra equation with self-consistent sources is constructed and solved by employing the source generation procedure, whose solutions are expressed in terms of pfaffians. As special cases of the pfaffian solutions, different types of explicit solutions are obtained, including dromions, soliton solutions and breather solutions.

Keyword: The symmetric $(2+1)$-dimensional Lotka–Volterra equation with self-consistent sources; pfaffian solutions; dromion solutions; soliton solutions; breather solutions

1 Introduction

In recent years, there has been extensive research on (2+1)-dimensional integrable systems [1–3]. In particular, one crucial aspect is to explore new (2+1)-dimensional soliton equations. In various fields such as fluid dynamics, nonlinear optics, particle physics, general relativity, differential and algebraic geometry, and topology, several well-known examples of multi-dimensional integrable systems have been identified. Currently, there are a couple of effective methods for discovering (2+1)-dimensional integrable systems. One of these methods is to find integrable extensions of known (2+1)-dimensional integrable systems. For example, two coupled KP equations were discovered in two different research directions for the well-known KP equation. One is the so-called KP equation with self-consistent sources. and the other is generated through what is now called "Pfaffianization" [4,5]. Following the leading work by Mel'nikov [6–12], much attention has been paid to soliton equations with self-consistent sources (SESCSs). A number of methods have been developed to study SESCSs, such as inverse scattering method, Darboux transformation, Hirota's bilinear method, dressing method and squared eigenfunction symmetry method [13–25]. Lately,

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Hu and Wang suggested the source generation procedure which provides an efficient and unified way to construct and solve SESCSs [14, 26–29]. The source generation procedure is in essence variation of constants and has been successfully applied to different types of soliton equations.

In literature, some work has been done on discrete soliton equations with sources. In [30], integrability of the differential-difference KdV equation with a source was investigated. In [31], the extended Toda lattice hierarchy was constructed by squared symmetric eigenfunctions, for which the non-autonomous Darboux transformation was derived. Furthermore, the two-dimensional Toda lattice equation, discrete KP equation and the semi-discrete BKP equation have been extended to their corresponding equations with self-consistent sources by source generation procedure, along with which determinant solutions and pfaffian solutions are derived, respectively [26–28].

It is well known that Lotka–Volterra (LV) equation

$$u_t(n) + e^{u(n)+u(n+1)} - e^{u(n)+u(n-1)} = 0 \tag{1.1}$$

is one of the most important lattices. In [32–35], several (2+1)-dimensional generalizations of equation (1.1) are presented. In [36], the symmetric (2+1)-dimensional Lotka–Volterra (2DLV) equation is proposed together with its bilinear Bäcklund transformation, Lax pair and Pfaffian solutions. Moreover, explicit solutions including dromions and soliton solutions are derived from the pfaffian solutions for the symmetric 2DLV equation. As is explained in [36], the property of strong two-dimensionality seems to be closely related to the existence of dromions which has been proved to be true for the DS equation [37–39], the NVN equation [40, 41] and the symmetric 2DLV equation. In this paper, we shall apply the source generation procedure to construct and solve the symmetric 2DLV equation with self-consistent sources (2DLV ESCS). It will be very interesting to explore the corresponding explicit solutions such as dromions as well.

This paper is organized as follows. In Section 2, by using the source generation procedure, the symmetric 2DLV ESCS as well as its DKP-type pfaffian solutions are presented. In Section 3, explicit solutions of the symmetric 2DLV ESCS including dromions, soliton solutions and breather solutions, are derived from the pfaffian solutions. Section 4 is devoted to conclusions and discussions.

2 The symmetric 2DLV ESCS and DKP-type pfaffian solutions

The symmetric 2DLV equation reads as [36]

$$\begin{aligned} 2u_t + e^{u+\Delta_m^2\phi_n} - e^{-u+\Delta_n^2\phi} + e^{u+\Delta_n^2\phi_m} - e^{-u+\Delta_m^2\phi} + e^{-u+\Delta_n^2\phi_{m\bar{n}}} - e^{u+\Delta_m^2\phi_{\bar{m}}} \\ + e^{-u+\Delta_m^2\phi_{\bar{m}n}} - e^{u+\Delta_n^2\phi_{\bar{n}}} = 0, \quad u = \Delta_m\Delta_n\phi \end{aligned} \tag{2.1}$$

where $u = u(m, n, t)$, $\phi = \phi(m, n, t)$, the subscript t denotes partial derivative as usual and the subscripts involving the discrete variables m or n denote shifts:

$$u_m \equiv u(m+1, n, t), \quad u_{\bar{n}} \equiv u(m, n-1, t), \quad u_{\bar{m}n} \equiv u(m-1, n+1, t).$$

The Δ_m and Δ_n are standard difference operators defined by

$$\Delta_m u = u_m - u, \quad \Delta_n u = u_n - u.$$

In the case that $m = n$, the symmetric 2DLV equation (2.1) reduces to (1.1). In this sense, (2.1) is regarded as a strong generalization of (1.1).

