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NEW EXOTIC EXAMPLES OF RICCI LIMIT SPACES

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ABSTRACT. For any integers $m \geqslant n \geqslant 3$, we construct a Ricci limit space $X_{m,n}$ such that for a fixed point, some tangent cones are $\mathbb{R}^m$ and some are $\mathbb{R}^n$. This is an improvement of Menguy's example [Men01].

CONTENTS

1. INTRODUCTION

Consider the measured Gromov-Hausdorff limit spaces as following:

$$(M_i^n, g_i, \nu_i, p_i) \xrightarrow{GH} (X^k, d, \nu, p), \quad \mathrm{Ric}_{g_i} \geqslant -\lambda, \quad \nu_i = \frac{1}{\mathrm{Vol}(B_1(p_i))} d\,\mathrm{Vol}_{g_i},$$

where $k \in \mathbb{N}$ is the rectifiable dimension of (X, d, ν), which is the unique integer k such that the limit is k-rectifiable. The existence of such a k is proved by Colding-Naber[CN12]. Moreover, the strong regular set $\mathcal{R}_k(X)$ is a ν-full measure set. Actually, there are two versions of regular sets on X [CC97]. For $l = 1, \cdots, k$, the weak regular set of (X, d) can be defined by

$$\mathcal{WR}_l(X) = \left\{ x \in X : \text{ there exists a tangent cone at } x \text{ isometric to } \mathbb{R}^l \right\},$$

and the strong regular set of (X, d) can be defined by

$$\mathcal{R}_l(X) = \left\{ x \in X : \text{ every tangent cone at } x \text{ isometric to } \mathbb{R}^l \right\}.$$

Cheeger-Colding[CC97] shows that in the noncollapsing case, i.e. $\mathrm{Vol}(B_1(p_i)) > v > 0$ uniformly, two versions coincide. Moreover, the rectifiable dimension and the Hausdorff dimension of the limit space are both equal to n. However, in collapsing case, i.e. $\mathrm{Vol}(B_1(p_i)) \to 0$, many things are quite different. Pan-Wei[PW22] shows that the Hausdorff dimension may be larger than the rectifiable dimension, and the Hausdorff dimension can be non-integers. Menguy[Men01] shows that the weak regular set may be not equal to the strong regular set. However, it is still not known whether the intersection of weakly regular sets of different dimensions can be non-empty. In this paper, we construct the first example that shows that the intersection of weakly regular sets of different dimensions can be non-empty. This is an improvement of Menguy's example [Men01].

Theorem 1.1. *Let $m \geqslant n \geqslant 2$ be integers. Then there exists a sequence of $(m+n+3)$-dimensional complete Riemannian manifolds (M_i, g_i, p_i) with $\mathrm{Ric}_{g_i} \geqslant 0$ converging to $(X_{m+1,n+1}, d, x)$, such that*

$$\mathcal{WR}_{m+1}(X) \cap \mathcal{WR}_{n+1}(X) \neq \emptyset.$$

Remark 1.2. For this example, the rectifiable dimension is $m+n+1$.

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2. Triple warped products

In this section, we recall the Ricci curvature of triple warped products.

Let φ, ϕ, ρ be smooth nonnegative functions on $[0, \infty)$ such that φ, ϕ, ρ are positive on $(0, \infty)$,

$$\phi(0) > 0,\ \phi^{(\mathrm{odd})}(0) = 0,\ \rho(0) > 0,\ \rho^{(\mathrm{odd})}(0) = 0, \tag{2.1}$$

and

$$\varphi(0) = 0,\ \varphi'(0) = 1,\ \varphi^{(\mathrm{even})}(0) = 0. \tag{2.2}$$

Then we can define a Riemannian metric on $\mathbb{R}^{m+1} \times \mathbb{S}^n \times \mathbb{S}^2$ by

$$g_{\varphi,\phi,\rho}(r) = dr^2 + \varphi(r)^2 g_{\mathbb{S}^m} + \phi(r)^2 g_{\mathbb{S}^n} + \rho(r)^2 g_{\mathbb{S}^2}.$$

See also [Pet16, Proposition 1.4.7] for more details.

Write $X_0 = \frac{\partial}{\partial r}$, $X_1 \in T_{\mathbb{S}^m}$, $X_2 \in T_{\mathbb{S}^n}$, and $X_3 \in T_{\mathbb{S}^2}$. Then the Ricci curvature of $g_{\varphi,\phi,\rho}$ can be expressed as following.

Lemma 2.1. *Let φ, ϕ, ρ and $g_{\varphi,\phi,\rho}$ be as above. Then the Ricci curvature tensors of $g_{\varphi,\phi,\rho}$ can be determined by*

$$\mathrm{Ric}_{g_{\varphi,\phi,\rho}}(X_0) = -\left(m\frac{\varphi''}{\varphi} + n\frac{\phi''}{\phi} + 2\frac{\rho''}{\rho}\right) X_0, \tag{2.3}$$

$$\mathrm{Ric}_{g_{\varphi,\phi,\rho}}(X_1) = \left[-\frac{\varphi''}{\varphi} + (m-1)\frac{1-(\varphi')^2}{\varphi^2} - n\frac{\varphi'\phi'}{\varphi\phi} - 2\frac{\varphi'\rho'}{\varphi\rho}\right] X_1, \tag{2.4}$$

$$\mathrm{Ric}_{g_{\varphi,\phi,\rho}}(X_2) = \left[-\frac{\phi''}{\phi} + (n-1)\frac{1-(\phi')^2}{\phi^2} - m\frac{\varphi'\phi'}{\varphi\phi} - 2\frac{\phi'\rho'}{\phi\rho}\right] X_2, \tag{2.5}$$

$$\mathrm{Ric}_{g_{\varphi,\phi,\rho}}(X_3) = \left[-\frac{\rho''}{\rho} + \frac{1-(\rho')^2}{\rho^2} - m\frac{\varphi'\rho'}{\varphi\rho} - n\frac{\phi'\rho'}{\phi\rho}\right] X_3. \tag{2.6}$$

