NON-POSITIVITY OF THE HEAT EQUATION WITH NON-LOCAL ROBIN BOUNDARY CONDITIONS

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Abstract. We study heat equations $\partial_t u + Lu = 0$ on bounded Lipschitz domains Ω, where $L = -\operatorname{div}(A\nabla\,\cdot\,)$ is a second-order uniformly elliptic operator with generalised Robin boundary conditions of the form $\nu \cdot A\nabla u + Bu = 0$, where $B \in \mathcal{L}(L^2(\partial\Omega))$ is a general operator. In contrast to large parts of the literature on non-local Robin boundary conditions we also allow for operators B that destroy the positivity preserving property of the solution semigroup $(e^{-tL})_{t\geq 0}$. Nevertheless, we obtain ultracontractivity of the semigroup under quite mild assumptions on B. For a certain class of operators B we demonstrate that the semigroup is in fact eventually positive rather than positivity preserving.

1. Introduction

1.1. **Main results and outline of paper.** Let L be a second-order differential operator in divergence form, meaning that L acts as $Lu = -\operatorname{div}(A\nabla u)$ on functions u defined on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, where the coefficient matrix $A = (a_{ij})$ satisfies $a_{ij} \in L^\infty(\Omega; \mathbb{C})$ and the uniform ellipticity condition

$$\operatorname{Re}\left(\overline{\xi}^\top A(x)\xi\right) \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{C}^n, x \in \Omega) \tag{1.1}$$

for some constant $\alpha > 0$. In this article, we study solutions $u = u(t,x)$ to the parabolic equation

$$\frac{\partial u(t,x)}{\partial t} + Lu(t,x) = 0 \qquad \text{for } (t,x) \in (0,\infty) \times \Omega \tag{1.2}$$

with generalised Robin boundary conditions of the form

$$\nu \cdot A\nabla u + Bu = 0 \qquad \text{on } \partial\Omega,$$

where B is a bounded linear operator on $L^2(\partial\Omega)$. The classical Robin boundary conditions are recovered by taking B to be a multiplication operator given by a function $b \in L^\infty(\partial\Omega)$. On the other hand, the general form of the boundary operator we consider allows the possibility of *non-local* boundary conditions, for instance if B is an integral operator

$$(Bf)(x) := \int_{\partial\Omega} k(x,y) f(y)\, dy, \qquad f \in L^2(\partial\Omega)$$

induced by a measurable function $k : \partial\Omega \times \partial\Omega \to \mathbb{C}$.

Many of our results are most easily formulated by considering the semigroup $(e^{-tL})_{t\geq 0}$ generated by $-L$. The main theme in our investigation is the lack of positivity of solutions. For classical boundary conditions (Dirichlet, Neumann, Robin, mixed) — which, to emphasise, are local — it is well-known that a *positivity preserving property* holds for the evolution equation. This means that given an

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initial function $u_0 \geq 0$, the solution $u(t,\cdot) = e^{-tL}u_0$ satisfies $u(t,x) \geq 0$ for all $x \in \Omega$ and for all $t > 0$. For non-local Robin boundary conditions, this does not hold in general, and in fact the positivity preserving property can be easily characterised in terms of B, see Proposition 4.1. For the positivity preserving case, there is a substantial body of work on non-local Robin boundary conditions — even for operators B that are unbounded on the boundary space $L^2(\partial\Omega)$, see Subsection 1.2 for details.

In contrast, we are mostly interested in the case where positivity is not preserved. We focus on two questions that arise in this situation. On the one hand, we study whether one still has *ultracontractivity* of the semigroup $(e^{-tL})_{t\geq 0}$, which is the property that for each $t > 0$, the operator e^{-tL} maps $L^2(\Omega)$ into $L^\infty(\Omega)$. This property is commonly shown by combining a Sobolev embedding theorem with an interpolation result which requires the semigroup to be bounded for small times on the spaces L^1 and L^∞. Without positivity, this boundedness is not straightforward to obtain. On the other hand, given that the semigroup $(e^{-tL})_{t\geq 0}$ will not preserve positivity in general, we will give sufficient conditions for the weaker property of *eventual positivity*. This property means that, given an initial function $u_0 \geq 0$, one has $u(t,\cdot) \geq 0$ not necessarily for all $t > 0$, but only for sufficiently large times. For this purpose, we employ recently developed tools in the theory of eventually positive C_0-semigroups, which we will describe briefly in Section 1.3.

To give the reader an overview of the main results, we state the following theorem, which is a simplified combination of Theorems 3.3 and 4.6.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and assume that the coefficient matrix A of L consists of real-valued functions. Let B be a self-adjoint linear operator on $L^2(\partial\Omega)$ that leaves $L^\infty(\partial\Omega)$ invariant.*

(i) (Ultracontractivity) *If B is order bounded on $L^2(\partial\Omega)$, then $e^{-tL}\big(L^2(\Omega)\big) \subseteq L^\infty(\Omega)$ for all $t > 0$.*

(ii) (Uniform eventual positivity) *If, in addition to* (i)*, B is positive semi-definite and $B\mathbf{1}_{\partial\Omega} = 0$, then solutions to* (1.2) *enjoy the following property: there exist $t_0 \geq 0$ and $\delta > 0$ such that for every initial function $0 \leq u_0 \in L^2(\Omega)$, the corresponding solution $u = u(t,x;u_0)$ satisfies*

$$u(t,x) \geq \delta \left(\int_\Omega u_0 \, dx \right) \qquad \text{for all } t \geq t_0, \ x \in \Omega.$$

In the context of operator theory on Banach lattices, the order boundedness condition on B in (i) — whose definition we recall in Subsection 2.2 — is quite natural, and moreover can be easily checked in practice.