Through the dependent variable transformation

$$u = \ln \frac{f_{mn} f}{f_m f_n}, \tag{2.2}$$

equation (2.1) is transformed into the multilinear form

$$\begin{aligned} &\sinh\left(\frac{1}{2}D_n\right)\left[\left(D_t e^{\frac{1}{2}D_m} - e^{D_n - \frac{1}{2}D_m} + e^{D_n + \frac{1}{2}D_m}\right) f\cdot f\right]\cdot\left(e^{\frac{1}{2}D_m} f\cdot f\right) \\ &\quad + \sinh\left(\frac{1}{2}D_m\right)\left[\left(D_t e^{\frac{1}{2}D_n} - e^{D_m - \frac{1}{2}D_n} + e^{D_m + \frac{1}{2}D_n}\right) f\cdot f\right]\cdot\left(e^{\frac{1}{2}D_n} f\cdot f\right) = 0, \end{aligned} \tag{2.3}$$

where the bilinear operators D_t^k and $\exp(D_n)$ are defined by [5]

$$D_t a\cdot b = (\partial_t - \partial_{t'}) a(t) b(t')|_{t'=t}, \quad \exp(\delta D_n) a(n)\cdot b(n) = a(n+\delta) b(n-\delta).$$

The multilibear equation (2.3) has the DKP-type Pfaffian solution

$$f = (1, 2, \cdots, 2N)$$

where the Pfaffian elements (i, j) are determined by the relations

$$(i, j)_n = (i, j) + \theta_{i,n}\theta_j - \theta_i\theta_{j,n}, \tag{2.4}$$

$$(i, j)_m = (i, j) - \theta_{i,m}\theta_j + \theta_i\theta_{j,m}, \tag{2.5}$$

$$(i, j)_t = \frac{1}{2}(\theta_{i,\bar{n}}\theta_{j,n} - \theta_{i,n}\theta_{j,\bar{n}} + \theta_{i,m}\theta_{j,\bar{m}} - \theta_{i,\bar{m}}\theta_{j,m}) \tag{2.6}$$

and θ_i $(i = 1, 2, \cdots, 2N)$ satisfy the linear dispersion relations

$$\theta_{i,mn} + \theta_i = \theta_{i,m} + \theta_{i,n}, \tag{2.7}$$

$$\theta_{i,t} = \frac{1}{2}(\theta_{i,\bar{n}} - \theta_{i,n} + \theta_{i,\bar{m}} - \theta_{i,m}). \tag{2.8}$$

For simplicity, the index d_i^j is introduced and defined by

$$(d_i^j, k) = \theta_k(m+i, n+j), \quad (d_i^j, d_k^l) = 0,$$

so that the (2.4)-(2.6) can be written as

$$
\begin{aligned}
(i,j)_n &= (i,j) + (d_0^0, d_0^1, i, j),\\
(i,j)_m &= (i,j) + (d_1^0, d_0^0, i, j),\\
(i,j)_t &= \frac{1}{2}((d_0^1, d_0^{-1}, i, j) + (d_{-1}^0, d_1^0, i, j)).
\end{aligned}
$$

With the known pfaffian solution f of the multilinear equation (2.1), we now construct the symmetric 2DLV ESCS. Following the source generation procedure [26], we change f into the following form

$$\tau = (1, 2, \cdots, 2N), \tag{2.9}$$

whose Pfaffian elements are defined by

$$(i,j)_n = (i,j) + (d_0^0, d_0^1, i, j), \tag{2.10}$$

$$(i,j)_m = (i,j) - (d_1^0, d_0^0, i, j), \tag{2.11}$$

$$(i,j)_t = \frac{1}{2}(\dot{C}_{i,j}(t) + (d_0^1, d_0^{-1}, i, j) + (d_{-1}^0, d_1^0, i, j)), \tag{2.12}$$

where $\dot{C}_{i,j}(t)$ is the t-derivative of $C_{i,j}(t)$ satisfying

$$C_{i,j}(t) = \begin{cases} C_i(t), & i<j \quad and \quad j = 2N+1-i, \quad 1 \le i \le K \le N \\ c_{i,j}, & i<j \quad and \quad j \ne 2N+1-i. \end{cases} \tag{2.13}$$