Proof. One can conclude it by a straightforward calculation. See also [Pet16, Subsection 4.2.4]. □

Lemma 2.2. *Define a Riemannian metric on $(0, \infty) \times \mathbb{S}^m \times \mathbb{S}^n \times \mathbb{S}^k$ by*

$$g_{\varphi,\phi,\rho}(r) = dr^2 + \varphi(r)^2 g_{\mathbb{S}^m} + \phi(r)^2 g_{\mathbb{S}^n} + \rho^2(r) g_{\mathbb{S}^k}.$$

If $\varphi(r) = a_1 r + b_1$ *and* $\phi(r) = \rho(r) = a_2 r + b_2$*, then the Ricci curvature is*

$$\begin{aligned}
\mathrm{Ric}(g_1)_{00} &= 0,\\
\mathrm{Ric}(g_1)_{11} &= \frac{[(m-1)-(m+n+k-1)a_1^2]a_2 r + (m-1)b_2 - [(m-1)a_1 b_2 + (n+k)a_2 b_1]a_1}{(a_1 r + b_1)^2(a_2 r + b_2)},\\
\mathrm{Ric}(g_1)_{22} &= \frac{[(n-1)-(m+n+k-1)a_2^2]a_1 r + (n-1)b_1 - [(n+k-1)a_2 b_1 + m a_1 b_2]a_2}{(a_1 r + b_1)(a_2 r + b_2)^2},\\
\mathrm{Ric}(g_1)_{33} &= \frac{[(k-1)-(m+n+k-1)a_2^2]a_1 r + (k-1)b_1 - [(n+k-1)a_2 b_1 + m a_1 b_2]a_2}{(a_1 r + b_1)(a_2 r + b_2)^2}
\end{aligned}$$

3. Construction of the local model spaces

In this section, we construct some local model spaces by triple warped products.

3.1. **Model I.** At first, we construct a metric $g_{\varphi,\phi,\rho}$ on $(0,\infty)\times\mathbb{S}^m\times\mathbb{S}^n\times\mathbb{S}^2$ such that the asymptotic cone is $C(\mathbb{S}^m_k\times\mathbb{S}^n_k)$, and the topology near $r=0$ is homeomorphic to $\mathbb{R}^{m+1}\times\mathbb{S}^n\times\mathbb{S}^2$.

Lemma 3.1. *Let* $m,n\geqslant 2$*. For any* $0<\epsilon\leq\frac{1}{100}$, $0<\delta\leq\delta_0(m,n,\epsilon)$ *and* $0<k<k_0(m,n)$*, there exist constants* $R(m,n,\epsilon,\delta,k)>0$ *and positive functions* φ,ϕ,ρ *on* $(0,\infty)$*, such that*

$$\begin{aligned}
&\varphi|_{(0,1)} = (1-\epsilon)r, \quad &&\phi|_{(0,1)} = \delta, \quad &&\rho'|_{(0,1)} = 0,\\
&\varphi|_{[R,+\infty)} = kr, \quad &&\phi|_{[R,+\infty)} = kr, \quad &&\rho'|_{[R,+\infty)} = 0,
\end{aligned}$$

and $\mathrm{Ric}_{g_{\varphi,\phi,\rho}}\geqslant 0$.

Proof. We denote $M=\mathbb{R}^{m+1}\times\mathbb{S}^n\times\mathbb{S}^2$. Let us begin with the initial metric

$$g_0(r) = dr^2 + (1-\epsilon)^2 r^2 g_{\mathbb{S}^m} + \delta^2 g_{\mathbb{S}^n} + \delta^2 g_{\mathbb{S}^2},$$

Step1: Constructing g_1**.** Set $U_1=\{r\leq 2\}$ and $R_1=100$. We will define a metric g_1 by modifying g_0 on $M\backslash U_1$ through the ansatz

$$g_1(r) = dr^2 + \varphi(r)^2 g_{\mathbb{S}^m} + \phi(r)^2 g_{\mathbb{S}^n} + \rho(r)^2 g_{\mathbb{S}^2}.$$

Let $0<\delta_1(m,n,\delta)<1$ be a constant to be determined later. By smoothing out the function $\min\{\delta,\delta+k(r-R_1)\}$ near $r=R_1$, we can build a smooth function $\phi(r)$

$$\phi(r) = \begin{cases} \delta & \text{if} \quad r\leq 10^{-1}R_1\\ 0<R_1\phi''<\delta_1 & \text{if} \quad 10^{-1}R_1\leq r\leq 10R_1\\ \delta+\delta_1(r-R_1) & \text{if} \quad 10R_1\leq r.\end{cases}$$

By smoothing out the function $\min\{\delta,\delta+\delta_1(r-R_1)\}$ near $r=R_1$, we can build a smooth function $\rho(r)$ satisfying

$$\rho(r) = \begin{cases} \delta & \text{if} \quad r\leq 10^{-1}R_1\\ 0<R_1\rho''<\delta_1 & \text{if} \quad 10^{-1}R_1\leq r\leq 10R_1\\ \delta+\delta_1(r-R_1) & \text{if} \quad 10R_1\leq r,\end{cases}$$

respectively. Similarly, by smoothing out the function $\min\{(1-\epsilon)r,\min\{(1-\epsilon)R_1+k(r-R_1)\}$ near $r=R_1$, we can build a smooth function $\varphi(r)$ satisfying

$$\varphi(r) = \begin{cases} (1-\epsilon)r & \text{if} \quad r\leq 20^{-1}R_1\\ \varphi''<0 & \text{if} \quad 20^{-1}R_1\leq r\leq 10^{-1}R_1\\ R_1\varphi''<-\frac{1-\epsilon-k}{100} & \text{if} \quad 10^{-1}R_1\leq r\leq 10R_1\\ (1-\epsilon)R_1+k(r-R_1) & \text{if} \quad 10R_1\leq r\end{cases}$$

For $r \leq 10^{-1}R_1$, we have $k \leq \varphi' \leq 1-\epsilon$ since $\varphi'' \leq 0$. And then

$$\begin{aligned}
\mathrm{Ric}(g_1)_{00} &= -m\frac{\varphi''}{\varphi} \geqslant 0,\\
\mathrm{Ric}(g_1)_{11} &\geqslant -\frac{\varphi''}{\varphi} + (m-1)\frac{\epsilon(2-\epsilon)}{\varphi^2} \geqslant 0,\\
\mathrm{Ric}(g_1)_{22} &= \frac{n-1}{\delta^2} \geqslant 0,\\
\mathrm{Ric}(g_1)_{33} &= \frac{1}{\delta^2} \geqslant 0,
\end{aligned}$$

where $\mathrm{Ric}(g_1)_{ii} = \mathrm{Ric}_{g_1}(X_i, X_i)$ defined in Lemma 2.1.