We recall some preliminaries and functional analytic properties of the operator L in Section 2. Section 3 is devoted to ultracontractivity of the semigroup (Theorem 3.3). For our results on eventual positivity, we require some spectral conditions on B, which turn out to be directly related to spectral properties of the differential operator L. In the present paper, we consider two simple conditions: firstly, the case $B\mathbf{1}_{\partial\Omega} = 0$ (as in the Theorem above) will be discussed in Section 4, and the condition $\langle B\mathbf{1}_{\partial\Omega}, \mathbf{1}_{\partial\Omega}\rangle < 0$ will be treated in Section 5. In the latter section, we focus the analysis on the special case that Ω is a ball.

1.2. **Earlier work on non-local Robin problems.** Non-local Robin boundary conditions appear in the literature as far back as the 1950's, due to Feller in his seminal work on Markov diffusions in one dimension [25]. Thanks to the probabilistic connection, positive semigroups arise very naturally in this context. Closer to the current day, specific examples of non-local Robin conditions have appeared in the study of Schrödinger operators [46], a model of Bose condensation [47], a reaction-diffusion equation [31], and in a model of a thermostat [32]. The latter paper demonstrated explicitly that the associated semigroup could fail to be positive. Possibly one of the earliest treatments of non-local Robin conditions in a general functional analytic framework appears in the work of Gesztesy and Mitrea [27]. The development of the abstract theory continued in collaboration with other co-authors in [28, 29, 30], with a particular focus on sesquilinear forms, positive semigroups and Gaussian estimates.

Non-local Robin conditions of a different type, closer to Feller's original inspiration, appear in the work of Arendt, Kunkel, and Kunze [7]. Using modern developments in semigroup theory, the authors were able to deduce smoothing properties (the strong Feller property and holomorphy), contractivity, and analyse the asymptotic behaviour of the semigroup. Further variations on the boundary operators and even extensions to nonlinear equations may be found in [4, 48, 49].

Except for [32, 30], the works mentioned above all feature Robin boundary conditions that produce positive semigroups. The more subtle property of eventual positivity was first analysed in the papers [18, 19], within a general theory of eventually positive semigroups in Banach lattices. The specific models of the thermostat and Bose condensation ([32] and [47] respectively) are revisited in [18, Section 6], where conditions on the boundary operator B are given such that the associated semigroup is non-positive but eventually positive.

1.3. **A taste of eventual positivity.** The phenomenon of eventually positive solutions to linear evolution equations was known for quite some time in finite dimensions [41] and in some concrete examples of partial differential equations, e.g. in fourth-order parabolic equations [26]. In [16], Daners investigated eventual positivity for the semigroup generated by the Dirichlet-to-Neumann operator on the unit disk. This case study motivated the development of the general theory of eventually positive semigroups using abstract techniques from Banach lattices and operator theory, in collaboration with Kennedy and the first-named of the present authors in the articles [17, 18, 19]. Since the publication of these works, the theory on eventual positivity has branched off in various directions. The interested reader may consult the survey article [34] for a 'bird's-eye view' of the subject of eventual positivity and more references to recent developments. In particular, we mention that the functional analytic approach to eventual positivity has proved to be especially useful for evolution equations with higher-order differential operators, see e.g. [3] and [22].

One of the core ingredients in the abstract study of eventual positivity is the spectral theory of positive operators, motivated by two celebrated results: the Perron-Frobenius theorems in finite dimensions and the Krein-Rutman theorem in infinite dimensions. Consequently, the existence of a positive leading eigenfunction of the differential operator and certain spectral considerations are crucial in order to apply the results of the theory. Another essential ingredient for the theory in infinite dimensions is a certain smoothing condition on the semigroup, which often translates to ultracontractivity in PDE applications. The present article will demonstrate both of the core ingredients in action. This feature of the general theory explains the specific spectral assumptions on the boundary operator B that we will consider in Sections 4 and 5, and also the need for an ultracontractivity result. An ultracontractivity result is implicitly contained in [30, Theorem 3.6], since Gaussian estimates are proved there. However, the assumptions in this theorem are considerably different from ours; see the discussion in Remark 3.9 for details.

Finally, one could wonder what happens if one or both of the core ingredients mentioned above are not available. This can happen, for instance, if one is interested in higher-order evolution equations on unbounded domains (spectral conditions fail) or non-smooth domains (smoothing condition fails). In such cases, the development of a general theory is still very much open; however, recent work of Arora [9] and the second author of the present paper [39] represents some steps in this direction.

2. Setting the stage

In this preliminary section, we collect some basic facts that will be essential to our analysis, and also fix notations and conventions.

2.1. **Lipschitz domains.** Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set, and define $U_\delta := \{x' \in \mathbb{R}^{n-1} : |x'| < \delta\}$ for $\delta > 0$. We say that Ω is a *Lipschitz domain* if its boundary $\partial\Omega$ is locally the graph of a Lipschitz function. Precisely, this means the following: for all $x_0 \in \partial\Omega$, there exist $\delta, \varepsilon > 0$, an orthogonal transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a Lipschitz function $\varphi : U_\delta \to \mathbb{R}$ such that

$$U := T^{-1}(\{(x', x_n) \in U_\delta \times \mathbb{R} : |x_n - \varphi(x')| < \varepsilon\})$$

is a neighbourhood of x_0, and

$$\Omega \cap U = T^{-1}(\{(x', x_n) \in U_\delta \times \mathbb{R} : 0 < x_n - \varphi(x') < \varepsilon\}).$$

We denote the $L^p(\Omega)$ norm simply as $\|\cdot\|_p$, and L^p norms on the boundary space $\partial\Omega$ will be denoted by $\|\cdot\|_{p,\partial\Omega}$.

