A LAW OF LARGE NUMBERS FOR VECTOR-VALUED LINEAR STATISTICS OF BERGMAN DPP

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ABSTRACT. We establish a law of large numbers for a certain class of vector-valued linear statistics for the Bergman determinantal point process on the unit disk. Our result seems to be the first LLN for vector-valued linear statistics in the setting of determinantal point processes. As an application, we prove that, for almost all configurations X with respect to the Bergman determinantal point process, the weighted Poincaré series (we denote by $d_{\mathrm{h}}(\cdot,\cdot)$ the hyperbolic distance on $\mathbb{D}$)

$$\sum_{k=0}^{\infty}\sum_{\substack{x\in X\\ k\leq d_{\mathrm{h}}(z,x)<k+1}} e^{-sd_{\mathrm{h}}(z,x)}f(x)$$

cannot be simultaneously convergent for all Bergman functions $f\in A^2(\mathbb{D})$ whenever $1<s<3/2$. This confirms a result announced without proof in Bufetov-Qiu [3].

1. INTRODUCTION

Let $A^2(\mathbb{D})$ denote the standard Bergman space on the unit disk $\mathbb{D}\subset\mathbb{C}$ with respect to the normalized Lebesgue measure $\mathrm{d}\mu=\frac{1}{\pi}\mathrm{d}x\mathrm{d}y$:

$$A^2(\mathbb{D})=\left\{f:\mathbb{D}\to\mathbb{C}\,\middle|\, f \text{ is holomorphic and } \int_{\mathbb{D}}|f(z)|^2\mathrm{d}\mu(z)<\infty\right\}.$$

The Hilbert space $A^2(\mathbb{D})$ admits a reproducing kernel (called Bergman kernel of $\mathbb{D}$) given by

$$K(z,w)=\frac{1}{(1-z\overline{w})^2},\quad z,w\in\mathbb{D}.$$

The reader is referred to [4, 15] for more details on the theory of Bergman spaces.

Let $\mathbb{P}_K$ be the determinantal point process on $\mathbb{D}$ induced by the kernel K. Namely, $\mathbb{P}_K$ is a probability measure on the space $\mathrm{Conf}(\mathbb{D})$ of locally finite configurations of $\mathbb{D}$ (see, e.g., [1, 2, 6, 7, 8, 12, 13, 14] for more details and backgrounds). It is well-known (see Peres-Virág [10]) that this determinantal point process $\mathbb{P}_K$ describes the probability distribution of the zero set of the hyperbolic Gaussian analytic function:

$$\sum_{n=0}^{\infty} g_n z^n \quad \text{with } g_n \text{ i.i.d. standard complex Gaussian random variables.}$$

In Bufetov-Qiu [3], the authors studies the random Patterson-Sullivan reconstruction of harmonic functions from a typical configuration sampled from the determinantal point process $\mathbb{P}_K$.

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One of the key ideas in Bufetov-Qiu [3] is a simultaneous law of large numbers for the following weighted Poincaré series (as the parameter s approaches the critical exponent $s=1$):

$$\sum_{k=0}^{\infty}\sum_{\substack{x\in X\\ k\leq d_{\mathrm{h}}(z,x)<k+1}} e^{-sd_{\mathrm{h}}(z,x)}f(x) \tag{1.1}$$

with f being any function in a certain function space (the reader is referred to [3, §1.3] for the reason of double summations in (1.1)). Here $d_{\mathrm{h}}(\cdot,\cdot)$ denotes the hyperbolic distance on $\mathbb{D}$:

$$d_{\mathrm{h}}(z,x)=\log\frac{1+|\varphi_z(x)|}{1-|\varphi_z(x)|} \text{ with } \varphi_z(x)=\frac{z-x}{1-\bar{z}x},\quad z,x\in\mathbb{D}.$$

It is shown in [3, Theorem 1.1] that for any fixed $f\in A^2(\mathbb{D})$, the series (1.1) converges for $\mathbb{P}_K$-almost every configuration $X\in\mathrm{Conf}(\mathbb{D})$. Moreover, such convergence is shown [3, Theorem 1.2] to hold simultaneously for all f being in a certain weighted Bergman space. However, when s approaches the critical exponent $s=1$, the series (1.1) cannot be defined simultaneously for all $f\in A^2(\mathbb{D})$. More precisely, we have

Theorem 1.1. *For any $1<s<3/2$ and any $z\in\mathbb{D}$, we have*

$$\mathbb{P}_K\Big(\Big\{X\in\mathrm{Conf}(\mathbb{D})\,\Big|\sum_{k=0}^{\infty}\sum_{\substack{x\in X\\ k\leq d_{\mathrm{h}}(z,x)<k+1}} e^{-sd_{\mathrm{h}}(z,x)}f(x)\text{ converges for all } f\in A^2(\mathbb{D})\Big\}\Big)=0.$$

The proof of Theorem 1.1 relies on a new law of large numbers obtained in Theorem 1.2 for the vector-valued linear statistics introduced below:

$$\Theta_N^{(s,z)}(X):=\sum_{k=0}^{N-1}\sum_{\substack{x\in X\\ k\leq d_{\mathrm{h}}(z,x)<k+1}} e^{-sd_{\mathrm{h}}(z,x)}K(\cdot,x)=\sum_{x\in X\cap\mathscr{U}_N(z)} e^{-sd_{\mathrm{h}}(z,x)}K(\cdot,x), \tag{1.2}$$

where $\mathscr{U}_N(z)$ denotes the hyperbolic disc $\mathscr{U}_N(z)=\{x\in\mathbb{D}:d_{\mathrm{h}}(z,x)<N\}$.

For simplifying the notations, throughout the whole paper, we shall fix the probability measure $\mathbb{P}_K$ on $\mathrm{Conf}(\mathbb{D})$ and use the simplified notations:

$$\mathbb{E}=\mathbb{E}_{\mathbb{P}_K} \text{ and } \mathrm{Var}=\mathrm{Var}_{\mathbb{P}_K}.$$

Theorem 1.2. *For any $1<s<3/2$ and any $z\in\mathbb{D}$, the sequence $\{\|\Theta_N^{(s,z)}\|^2_{A^2(\mathbb{D})}\}_{N\geq1}$ satisfies the law of large numbers, that is,*

$$\lim_{N\to\infty}\frac{\|\Theta_N^{(s,z)}\|^2_{A^2(\mathbb{D})}}{\mathbb{E}\big[\|\Theta_N^{(s,z)}\|^2_{A^2(\mathbb{D})}\big]}=1\quad a.s.$$

Our proof of the law of large numbers in Theorem 1.2 is based on the following key lemmas.

Lemma A. *For any $1<s<3/2$, there exist $C_1(s),C_2(s)>0$, such that for any $z\in\mathbb{D}$ and sufficiently large N,*

$$C_1(s)\frac{|z|^4+4|z|^2+1}{(1-|z|^2)^2}e^{(3-2s)N}\leq\mathbb{E}\big[\|\Theta_N^{(s,z)}\|^2_{A^2(\mathbb{D})}\big]\leq C_2(s)\frac{|z|^4+4|z|^2+1}{(1-|z|^2)^2}e^{(3-2s)N}.$$

Lemma B. *For any $1 < s < 3/2$, there exists $C(s) > 0$, such that for any $z \in \mathbb{D}$ and sufficiently large N,*

$$\mathrm{Var}\big(\|\Theta_N^{(s,z)}\|_{A^2(\mathbb{D})}^2\big) \le C(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

We end this section by providing the standard derivation of Theorem 1.2 from Lemmas A and B, and the standard derivation of Theorem 1.1 from Theorem 1.2.

The derivation of Theorem 1.2 from Lemmas A and B. Fix any $1 < s < 3/2$ and any $z \in \mathbb{D}$. Lemmas A and B together imply that

$$\mathbb{E}\left[\sum_{N=1}^{\infty}\left|\frac{\|\Theta_N^{(s,z)}\|_{A^2(\mathbb{D})}^2}{\mathbb{E}\big[\|\Theta_N^{(s,z)}\|_{A^2(\mathbb{D})}^2\big]}-1\right|^2\right] = \sum_{N=1}^{\infty}\frac{\mathrm{Var}\big(\|\Theta_N^{(s,z)}\|_{A^2(\mathbb{D})}^2\big)}{\Big(\mathbb{E}\big[\|\Theta_N^{(s,z)}\|_{A^2(\mathbb{D})}^2\big]\Big)^2} < \infty.$$

This clearly completes the proof of Theorem 1.2.