To estimate the Ricci curvature in the interval $[R_1/10, 10R_1]$, we will use the following facts

$$0 < \rho'' < \frac{\delta_1}{R_1}, \quad 0 \leq \rho' \leq \delta_1, \quad \delta \leq \rho \leq \delta + 9\delta_1 R_1,$$

$$0 < \phi'' < \frac{\delta_1}{R_1}, \quad 0 \leq \phi' \leq \delta_1, \quad \delta \leq \phi \leq \delta + 9\delta_1 R_1,$$

$$\frac{1-\epsilon-k}{100R_1} < -\varphi'', \quad k \leq \varphi' \leq 1-\epsilon, \quad \frac{(1-\epsilon)R_1}{20} \leq \varphi \leq (1-\epsilon+9k)R_1.$$

Then we have

$$\begin{aligned}
\mathrm{Ric}(g_1)_{00} &\geqslant \frac{1}{R_1}\left[\frac{m(1-\epsilon-k)}{100R_1(1-\epsilon+9k)} - \frac{(n+2)\delta_1}{\delta}\right],\\
\mathrm{Ric}(g_1)_{11} &\geqslant \frac{1}{R_1}\left[\frac{1-\epsilon-k}{100R_1(1-\epsilon+9k)} + \frac{(m-1)\epsilon(2-\epsilon)}{R_1(1-\epsilon+9k)^2} - \frac{20(n+2)\delta_1}{(1-\epsilon)\delta}\right],\\
\mathrm{Ric}(g_1)_{22} &\geqslant \frac{(n-1)(1-\delta_1^2)}{(\delta+9\delta_1R_1)^2} - \frac{\delta_1}{R_1\delta}(10m+1+2\frac{R_1\delta_1}{\delta}),\\
\mathrm{Ric}(g_1)_{33} &\geqslant \frac{1-\delta_1^2}{(\delta+9\delta_1R_1)^2} - \frac{\delta_1}{R_1\delta}(10m+1+2\frac{R_1\delta_1}{\delta}).
\end{aligned}$$

If $0 < k < 10^{-2}$, $\delta_1 \leqslant \delta_1(m,n,\delta)$, then we have $\mathrm{Ric} \geqslant 0$ in $[10^{-1}R_1, 10R_1]$.

Apply Lemma 2.2, where $a_1 = k$, $a_2 = \delta_1$, $b_1 = R_1(1-\epsilon-k)$, $b_2 = \delta - R_1\delta_1$, then we know that the Ricci curvature is non-negative for all $r > 0$ if $k \leqslant k(m,n)$, $\delta_1 \leqslant \delta_1(m,n,\delta)$.

Now we build a metric g_1 satisfying the initial condition we stated and have the property that

$$g_1(r) = dr^2 + [kr + R_1(1-\epsilon-k)]^2 g_{\mathbb{S}^m} + (\delta_1 r + \delta - R_1\delta_1)^2 g_{\mathbb{S}^n} + (\delta_1 r + \delta - R_1\delta_1)^2 g_{\mathbb{S}^2},$$

for $r \geqslant 10R_1$.

Step2: Constructing g_2**.** Set $R_2 = 10^3 R_1$, and $U_2 = \{r \leqslant 10R_1\}$. We will define a metric g_2 by modifying g_1 on $M \backslash U_2$ through the ansatz

$$g_2(r) = dr^2 + [kr + R_1(1-\epsilon-k)]^2 g_{\mathbb{S}^m} + (\delta_1 r + \delta - R_1\delta_1)^2 g_{\mathbb{S}^n} + \rho(r)^2 g_{\mathbb{S}^2},$$

For $0 < s << 1$ to be determined later, we consider $\rho(r)$ by smoothing the function $\min\{\delta_1 r + \delta - R_1\delta_1, (\delta_1 R_2 + \delta - R_1\delta_1)(\frac{r}{R_2})^s)\}$ at R_2 with the following properties

$$\rho(r) = \begin{cases} \delta_1 r + \delta - R_1\delta_1 & \text{if } r \leqslant 10^{-1}R_2,\\ \rho'' \leqslant 0 & \text{if } 10^{-1}R_2 \leqslant r \leqslant 10R_2,\\ ar^s & \text{if } r \geqslant 10R_2,\end{cases}$$

where $a = (\delta_1 R_2 + \delta - R_1\delta_1)R_2^{-s}$.