2.2. **Banach lattices and positive operators.** We assume some familiarity with basic aspects in the theory of Banach lattices and recall here some key definitions. The *principal ideal* generated by a positive vector u in a Banach lattice E is defined as

$$E_u := \{v \in E : |v| \leq cu \text{ for some constant } c \geq 0\}.$$

In fact, E_u is itself a Banach lattice when equipped with the *gauge norm*

$$\|f\|_u := \inf\{c \geq 0 : |f| \leq cu\} \qquad (f \in E_u),$$

and the embedding $E_u \hookrightarrow E$ is continuous; see [45, Proposition II.7.2 and its Corollary] for details. A positive vector $u \in E$ is called *quasi-interior* if E_u is dense in E. The following characterisation [11, Example 10.16] is well-known in Banach lattice theory: for a σ-finite measure space (Ω, μ) and $E = L^p(\Omega, \mu)$ with $1 \leq p < \infty$, a positive vector $u \in E$ is quasi-interior if and only if $u(x) > 0$ for μ-almost every $x \in \Omega$.

We also recall that the notation $[f, g]$ in a Banach lattice denotes the *order interval*

$$[f, g] := \{u \in E : f \leq u \leq g\}.$$

A subset S of a Banach lattice E is called *order bounded* if there exists a positive vector $g \in E$ such that $|f| \leq g$ for all $f \in S$. A linear operator $T : E \to E$ on a Banach lattice E is called *order bounded* if it maps order bounded sets to order bounded sets. One sees immediately from the definitions that a positive operator is order bounded. Note that in a Banach lattice, every order bounded operator (hence every positive operator) is automatically continuous, see for example [50, Theorem 18.4].

A *complex Banach lattice* is, by definition, the complexification of a real Banach lattice. Thus, if E is a complex Banach lattice, the underlying *real part* is denoted by $E_\mathbb{R}$, and then $E = E_\mathbb{R} \oplus iE_\mathbb{R}$. Some technicalities are required to extend the underlying lattice norm to the complexification. However, we do not require the details in this article, and refer the interested reader to [33, Appendix C] and the references therein.

A closed operator $A : D(A) \subseteq E \to E$ on a complex Banach lattice is said to be *real* if

$$D(A) = D(A) \cap E_{\mathbb{R}} + iD(A) \cap E_{\mathbb{R}} \quad \text{and} \quad A(D(A) \cap E_{\mathbb{R}}) \subseteq E_{\mathbb{R}}.$$

Clearly, positive operators are real.

2.3. **Semigroup theory.** Recall that the *spectral bound* of a closed operator $A : D(A) \subseteq X \to X$ on a Banach space X is defined by

$$\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in [-\infty, \infty],$$

where $\sigma(A)$ denotes the *spectrum* of A. If A is the generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on X, then the *growth bound* of the semigroup is the quantity

$$\omega_0(A) := \inf\left\{\omega \in \mathbb{R} : \exists\, M_\omega \geq 1 \text{ s.t. } \|e^{tA}\|_{\mathcal{L}(X)} \leq M_\omega e^{\omega t}\, \forall\, t \geq 0\right\}.$$

It is a standard fact that the resolvent operator $R(\lambda, A) := (\lambda - A)^{-1}$ can be represented by the Laplace transform of the semigroup whenever $\operatorname{Re}\lambda > \omega_0(A)$; namely

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} e^{tA} f\, dt$$

converges as an improper Riemann integral (and in fact even as a Bochner integral) for all $f \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(A)$. We will use this fact in the proof of Theorem 3.3 below.

In general we have $\mathrm{s}(A) \leq \omega_0(A)$, and strict inequality is possible. If E is a Banach lattice, a semigroup $(e^{tA})_{t\geq 0}$ on E is called *positive* if each operator e^{tA} is a positive operator on E. In that case, the Laplace transform representation is valid even for $\operatorname{Re}\lambda > \mathrm{s}(A)$ (but now only as an improper Riemann integral and not as a Bochner integral, in general), see [11, Theorem 12.7] for a direct proof.

Finally, if A is the generator of a C_0-semigroup on a complex Banach lattice E, then the semigroup $(e^{tA})_{t\geq 0}$ is real (i.e. each operator e^{tA} is real) if and only if A is real. For our purposes, this is the only aspect that involves complex Banach lattices, since the abstract results on eventual positivity used in Sections 4 and 5 are formulated in this setting.

2.4. **Duality.** If $(e^{tA})_{t\geq 0}$ is a C_0-semigroup on a Banach space E, then the *dual semigroup* $((e^{tA})')_{t\geq 0}$ on E' is always weak*-continuous, and the dual operator A' is the *weak* generator* of the dual semigroup. Some details can be found in [23, p. 61]. Hence it is justified to use the notation $e^{tA'} := (e^{tA})'$. Moreover, if E is *reflexive*, then the dual semigroup is also strongly continuous as a consequence of the fact that the weak and weak* topologies on E' coincide and the (non-trivial) result [23, Theorem I.5.8] that weak continuity already implies strong continuity for operator semigroups on Banach spaces.