It is obvious that τ no longer satisfies the symmetric 2DLV equation (2.3) since $C_{i,j}(t)$ becomes dependent of t. In fact, we can prove that τ given by (2.9) satisfies the new equation

$$
\begin{aligned}
&\sinh\left(\frac{1}{2}D_n\right)\left[\left(D_t e^{\frac{1}{2}D_m} - e^{D_n - \frac{1}{2}D_m} + e^{D_n + \frac{1}{2}D_m}\right)\tau\cdot\tau - \sum_{i=1}^{K}\sinh\left(\frac{1}{2}D_m\right) h_i \cdot g_i\right]\\
&\quad \cdot\left(e^{\frac{1}{2}D_m}\tau\cdot\tau\right) + \sinh\left(\frac{1}{2}D_m\right)\left[\left(D_t e^{\frac{1}{2}D_n} - e^{D_m - \frac{1}{2}D_n} + e^{D_m + \frac{1}{2}D_n}\right)\tau\cdot\tau\right.\\
&\quad \left. - \sum_{i=1}^{K}\sinh\left(\frac{1}{2}D_n\right) g_i \cdot h_i\right]\cdot\left(e^{\frac{1}{2}D_n}\tau\cdot\tau\right) = 0
\end{aligned} \tag{2.14}
$$

with

$$g_i = \sqrt{\dot{C}_i(t)}(d_0^0, 1, \cdots, \hat{i}, \cdots, 2N), \quad i = 1, 2, \cdots, K, \tag{2.15}$$

$$h_i = \sqrt{\dot{C}_i(t)}(d_0^0, 1, \cdots, \widehat{2N+1-i}, \cdots, 2N), \quad i = 1, 2, \cdots, K. \tag{2.16}$$

Meanwhile, τ, g_i and h_i satisfy the following two equations

$$e^{\frac{1}{2}D_m + \frac{1}{2}D_n} g_i \cdot \tau = \left(e^{\frac{1}{2}D_m - \frac{1}{2}D_n} - e^{-\frac{1}{2}D_m - \frac{1}{2}D_n} + e^{-\frac{1}{2}D_m + \frac{1}{2}D_n}\right) g_i \cdot \tau, \tag{2.17}$$

$$e^{\frac{1}{2}D_m + \frac{1}{2}D_n} h_i \cdot \tau = \left(e^{\frac{1}{2}D_m - \frac{1}{2}D_n} - e^{-\frac{1}{2}D_m - \frac{1}{2}D_n} + e^{-\frac{1}{2}D_m + \frac{1}{2}D_n}\right) h_i \cdot \tau. \tag{2.18}$$

Actually, by detailed calculations we have

$$(i,j)_{\bar{n}} = (i,j) + (d_0^0, d_0^{-1}, i, j), \qquad (i,j)_{\bar{m}} = (i,j) + (d_{-1}^0, d_0^0, i, j), \tag{2.19}$$

$$(i,j)_{mn} = (i,j) + (d_1^0, d_0^1, i, j), \qquad (i,j)_{m\bar{n}} = (i,j) + (d_1^0, d_0^{-1}, i, j), \tag{2.20}$$

$$(i,j)_{\bar{m}n} = (i,j) + (d_{-1}^0, d_0^1, i, j). \tag{2.21}$$

These results are then used to give expressions for the derivatives and differences of τ, g_i and h_i. For simplicity, denote $(1,2,\cdots,2N) = (\bullet)$, $(1,\cdots,\hat{i},\cdots,\widehat{2N+1-i},\cdots,2N) = (\circ)$, $(1,\cdots,\hat{i},\cdots,2N) = (\star)$ and $(1,\cdots,\widehat{2N+1-i},\cdots,2N) = (*)$ for short and extend these notations to write $(d_0^1, d_1^0, 1, 2, \cdots, 2N) = (d_0^1, d_1^0, \bullet)$ and so on. Using equations (2.19)-(2.21), we obtain