For $r \in [10^{-1}R_2, 10R_2]$, we have

$$\rho'' \leqslant 0, \quad sa(10R_2)^{s-1} \leqslant \rho' \leqslant \delta_1, \quad \delta \leqslant \rho \leqslant 20\delta.$$

Then we have

$$\begin{aligned}
\mathrm{Ric}(g_2)_{00} &= -2\frac{\rho''}{\rho} \geqslant 0,\\
\mathrm{Ric}(g_2)_{11} &\geqslant (m-1)\frac{1-k^2}{\varphi^2} - (n+2)\frac{k\delta_1}{\delta\varphi} \geqslant \varphi^{-2}\left[\frac{m-1}{2} - k(n+2)(10kR_2 + R_1)\frac{\delta_1}{\delta}\right],\\
\mathrm{Ric}(g_2)_{22} &\geqslant (n-1)\frac{1-\delta_1^2}{\phi^2} - \frac{mk\delta_1}{\phi} - \frac{2\delta_1^2}{\delta\phi} \geqslant \phi^{-2}\left[\frac{n-1}{2} - (mk + 2\frac{\delta_1}{\delta})\delta_1(10\delta_1 R_2 + \delta)\right],\\
\mathrm{Ric}(g_2)_{33} &\geqslant \frac{1-\delta_1^2}{\rho^2} - \frac{mk\delta_1}{\rho} - \frac{n\delta_1^2}{\delta\rho} \geqslant \rho^{-2}\left[\frac{1}{2} - 20(mk\delta_1\delta + n\delta_1^2)\right].
\end{aligned}$$

So $\mathrm{Ric} \geqslant 0$ for $r \in [10^{-1}R_2, 10R_2]$ if $\delta_1/\delta \leqslant c(m,n)$.

For $r \geqslant 10R_2$,

$$\begin{aligned}
\mathrm{Ric}(g_2)_{00} &= -2\frac{\rho''}{\rho} \geqslant 0,\\
\mathrm{Ric}(g_2)_{11} &\geqslant (m-1)\frac{1-k^2}{\varphi^2} - \frac{nk}{r\varphi} - \frac{2sk}{r\varphi} \geqslant (r\varphi^2)^{-1}\left[\frac{m-1}{2}r - k(n+2s)(kr + R_1)\right],\\
\mathrm{Ric}(g_2)_{22} &\geqslant (n-1)\frac{1-\delta_1^2}{\phi^2} - \frac{m\delta_1}{r\phi} - \frac{2s\delta_1}{r\phi} \geqslant (r\phi^2)^{-1}\left[\frac{n-1}{2}r - \delta_1(m+2s)(\delta_1 r + \delta)\right],\\
\mathrm{Ric}(g_2)_{33} &\geqslant \frac{s(1-s)}{r^2} + \frac{1-\delta_1^2}{a^2r^{2s}} - \frac{ms}{r^2} - \frac{ns}{r^2} \geqslant \frac{R_2^{2-2s}}{2a^2r^2} - \frac{(m+n)s}{r^2}.
\end{aligned}$$

If $0 < s < s(m,n)$, $\mathrm{Ric}(g_2) \geqslant 0$ for all $r > 0$.

Now we build a metric g_2 satisfying the initial condition we stated and have the property that

$$g_2(r) = dr^2 + [kr + R_1(1-\epsilon-k)]^2 g_{\mathbb{S}^m} + (\delta_1 r + \delta - R_1\delta_1)^2 g_{\mathbb{S}^n} + (ar^s)^2 g_{\mathbb{S}^2},$$

for $r \geqslant 10R_2$.

Step3: Constructing g_3**.** Set $U_3 = \{r \leqslant 10R_2\}$. We will define a metric g_3 by modifying g_2 on $M\backslash U_3$ through the ansatz

$$g_3(r) = dr^2 + \varphi(r)^2 g_{\mathbb{S}^m} + \phi(r)^2 g_{\mathbb{S}^n} + (ar^s)^2 g_{\mathbb{S}^2}.$$

For $R_3 = R_3(m,n,\delta,\delta_1,k,s)$, we can choose smooth functions $\varphi(r)$ and $\phi(r)$ satisfying

$$\varphi(r) = \begin{cases} kr + b_1 & \text{if} \quad r \leq 10^{-1}R_3 \\ |\varphi'| < 2k, |r\varphi''| < R_3^{-1} & \text{if} \quad 10^{-1}R_3 \leq r \leq 10R_3 \\ kr & \text{if} \quad 10R_3 \leq r. \end{cases}$$

$$\phi(r) = \begin{cases} \delta_1 r + b_2 & \text{if} \quad r \leq 10^{-1}R_3 \\ |\phi'| < 2k, |r\phi''| < R_3^{-1}, |\phi'/\phi| < 10r^{-1} & \text{if} \quad 10^{-1}R_3 \leq r \leq 10R_3 \\ kr & \text{if} \quad 10R_3 \leq r, \end{cases}$$

respectively, where $b_1 = R_1(1-\epsilon-k)$, $b_2 = \delta - R_1\delta_1$.

For $r \in [10^{-1}R_3, 10R_3]$,

$$\mathrm{Ric}(g_3)_{00} \geqslant \frac{2s(1-s)}{r^2} - \frac{mR_3^{-1}}{kr^2} - \frac{nR_3^{-1}}{\delta_1 r^2},$$

$$\mathrm{Ric}(g_3)_{11} \geqslant (m-1)\frac{1-4k^2}{\varphi^2} - \frac{R_3^{-1}}{r\varphi} - \frac{20nk}{r\varphi} - \frac{4ks}{r\varphi} \geqslant (r\varphi^2)^{-1}\left[\frac{m-1}{2}r - 100nk(kr+b_1)\right],$$

$$\mathrm{Ric}(g_3)_{22} \geqslant (n-1)\frac{1-4k^2}{\phi^2} - \frac{R_3^{-1}}{r\phi} - \frac{20mk}{r\phi} - \frac{4ks}{r\phi} \geqslant (r\phi^2)^{-1}\left[\frac{n-1}{2}r - 100mk^2 r\right],$$

$$\mathrm{Ric}(g_3)_{33} \geqslant \frac{1-\delta_1^2}{a^2r^{2s}} - \frac{20(m+n)s}{r^2}.$$

Then Ric $\geqslant 0$ for $r \in [10^{-1}R_3, 10R_3]$ after choosing $R_3^{-1} < \delta_1/(100\delta)$.