If E is the Hilbert space $L^2(\Omega)$, we can identify E with E' via the Riesz isomorphism, which is an anti-linear map. Then A' induces an operator $A^* : D(A^*) \subseteq E \to E$ called the *adjoint* of A. On the other hand, we can identify E with E' in a 'Banach space way', namely by sending each $u \in E$ to the functional $v \mapsto \int_\Omega uv\, dx$. This map is linear (instead of anti-linear), and is compatible with the standard identification of $(L^p)'$ with $L^{p/(p-1)}$. Under the 'Banach' identification, A' induces another operator on E, denoted again by A' by abuse of notation, and one can show that

$$D(A^*) = \{u \in E : \overline{u} \in D(A')\}, \quad A^*u = \overline{A'\overline{u}} \quad \forall\, u \in D(A^*).$$

However, this complication disappears if we consider real operators, since $A^* = A'$ in that case.

2.5. **Non-local Robin boundary conditions via forms.** In the sequel, we employ general results in the theory of sesquilinear forms, and hence will consider complex-valued functions unless otherwise stated. For example, $L^2(\Omega)$ will mean $L^2(\Omega;\mathbb{C})$, and so on. We define a sesquilinear form on $L^2(\Omega)$ by

$$\begin{aligned} &\operatorname{dom}(\mathfrak{a}_B) := H^1(\Omega) \\ &\mathfrak{a}_B[u,v] := \int_\Omega A\nabla u \cdot \overline{\nabla v}\, dx + \int_{\partial\Omega} (B\gamma(u))\overline{\gamma(v)}\, d\sigma \qquad (u,v \in H^1(\Omega)) \end{aligned} \tag{2.1}$$

where A, B, and Ω have been introduced at the beginning of the introduction, σ denotes the surface measure on $\partial\Omega$, and $\gamma \in \mathcal{L}(H^1(\Omega), L^2(\partial\Omega))$ is the trace operator. The closedness and continuity of the form on $H^1(\Omega)$ follow easily from the uniform ellipticity (1.1) and the trace inequality

$$\|\gamma(u)\|_{2,\partial\Omega}^2 \le C\|u\|_{H^1(\Omega)}^2.$$

Clearly the form $\mathfrak{a}$ is densely defined. We then obtain its associated operator L_B, defined by

$$\begin{aligned} &L_B u = Lu \\ &\operatorname{dom}(L_B) = \Big\{ u \in H^1(\Omega) : \exists\, v \in L^2(\Omega) \text{ s.t. } \mathfrak{a}_B[u,\phi] = \langle v, \phi\rangle_{L^2(\Omega)}\ \forall\, \phi \in H^1(\Omega) \Big\}. \end{aligned} \tag{2.2}$$

When A is the identity matrix everywhere on Ω, then L_B acts on its domain as the differential operator $L = -\Delta$ and we call L_B the *non-local Robin Laplacian* associated to B in this case. For ease of expression, we will call L_B a 'generalised Laplacian' in the general case where it acts as $L = -\operatorname{div}(A\nabla\cdot)$. It will also be convenient to define

$$\mathfrak{b}[f,g] := \int_{\partial\Omega} (Bf)\overline{g}\, d\sigma = \langle Bf, g\rangle_{L^2(\partial\Omega)} \qquad (f,g \in L^2(\partial\Omega)). \tag{2.3}$$

We follow the convention that $\mathfrak{a}[u] := \mathfrak{a}[u,u]$ denotes the quadratic form corresponding to a sesquilinear form $\mathfrak{a}$.

Remark 2.1. For bounded Lipschitz domains, it is a well-known but non-trivial fact that the norm

$$\|u\|_V := \left(\|\nabla u\|_2^2 + \|\gamma(u)\|_{2,\partial\Omega}^2\right)^{1/2} \tag{2.4}$$

is equivalent to the usual H^1 norm. However, the V-norm is necessary to develop a theory of Robin boundary value problems on *arbitrary* domains, and in this general setting the V-norm is stronger. See [13, 14] for much more on this subject. In order to keep the functional analytic setting simple in this paper, we will focus on Lipschitz domains.

Further fundamental properties of $\mathfrak{a}_B$ and L_B are collected below.

Proposition 2.2. *The form (2.1) and the associated generalised Laplacian L_B (2.2) satisfy the following properties.*

(i) *The operator L_B is densely defined, closed, and has compact resolvent. It is self-adjoint if B is self-adjoint and the matrix $A(x)$ is symmetric for each $x \in \Omega$.*
(ii) *The form (2.1) is H^1-elliptic; that is, there exist constants $c, \omega > 0$ such that*

$$\operatorname{Re}\mathfrak{a}_B[u] + \omega\|u\|_2^2 \ge c\|\nabla u\|_2^2. \tag{2.5}$$

Moreover, the semigroup $(e^{-tL_B})_{t\ge 0}$ is analytic on $L^2(\Omega)$.

Proof. We freely use standard facts from the theory of sesquilinear forms, e.g. [42, Section 1.2].

(i) The compactness of the resolvent is a direct consequence of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, which is compact since Ω is a bounded Lipschitz domain. Clearly, if B is self-adjoint and $A(x)$ is symmetric for each $x \in \Omega$, the form $\mathfrak{a}_B$ is then symmetric ($\mathfrak{a}_B[u,v] = \overline{\mathfrak{a}_B[v,u]}$ for all $u, v \in H^1(\Omega)$), and thus the associated operator (2.2) is self-adjoint. The other properties were already discussed above.