The derivation of Theorem 1.1 from Theorem 1.2. Fix any $1 < s < 3/2$ and any $z \in \mathbb{D}$. Theorem 1.2 and Lemma A together imply that for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(\mathbb{D})$,

$$\lim_{N\to\infty}\|\Theta_N^{(s,z)}(X)\|_{A^2(\mathbb{D})} = \infty.$$

Note that, if $X \in \mathrm{Conf}(\mathbb{D})$ is such that the series (1.1) converges for all $f \in A^2(\mathbb{D})$, then by the uniform boundedness principle (see, e.g., [11, Theorem 2.5]), we must have

$$\sup_{N\ge 1}\|\Theta_N^{(s,z)}(X)\|_{A^2(\mathbb{D})} < \infty.$$

This clearly completes the proof of Theorem 1.1.

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2. The proof of Lemma A

From now on, we shall always fix $1 < s < 3/2$ and $z \in \mathbb{D}$. Recall the definition (1.2) of the vector-valued linear statistics $\Theta_N^{(s,z)}(X)$. For any integer $N \ge 1$ and any configuration $X \in \mathrm{Conf}(\mathbb{D})$, we set

$$S_N(X) = S_N^{(s,z)}(X) := \|\Theta_N^{(s,z)}(X)\|_{A^2(\mathbb{D})}^2.$$

Using the reproducing property of the Bergman kernel

$$\big\langle K(\cdot,x_1), K(\cdot,x_2)\big\rangle_{A^2(\mathbb{D})} = K(x_2,x_1),$$

we have

$$S_N(X) = \sum_{x_1,x_2\in X\cap\mathscr{U}_N(z)} e^{-sd_{\mathrm{h}}(z,x_1)-sd_{\mathrm{h}}(z,x_2)}K(x_1,x_2). \tag{2.3}$$

As usual, for using the explicit correlation functions (due to the determinantal structure) of our point process $\mathbb{P}_K$, we decompose S_N as

$$S_N(X) = \underbrace{\sum_{x\in X\cap \mathscr{U}_N(z)} e^{-2sd_{\mathrm{h}}(z,x)}K(x,x)}_{\text{denoted } S_{N,1}(X)} + \underbrace{\sum_{\substack{x_1,x_2\in X\cap\mathscr{U}_N(z)\\ x_1\neq x_2}} e^{-sd_{\mathrm{h}}(z,x_1)-sd_{\mathrm{h}}(z,x_2)}K(x_1,x_2)}_{\text{denoted } S_{N,2}(X)},$$

and hence

$$\mathbb{E}[S_N] = \mathbb{E}[S_{N,1}] + \mathbb{E}[S_{N,2}]. \tag{2.4}$$

2.1. **The estimate for $\mathbb{E}[S_{N,1}]$.**

Lemma 2.1. *We have*

$$\lim_{N\to\infty}\frac{\mathbb{E}[S_{N,1}]}{e^{(3-2s)N}} = \frac{|z|^4+4|z|^2+1}{2^6(3-2s)(1-|z|^2)^2}.$$

Proof. By the definition of Bergman determinantal point process $\mathbb{P}_K$, we have

$$\mathbb{E}[S_{N,1}] = \int_{\mathscr{U}_N(z)} e^{-2sd_{\mathrm{h}}(z,x)}K(x,x)^2\mathrm{d}\mu(x).$$

We introduce the following notations:

$$r_N = \frac{e^N-1}{e^N+1} \ \text{ and } \ T_z(x) = e^{-d_{\mathrm{h}}(z,x)} = \frac{1-|\varphi_z(x)|}{1+|\varphi_z(x)|}. \tag{2.5}$$

Then $\mathscr{U}_N(z) = \{x\in\mathbb{D} : |\varphi_z(x)| < r_N\}$ and

$$\mathbb{E}[S_{N,1}] = \int_{|\varphi_z(x)|<r_N} T_z(x)^{2s}K(x,x)^2\mathrm{d}\mu(x).$$

We are going to make a change-of-variable $w = \varphi_z(x)$, then

$$x = \varphi_z(w) \ \text{ and } \ \mathrm{d}\mu(x) = \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\mathrm{d}\mu(w).$$

Using the standard equality (see, e.g., [4, 15])

$$\frac{1}{1-|\varphi_z(w)|^2} = \frac{|1-\bar{z}w|^2}{(1-|z|^2)(1-|w|^2)},$$

we obtain

$$\mathbb{E}[S_{N,1}] = \frac{1}{(1-|z|^2)^2}\int_{D(0,r_N)}\frac{(1-|w|)^{2s-4}}{(1+|w|)^{2s+4}}(1-\bar{z}w)^2(1-z\overline{w})^2\mathrm{d}\mu(w),$$

where $D(0,r_N) = \{w\in\mathbb{D} : |w| < r_N\}$. Under the polar coordinates, we get

$$\mathbb{E}[S_{N,1}] = \frac{2}{(1-|z|^2)^2}\int_0^{r_N}\frac{(1-r)^{2s-4}}{(1+r)^{2s+4}}r\big(|z|^4r^4+4|z|^2r^2+1\big)\mathrm{d}r.$$

Since $1 < s < 3/2$, we have

$$\lim_{u\to 1^-}\frac{\int_0^u\frac{(1-r)^{2s-4}}{(1+r)^{2s+4}}r(|z|^4r^4+4|z|^2r^2+1)\mathrm{d}r}{\frac{|z|^4+4|z|^2+1}{2^{2s+4}(3-2s)}(1-u)^{2s-3}} = 1.$$

Consequently, when $N \to \infty$, we have

$$\mathbb{E}[S_{N,1}] \sim \frac{2}{(1-|z|^2)^2} \frac{|z|^4+4|z|^2+1}{2^{2s+4}(3-2s)}(1-r_N)^{2s-3} = \frac{|z|^4+4|z|^2+1}{2^6(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s}.$$

This completes the proof of Lemma 2.1. □

2.2. The estimate for $\mathbb{E}[S_{N,2}]$.

Lemma 2.2. *There exist $B(s,z), C(s,z) \in \mathbb{R}$, such that for large enough N, we have*

$$-\frac{5\cdot 2^{s-9}(|z|^4+4|z|^2+1)}{(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + B(s,z) \le \mathbb{E}[S_{N,2}] \le C(s,z).$$

Remark. *In Lemma 2.2, we do not obtain a correct asymptotic order of $\mathbb{E}[S_{N,2}]$. However, the above two-sided estimates of $\mathbb{E}[S_{N,2}]$, combined with the asymptotic order of $\mathbb{E}[S_{N,1}]$ obtained in Lemma 2.1, is sufficient for us to obtain the correct asymptotic order of $\mathbb{E}[S_N] = \mathbb{E}[S_{N,1}] + \mathbb{E}[S_{N,2}]$. See §2.3 below.*

Recall that the correlation functions of the determinantal point process $\mathbb{P}_K$ are given by

$$\rho_k(x_1,\cdots,x_k) = \det[K(x_i,x_j)]_{1\le i,j\le k}, \quad k \ge 1.$$

More precisely, for any bounded measurable compactly supported function $\phi : \mathbb{D} \to \mathbb{C}$, we have

$$\mathbb{E}\Big[\sum_{x_1,\cdots,x_k\in X}^{\sharp} \phi(x_1,\cdots,x_k)\Big] = \int_{\mathbb{D}^k} \phi(x_1,\cdots,x_k)\det[K(x_i,x_j)]_{1\le i,j\le k}\prod_{j=1}^{k}\mu(x_j), \tag{2.6}$$

here and below we use the notation:

$$\sum_{x_1,\cdots,x_k\in X}^{\sharp} = \sum_{\substack{x_1,\cdots,x_k\in X\\ x_1,\cdots,x_k \text{ distinct}}}. \tag{2.7}$$

In particular,

$$\rho_2(x_1,x_2) = \det[K(x_i,x_j)]_{1\le i,j\le 2} = K(x_1,x_1)K(x_2,x_2) - K(x_1,x_2)K(x_2,x_1).$$