For $r \geqslant 10R_3$,

$$\mathrm{Ric}(g_3)_{00} = \frac{2s(1-s)}{r^2} > 0,$$
$$\mathrm{Ric}(g_3)_{11} = \frac{(m-1)(1-k^2)}{k^2r^2} - \frac{n+2s}{r^2} > 0,$$
$$\mathrm{Ric}(g_3)_{22} = \frac{(n-1)(1-k^2)}{k^2r^2} - \frac{m+2s}{r^2} > 0,$$
$$\mathrm{Ric}(g_3)_{33} = \frac{s(1-s)}{r^2} + \frac{1-s^2a^2r^{2s-2}}{a^2r^{2s}} - \frac{s(m+n)}{r^2} > 0.$$

Now we build a metric g_3 with $\mathrm{Ric}(g_3) \geqslant 0$ satisfying the initial condition we stated and have the property that

$$g_3(r) = dr^2 + (kr)^2 g_{\mathbb{S}^m} + (kr)^2 g_{\mathbb{S}^n} + (ar^s)^2 g_{\mathbb{S}^2},$$

for $r \geqslant 10R_3$.

Step4: Constructing g_4**.** Set $U_4 = \{r \leqslant 10R_3\}$. We will define a metric g_4 by modifying g_3 on $M \backslash U_4$ through the ansatz

$$g_4(r) = dr^2 + (kr)^2 g_{\mathbb{S}^m} + (kr)^2 g_{\mathbb{S}^n} + \rho(r)^2 g_{\mathbb{S}^2}.$$

Then the Ricci curvature of this ansatz is

$$\mathrm{Ric}(g_4)_{00} = -\frac{2\rho''}{\rho},$$
$$\mathrm{Ric}(g_4)_{11} = \frac{(m-1)(1-k^2)}{k^2r^2} - \frac{n}{r^2} - \frac{2\rho'}{r\rho},$$
$$\mathrm{Ric}(g_4)_{22} = \frac{(n-1)(1-k^2)}{k^2r^2} - \frac{m}{r^2} - \frac{2\rho'}{r\rho},$$
$$\mathrm{Ric}(g_4)_{33} = -\frac{2\rho''}{\rho} + \frac{1-\rho'^2}{\rho^2} - \frac{(m+n)\rho'}{r\rho}.$$

We can choose $\rho(r)$ of the form

$$\rho(r) = \begin{cases} ar^s & \text{if} \quad r \leq 10R_3 \\ \rho'' \leqslant 0 & \text{if} \quad 10R_3 \leq r \leq 10^3 R_3 \\ \lambda & \text{if} \quad 10^3 R_3 \leq r, \end{cases}$$

for some $\lambda = \lambda(a, R_3)$. Then it's easy to see that Ric $\geqslant 0$ for any $r > 0$. Moreover, for $R = 10^4 R_3$, the last metric g_4 satisfies all the properties we stated.

□

3.2. **Model II.** Next we construct a metric $g_{\varphi,\phi,\rho}$ on $(0,\infty)\times\mathbb{S}^m\times\mathbb{S}^n\times\mathbb{S}^2$ such that the metric around ∞ is isometric to $C(\mathbb{S}^m_{1-\epsilon})\times\mathbb{S}^n_\delta\times\mathbb{S}^2_\rho$

Lemma 3.2. *Let $m,n\geqslant 2$. Then for any $0<\epsilon\leq\frac{1}{100}$, $\lambda>0$, $0<k<k_0(m,n)$, there are constants $R(m,n,k,\epsilon)>0$, $\delta(m,n,k,\epsilon)>0$ and positive functions φ,ϕ,ρ on $(0,\infty)$, such that*

$$\begin{aligned}
\varphi|_{(0,1)} &= kr, & \phi|_{(0,1)} &= kr, & \rho|_{(0,1)} &= \lambda,\\
\varphi|_{[R,+\infty)} &= (1-\epsilon)r, & \phi|_{[R,+\infty)} &= \delta, & \rho|_{[R,+\infty)} &= \lambda,
\end{aligned}$$

and $\mathrm{Ric}_{g_{\varphi,\phi,\rho}}\geqslant 0$.

Proof. Let us begin with the initial metric

$$g_0(r)=dr^2+(kr)^2g_{\mathbb{S}^m}+(kr)^2g_{\mathbb{S}^n}+\lambda^2g_{\mathbb{S}^2},$$

Step1: Constructing g_1**.** Set $U_1=\{r\leq 2\}$ and $R_1=100$. We will define a metric g_1 by modifying g_0 on $M\backslash U_1$ through the ansatz

$$g_1(r)=dr^2+(kr)^2g_{\mathbb{S}^m}+\phi(r)^2g_{\mathbb{S}^n}+\lambda^2g_{\mathbb{S}^2}.$$

Set $s=\epsilon/(10^6mn)$. By smoothing out the function $\min\{kr,kR_1^{1-s}r^s\}$ near $r=R_1$, we can build a smooth function $\phi(r)$ of the form

$$\phi(r)=\begin{cases} kr & \text{if} \quad r\leq 10^{-1}R_1\\ \phi''\leqslant 0 & \text{if} \quad 10^{-1}R_1\leq r\leq 10R_1\\ kR_1(\frac{r}{R_1})^s & \text{if} \quad 10R_1\leq r.\end{cases}$$

The Ricci curvature of the ansatz is

$$\begin{aligned}
\mathrm{Ric}(g_1)_{00} &= -\frac{n\phi''}{\phi},\\
\mathrm{Ric}(g_1)_{11} &= \frac{(m-1)(1-k^2)}{k^2r^2}-\frac{n\phi'}{r\phi},\\
\mathrm{Ric}(g_1)_{22} &= -\frac{\phi''}{\phi}+\frac{(n-1)(1-\phi'^2)}{\phi^2}-\frac{m\phi'}{r\phi},\\
\mathrm{Ric}(g_1)_{33} &= \frac{1}{\lambda^2}>0.
\end{aligned}$$

By direct computation, we have $\mathrm{Ric}(g_1)\geqslant 0$ for any $r>0$ if $0<k<k_0(m,n)$.

Now we build a metric g_1 satisfying the initial condition we stated and have the property that

$$g_1(r)=dr^2+(kr)^2g_{\mathbb{S}^m}+(ar^s)^2g_{\mathbb{S}^n}+\lambda^2g_{\mathbb{S}^2},$$

for $r\geqslant 10R_1$, where $a=kR_1^{1-s}$.