(ii) By [27, Lemma 2.5], for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|\gamma(u)\|_{2,\partial\Omega}^2 \le \varepsilon \|\nabla u\|_2^2 + C_\varepsilon \|u\|_2^2 \qquad \forall\, u \in H^1(\Omega). \tag{2.6}$$

From this result, we obtain

$$|\mathfrak{b}[\gamma(u)]| \le \|B\|_{\mathcal{L}(L^2(\partial\Omega))}(\varepsilon\|\nabla u\|_2^2 + C_\varepsilon\|u\|_2^2) \quad \forall\, u \in H^1(\Omega).$$

If α is the ellipticity constant of L from (1.1), it consequently holds that

$$\begin{aligned} \operatorname{Re} \mathfrak{a}_B[u] &\ge \alpha\|\nabla u\|_2^2 + \operatorname{Re}\mathfrak{b}[\gamma(u)] \\ &\ge \alpha\|\nabla u\|_2^2 - \|B\|(\varepsilon\|\nabla u\|_2^2 + C_\varepsilon\|u\|_2^2). \end{aligned}$$

For concreteness, we choose $\varepsilon \le \alpha(2\|B\|)^{-1}$ so that

$$\operatorname{Re} \mathfrak{a}_B[u] + \omega\|u\|_2^2 \ge \frac{\alpha}{2}\|\nabla u\|_2^2 \qquad \forall\, u \in H^1(\Omega)$$

with $\omega := \|B\| C_\varepsilon$. This proves the H^1-ellipticity, and the analyticity of the semigroup $(e^{-tL_B})_{t\ge 0}$ follows by standard results. □

The following corollary on well-posedness of the non-local Robin boundary value problem now follows immediately from the preceding proposition and the Lax-Milgram theorem.

Corollary 2.3. *Let $B \in \mathcal{L}(L^2(\partial\Omega))$. Then there exists $\lambda_0 > 0$ (depending only on B and Ω) such that for every $f \in L^2(\Omega)$ and every $\lambda \ge \lambda_0$, the boundary value problem*

$$\begin{aligned} \lambda u + Lu &= f &&\text{in } \Omega \\ \nu \cdot A\nabla u + B\gamma(u) &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{2.7}$$

has a unique weak solution $u \in H^1(\Omega)$.

Remark 2.4. We briefly remark on higher regularity of solutions. Since the weak solution of (2.7) belongs to $H^1(\Omega)$, we have $B\gamma(u) \in L^2(\partial\Omega)$, and thus the problem of higher regularity reduces to the study of the inhomogeneous Neumann problem

$$\begin{aligned} \lambda u + Lu &= f &&\text{in } \Omega \\ \nu \cdot A\nabla u &= g &&\text{on } \partial\Omega \end{aligned}$$

with $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. It is well-known that even in the case where $L = -\Delta$, $f \in C^\infty(\overline{\Omega})$ and $g \equiv 0$, one cannot expect the 'usual' result $u \in H^2(\Omega)$. Counterexamples can be constructed on suitable cones in $\mathbb{R}^2$ — some details are given in [36, Theorem 1.4.5.3]. Thus the regularity problem is highly non-trivial, and has been extensively studied, notably in the works of Jerison and Kenig [37], Fabes et al. [24], and Savaré [44]. In the special case that $L = -\Delta$, one has the precise result

$$u \in H^{3/2}_\Delta(\Omega) := \{u \in H^{3/2}(\Omega) : \Delta u \in L^2(\Omega)\};$$

see [22, Proposition 2.4] for a proof.

On the other hand, if we consider f, g with sufficiently high integrability, then Nittka has shown in [40, Proposition 3.6] that the solution to the Neumann problem belongs to the Hölder space $C^{0,\gamma}(\overline{\Omega})$ for some $\gamma > 0$.

3. Ultracontractivity

In this and subsequent sections, we will use some recurring assumptions.

Assumption 3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain in the sense described in Section 2.1, and let L_B be the generalised Laplacian with coefficient matrix $A = (a_{ij})$ and boundary operator B, arising from the form $\mathfrak{a}_B$ defined in Section 2.5. We assume that

(A1) $a_{ij} \in L^\infty(\Omega; \mathbb{R})$, i.e. the entries of A consist of bounded real-valued functions.
(A2) A is uniformly elliptic with lower bound $\alpha > 0$, i.e. it satisfies (1.1).
(A3) B is a real and order bounded linear operator on $L^2(\partial\Omega)$.

We begin with a simple sufficient condition for the contractivity of the semigroup $(e^{-tL_B})_{t\geq 0}$.

Proposition 3.2. *Suppose A satisfies (A1) and (A2). If $B + B^*$ is positive semi-definite on $L^2(\partial\Omega)$, then the following equivalent assertions hold:*

(i) *The form $\mathfrak{a}_B$ is accretive.*
(ii) *$(e^{-tL_B})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(\Omega)$.*

Proof. We first establish the equivalence of (i) and (ii). As discussed in the previous section, $\mathfrak{a}_B$ is densely defined, closed, and continuous. If (i) also holds, the contractivity and strong continuity of the semigroup $(e^{-tL_B})_{t\geq 0}$ on $L^2(\partial\Omega)$ then follows from [42, Proposition 1.51], which uses the well-known Lumer-Phillips theorem. Conversely, if (ii) holds, then

$$0 \geq \lim_{t\downarrow 0} \frac{1}{t} \operatorname{Re} \langle e^{-tL_B} u - u, u\rangle = -\operatorname{Re} \langle L_B u, u\rangle = -\operatorname{Re} \mathfrak{a}_B[u]$$

for all $u \in \operatorname{dom}(L_B) \subset \operatorname{dom}(\mathfrak{a}_B)$. Since $\operatorname{dom}(L_B)$ is a core of $\mathfrak{a}_B$ — see for example [42, Lemma 1.25] — the result extends by density to all $u \in \operatorname{dom}(\mathfrak{a}_B)$, and $\mathfrak{a}_B$ is therefore accretive.