Therefore, using the notations (2.5), we obtain

$$\begin{aligned}\mathbb{E}[S_{N,2}] =& \mathbb{E}\Big[\sum_{x_1,x_2\in X\cap\mathscr{U}_N(z)}^{\sharp} T_z(x_1)^s T_z(x_2)^s K(x_1,x_2)\Big]\\ =& \int_{\mathscr{U}_N(z)^2} T_z(x_1)^s T_z(x_2)^s K(x_1,x_2)\rho_2(x_1,x_2)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2).\end{aligned}$$

Hence $\mathbb{E}[S_{N,2}]$ has the following decomposition:

$$\mathbb{E}[S_{N,2}] = I_{N,1} - I_{N,2} \tag{2.8}$$

with

$$I_{N,1} = \int_{\mathscr{U}_N(z)^2} T_z(x_1)^s T_z(x_2)^s K(x_1,x_1)K(x_2,x_2)K(x_1,x_2)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)$$

and

$$I_{N,2} = \int_{\mathscr{U}_N(z)^2} T_z(x_1)^s T_z(x_2)^s K(x_1,x_2)^2 K(x_2,x_1)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2).$$

2.2.1. *The estimate for $I_{N,1}$.*

Lemma 2.3. *We have*

$$\lim_{N\to\infty} I_{N,1} = \frac{4}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2.$$

Proof. For $k=1,2$, make a change-of-variable $w_k = \varphi_z(x_k)$, then

$$x_k = \varphi_z(w_k) \text{ and } \mathrm{d}\mu(x_k) = \frac{(1-|z|^2)^2}{|1-\bar{z}w_k|^4}\mathrm{d}\mu(w_k).$$

Using the standard equalities (see, e.g., [4, 15])

$$\frac{1}{1-|\varphi_z(w_k)|^2} = \frac{|1-\bar{z}w_k|^2}{(1-|z|^2)(1-|w_k|^2)}$$

and

$$\frac{1}{1-\varphi_z(w_1)\overline{\varphi_z(w_2)}} = \frac{(1-\bar{z}w_1)(1-z\overline{w}_2)}{(1-|z|^2)(1-w_1\overline{w}_2)},$$

we obtain

$$I_{N,1} = \frac{1}{(1-|z|^2)^2}\int_{D(0,r_N)^2} \frac{(1-|w_1|)^{s-2}}{(1+|w_1|)^{s+2}}\frac{(1-|w_2|)^{s-2}}{(1+|w_2|)^{s+2}}\frac{(1-\bar{z}w_1)^2(1-z\overline{w}_2)^2}{(1-w_1\overline{w}_2)^2}\mathrm{d}\mu(w_1)\mathrm{d}\mu(w_2).$$

Under the polar coordinates, we get

$$I_{N,1} = \frac{4}{(1-|z|^2)^2}\Big(\int_0^{r_N} \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2.$$

Therefore, under the assumption $1<s<3/2$, we obtain

$$\lim_{N\to\infty} I_{N,1} = \frac{4}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2 < \infty.$$

This completes the proof of Lemma 2.3. □

2.2.2. *The estimate for $I_{N,2}$.*

Lemma 2.4. *There exists $A(s,z)\in\mathbb{R}$, such that for large enough N, we have*

$$0 \le I_{N,2} \le \frac{5\cdot 2^{s-9}(|z|^4+4|z|^2+1)}{(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + \frac{16(|z|^4+4|z|^2+1)}{(1-|z|^2)^2}A(s,z).$$

Proof. Similar to the proof of Lemma 2.3, by making a change-of-variable $w_k=\varphi_z(x_k)$, $k=1,2$, we get

$$I_{N,2} = \frac{1}{(1-|z|^2)^2}\int_{D(0,r_N)^2}\Big(\frac{1-|w_1|}{1+|w_1|}\Big)^s\Big(\frac{1-|w_2|}{1+|w_2|}\Big)^s\frac{(1-\bar{z}w_1)^2(1-z\overline{w}_2)^2}{(1-w_1\overline{w}_2)^4(1-\overline{w}_1w_2)^2}\mathrm{d}\mu(w_1)\mathrm{d}\mu(w_2).$$

Then, under the polar coordinates, we have

$$I_{N,2} = \frac{4}{(1-|z|^2)^2}\int_0^{r_N}\int_0^{r_N}\Big(\frac{1-r_1}{1+r_1}\Big)^s\Big(\frac{1-r_2}{1+r_2}\Big)^s\frac{r_1r_2}{(1-r_1^2r_2^2)^5}\mathscr{R}(r_1,r_2)\mathrm{d}r_1\mathrm{d}r_2,$$

where

$$\mathscr{R}(r_1,r_2) = |z|^4r_1^6r_2^6 + (3|z|^4+8|z|^2)r_1^4r_2^4 + (8|z|^2+3)r_1^2r_2^2 + 1.$$

It follows that

$$0 \le I_{N,2} \le \frac{16(|z|^4+4|z|^2+1)}{(1-|z|^2)^2} \int_0^{r_N} \Big(\frac{1-r_2}{1+r_2}\Big)^s \int_0^{r_N} \Big(\frac{1-r_1}{1+r_1}\Big)^s \frac{1}{(1-r_1^2 r_2^2)^5} \mathrm{d}r_1 \mathrm{d}r_2.$$

Let

$$F(u) = \int_0^{u} \Big(\frac{1-r_2}{1+r_2}\Big)^s \int_0^{u} \Big(\frac{1-r_1}{1+r_1}\Big)^s \frac{1}{(1-r_1^2 r_2^2)^5} \mathrm{d}r_1 \mathrm{d}r_2, \quad u \in (0,1),$$

then

$$F'(u) = 2\Big(\frac{1-u}{1+u}\Big)^s \int_0^{u} \Big(\frac{1-r}{1+r}\Big)^s \frac{1}{(1-u^2 r^2)^5} \mathrm{d}r,$$

and make a change-of-variable $t = ur$, we get

$$F'(u) = \frac{2}{u}\Big(\frac{1-u}{1+u}\Big)^s \int_0^{u^2} \Big(\frac{1-\frac{t}{u}}{1+\frac{t}{u}}\Big)^s \frac{1}{(1-t^2)^5} \mathrm{d}t \le \frac{2}{u}\Big(\frac{1-u}{1+u}\Big)^s \int_0^{u^2} \Big(\frac{1-t}{1+t}\Big)^s \frac{1}{(1-t^2)^5} \mathrm{d}t.$$

Since $1 < s < 3/2$, we have

$$\lim_{u \to 1^-} \frac{\frac{2}{u}(\frac{1-u}{1+u})^s \int_0^{u^2} (\frac{1-t}{1+t})^s \frac{1}{(1-t^2)^5} \mathrm{d}t}{\frac{(1-u)^{2s-4}}{2^{s+8}(4-s)}} = 1.$$

Therefore, there exists $u_0 \in (0,1)$, such that for any $u \in (u_0, 1)$,

$$F'(u) \le \Big(1+\frac{1}{4}\Big)\frac{(1-u)^{2s-4}}{2^{s+8}(4-s)} = \frac{5(1-u)^{2s-4}}{2^{s+10}(4-s)}.$$

Hence for any $u \in (u_0, 1)$,

$$F(u) = \int_{u_0}^{u} F'(t)\mathrm{d}t + F(u_0) \le \frac{5(1-u)^{2s-3}}{2^{s+10}(4-s)(3-2s)} + A(s,z),$$

where

$$A(s,z) = -\frac{5(1-u_0)^{2s-3}}{2^{s+10}(4-s)(3-2s)} + F(u_0).$$

Consequently, for large enough N satisfying $r_N > u_0$, we have

$$I_{N,2} \le \frac{16(|z|^4+4|z|^2+1)}{(1-|z|^2)^2}\Big[\frac{5(1-r_N)^{2s-3}}{2^{s+10}(4-s)(3-2s)} + A(s,z)\Big].$$

That is,

$$0 \le I_{N,2} \le \frac{5 \cdot 2^{s-9}(|z|^4+4|z|^2+1)}{(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + \frac{16(|z|^4+4|z|^2+1)}{(1-|z|^2)^2} A(s,z).$$