$$\begin{aligned}
&\tau_n = (\bullet) + (d_0^0, d_0^1, \bullet), \quad \tau_m = (\bullet) + (d_1^0, d_0^0, \bullet), \quad \tau_{\bar{n}} = (\bullet) + (d_0^0, d_0^{-1}, \bullet),\\
&\tau_{\bar{m}} = (\bullet) + (d_{-1}^0, d_0^0, \bullet), \quad \tau_{mn} = (\bullet) + (d_1^0, d_0^1, \bullet), \quad \tau_{m\bar{n}} = (\bullet) + (d_1^0, d_0^{-1}, \bullet),\\
&\tau_{\bar{m}n} = (\bullet) + (d_{-1}^0, d_0^1, \bullet), \quad \tau_t = \frac{1}{2}((d_0^1, d_0^{-1}, \bullet) + (d_{-1}^0, d_1^0, \bullet)) + \frac{1}{2}\sum_{i=1}^{K} \dot{C}_i(t)(\circ),\\
&\tau_{m,t} = \frac{1}{2}((d_0^1, d_0^{-1}, \bullet) + (d_0^{-1}, d_0^0, \bullet) - (d_0^1, d_0^0, \bullet) - (d_2^0, d_0^0, \bullet) + (d_1^0, d_0^{-1}, \bullet)\\
&\qquad - (d_1^0, d_0^1, \bullet) + (d_0^1, d_0^{-1}, d_1^0, d_0^0, \bullet)) + \frac{1}{2}\sum_{i=1}^{K} \dot{C}_i(t)((\circ) + (d_1^0, d_0^0, \circ)),\\
&\tau_{n,t} = \frac{1}{2}((d_{-1}^0, d_1^0, \bullet) + (d_{-1}^0, d_0^1, \bullet) - (d_{-1}^0, d_0^1, \bullet) + (d_0^0, d_{-1}^0, \bullet) - (d_0^0, d_1^0, \bullet)\\
&\qquad - (d_0^0, d_0^2, \bullet) + (d_{-1}^0, d_1^0, d_0^0, d_0^1, \bullet)) + \frac{1}{2}\sum_{i=1}^{K} \dot{C}_i(t)((\circ) + (d_0^0, d_0^1, \circ)).\\
&g_{i,n} = (d_0^1, \star), \quad g_{i,m} = (d_1^0, \star), \quad g_{i,mn} = (d_1^0, \star) + (d_0^1, \star) - (d_0^0, \star) + (d_0^1, d_1^0, d_0^0, \star),\\
&h_{i,n} = (d_0^1, *), \quad h_{i,m} = (d_1^0, *), \quad h_{i,mn} = (d_1^0, *) + (d_0^1, *) - (d_0^0, *) + (d_0^1, d_1^0, d_0^0, *).
\end{aligned}$$

On one hand, by direct substitution, equation (2.14) turns into the combination of the following two Pfaffian identities

$$\begin{aligned}
(d_1^0, d_0^0, \circ)(\bullet) - (d_1^0, d_0^0, 1, \bullet)(\circ) = (d_1^0, *)(d_0^0, \star) - (d_1^0, \star)(d_0^0, *),\\
(d_0^1, d_0^0, \circ)(\bullet) - (d_0^1, d_0^0, 1, \bullet)(\circ) = (d_0^1, *)(d_0^0, \star) - (d_0^1, \star)(d_0^0, *).
\end{aligned}$$

On the other hand, substituting these results into (2.17) and (2.18) will lead to

$$\begin{aligned}
(d_0^1, d_1^0, d_0^0, \star)(\bullet) =& (d_0^1, \star)(d_1^0, d_0^0, \bullet) - (d_1^0, \star)(d_0^1, d_0^0, \bullet) + (d_0^0, \star)(d_0^1, d_1^0, \bullet),\\
(d_0^1, d_1^0, d_0^0, *)(\bullet) =& (d_0^1, *)(d_1^0, d_0^0, \bullet) - (d_1^0, *)(d_0^1, d_0^0, \bullet) + (d_0^0, *)(d_0^1, d_1^0, \bullet),
\end{aligned}$$

respectively, which are nothing but Pfaffian identities.

To sum up, equations (2.14), (2.17) and (2.18) constitute a coupled system with K pairs of self-consistent sources which can be viewed as the symmetric 2DLV ESCS. At the same time, τ, g_i and h_i given by (2.9), (2.15) and (2.16) provide the associated Pfaffian solutions.

With the help of the dependent variable transformations

$$u = \ln \frac{\tau_{mn}\tau}{\tau_m \tau_n}, \quad q_i = \frac{g_i}{\tau}, \quad r_i = \frac{h_i}{\tau},$$

equations (2.14)-(2.18) are transformed into the nonlinear symmetric 2DLV ESCS

$$
\begin{aligned}
&2u_t+e^{u+\Delta_m^2\phi_n}-e^{-u+\Delta_n^2\phi}+e^{u+\Delta_n^2\phi_m}-e^{-u+\Delta_m^2\phi}+e^{-u+\Delta_n^2\phi_{m\bar{n}}}-e^{u+\Delta_m^2\phi_{\bar{m}}}+e^{-u+\Delta_m^2\phi_{\bar{m}n}}\\
&\qquad -e^{u+\Delta_n^2\phi_{\bar{n}}}=\frac{1}{4}\sum_{i=1}^{K}[(q_{i,mn}-q_i)(r_{i,m}-r_{i,n})+(q_{i,m}-q_{i,n})(r_i-r_{i,mn})], \qquad (2.22)\\
&q_{i,mn}e^u-q_{i,m}+q_ie^u-q_{i,n}=0, \qquad i=1,2,\cdots,K, \qquad (2.23)\\
&r_{i,mn}e^u-r_{i,m}+r_ie^u-r_{i,n}=0, \qquad i=1,2,\cdots,K. \qquad (2.24)
\end{aligned}
$$

3 Explicit solutions of the symmetric 2DLV ESCS

In the previous section, we have constructed the symmetric 2DLV ESCS and obtained its Pfaffian solutions. In this section, we shall follow the the method in [42, 43] to derive $(M, N-M)$-dromions, N soliton solutions and therefore multi-breather solutions for the symmetric 2DLV ESCS.