Now observe that

$$\operatorname{Re} \mathfrak{a}_B[u] \geq \alpha \|\nabla u\|^2_{L^2(\Omega)} + \operatorname{Re} \langle B\gamma(u), \gamma(u)\rangle_{L^2(\partial\Omega)}$$

for all $u \in \operatorname{dom}(\mathfrak{a}_B) = H^1(\Omega)$, where $\alpha > 0$ is the ellipticity constant of L from (1.1). It follows that $\mathfrak{a}_B$ is accretive if the form $\mathfrak{b}$ defined in (2.3) is accretive. However, this holds if and only if $B + B^*$ is positive semi-definite, due to the identity

$$\langle (B + B^*) f, f\rangle_{L^2(\partial\Omega)} = \langle Bf, f\rangle_{L^2(\partial\Omega)} + \langle f, Bf\rangle_{L^2(\partial\Omega)} = 2 \operatorname{Re} \langle Bf, f\rangle_{L^2(\partial\Omega)}$$

for all $f \in L^2(\partial\Omega)$. This completes the proof. □

If $A = -\Delta$ with local Robin boundary conditions, it is well-known that the semigroup $(e^{-tA})_{t\geq 0}$ is *ultracontractive*, which means

$$e^{-tA}(L^2(\Omega)) \subset L^\infty(\Omega) \quad \text{for all } t > 0.$$

In fact, this regularity property holds for semigroups generated by quite general divergence-form uniformly elliptic operators. The well-known monograph of Davies [20] covers the classical case of symmetric Markov semigroups, while more recent developments specifically for the local Robin boundary value problem are found in [14]. We now discuss sufficient conditions for the non-local Robin Laplacian to generate an ultracontractive semigroup. For this purpose, it is natural to assume that the operator B appearing in the boundary conditions acts boundedely on $L^1(\partial\Omega)$ and $L^\infty(\partial\Omega)$. As we will see, this ensures that the associated semigroup $(e^{-tL_B})_{t\geq 0}$ is also bounded on $L^1(\Omega)$ and $L^\infty(\Omega)$, which then enables us to apply existing results on ultracontractivity.

We now state the main result of this section.

Theorem 3.3 (Ultracontractivity)**.** *Let L_B satisfy Assumption (3.1). Suppose in addition that*

$$\begin{aligned}&B(L^\infty(\partial\Omega)) \subseteq L^\infty(\partial\Omega) \text{ and}\\ &B \text{ extrapolates to a bounded operator on } L^1(\partial\Omega).\end{aligned} \tag{3.1}$$

Then the semigroup $(e^{-tL_B})_{t\geq 0}$ satisfies $e^{-tL_B}(L^2(\Omega)) \subset L^\infty(\Omega)$ for all $t > 0$, and there exist constants $c, \mu > 0$ such that

$$\|e^{-tL_B}\|_{\mathcal{L}(L^2,L^\infty)} \leq ct^{-\mu/4}, \qquad 0 < t \leq 1 \tag{3.2}$$

and such that the dual semigroup $e^{-t(L_B)'} = (e^{-tL_B})'$ also satisfies (3.2).

At this point, it is worth commenting on the assumptions of the theorem and a few aspects of Banach lattice theory.

Remark 3.4. (i) Since B is a real operator and the coefficients of A are also real, it follows that L_B and hence the semigroup operators e^{-tL_B} are real operators. As a general rule, an elliptic operator L produces a real semigroup if and only if its coefficients are real; see [42, Proposition 4.1] for a precise statement and proof.

(ii) On L^p spaces — more generally on Dedekind complete Banach lattices — it is easy to describe all order bounded operators. Indeed, it follows from the Riesz-Kantorovich theorem [1, Theorem 1.18] that a linear operator T on L^p is order bounded if and only if it is the difference of two positive operators. Moreover, every order bounded operator T admits a *modulus* $|T|$ [1, Definition 1.12, Theorem 1.14]. The proof of the Riesz-Kantorovich theorem also yields a decomposition $T = T^+ - T^-$ where $T^\pm$ are positive operators, T^+ is the smallest positive operator that dominates T, and $|T| = T^+ + T^-$.

If B is an order bounded operator on E, then its adjoint B^* is also order bounded on E. Indeed, as explained above, $B = B^+ - B^-$ is necessarily the difference of two positive operators, and it is easily seen that the adjoint of a positive operator on a Banach lattice is again a positive operator defined on the dual space. Consequently $B^* = (B^+)^* - (B^-)^*$ is also the difference of two positive operators and thus order bounded.

(iii) It follows from (3.1) and the closed graph theorem that B extrapolates to a bounded linear operator on $L^1(\partial\Omega)$ and $L^\infty(\partial\Omega)$. The same then holds for $|B| = B^+ + B^-$; this is also a consequence of the formula for $|B|$ in the Riesz-Kantorovich theorem. Assumption (3.1) is of course satisfied when B is a multiplication operator associated to a function $\beta \in L^\infty(\partial\Omega)$, and hence the theorem includes the case of local Robin boundary conditions. Actually, both condition (3.1) and order boundedness are redundant in the local case, which is straightforward to check.

Let us also comment on some aspects of duality.