Step2: Constructing g_2**.** Set $U_2=\{r\leqslant 10R_1\}$. We will define a metric g_2 by modifying g_1 on $M\backslash U_2$ through the ansatz

$$g_2(r)=dr^2+\varphi(r)^2g_{\mathbb{S}^m}+(ar^s)^2\,g_{\mathbb{S}^n}+\lambda^2g_{\mathbb{S}^2},$$

For $R_2=R_2(\epsilon)$, we can choose a smooth function $\varphi(r)$ with the following properties

$$\varphi(r)=\begin{cases} kr & \text{if } r\leqslant 10^{-1}R_2,\\ |\varphi'|\leqslant(1-10^{-1}\epsilon), |r\varphi''|<10^{-20}\epsilon & \text{if } 10^{-1}R_2\leqslant r\leqslant 10R_2,\\ (1-\epsilon)r & \text{if } r\geqslant 10R_2.\end{cases}$$

The Ricci curvature of the ansatz is

$$\begin{aligned}
\mathrm{Ric}(g_2)_{00} &= -\frac{m\varphi''}{\varphi} + n\frac{s(1-s)}{r^2},\\
\mathrm{Ric}(g_2)_{11} &= -\frac{\varphi''}{\varphi} + (m-1)\frac{(1-\varphi'^2)}{\varphi^2} - \frac{ns\varphi'}{r\varphi},\\
\mathrm{Ric}(g_2)_{22} &= \frac{s(1-s)}{r^2} + \frac{n-1}{a^2r^{2s}} - \frac{(n-1)s^2}{r^2} - \frac{ms\varphi'}{r\varphi},\\
\mathrm{Ric}(g_2)_{33} &= \frac{1}{\lambda^2} > 0.
\end{aligned}$$

By direct computation, we have $\mathrm{Ric}(g_2) \geqslant 0$ for any $r > 0$.

Now we build a metric g_2 satisfying the initial condition we stated and have the property that

$$g_2(r) = dr^2 + [(1-\epsilon)r]^2 g_{\mathbb{S}^m} + (ar^s)^2 g_{\mathbb{S}^n} + \lambda^2 g_{\mathbb{S}^2},$$

for $r \geqslant 10R_2$,

Step3: Constructing g_3. Set $U_3 = \{r \leqslant 10R_2\}$. We will define a metric g_3 by modifying g_2 on $M\backslash U_3$ through the ansatz

$$g_3(r) = dr^2 + [(1-\epsilon)r]^2 g_{\mathbb{S}^m} + \phi(r)^2 g_{\mathbb{S}^n} + \lambda^2 g_{\mathbb{S}^2},$$

Then the Ricci curvature of this ansatz is

$$\begin{aligned}
\mathrm{Ric}(g_3)_{00} &= -\frac{n\phi''}{\phi},\\
\mathrm{Ric}(g_3)_{11} &= \frac{(m-1)\epsilon(2-\epsilon)}{(1-\epsilon)^2r^2} - \frac{n\phi'}{r\phi},\\
\mathrm{Ric}(g_3)_{22} &= -\frac{\phi''}{\phi} + \frac{(n-1)(1-\phi'^2)}{\phi^2} - \frac{m\phi'}{r\phi},\\
\mathrm{Ric}(g_3)_{33} &= \frac{1}{\lambda^2} > 0.
\end{aligned}$$

We can choose $\phi(r)$ of the form

$$\phi(r) = \begin{cases} ar^s & \text{if} \quad r \leq 10R_2 \\ \phi'' \leqslant 0 & \text{if} \quad 10R_2 \leq r \leq 10^3R_2 \\ \delta & \text{if} \quad 10^3R_2 \leq r, \end{cases}$$

for some $\delta = \delta(a, R_2)$. Then by direct computation we have $\mathrm{Ric}(g_3) \geqslant 0$ for any $r > 0$. Moreover, for $R = 10^4R_2$, the last metric g_3 satisfies all the properties we stated.

□

4. Connecting $\mathbb{R}^m$ and $\mathbb{R}^n$

Lemma 4.1. *For any $m, n \geqslant 2$, $\epsilon > 0$ and $L > 1$, then there exists $k = k(m,n)$, $R = R(m,n,\epsilon,L) > 1$, $0 < \delta < c(m,n,\epsilon)L^{-1}$, and positive smooth functions φ, ϕ, ρ on $r \in (0,\infty)$ such that*

$$\begin{aligned}
&\varphi|_{(0,(LR)^{-1})} = kr, && \phi|_{(0,(LR)^{-1})} = kr, && \rho|_{(0,(LR)^{-1})} = \lambda_1,\\
&\varphi|_{[L^{-1},1]} = (1-\epsilon)r, && \phi|_{[L^{-1},1]} = \delta, && \rho|_{[L^{-1},1]} = \lambda_1,\\
&\varphi|_{[R,\infty)} = kr, && \phi|_{[R,\infty)} = kr, && \rho|_{[R,\infty)} = \lambda_2,
\end{aligned}$$

and $\mathrm{Ric}_{\varphi,\phi,\rho} \geqslant 0$.