This completes the proof of Lemma 2.4. □

2.2.3. *The conclusion for $\mathbb{E}[S_{N,2}]$: the proof of Lemma 2.2.*

Proof of Lemma 2.2. By Lemma 2.3, when N is large enough, we have

$$\frac{3}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2 \le I_{N,1} \le \frac{5}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2. \tag{2.9}$$

Combine (2.8) with (2.9) and Lemma 2.4, for large enough N, we have

$$-\frac{5\cdot 2^{s-9}(|z|^4+4|z|^2+1)}{(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + B(s,z) \le \mathbb{E}[S_{N,2}] \le C(s,z),$$

where

$$B(s,z) = \frac{3}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2 - \frac{16(|z|^4+4|z|^2+1)}{(1-|z|^2)^2}A(s,z)$$

and

$$C(s,z) = \frac{5}{(1-|z|^2)^2}\Big(\int_0^1 \frac{(1-r)^{s-2}}{(1+r)^{s+2}} r\mathrm{d}r\Big)^2.$$

This completes the proof of Lemma 2.2. □

2.3. **The conclusion for $\mathbb{E}[S_N]$: the proof of Lemma A.**

Proof of Lemma A. By Lemma 2.1, when N is large enough, we have

$$\frac{3(|z|^4+4|z|^2+1)}{2^8(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} \le \mathbb{E}[S_{N,1}] \le \frac{5(|z|^4+4|z|^2+1)}{2^8(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s}. \tag{2.10}$$

By (2.4), (2.10) and Lemma 2.2, when N is large enough,

$$\mathbb{E}[S_N] \le \frac{5(|z|^4+4|z|^2+1)}{2^8(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + C(s,z)$$

and

$$\begin{aligned}\mathbb{E}[S_N] &\ge \frac{3(|z|^4+4|z|^2+1)}{2^8(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} - \frac{5\cdot 2^{s-9}(|z|^4+4|z|^2+1)}{(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + B(s,z)\\ &= \frac{(12-3s-5\cdot 2^{s-1})(|z|^4+4|z|^2+1)}{2^8(4-s)(3-2s)(1-|z|^2)^2}(e^N+1)^{3-2s} + B(s,z).\end{aligned}$$

Finally, it is easy to check that $12-3s-5\cdot 2^{s-1} > 0$ for any $1 < s < 3/2$. Thus for large enough N, we have

$$\frac{(12-3s-5\cdot 2^{s-1})(|z|^4+4|z|^2+1)}{2^9(4-s)(3-2s)(1-|z|^2)^2}e^{(3-2s)N} \le \mathbb{E}[S_N] \le \frac{5(|z|^4+4|z|^2+1)}{2^7(3-2s)(1-|z|^2)^2}e^{(3-2s)N}.$$

This completes the whole proof of Lemma A. □

3. THE PROOF OF LEMMA B

In this section, we are going to estimate the variance

$$\mathrm{Var}(S_N) = \mathbb{E}[S_N^2] - \big(\mathbb{E}[S_N]\big)^2.$$

In §3.1 and §3.2 respectively, we are going to decompose both terms $\mathbb{E}[S_N^2]$ and $\big(\mathbb{E}[S_N]\big)^2$ into sums of integrals of different orders. Then, in §3.3, we regroup these integrals in a suitable way into the formula (3.16) for $\mathrm{Var}(S_N)$, some major cancellations among these integrals will be used in the estimate of $\mathrm{Var}(S_N)$. See the discusssion at the end of §3.3.

Recall that we have fixed $1 < s < 3/2$ and $z \in \mathbb{D}$, and we will use the notations (2.5):

$$r_N = \frac{e^N - 1}{e^N + 1} \quad \text{and} \quad T_z(x) = e^{-d_{\mathrm{h}}(z,x)} = \frac{1 - |\varphi_z(x)|}{1 + |\varphi_z(x)|}.$$

3.1. **The expression for $\mathbb{E}[S_N^2]$.** Recall the formula (2.3) for $S_N(X)$. Then, under the notations (2.5),

$$S_N(X)^2 = \sum_{x_1,x_2,x_3,x_4 \in X \cap \mathscr{U}_N(z)} \Big(\prod_{j=1}^{4} T_z(x_j)^s\Big) K(x_1,x_2)K(x_3,x_4).$$

We will need the following decomposition:

$$S_N(X)^2 = \sum_{k=1}^{4} L_{N,k}(X) \tag{3.11}$$

with

$$L_{N,k}(X) = \sum_{\substack{x_1,x_2,x_3,x_4 \in X \cap \mathscr{U}_N(z) \\ |\{x_1,x_2,x_3,x_4\}| = k}} \Big(\prod_{j=1}^{4} T_z(x_j)^s\Big) K(x_1,x_2)K(x_3,x_4), \quad k = 1,2,3,4.$$

3.1.1. *The expressions of $L_{N,k}$.* In what follows, for simplifying the notation, we write

$$K_{ij} = K(x_i,x_j), \quad 1 \le i,j \le 4.$$

In order to use the defining formula (2.6) of the correlation functions of our determinantal point process $\mathbb{P}_K$, we need to decompose all terms $L_{N,k}$ into summations of the form (2.7):

$$\sum_{x_1,\cdots,x_k \in X \cap \mathscr{U}_N(z)}^{\sharp} = \sum_{\substack{x_1,\cdots,x_k \in X \cap \mathscr{U}_N(z) \\ x_1,\cdots,x_k \text{ distinct}}}.$$

1. The expressions of $L_{N,1}$ and $L_{N,4}$. It is easy to see

$$L_{N,1}(X) = \sum_{x_1 \in X \cap \mathscr{U}_N(z)} T_z(x_1)^{4s} K_{11}^2$$

and

$$L_{N,4}(X) = \sum_{x_1,x_2,x_3,x_4 \in X \cap \mathscr{U}_N(z)}^{\sharp} \Big(\prod_{j=1}^{4} T_z(x_j)^s\Big) K_{12}K_{34}.$$

2. The expression of $L_{N,2}$. Note first that

$$L_{N,2}(X) = \sum_{\substack{x_1,\cdots,x_4 \in X\cap \mathscr{U}_N(z)\\ |\{x_1,\cdots,x_4\}|=2}} = \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ |\{x_1,\cdots,x_4\}|=2}}.$$

Therefore, we have the following decomposition:

$$\begin{aligned} L_{N,2}(X) = & \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_2=x_3\neq x_4}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_2=x_4\neq x_3}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_3=x_4\neq x_2}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_2=x_3=x_4\neq x_1}} \\ & + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_2\neq x_3=x_4}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_3\neq x_2=x_4}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1=x_4\neq x_2=x_3}} \end{aligned}$$

and thus obtain

$$\begin{aligned} L_{N,2}(X) = & \underbrace{2 \sum_{x_1,x_2\in X\cap\mathscr{U}_N(z)}^{\sharp} T_z(x_1)^{3s}T_z(x_2)^s[K_{11}K_{12}+K_{11}K_{21}]}_{\text{denoted } L'_{N,2}(X)} \\ & + \underbrace{\sum_{x_1,x_2\in X\cap\mathscr{U}_N(z)}^{\sharp} T_z(x_1)^{2s}T_z(x_2)^{2s}[K_{11}K_{22}+K_{12}^2+K_{12}K_{21}]}_{\text{denoted } L''_{N,2}(X)}. \end{aligned} \tag{3.12}$$

3. The expression of $L_{N,3}$. Similar to the above decomposition, we write

$$\begin{aligned} L_{N,3}(X) = & \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_3,x_4 \text{ distinct}\\ x_1=x_2}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_2,x_4 \text{ distinct}\\ x_1=x_3}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_2,x_3 \text{ distinct}\\ x_1=x_4}} \\ & + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_2,x_4 \text{ distinct}\\ x_2=x_3}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_2,x_3 \text{ distinct}\\ x_2=x_4}} + \sum_{\substack{(x_1,\cdots,x_4)\in(X\cap\mathscr{U}_N(z))^4\\ x_1,x_2,x_3 \text{ distinct}\\ x_3=x_4}} \end{aligned}$$

and thus obtain

$$\begin{aligned} L_{N,3}(X) = & \underbrace{\sum_{x_1,x_2,x_3\in X\cap\mathscr{U}_N(z)}^{\sharp} T_z(x_1)^{2s}T_z(x_2)^sT_z(x_3)^s[K_{12}K_{13}+K_{21}K_{31}+2K_{12}K_{31}]}_{\text{denoted } L'_{N,3}(X)} \\ & + \underbrace{2\sum_{x_1,x_2,x_3\in X\cap\mathscr{U}_N(z)}^{\sharp} T_z(x_1)^{2s}T_z(x_2)^sT_z(x_3)^sK_{11}K_{23}}_{\text{denoted } L''_{N,3}(X)}. \end{aligned} \tag{3.13}$$