Notice that (2.7) may be rewritten as

$$\Delta_m\Delta_n\theta_i=0,$$

which implies that each θ_i can be decomposed as

$$\theta_i(m,n,t)=\phi_i(n,t)+\psi_i(m,t). \qquad (3.1)$$

Substituting (3.1) into (2.8), we have

$$
\begin{aligned}
2\phi_{i,t}&=\phi_{i,\bar{n}}-\phi_{i,n},\\
2\psi_{i,t}&=\psi_{i,\bar{m}}-\psi_{i,m}.
\end{aligned} \qquad (3.2)
$$

Based on the above calculations, the Pfaffian element (i,j) in f determined by (2.10), (2.11) and (2.12) can be established as

$$(i,j)=C_{i,j}(t)+\phi_i\psi_j-\psi_i\phi_j+\int(\phi_{i,\bar{n}}\phi_{j,n}-\phi_{i,n}\phi_{j,\bar{n}}+\psi_{i,m}\psi_{j,\bar{m}}-\psi_{i,\bar{m}}\psi_{j,m})dt. \qquad (3.3)$$

Following the method proposed in [42], we choose the following appropriate functions for ϕ_i and ψ_i for the Pfaffian element (i,j) in $f=(1,2,\cdots,2N)$. We take

$$
\begin{aligned}
&\phi_i=P_ie^{\eta_i}, && 1\le i\le 2M,\\
&\phi_i=0, && 2M+1\le i\le 2N,\\
&\psi_i=0, && 1\le i\le 2M,\\
&\psi_i=Q_{2N+1-i}e^{\xi_{2N+1-i}}, && 2M+1\le i\le 2N
\end{aligned}
$$

where

$$\eta_i=\frac{-2p_it}{1-p_i^2}, \quad \xi_i=\frac{-2q_it}{1-q_i^2}, \quad P_i=\alpha_i\left(\frac{1-p_i}{1+p_i}\right)^{-n}, \quad Q_i=\beta_i\left(\frac{1-q_i}{1+q_i}\right)^{-m}$$

and α_i, β_i, p_i and q_i are constants. With these assumptions, explicit solutions such as dromion solutions, soliton solutions and breather solutions are able to be derived.

3.1 Dromion solutions

By taking $0 < M < N$, we have from (3.3) that

$$
\begin{aligned}
&(i,j) = C_{i,j}(t) + \frac{p_i - p_j}{p_i + p_j} P_i P_j e^{\eta_i+\eta_j}, && 1 \le i < j \le 2M,\\
&(i, 2N+1-j) = C_{i,2N+1-j}(t) + P_i Q_j e^{\eta_i+\xi_j}, && 1 \le i \le 2M, 1 \le j \le 2N-2M,\\
&(2N+1-j, 2N+1-i) = C_{2N+1-j,2N+1-i}(t) + \frac{q_i - q_j}{q_i + q_j} Q_i Q_j e^{\xi_i+\xi_j}, && 1 \le i < j \le 2N-2M.
\end{aligned}
$$

In this case, τ, g_i and h_i give the $M \times (N-M)$-dromion solutions.

Consider the simplest case $M = 1$, $N = 2$ and $K = 1$. According to the definition of $C_{i,j}(t)$ in (2.13), we have