Proof. For $m, n \geqslant 2$, take $k = k_0(m, n)$ to be the smaller one in lemma 3.1 and lemma 3.2. By lemma 3.1, we have $\delta_0 = \delta_0(m, n, \epsilon)$. By lemma 3.2, we have $\delta_2(m, n, k_0, \epsilon)$ and $R_2(m, n, k_0, \epsilon)$. After possibly increasing R_2, we can assume $\delta_2/R_2 < \delta_0$. Then we take $\delta = \frac{\delta_2}{LR_2} < \delta_0$. Applying lemma 3.1, we get $R = R_1(m, n, \epsilon, \delta, k_0) > R_2$ and functions φ_1, ϕ_1, ρ_1 satisfying

$$\begin{aligned} &\varphi_1|_{(0,1)} = (1-\epsilon)r, \quad &&\phi_1|_{(0,1)} = \delta, \quad &&\rho_1|_{(0,1)} = \lambda_1, \\ &\varphi_1|_{[R,\infty)} = kr, &&\phi_1|_{[R,\infty)} = kr, &&\rho_1|_{[R,\infty)} = \lambda_2. \end{aligned}$$

Applying lemma 3.2, we get functions φ_2, ϕ_2, ρ_2. We rescale the functions by $\tilde{\varphi}(r) := (LR_2)^{-1}\varphi(LR_2 r)$. Similarly we get $\tilde{\phi}$ and $\tilde{\rho}$, then they satisfy the following

$$\begin{aligned} &\tilde{\varphi}_2|_{(0,(LR)^{-1})} = kr, &&\tilde{\phi}_2|_{(0,(LR)^{-1})} = kr, &&\tilde{\rho}_2|_{(0,(LR)^{-1})} = \lambda_1, \\ &\tilde{\varphi}_2|_{[L^{-1},\infty)} = (1-\epsilon)r, \quad &&\tilde{\phi}_2|_{[L^{-1},\infty)} = (LR_2)^{-1}\delta_2, \quad &&\tilde{\rho}_2|_{[L^{-1},\infty)} = \lambda_1. \end{aligned}$$

Note that since $(LR_2)^{-1}\delta_2 = \delta$, two groups of functions agree in $r \in [L^{-1}, 1]$ respectively. Then we can glue them to get the new functions φ, ϕ, ρ. These functions satisfy all the properties we stated.

∎

Proposition 4.2. *For any $m, n \geqslant 2$, $\epsilon > 0$ and $L > 1$, then there exists $k = k(m, n)$, $R = R(m, n, \epsilon, L) > 1$ and positive smooth functions φ, ϕ, ρ on $r \in (0, \infty)$ such that*

$$\begin{aligned} &\varphi|_{(0,(2L^2R^3)^{-1})} = kr, &&\phi|_{(0,(2L^2R^3)^{-1})} = kr, &&\rho|_{(0,(2L^2R^3)^{-1})} = \lambda_1, \\ &\varphi|_{[(2L^2R^2)^{-1},(2LR^2)^{-1}]} = \delta_1, \quad &&\phi|_{[(2L^2R^2)^{-1},(2LR^2)^{-1}]} = (1-\epsilon)r, \quad &&\rho|_{[(2L^2R^2)^{-1},(2LR^2)^{-1}]} = \lambda_1, \\ &\varphi|_{[(2LR)^{-1},(LR)^{-1}]} = kr, &&\phi|_{[(2LR)^{-1},(LR)^{-1}]} = kr, &&\rho|_{[(2LR)^{-1},(LR)^{-1}]} = \lambda_2, \\ &\varphi|_{[L^{-1},1]} = (1-\epsilon)r, &&\phi|_{[L^{-1},1]} = \delta_2, &&\rho|_{[L^{-1},1]} = \lambda_2 \\ &\varphi|_{[R,\infty)} = kr, &&\phi|_{[R,\infty)} = kr, &&\rho|_{[R,\infty)} = \lambda_3, \end{aligned}$$

where $0 < \delta_1 < c(m, n, \epsilon)(L^2R^2)^{-1}$, $0 < \delta_2 < c(m, n, \epsilon)L^{-1}$ and $\mathrm{Ric}_{\varphi,\phi,\rho} \geqslant 0$.

Proof. First apply Lemma 4.1 to get φ_1, ϕ_1, ρ_1. Next we exchange m and n and then apply Lemma 4.1 again to get φ_2, ϕ_2, ρ_2. Rescale the second metric $\tilde{\varphi}(r) = (2LR^2)^{-1}\varphi(2LR^2 r)$, $\tilde{\phi}$ and $\tilde{\rho}$ likewise. Then two metrics agree on $r \in [(2LR)^{-1}, (LR)^{-1}]$. We can glue them to get the desired functions. ∎

5. Proof of the Main Theorem

Lemma 5.1 (smoothing)**.** *For any $m, n \geqslant 2$, $\epsilon > 0$ and $L > 2$, $\varphi|_{[L^{-1},1]} = (1-\epsilon)r$, $\phi|_{[L^{-1},1]} = \delta$, $\rho|_{[L^{-1},1]} = \lambda$, $\mathrm{Ric}_{\varphi,\phi,\rho} \geqslant 0$, then we can take smooth modified functions $\hat{\varphi}$, $\hat{\phi}$, $\hat{\rho}$ such that*

$$\begin{aligned} &\hat{\varphi}|_{(0,L^{-1})} = r, &&\hat{\phi}|_{(0,L^{-1})} = \delta, &&\hat{\rho}|_{(0,L^{-1})} = \lambda, \\ &\hat{\varphi}|_{[2L^{-1},\infty)} = \varphi, \quad &&\hat{\phi}|_{[2L^{-1},\infty)} = \phi, \quad &&\hat{\rho}|_{[2L^{-1},\infty)} = \rho, \end{aligned}$$

and $\mathrm{Ric}_{\hat{\varphi},\hat{\phi},\hat{\rho}} \geqslant 0$.

Proof. We construct $\hat{g}$ by modifying g on $r \in (0, 2L^{-1})$ through the ansatz

$$\hat{g} = dr^2 + \varphi(r)^2 g_{\mathbb{S}^m} + \delta^2 g_{\mathbb{S}^n} + \lambda^2 g_{\mathbb{S}^2}.$$

The Ricci curvature of this ansatz is

$$\begin{aligned}\mathrm{Ric}(\hat g)_{00} &= -\frac{m\varphi''}{\varphi},\\ \mathrm{Ric}(\hat g)_{11} &= -\frac{\varphi''}{\varphi} + \frac{(m-1)(1-\varphi'^2)}{\varphi^2},\\ \mathrm{Ric}(\hat g)_{22} &= \frac{n-1}{\delta^2} > 0,\\ \mathrm{Ric}(\hat g)_{33} &= \frac{1}{\lambda^2} > 0.\end{aligned}$$

So we can choose smooth $\varphi(r)$ such that $\mathrm{Ric}(\hat g) \geqslant 0$ of the form

$$\varphi(r) = \begin{cases} r & \text{if} \quad r \leqslant L^{-1},\\ \varphi'' \leqslant 0 & \text{if} \quad L^{-1} \leqslant r \leqslant 2L^{-1},\\ (1-\epsilon)r & \text{if} \quad 2L^{-1} \leqslant r \leqslant 1.\end{cases}$$

□

Now we are ready to prove the main theorem.