3.1.2. *The decomposition of $\mathbb{E}[S_N^2]$.* Using the decompositions (3.11), (3.12) and (3.13), we can write

$$S_N(X)^2 = L_{N,1}(X) + L'_{N,2}(X) + L''_{N,2}(X) + L'_{N,3}(X) + L''_{N,3}(X) + L_{N,4}(X).$$

Therefore, we have the following expression:

$$\mathbb{E}[S_N^2] = J_{N,1} + J'_{N,2} + J''_{N,2} + J'_{N,3} + J''_{N,3} + J_{N,4} \tag{3.14}$$

with

$$J_{N,k} = \mathbb{E}[L_{N,k}],\ J'_{N,k} = \mathbb{E}[L'_{N,k}] \text{ and } J''_{N,k} = \mathbb{E}[L''_{N,k}].$$

Note that for each summand $J_{N,k}$, $J'_{N,k}$, $J''_{N,k}$ in (3.14), we can now use the defining formula (2.6) of the correlation functions. For instance, we have

$$J_{N,1} = \int_{\mathscr{U}_N(z)} T_z(x_1)^{4s} K_{11}^3 \mathrm{d}\mu(x_1),$$

$$J'_{N,2} = 2\int_{\mathscr{U}_N(z)^2} T_z(x_1)^{3s} T_z(x_2)^s [K_{11}K_{12} + K_{11}K_{21}] \det[K_{ij}]_{1\le i,j\le 2} \prod_{k=1}^{2} \mathrm{d}\mu(x_k)$$

and

$$J'_{N,3} = \int_{\mathscr{U}_N(z)^3} T_z(x_1)^{2s} T_z(x_2)^s T_z(x_3)^s [K_{12}K_{13} + K_{21}K_{31} + 2K_{12}K_{31}] \det[K_{ij}]_{1\le i,j\le 3} \prod_{k=1}^{3} \mathrm{d}\mu(x_k).$$

The other terms $J''_{N,2}$, $J''_{N,3}$ and $J_{N,4}$ also have explicit integral forms.

3.2. **The expression for $\big(\mathbb{E}[S_N]\big)^2$.** From the definition of Bergman determinantal point process $\mathbb{P}_K$, we have

$$\mathbb{E}[S_N] = \underbrace{\int_{\mathscr{U}_N(z)} T_z(x_1)^{2s} K_{11}^2 \mathrm{d}\mu(x_1)}_{\text{denoted } Q_{N,1}} + \underbrace{\int_{\mathscr{U}_N(z)^2} T_z(x_1)^s T_z(x_2)^s K_{12} \det[K_{ij}]_{1\le i,j\le 2} \prod_{k=1}^{2} \mathrm{d}\mu(x_k)}_{\text{denoted } Q_{N,2}}.$$

Therefore, we can decompose $\big(\mathbb{E}[S_N]\big)^2$ as

$$\big(\mathbb{E}[S_N]\big)^2 = \sum_{k=1}^{3} V_{N,k} \tag{3.15}$$

with

$$V_{N,1} = Q_{N,1}^2 = \int_{\mathscr{U}_N(z)^2} T_z(x_1)^{2s} T_z(x_2)^{2s} K_{11}^2 K_{22}^2 \prod_{k=1}^{2} \mathrm{d}\mu(x_k),$$

$$V_{N,2} = 2Q_{N,1}Q_{N,2} = 2\int_{\mathscr{U}_N(z)^3} T_z(x_1)^{2s} T_z(x_2)^s T_z(x_3)^s K_{11}^2 K_{23} \det[K_{ij}]_{2\le i,j\le 3} \prod_{k=1}^{3} \mathrm{d}\mu(x_k)$$

and

$$V_{N,3} = Q_{N,2}^2 = \int_{\mathscr{U}_N(z)^4} \Big(\prod_{k=1}^{4} T_z(x_k)^s\Big) K_{12}K_{34} \det[K_{ij}]_{1\le i,j\le 2} \det[K_{ij}]_{3\le i,j\le 4} \prod_{k=1}^{4} \mathrm{d}\mu(x_k).$$

3.3. **The expression for** $\mathrm{Var}(S_N)$**.** By (3.14) and (3.15), we have

$$\begin{aligned}\mathrm{Var}(S_N) &= \mathbb{E}[S_N^2] - \big(\mathbb{E}[S_N]\big)^2 \\ &= J_{N,1} + J'_{N,2} + J''_{N,2} + J'_{N,3} + J''_{N,3} + J_{N,4} - (V_{N,1} + V_{N,2} + V_{N,3}) \\ &= J_{N,1} + J'_{N,2} + \underbrace{(J''_{N,2} - V_{N,1})}_{\text{denoted } \widehat{J}_{N,2}} + J'_{N,3} + \underbrace{(J''_{N,3} - V_{N,2})}_{\text{denoted } \widehat{J}_{N,3}} + \underbrace{(J_{N,4} - V_{N,3})}_{\text{denoted } \widehat{J}_{N,4}} \\ &= J_{N,1} + J'_{N,2} + \widehat{J}_{N,2} + J'_{N,3} + \widehat{J}_{N,3} + \widehat{J}_{N,4},\end{aligned} \tag{3.16}$$

where the explicit expressions of $J_{N,1}$, $J'_{N,2}$ and $J'_{N,3}$ are given in §3.1.2, and by direct computation, the explicit expressions of $\widehat{J}_{N,2}$, $\widehat{J}_{N,3}$ and $\widehat{J}_{N,4}$ are as follows:

$$\widehat{J}_{N,2} = \int_{\mathscr{U}_N(z)^2} T_z(x_1)^{2s} T_z(x_2)^{2s} \mathscr{J}_2(x_1,x_2) \prod_{k=1}^{2} \mathrm{d}\mu(x_k),$$

$$\widehat{J}_{N,3} = 2\int_{\mathscr{U}_N(z)^3} T_z(x_1)^{2s} T_z(x_2)^{s} T_z(x_3)^{s} \mathscr{J}_3(x_1,x_2,x_3) \prod_{k=1}^{3} \mathrm{d}\mu(x_k),$$

$$\widehat{J}_{N,4} = \int_{\mathscr{U}_N(z)^4} \Big(\prod_{k=1}^{4} T_z(x_k)^s\Big) \mathscr{J}_4(x_1,x_2,x_3,x_4) \prod_{k=1}^{4} \mathrm{d}\mu(x_k),$$

with

$$\mathscr{J}_2(x_1,x_2) = [K_{11}K_{22} + K_{12}^2 + K_{12}K_{21}] \det[K_{ij}]_{1\le i,j\le 2} - K_{11}^2 K_{22}^2,$$

$$\mathscr{J}_3(x_1,x_2,x_3) = K_{11}K_{23}\big(\det[K_{ij}]_{1\le i,j\le 3} - K_{11}\det[K_{ij}]_{2\le i,j\le 3}\big),$$

$$\mathscr{J}_4(x_1,x_2,x_3,x_4) = K_{12}K_{34}\big(\det[K_{ij}]_{1\le i,j\le 4} - \det[K_{ij}]_{1\le i,j\le 2}\det[K_{ij}]_{3\le i,j\le 4}\big).$$