Remark 3.5. (i) Note that the (norm) dual E' of a Banach lattice E is again a Banach lattice, with the functionals ordered in the obvious way — namely, $\varphi \geq \psi$ in E' if and only if $\varphi(x) \geq \psi(x)$ for all $0 \leq x \in E$. A proof of this fact may be found in [1, Theorem 4.1]. We mention for the sake of the curious reader that the Riesz-Kantorovich theorem is invoked to define the lattice operations, but we will not require this further detail.

(ii) By the assumption that A has real coefficients, the adjoint form $\mathfrak{a}_B^*$, defined by $a_B^*[u,v] := \overline{\mathfrak{a}_B[v,u]}$ for all $u, v \in \operatorname{dom}(\mathfrak{a}_B)$, is given by

$$\mathfrak{a}_B^*[u,v] := \int_\Omega A^\top \nabla u \cdot \overline{\nabla v}\, dx + \langle B^*\gamma(u), \gamma(v)\rangle_{L^2(\partial\Omega)}$$

so that the adjoint operator of $-L_B$ is given by an analogous differential operator with coefficients $A^\top$ and boundary operator B^*. It follows from (i) and Section 2.4 that this operator coincides with the generator of the dual semigroup, so that we have $(e^{-tL_B})' = e^{-tL'_B}$. (With slightly more pedantic notation, we could write L_B as $L_{A,B}$, and hence $L'_{A,B} = L_{A^\top,B^*}$).

The core idea for the proof of Theorem 3.3 is as follows: we will construct a *positive* semigroup $(S(t))_{t\geq 0}$ such that $|e^{-tL_B}f| \leq S(t)|f|$ for all $f \in L^2(\Omega)$, and show that $S(t)$ is bounded on $L^\infty(\Omega)$ for small times. An analogous statement holds for the dual semigroup, which then implies that $(e^{-tL_B})_{t\geq 0}$ extrapolates to a consistent family of semigroups acting on the L^p scale, $1 \leq p \leq \infty$. This is the starting point from which we can then apply known results [6, Sections 7.2, 7.3] on ultracontractivity.

Domination of semigroups can be effectively checked using the Ouhabaz criterion [42, Theorem 2.21], which we will use in the following simplified form.

Proposition 3.6 (Ouhabaz)**.** *Let $\mathfrak{a}$, $\mathfrak{b}$ be two densely defined, accretive, continuous and closed sesquilinear forms on $L^2(\Omega)$ with common domain $D(\mathfrak{a}) = D(\mathfrak{b}) = H^1(\Omega)$, and let $(e^{-tA})_{t\geq 0}$ and $(e^{-tB})_{t\geq 0}$ be their associated semigroups. Assume that $(e^{-tB})_{t\geq 0}$ is positive and $(e^{-tA})_{t\geq 0}$ is real. Then the following assertions are equivalent:*

(i) *$|e^{-tA}f| \leq e^{-tB}|f|$ for all $f \in L^2(\Omega)$;*
(ii) *$\mathfrak{b}[|u|, |v|] \leq \mathfrak{a}[u, v]$ for all $u, v \in D(\mathfrak{a})_\mathbb{R} = H^1(\Omega; \mathbb{R})$ such that $uv \geq 0$.*

We require some facts about the Robin Laplacian with local boundary conditions on Lipschitz domains.

Lemma 3.7. *Let $\beta \in L^\infty(\partial\Omega; \mathbb{R})$ be given and consider the generalised (local) Robin Laplacian L_β which is assumed to satisfy Assumption 3.1. For all sufficiently large $\lambda > 0$, the function $u := \lambda R(\lambda, -L_\beta)\mathbf{1}$ satisfies $\frac{1}{2} \leq u \leq 2$ on $\overline{\Omega}$.*

Proof. It was shown by Nittka [40, Theorem 4.3] that the part $-L_{\beta,C}$ of $-L_\beta$ in $C(\overline{\Omega})$ generates a (positive and analytic) C_0-semigroup on $C(\overline{\Omega})$. It follows that the function $\lambda R(\lambda, -L_\beta)\mathbf{1} = \lambda R(\lambda, -L_{\beta,C})\mathbf{1}$ is in $C(\overline{\Omega})$ and converges to $\mathbf{1}$ with respect to the sup norm as $\lambda \to \infty$. This gives the claim for sufficiently large λ. We note that the inequality makes sense on $\overline{\Omega}$ rather than only on Ω since u is in $C(\overline{\Omega})$. ☐

Remark 3.8. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and consider the Banach lattice $E = L^2(\Omega)$. The function u arising from Lemma 3.7 satisfies

$$E_u = L^\infty(\Omega) \quad \text{(with equivalent norms)}.$$

Indeed, since there exist $\delta, \delta' > 0$ such that $\delta\mathbf{1} \leq u \leq \delta'\mathbf{1}$ in E, we immediately obtain $E_u = E_\mathbf{1} = L^\infty(\Omega)$. It is also clear that any bounded linear operator T on E that satisfies $T([-u, u]) \subseteq [-Cu, Cu]$ for some constant $C > 0$ is bounded on $L^\infty(\Omega)$. Then by interpolation, it follows that T is bounded on $L^p(\Omega)$ for all $2 \leq p \leq \infty$. If the dual operator T' also satisfies $T'([-u, u]) \subseteq [-Cu, Cu]$, one even obtains that T is bounded on $L^p(\Omega)$ for all $1 \leq p \leq \infty$. Moreover, we can choose a constant M independent of p such that $\|T\|_{\mathcal{L}(L^p(\Omega))} \leq M$ for all $1 \leq p \leq \infty$.

All the ingredients are now in place to prove our theorem on ultracontractivity.