$$
\begin{aligned}
\tau =& c_{1,2}c_{3,4} - c_{1,3}c_{2,4} + C_1(t)C_2(t) - c_{1,3}P_2Q_1e^{\eta_2+\xi_1} - c_{2,4}P_1Q_2e^{\eta_1+\xi_2}\\
&+ C_2(t)P_1Q_1e^{\eta_1+\xi_1} + C_1(t)P_2Q_2e^{\eta_2+\xi_2} + c_{3,4}\frac{p_1-p_2}{p_1+p_2}P_1P_2e^{\eta_1+\eta_2}\\
&+ c_{1,2}\frac{q_1-q_2}{q_2+q_1}Q_1Q_2e^{\xi_1+\xi_2} + \frac{p_1-p_2}{p_1+p_2}\frac{q_1-q_2}{q_1+q_2}P_1P_2Q_1Q_2e^{\eta_1+\eta_2+\xi_1+\xi_2},\\
g_1 =& \sqrt{\dot{C}_1(t)}(C_2(t)Q_1e^{\xi_1} - c_{2,4}Q_2e^{\xi_2} + (c_{3,4} + \frac{q_1-q_2}{q_1+q_2}Q_1Q_2e^{\xi_1+\xi_2})P_2e^{\eta_2}),\\
h_1 =& \sqrt{\dot{C}_1(t)}(C_2(t)P_1e^{\eta_1} - c_{1,3}P_2e^{\eta_2} + (c_{1,2} + \frac{p_1-p_2}{p_1+p_2}P_1P_2e^{\eta_1+\eta_2})Q_2e^{\xi_2}).
\end{aligned}
\tag{3.4}
$$

Furthermore, by setting $p_2 = q_1 = 0$, we have

$$
\begin{aligned}
\tau =& c_{1,2}c_{3,4} - c_{1,3}c_{2,4} - c_{1,3}\alpha_2\beta_1 + C_1(t)C_2(t) + (c_{3,4}\alpha_2 + \beta_1C_2(t))P_1e^{\eta_1}\\
&+ (\alpha_2C_1(t) - c_{1,2}\beta_1)Q_2e^{\xi_2} - (\alpha_2\beta_1 + c_{2,4})P_1Q_2e^{\eta_1+\xi_2},\\
g_1 =& \sqrt{\dot{C}_1(t)}(\alpha_2c_{3,4} + \beta_1C_2(t) - (\alpha_2\beta_1 + c_{2,4})Q_2e^{\xi_2}),\\
h_1 =& \sqrt{\dot{C}_1(t)}(-c_{1,3}\alpha_2 + c_{1,2}Q_2e^{\xi_2} + C_2(t)P_1e^{\eta_1} + \alpha_2P_1Q_2e^{\eta_1+\xi_2}),
\end{aligned}
$$

which gives the $(1,1)$-dromion solution of the symmetric 2DLV ESCS (2.22) (see Fig.1).

3.2 Soliton solutions

By taking $2M = N$ and $C_{i,j}(t) = \delta_{i,2N+1-j}C_i(t)$, we have

$$
\begin{aligned}
&(i,j) = \frac{p_i - p_j}{p_i + p_j} P_i P_j e^{\eta_i+\eta_j}, && 1 \le i < j \le N,\\
&(i, 2N+1-j) = \delta_{i,j}C_i(t) + P_i Q_j e^{\eta_i+\xi_j}, && 1 \le i, j \le N,\\
&(2N+1-j, 2N+1-i) = \frac{q_i - q_j}{q_i + q_j} Q_i Q_j e^{\xi_i+\xi_j}, && 1 \le i < j \le N.
\end{aligned}
$$

In this case, τ, g_i and h_i give the N-soliton solution.

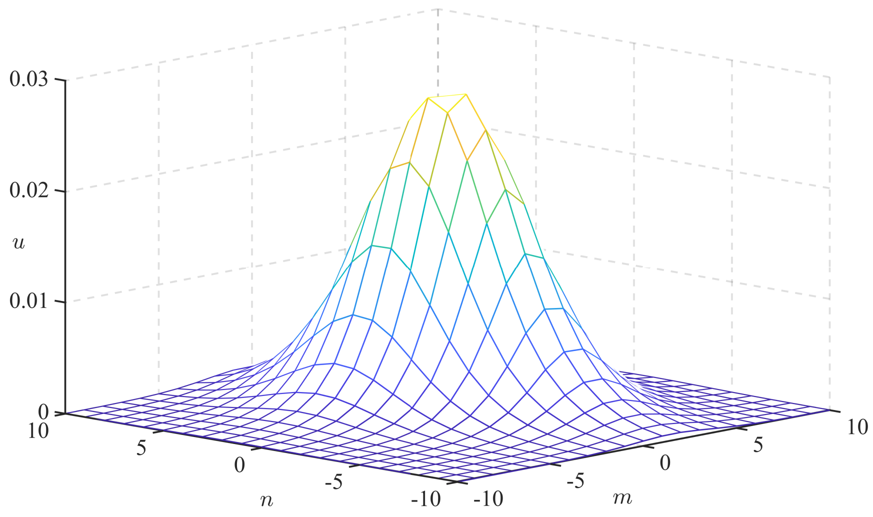

Figure 1: $(1,1)$-dromion solution of (2.22) with $t=1,\ C_1(t)=t^2,\ C_2(t)=t,\ \alpha_1=\alpha_2=\beta_1=\beta_2=1,\ p_1=-\frac{1}{4},\ q_2=\frac{1}{3},\ c_{1,2}=-1,$
$c_{1,3}=1,\ c_{2,4}=-2,\ c_{2,4}=-2.$