Remark. *For obtaining the up-estimate of* $\mathrm{Var}(S_N)$*, we are going to give up-estimate for all the terms* $J_{N,1}$*,* $J'_{N,2}$*,* $\widehat{J}_{N,2}$*,* $J'_{N,3}$*,* $\widehat{J}_{N,3}$ *and* $\widehat{J}_{N,4}$*. Note that it is crucial for us to give the up-estimate for* $\widehat{J}_{N,2} = J''_{N,2} - V_{N,1}$*, but not direct up-estimate for both terms* $J''_{N,2}$ *and* $V_{N,1}$*. More precisely, the following naive up-estimate is not sufficient for our purpose*:

$$|J''_{N,2} - V_{N,1}| \le |J''_{N,2}| + |V_{N,1}|.$$

Our proof relies on the following cancellation of main parts of $J''_{N,2}$ *and* $V_{N,1}$:

$$J''_{N,2} - V_{N,1} = o(\min\{J''_{N,2}, V_{N,1}\}).$$

Similarly, major cancellations arise in the terms $J''_{N,3} - V_{N,2}$ *and* $J_{N,4} - V_{N,3}$:

$$J''_{N,3} - V_{N,2} = o(\min\{J''_{N,3}, V_{N,2}\}) \text{ and } J_{N,4} - V_{N,3} = o(\min\{J_{N,4}, V_{N,3}\}).$$

3.4. The estimate for $J_{N,1}$.

Lemma 3.1. *There exists $c_1(s) > 0$, such that for large enough N, we have*

$$|J_{N,1}| \leq c_1(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

Proof. Since

$$J_{N,1} = \int_{\mathscr{U}_N(z)} T_z(x_1)^{4s} K_{11}^3 \mathrm{d}\mu(x_1),$$

by setting $w_1 = \varphi_z(x_1)$, we have

$$\begin{aligned} J_{N,1} &= \frac{1}{(1-|z|^2)^4}\int_{D(0,r_N)} \frac{(1-|w_1|)^{4s-6}}{(1+|w_1|)^{4s+6}}|1-\overline{z}w_1|^8 \mathrm{d}\mu(w_1) \\ &\leq \frac{(1+|z|)^4}{(1-|z|)^4}\int_{D(0,r_N)} (1-|w_1|)^{4s-6}\mathrm{d}\mu(w_1). \end{aligned}$$

Then, by using the polar coordinates,

$$|J_{N,1}| \leq \frac{2(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}(1-r)^{4s-6}\mathrm{d}r.$$

It is easy to verify that there exists $A_1(s) > 0$, such that when N is large enough,

$$\int_0^{r_N}(1-r)^{4s-6}\mathrm{d}r \leq A_1(s)e^{(3-2s)N}.$$

Therefore, there exists $c_1(s) > 0$, such that for large enough N,

$$|J_{N,1}| \leq c_1(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

This completes the proof of Lemma 3.1. □

3.5. The estimate for $J'_{N,2}$.

Lemma 3.2. *There exists $c'_2(s) > 0$, such that for large enough N, we have*

$$|J'_{N,2}| \leq c'_2(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

Proof. By the Hadamard inequality (see, e.g., [5, Formula (4.2.15)] or [9, Theorem 13.5.5])

$$\det[K_{ij}]_{1\leq i,j\leq 2} \leq K_{11}K_{22},$$

we get

$$|J'_{N,2}| \leq 4\int_{\mathscr{U}_N(z)^2} T_z(x_1)^{3s}T_z(x_2)^s K_{11}^2 K_{22}|K_{12}|\prod_{k=1}^2 \mathrm{d}\mu(x_k).$$

By setting $w_k = \varphi_z(x_k)$, $k = 1,2$, we have

$$\begin{aligned} |J'_{N,2}| &\leq \frac{4}{(1-|z|^2)^4}\int_{D(0,r_N)^2} \frac{(1-|w_1|)^{3s-4}}{(1+|w_1|)^{3s+4}}\frac{(1-|w_2|)^{s-2}}{(1+|w_2|)^{s+2}}\frac{|1-\overline{z}w_1|^6|1-\overline{z}w_2|^2}{|1-w_1\overline{w}_2|^2}\prod_{k=1}^2 \mathrm{d}\mu(w_k) \\ &\leq \frac{4(1+|z|)^4}{(1-|z|)^4}\int_{D(0,r_N)^2}\frac{(1-|w_1|)^{3s-4}(1-|w_2|)^{s-2}}{|1-w_1\overline{w}_2|^2}\prod_{k=1}^2 \mathrm{d}\mu(w_k). \end{aligned}$$

Then, by using the polar coordinates,

$$|J'_{N,2}| \leq \frac{16(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}\int_0^{r_N}\frac{(1-r_1)^{3s-4}(1-r_2)^{s-2}r_1r_2}{1-r_1^2r_2^2}\mathrm{d}r_1\mathrm{d}r_2.$$

Noticing that

$$1-r_1^2r_2^2 \geq 1-r_1r_2 = (1-r_1r_2)^{\frac{1}{2}}(1-r_1r_2)^{\frac{1}{2}} \geq (1-r_1)^{\frac{1}{2}}(1-r_2)^{\frac{1}{2}}, \tag{3.17}$$

we get

$$|J'_{N,2}| \leq \frac{16(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}(1-r)^{3s-\frac{9}{2}}\mathrm{d}r\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r.$$

It is easy to verify that there exist $A_2(s), A_3(s) > 0$, such that when N is large enough,

$$\int_0^{r_N}(1-r)^{3s-\frac{9}{2}}\mathrm{d}r \leq A_2(s)e^{(\frac{3}{2}-s)N}, \quad \int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r \leq A_3(s)e^{(\frac{3}{2}-s)N}.$$

Therefore, there exists $c'_2(s) > 0$, such that for large enough N,

$$|J'_{N,2}| \leq c'_2(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

This completes the proof of Lemma 3.2. □

3.6. **The estimate for $\widehat{J}_{N,2}$.**

Lemma 3.3. *There exists $\widehat{c}_2(s) > 0$, such that for large enough N, we have*

$$|\widehat{J}_{N,2}| \leq \widehat{c}_2(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

Proof. By a direct calculation, we have

$$\mathscr{J}_2(x_1,x_2) = K_{11}K_{22}K_{12}^2 - K_{12}^3K_{21} - K_{12}^2K_{21}^2.$$

Noticing that

$$|K_{12}| = |K_{21}| \leq K_{11}^{\frac{1}{2}}K_{22}^{\frac{1}{2}},$$

we get

$$|\mathscr{J}_2(x_1,x_2)| \leq 3K_{11}^{\frac{3}{2}}K_{22}^{\frac{3}{2}}|K_{12}|.$$

Hence

$$|\widehat{J}_{N,2}| \leq 3\int_{\mathscr{U}_N(z)^2}T_z(x_1)^{2s}T_z(x_2)^{2s}K_{11}^{\frac{3}{2}}K_{22}^{\frac{3}{2}}|K_{12}|\prod_{k=1}^{2}\mathrm{d}\mu(x_k).$$

By setting $w_k = \varphi_z(x_k)$, $k = 1,2$, we have

$$\begin{aligned}|\widehat{J}_{N,2}| &\leq \frac{3}{(1-|z|^2)^4}\int_{D(0,r_N)^2}\frac{(1-|w_1|)^{2s-3}}{(1+|w_1|)^{2s+3}}\frac{(1-|w_2|)^{2s-3}}{(1+|w_2|)^{2s+3}}\frac{|1-\bar{z}w_1|^4|1-\bar{z}w_2|^4}{|1-w_1\overline{w}_2|^2}\prod_{k=1}^{2}\mathrm{d}\mu(w_k)\\ &\leq \frac{3(1+|z|)^4}{(1-|z|)^4}\int_{D(0,r_N)^2}\frac{(1-|w_1|)^{2s-3}(1-|w_2|)^{2s-3}}{|1-w_1\overline{w}_2|^2}\prod_{k=1}^{2}\mathrm{d}\mu(w_k).\end{aligned}$$

Then, by using the polar coordinates,

$$|\widehat{J}_{N,2}| \leq \frac{12(1+|z|)^4}{(1-|z|)^4} \int_0^{r_N} \int_0^{r_N} \frac{(1-r_1)^{2s-3}(1-r_2)^{2s-3} r_1 r_2}{1-r_1^2 r_2^2} \mathrm{d}r_1 \mathrm{d}r_2.$$