Proof of the Main Theorem. *Let $m \geqslant n \geqslant 2$ be integers. For any $i \geqslant 1$, we will construct smooth metrics $g_i = (\varphi_i, \phi_i, \rho_i)$ with $\mathrm{Ric}(g_i) \geqslant 0$ on $M = (0,\infty) \times \mathbb{S}^m \times \mathbb{S}^n \times \mathbb{S}^2$. Moreover, we will find a sequence of numbers $N_i \geqslant 10N_{i-1}$ such that (φ_i, ϕ_i) and $(\varphi_{i+1}, \phi_{i+1})$ coincide on $r \in [N_i^{-1}, \infty)$ and $\rho_i \leqslant N_i^{-10}$.*

Set $\epsilon_i = 100^{-i}$. Apply Proposition 4.2 with $\epsilon = \epsilon_1$ and $L_1 = 10$, then we get $g_1 = (\varphi_1, \phi_1, \rho_1)$ and R_1. Set $N_1 = 2L_1^2 R_1^3$. We also have $\rho_1 \leqslant N_1^{-4}$ after possibly scaling $N_1^{-4}\rho_1(r)$. Note that the Ricci curvature will increase if we change $\rho(r)$ into $N^{-1}\rho(r)$.

We construct g_{i+1} by induction. Assume we have already constructed g_i and N_i, and $\varphi_i(r) = \phi_i(r) = kr$, $\rho(r) = \lambda_i$ on $r \in (0, N_i^{-1})$. Again apply Proposition 4.2 with ϵ_{i+1} and $L_{i+1} = 10^{i+1}$, then we get $(\tilde\varphi_{i+1}, \tilde\phi_{i+1}, \tilde\rho_{i+1})$ and R_{i+1}. After scaling $\varphi(r) = (2N_i R_{i+1})^{-1}\tilde\varphi_{i+1}(2N_i R_{i+1} r)$, $(\tilde\varphi_{i+1}, \tilde\phi_{i+1})$ agree with (φ_i, ϕ_i) on $r \in [(2N_i)^{-1}, N_i^{-1}]$. Although $\tilde\rho_{i+1}$ may not agree with ρ_i, we can make them equal by scaling both. So we can glue them to get g_{i+1}. Set $N_{i+1} = 4N_i L_{i+1}^2 R_{i+1}^4$, then $\rho_{i+1} \leqslant N_{i+1}^{-4}$ after possible scaling.

Next we modify g_i on $r \in (0, 2R_i N_i^{-1})$ by Lemma 5.1, then we get $\hat g_i$, which is also smooth at $r = 0$. Set $\hat M = \mathbb{R}^{n+1} \times \mathbb{S}^m \times \mathbb{S}^2$. Now $(\hat M^{m+n+3}, \hat g_i, 0)$ are a sequence of pointed complete smooth metric with $\mathrm{Ric}(\hat g_i) \geqslant 0$, then by Gromov's precompactness theorem, up to subsequence, there exists a metric space (X, d) such that

$$(\hat M, \hat g_i, 0) \xrightarrow{GH} (X, d, p).$$

On one hand, first note that for $A_i := N_i R_i^{-1} L_i^{-1/2} \to \infty$, the rescaled metrics $(\hat M, A_i \hat g_j, 0)$ for $j \geqslant i$ become

$$\varphi_j|_{(L_i^{-1/2}, L_i^{1/2})} \leqslant cL_i^{-1/2}, \quad \phi_j|_{(L_i^{-1/2}, L_i^{1/2})} = (1-\epsilon_i)r, \quad \rho_j|_{(L_i^{-1/2}, L_i^{1/2})} \leqslant N_i^{-5}.$$

Let $j \to \infty$ and denote $A_{a,b}(X, d, p) := \{x \in X : a < d(x,p) < b\}$, then

$$d_{GH}\left(A_{L_i^{-1/2}, L_i^{1/2}}(X, A_i d, p), A_{L_i^{-1/2}, L_i^{1/2}}(\mathbb{R}^{n+1}, g_0, 0^{n+1})\right) \leqslant \Psi(i^{-1}).$$

Then let $i \to \infty$, we have

$$(X, A_i d, p) \xrightarrow{GH} (\mathbb{R}^{n+1}, g_0, 0^{n+1}).$$

On the other hand, note that for $B_i := \frac{1}{2}N_i R_i^{-3} L_i^{-3/2} \to \infty$*, the rescaled metrics* $(\hat{M}, B_i \hat{g}_j, 0)$ *for* $j \geqslant i$ *become*

$$\varphi_j|_{(L_i^{-1/2}, L_i^{1/2})} = (1-\epsilon_i)r, \quad \phi_j|_{(L_i^{-1/2}, L_i^{1/2})} \leqslant c L_i^{-1/2}, \quad \rho_j|_{(L_i^{-1/2}, L_i^{1/2})} \leqslant N_i^{-5}.$$

Then

$$d_{GH}\left(A_{L_i^{-1/2}, L_i^{1/2}}(X, B_i d, p), A_{L_i^{-1/2}, L_i^{1/2}}(\mathbb{R}^{m+1}, g_0, 0^{m+1})\right) \leqslant \Psi(i^{-1}).$$

Then let $i \to \infty$*, we have*

$$(X, B_i d, p) \xrightarrow{GH} (\mathbb{R}^{m+1}, g_0, 0^{m+1}).$$

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