Consider the case $N=2$ and $K=1$, we can obtain the 2-soliton solution (see Fig. 2)

$$\begin{aligned}
\tau &= C_1(t)C_2(t)+C_1(t)P_2Q_2e^{\eta_2+\xi_2}+C_2(t)P_1Q_1e^{\eta_1+\xi_1}\\
&\quad+\frac{p_1-p_2}{p_1+p_2}\frac{q_1-q_2}{q_1+q_2}P_1P_2Q_1Q_2e^{\eta_1+\eta_2+\xi_1+\xi_2},\\
g_1 &= \sqrt{\dot{C}_1(t)}(C_2(t)Q_1e^{\xi_1}+\frac{q_1-q_2}{q_1+q_2}P_2Q_1Q_2e^{\eta_2+\xi_1+\xi_2}),\\
h_1 &= \sqrt{\dot{C}_1(t)}(C_2(t)P_1e^{\eta_1}+\frac{p_1-p_2}{p_1+p_2}P_1P_2Q_2e^{\eta_1+\eta_2+\xi_2}).
\end{aligned} \tag{3.5}$$

Note that if we further set $p_2=q_2=0$, we have

$$\begin{aligned}
\tau &= \alpha_2\beta_2C_1(t)+C_1(t)C_2(t)+(\alpha_2\beta_2+C_2(t))P_1Q_1e^{\eta_1+\xi_1},\\
g_1 &= \sqrt{\dot{C}_1(t)}(C_2(t)+\alpha_2\beta_2)Q_1e^{\xi_1},\\
h_1 &= \sqrt{\dot{C}_1(t)}(C_2(t)+\alpha_2\beta_2)P_1e^{\eta_1},
\end{aligned}$$

which is the one-soliton solution.

3.3 Breather solutions

In the preceding subsections, we have obtained soliton solutions for the symmetric 2DLV ESCS. In what follows, we are going to derive breather solutions from soliton solutions.

For the sake of convenience, we set $\alpha_i=\beta_i=1$. Consider the case $N=2$ and $K=1$. Let * denote complex conjugate. By taking $p_1=p_2^*=a+bi$, $q_1=q_2^*=c+di$ and

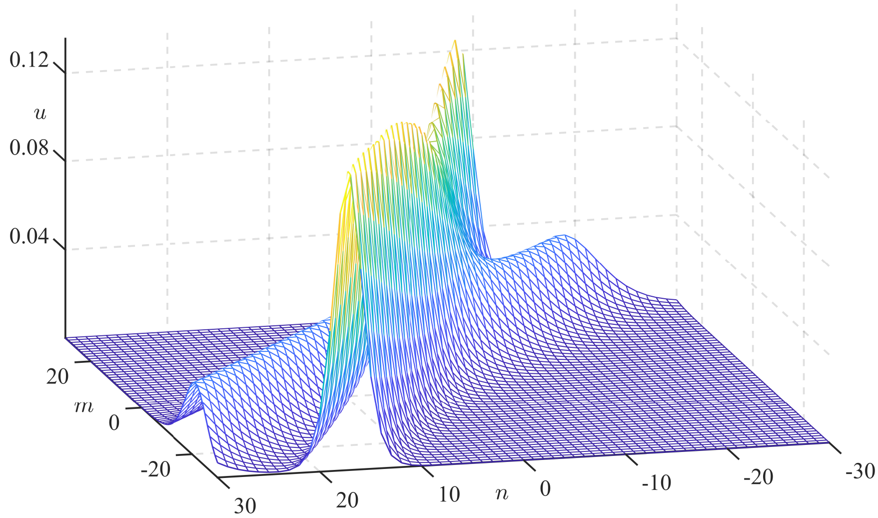

Figure 2: 2-soliton solution of (2.22) with $t = 2,\ C_1(t) = t^2,\ C_2(t) = t,\ \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1,\ p_1 = \frac{1}{5},\ p_2 = \frac{1}{2},\ q_1 = \frac{1}{6},\ q_2 = \frac{1}{4}$.