Thus by the inequality (3.17), we get

$$|\widehat{J}_{N,2}| \leq \frac{12(1+|z|)^4}{(1-|z|)^4} \Big(\int_0^{r_N} (1-r)^{2s-\frac{7}{2}} \mathrm{d}r \Big)^2.$$

It is easy to verify that there exists $A_4(s) > 0$, such that when N is large enough,

$$\int_0^{r_N} (1-r)^{2s-\frac{7}{2}} \mathrm{d}r \leq A_4(s) e^{(\frac{3}{2}-s)N}.$$

Therefore, there exists $\widehat{c}_2(s) > 0$, such that for large enough N,

$$|\widehat{J}_{N,2}| \leq \widehat{c}_2(s) \frac{(1+|z|)^4}{(1-|z|)^4} e^{(3-2s)N}.$$

This completes the proof of Lemma 3.3. □

3.7. **The estimate for $J'_{N,3}$.**

Lemma 3.4. *There exists $c'_3(s) > 0$, such that for large enough N, we have*

$$|J'_{N,3}| \leq c'_3(s) \frac{(1+|z|)^4}{(1-|z|)^4} e^{(3-2s)N}.$$

Proof. By the Hadamard inequality (see, e.g., [5, Formula (4.2.15)] or [9, Theorem 13.5.5])

$$\det[K_{ij}]_{1\leq i,j\leq 3} \leq K_{11}K_{22}K_{33},$$

we get

$$|J'_{N,3}| \leq 4 \int_{\mathscr{U}_N(z)^3} T_z(x_1)^{2s} T_z(x_2)^s T_z(x_3)^s K_{11}K_{22}K_{33} |K_{12}||K_{13}| \prod_{k=1}^3 \mathrm{d}\mu(x_k).$$

By setting $w_k = \varphi_z(x_k)$, $k = 1,2,3$, we have

$$|J'_{N,3}| \leq \frac{4}{(1-|z|)^4} \int_{D(0,r_N)^3} \mathscr{M}(w_1,w_2,w_3) \prod_{k=1}^3 \mathrm{d}\mu(w_k),$$

where

$$\mathscr{M}(w_1,w_2,w_3) = \frac{(1-|w_1|)^{2s-2}}{(1+|w_1|)^{2s+2}} \frac{(1-|w_2|)^{s-2}}{(1+|w_2|)^{s+2}} \frac{(1-|w_3|)^{s-2}}{(1+|w_3|)^{s+2}} \frac{|1-\bar{z}w_1|^4 |1-\bar{z}w_2|^2 |1-\bar{z}w_3|^2}{|1-w_1\overline{w}_2|^2 |1-w_1\overline{w}_3|^2}.$$

This yields that

$$|J'_{N,3}| \leq \frac{4(1+|z|)^4}{(1-|z|)^4} \int_{D(0,r_N)^3} \frac{(1-|w_1|)^{2s-2}(1-|w_2|)^{s-2}(1-|w_3|)^{s-2}}{|1-w_1\overline{w}_2|^2 |1-w_1\overline{w}_3|^2} \prod_{k=1}^3 \mathrm{d}\mu(w_k).$$

Then, by using the polar coordinates,

$$|J'_{N,3}| \leq \frac{32(1+|z|)^4}{(1-|z|)^4} \int_0^{r_N} \int_0^{r_N} \int_0^{r_N} \frac{(1-r_1)^{2s-2}(1-r_2)^{s-2}(1-r_3)^{s-2} r_1 r_2 r_3}{(1-r_1^2 r_2^2)(1-r_1^2 r_3^2)} \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}r_3.$$

By the same inequality as (3.17), for any $k,j\in\{1,2,3\}$,

$$1-r_k^2r_j^2\geq(1-r_k)^{\frac{1}{2}}(1-r_j)^{\frac{1}{2}},\tag{3.18}$$

so we get

$$|J'_{N,3}|\leq\frac{32(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}(1-r)^{2s-3}\mathrm{d}r\Big(\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r\Big)^2.$$

It is easy to verify that there exist $A_5(s),A_6(s)>0$, such that when N is large enough,

$$\int_0^{r_N}(1-r)^{2s-3}\mathrm{d}r\leq A_5(s),\qquad\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r\leq A_6(s)e^{(\frac{3}{2}-s)N}.$$

Therefore, there exists $c'_3(s)>0$, such that for large enough N,

$$|J'_{N,3}|\leq c'_3(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

This completes the proof of Lemma 3.4. □

3.8. The estimate for $\widehat{J}_{N,3}$.

Lemma 3.5. *There exists $\widehat{c}_3(s)>0$, such that for large enough N, we have*

$$|\widehat{J}_{N,3}|\leq\widehat{c}_3(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

Proof. A direct calculation gives that

$$\det[K_{ij}]_{1\leq i,j\leq3}-K_{11}\det[K_{ij}]_{2\leq i,j\leq3}=-K_{12}K_{21}K_{33}+K_{12}K_{23}K_{31}+K_{13}K_{21}K_{32}-K_{13}K_{22}K_{31}.$$

Noticing that for any $k,j\in\{1,2,3\}$,

$$|K_{kj}|=|K_{jk}|\leq K_{kk}^{\frac{1}{2}}K_{jj}^{\frac{1}{2}},$$

we get

$$|\mathscr{J}_3(x_1,x_2,x_3)|\leq2K_{11}^{\frac{3}{2}}K_{22}^{\frac{1}{2}}K_{33}|K_{12}||K_{23}|+2K_{11}^{\frac{3}{2}}K_{33}^{\frac{1}{2}}K_{22}|K_{13}||K_{32}|.$$

Hence

$$|\widehat{J}_{N,3}|\leq8\int_{\mathscr{U}_N(z)^3}T_z(x_1)^{2s}T_z(x_2)^sT_z(x_3)^sK_{11}^{\frac{3}{2}}K_{22}^{\frac{1}{2}}K_{33}|K_{12}||K_{23}|\prod_{k=1}^3\mathrm{d}\mu(x_k).$$

By setting $w_k=\varphi_z(x_k)$, $k=1,2,3$, we have

$$|\widehat{J}_{N,3}|\leq\frac{8}{(1-|z|)^4}\int_{D(0,r_N)^3}\mathscr{N}(w_1,w_2,w_3)\prod_{k=1}^3\mathrm{d}\mu(w_k),$$

where

$$\mathscr{N}(w_1,w_2,w_3)=\frac{(1-|w_1|)^{2s-3}}{(1+|w_1|)^{2s+3}}\frac{(1-|w_2|)^{s-1}}{(1+|w_2|)^{s+1}}\frac{(1-|w_3|)^{s-2}}{(1+|w_3|)^{s+2}}\frac{|1-\bar{z}w_1|^4|1-\bar{z}w_2|^2|1-\bar{z}w_3|^2}{|1-w_1\overline{w}_2|^2|1-w_2\overline{w}_3|^2}.$$

This yields that

$$|\widehat{J}_{N,3}|\leq\frac{8(1+|z|)^4}{(1-|z|)^4}\int_{D(0,r_N)^3}\frac{(1-|w_1|)^{2s-3}(1-|w_2|)^{s-1}(1-|w_3|)^{s-2}}{|1-w_1\overline{w}_2|^2|1-w_2\overline{w}_3|^2}\prod_{k=1}^3\mathrm{d}\mu(w_k).$$

Then, by using the polar coordinates,

$$|\widehat{J}_{N,3}| \leq \frac{64(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}\int_0^{r_N}\int_0^{r_N}\frac{(1-r_1)^{2s-3}(1-r_2)^{s-1}(1-r_3)^{s-2}r_1r_2r_3}{(1-r_1^2r_2^2)(1-r_2^2r_3^2)}\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3.$$

Thus by the inequality (3.18), we get

$$|\widehat{J}_{N,3}| \leq \frac{64(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}(1-r)^{2s-\frac{7}{2}}\mathrm{d}r\int_0^{r_N}(1-r)^{s-2}\mathrm{d}r\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r.$$

It is easy to verify that there exist $A_7(s), A_8(s), A_9(s) > 0$, such that when N is large enough,

$$\int_0^{r_N}(1-r)^{2s-\frac{7}{2}}\mathrm{d}r \leq A_7(s)e^{(\frac{3}{2}-s)N}, \quad \int_0^{r_N}(1-r)^{s-2}\mathrm{d}r \leq A_8(s)$$

and

$$\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r\mathrm{d}r \leq A_9(s)e^{(\frac{3}{2}-s)N}.$$