$C_1(t) = C_2^*(t) = \gamma(t) + \delta(t)i$, we have the 1-breather

$$\begin{aligned}
\tau = {} & \gamma^2(t) + \delta^2(t) + 2\gamma(t)R(a,b)^{-\frac{n}{2}}R(c,d)^{-\frac{m}{2}}e^{2(I(a,b)+I(c,d))t}\cos(-\mathrm{Arg}(S(a,b))n \\
& - \mathrm{Arg}(S(c,d))m - 2(T(a,b)+T(c,d))t + \mathrm{Arg}(\delta(t)i)) \\
& + a_{12}R(a,b)^{-n}R(c,d)^{-m}e^{4(I(a,b)+I(c,d))t}, \\
g_1 = {} & \sqrt{(\dot\gamma(t)+\dot\delta(t)i)}M(c,d)^{-m}e^{(I(c,d)-T(c,d)i)t}(\gamma(t)-\delta(t)i+\frac{di}{c}S(a,b)^{-n} \\
& S(c,d)^{-m}e^{(I(a,b)+I(c,d)-(T(a,b)+T(c,d))i)t}), \\
h_1 = {} & \sqrt{(\dot\gamma(t)+\dot\delta(t)i)}M(a,b)^{-m}e^{(I(a,b)-T(a,b)i)t}(\gamma(t)-\delta(t)i+\frac{bi}{a}S(a,b)^{-n} \\
& S(c,d)^{-m}e^{(I(a,b)+I(c,d)-(T(a,b)+T(c,d))i)t})
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
R(x,y) = {} & \frac{(x-1)^2+y^2}{(x+1)^2+y^2}, \quad I(x,y) = \frac{2x}{x^2+y^2-1}, \quad S(x,y) = \frac{-x^2-y^2+1+2yi}{(x+1)^2+y^2}, \\
T(x,y) = {} & \frac{2y}{x^2+y^2-1}, \quad M(x,y) = \frac{-x^2-y^2+1-2yi}{(x+1)^2+y^2}, \quad a_{12} = -\frac{bd}{ac}
\end{aligned}$$

with a, b, c, d being arbitrary real-valued constants (see Fig.3). Remarkably, $a_{12} > 1$, $\gamma \leq 1$ and $\gamma^2(t) + \delta^2(t) \geq 1$ imply that τ is always positive, which accounts for the nonsingular solution. The general breather solutions can be obtained from the soliton solutions in a similar way.

4 Conclusions and discussions

The symmetric 2DLV ESCS is constructed by employing the source generation procedure. DKP-type Pfaffian solutions of the system have also been derived. In the case that $C_i(t)$

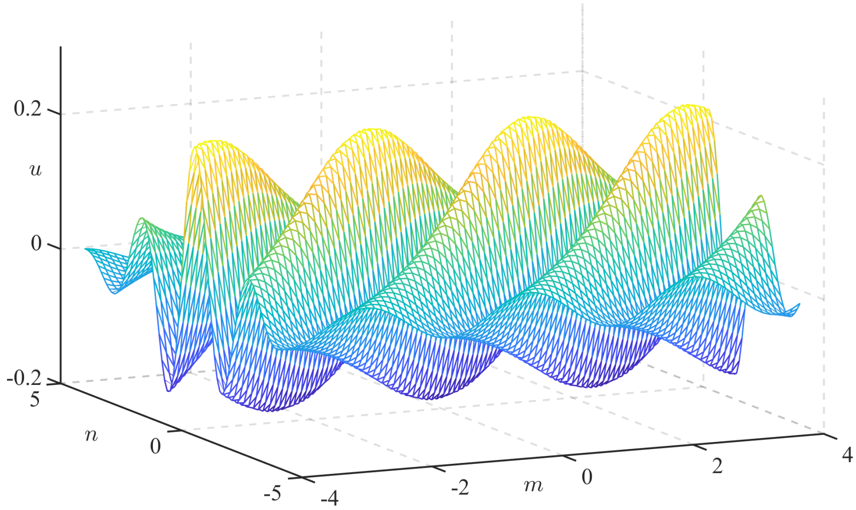

Figure 3: 1-breather solution of (2.22) with $t = 0.01,\ a = -1,\ b = 1,\ c = -0.2,\ d = -3,\ \delta(t) = t^2 + 2,\ \gamma(t) = -t^2 - 1$.

is independent of t, the pairs of sources g_i and h_i become zero identically. Consequently, the symmetric bilinear (or nonlinear) 2DLV ESCS is reduced to the symmetric bilinear (or nonlinear) 2DLV equation. Meanwhile, DKP-type Pfaffian solutions of the symmetric 2DLV ESCS is reduced to the ones of the symmetric 2DLV equation. Due to the property of strong two-dimensionality, the symmetric 2DLV ESCS has been proved to have dromion solutions in addition to soliton solutions and breather solutions.

It has been shown that the NVN equation is a continuous analogue of the symmetric 2DLV equation [36]. Since the NVN equation has nonsingular lump solutions [44, 45], it is believed that the symmetric 2DLV equation and the symmetric 2DLV ESCS should have nonsingular lump solutions too. We will discuss such problems elsewhere.

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