Therefore, there exists $\widehat{c}_3(s) > 0$, such that for large enough N,

$$|\widehat{J}_{N,3}| \leq \widehat{c}_3(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

This completes the proof of Lemma 3.5. □

3.9. The estimate for $\widehat{J}_{N,4}$.

Lemma 3.6. *There exists $\widehat{c}_4(s) > 0$, such that for large enough N, we have*

$$|\widehat{J}_{N,4}| \leq \widehat{c}_4(s)\frac{(1+|z|)^4}{(1-|z|)^4}e^{(3-2s)N}.$$

Proof. A direct calculation gives that

$$\det[K_{ij}]_{1\leq i,j\leq 4} - \det[K_{ij}]_{1\leq i,j\leq 2}\det[K_{ij}]_{3\leq i,j\leq 4} = \sum_{k=1}^{4}\mathscr{W}_k(x_1,x_2,x_3,x_4)$$

with

$$\mathscr{W}_1(x_1,x_2,x_3,x_4) = -K_{11}K_{23}K_{32}K_{44} + K_{11}K_{23}K_{34}K_{42} + K_{12}K_{23}K_{31}K_{44} - K_{12}K_{23}K_{34}K_{41},$$

$$\mathscr{W}_2(x_1,x_2,x_3,x_4) = K_{11}K_{24}K_{32}K_{43} - K_{11}K_{24}K_{33}K_{42} - K_{12}K_{24}K_{31}K_{43} + K_{12}K_{24}K_{33}K_{41},$$

and

$$\begin{aligned}\mathscr{W}_3(x_1,x_2,x_3,x_4) = {} & K_{13}K_{21}K_{32}K_{44} - K_{13}K_{21}K_{34}K_{42} - K_{13}K_{22}K_{31}K_{44} \\ & + K_{13}K_{22}K_{34}K_{41} + K_{13}K_{24}K_{31}K_{42} - K_{13}K_{24}K_{32}K_{41},\end{aligned}$$

$$\begin{aligned}\mathscr{W}_4(x_1,x_2,x_3,x_4) = {} & -K_{14}K_{21}K_{32}K_{43} + K_{14}K_{21}K_{33}K_{42} + K_{14}K_{22}K_{31}K_{43} \\ & - K_{14}K_{22}K_{33}K_{41} - K_{14}K_{23}K_{31}K_{42} + K_{14}K_{23}K_{32}K_{41}.\end{aligned}$$

Noticing that for any $k, j \in \{1,2,3,4\}$,

$$|K_{kj}| = |K_{jk}| \leq K_{kk}^{\frac{1}{2}}K_{jj}^{\frac{1}{2}},$$

we get

$$
\begin{aligned}
|\mathscr{J}_4(x_1,x_2,x_3,x_4)| \leq\ & 4K_{11}K_{22}^{\frac{1}{2}}K_{33}^{\frac{1}{2}}K_{44}|K_{12}||K_{23}||K_{34}| + 4K_{11}K_{22}^{\frac{1}{2}}K_{44}^{\frac{1}{2}}K_{33}|K_{12}||K_{24}||K_{43}| \\
& + 6K_{22}K_{11}^{\frac{1}{2}}K_{33}^{\frac{1}{2}}K_{44}|K_{21}||K_{13}||K_{34}| + 6K_{22}K_{11}^{\frac{1}{2}}K_{44}^{\frac{1}{2}}K_{33}|K_{21}||K_{14}||K_{43}|.
\end{aligned}
$$

Hence

$$
|\widehat{J}_{N,4}| \leq 20\int_{\mathscr{U}_N(z)^4}\Big(\prod_{k=1}^4 T_z(x_j)^s\Big)K_{11}K_{22}^{\frac{1}{2}}K_{33}^{\frac{1}{2}}K_{44}|K_{12}||K_{23}||K_{34}|\prod_{k=1}^4 \mathrm{d}\mu(x_k).
$$

By setting $w_k = \varphi_z(x_k)$, $k = 1,2,3,4$, we have

$$
|\widehat{J}_{N,4}| \leq \frac{20}{(1-|z|^2)^4}\int_{D(0,r_N)^4}\mathscr{F}_1(w_1,w_2,w_3,w_4)\mathscr{F}_2(w_1,w_2,w_3,w_4)\prod_{k=1}^4 \mathrm{d}\mu(w_k),
$$

where

$$
\mathscr{F}_1(w_1,w_2,w_3,w_4) = \frac{(1-|w_1|)^{s-2}}{(1+|w_1|)^{s+2}}\frac{(1-|w_2|)^{s-1}}{(1+|w_2|)^{s+1}}\frac{(1-|w_3|)^{s-1}}{(1+|w_3|)^{s+1}}\frac{(1-|w_4|)^{s-2}}{(1+|w_4|)^{s+2}}
$$

and

$$
\mathscr{F}_2(w_1,w_2,w_3,w_4) = \frac{|1-\bar{z}w_1|^2|1-\bar{z}w_2|^2|1-\bar{z}w_3|^2|1-\bar{z}w_4|^2}{|1-w_1\overline{w}_2|^2|1-w_2\overline{w}_3|^2|1-w_3\overline{w}_4|^2}.
$$

This yields that

$$
|\widehat{J}_{N,4}| \leq \frac{20(1+|z|)^4}{(1-|z|)^4}\int_{D(0,r_N)^4}\mathscr{G}(w_1,w_2,w_3,w_4)\prod_{k=1}^4 \mathrm{d}\mu(w_k),
$$

where

$$
\mathscr{G}(w_1,w_2,w_3,w_4) = \frac{(1-|w_1|)^{s-2}(1-|w_2|)^{s-1}(1-|w_3|)^{s-1}(1-|w_4|)^{s-2}}{|1-w_1\overline{w}_2|^2|1-w_2\overline{w}_3|^2|1-w_3\overline{w}_4|^2}.
$$

Then, by using the polar coordinates,

$$
|\widehat{J}_{N,4}| \leq \frac{320(1+|z|)^4}{(1-|z|)^4}\int_0^{r_N}\int_0^{r_N}\int_0^{r_N}\int_0^{r_N}\mathscr{H}(r_1,r_2,r_3,r_4)\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3\mathrm{d}r_4,
$$

where

$$
\mathscr{H}(r_1,r_2,r_3,r_4) = \frac{(1-r_1)^{s-2}(1-r_2)^{s-1}(1-r_3)^{s-1}(1-r_4)^{s-2}r_1r_2r_3r_4}{(1-r_1^2r_2^2)(1-r_2^2r_3^2)(1-r_3^2r_4^2)}.
$$

By the same inequality as (3.17), for any $k,j \in \{1,2,3,4\}$,

$$
1-r_k^2r_j^2 \geq (1-r_k)^{\frac{1}{2}}(1-r_j)^{\frac{1}{2}},
$$

so we get

$$
|\widehat{J}_{N,4}| \leq \frac{320(1+|z|)^4}{(1-|z|)^4}\Big(\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r\Big)^2\Big(\int_0^{r_N}(1-r)^{s-2}\mathrm{d}r\Big)^2.
$$

It is easy to verify that there exist $A_{10}(s), A_{11}(s) > 0$, such that when N is large enough,

$$
\int_0^{r_N}(1-r)^{s-\frac{5}{2}}\mathrm{d}r \leq A_{10}(s)e^{(\frac{3}{2}-s)N}, \quad \int_0^{r_N}(1-r)^{s-2}\mathrm{d}r \leq A_{11}(s).
$$

Therefore, there exists $\widehat{c}_4(s) > 0$, such that for large enough N,

$$|\widehat{J}_{N,4}| \leq \widehat{c}_4(s) \frac{(1+|z|)^4}{(1-|z|)^4} e^{(3-2s)N}.$$

This completes the proof of Lemma 3.6. $\square$

3.10. The conclusion for $\mathrm{Var}(S_N)$: the proof of Lemma B.

Proof of Lemma B. By (3.16), we obtain Lemma B from Lemmas 3.1-3.6 directly. This completes the whole proof of Lemma B. $\square$

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