Proof of Theorem 3.3. As explained in Remark 3.4, the modulus operator $|B|$ exists and leaves $L^\infty(\partial\Omega)$ invariant, so there exists a constant $c > 0$ such that $|B|\mathbf{1}_{\partial\Omega} \leq c\mathbf{1}_{\partial\Omega}$. Now we choose $\beta := 4c\mathbf{1}_{\partial\Omega} \in L^\infty(\partial\Omega; \mathbb{R})$. Choose a number $\lambda > 0$ that is larger than the growth bound of the semigroup generated by $-L_{-\beta}$ and that is so large that the function $u := \lambda R(\lambda, -L_{-\beta})\mathbf{1}$ satisfies $\frac{1}{2} \leq u \leq 2$ on $\overline{\Omega}$; such a λ exists by Lemma 3.7 (applied to the function $-\beta$ instead of β). On the boundary $\partial\Omega$ we thus get

$$\beta u = 4cu \geq 2c\mathbf{1}_{\partial\Omega} \geq 2|B|\mathbf{1}_{\partial\Omega} \geq |B|u,$$

so in short, $|B|u \leq \beta u$ on $\partial\Omega$. We can thus increase $|B|$ to obtain a positive operator $\tilde{B} \in \mathcal{L}(L^2(\Omega))$ such that $|B| \leq \tilde{B}$ and $\tilde{B}u = \beta u$ on $\partial\Omega$.

It follows from $\tilde{B}u = \beta u$ on $\partial\Omega$ that $\mathfrak{a}_{-\tilde{B}}[u,v] = \mathfrak{a}_{-\beta}[u,v]$ for all $v \in H^1(\Omega)$, so we conclude that also $u \in \operatorname{dom}(-L_{-\tilde{B}})$ and $-L_{-\tilde{B}}u = -L_{-\beta}u$. Now choose $\nu \geq \lambda$ such that ν also dominates the growth bound of $-L_{-\tilde{B}}$. One has

$$\nu u + L_{-\tilde{B}}u = \nu u + L_{-\beta}u = \lambda\mathbf{1} + (\nu - \lambda)u =: w \geq 0,$$

where the second inequality follows from the definition of u. Since ν is larger than the growth bound of $-L_{-\tilde{B}}$, the Laplace transform representation of the resolvent $R(\nu, -L_{-\tilde{B}})$ yields that $u = R(\nu, -L_{-\tilde{B}})w = \int_0^\infty e^{-\nu s}e^{-sL_{-\tilde{B}}}w\,ds$. Since $\tilde{B}$ is positive, Proposition 4.1 below shows that the semigroup $(e^{tsL_{-\tilde{B}}})_{t\geq 0}$ is positive on $L^2(\Omega)$ (note carefully the minus signs!). This together with the positivity of w gives

$$e^{-tL_{-\tilde{B}}}u = \int_0^\infty e^{-\nu s}e^{-(t+s)L_{-\tilde{B}}}w\,ds = e^{\nu t}\int_t^\infty e^{-\nu\tau}e^{-\tau L_{-\tilde{B}}}w\,d\tau \leq e^{\nu t}u$$

for all $t \geq 0$. By Remark 3.8 we thus conclude that the operators $e^{-tL_{-\tilde{B}}}$ act boundedly on $L^\infty(\Omega)$ and that there exists a constant $d > 0$ such that

$$\|e^{-tL_{-\tilde{B}}}\|_{L^\infty\to L^\infty} \leq d \qquad \forall t \in [0,1]. \tag{3.3}$$

Next we show by means of Proposition 3.6 that the semigroup generated by $-L_B$ is dominated by the semigroup generated by $-L_{-\tilde{B}}$. For all $f, g \in L^2(\partial\Omega; \mathbb{R})$, observe that

$$\langle -Bf, g\rangle_{\partial\Omega} \leq \langle |Bf|, |g|\rangle_{\partial\Omega} \leq \langle \tilde{B}|f|, |g|\rangle_{\partial\Omega}\,,$$

since $\tilde{B}$ is a positive operator and $\pm B \leq \tilde{B}$ on $L^2(\partial\Omega)$ by design. Thus we have

$$\langle -\tilde{B}|f|, |g|\rangle_{\partial\Omega} \leq \langle Bf, g\rangle_{\partial\Omega}\,. \tag{3.4}$$

If D is any of the partial derivatives $\frac{\partial}{\partial x_i}$, it holds that

$$D|u| = Du\,\operatorname{sgn}(u)$$

for all $u \in H^1(\Omega; \mathbb{R})$ (see e.g. [35, Lemma 7.6]). From this we obtain

$$\int_\Omega a_{ij}(x)\partial_i|u|\partial_j|v|\,dx = \int_\Omega a_{ij}(x)|\partial_i u||\partial_j u|\,dx$$

for all $i, j \in \{1, \ldots, n\}$ and $u, v \in H^1(\Omega; \mathbb{R})$ such that $uv \geq 0$, thanks to [35, Corollary 7.7]. By combining this above with (3.4), we deduce

$$\begin{aligned}\mathfrak{a}_{-\tilde{B}}[|u|,|v|] &= \int_\Omega A\nabla|u|\cdot\nabla|v|\,dx - \int_{\partial\Omega}(\tilde{B}\gamma(|u|))\gamma(|v|)\,d\sigma \\ &\leq \int_\Omega A\nabla u\cdot\nabla v\,dx + \int_{\partial\Omega}(B\gamma(u))\gamma(v)\,d\sigma = \mathfrak{a}_B[u,v]\end{aligned}$$

for all $u, v \in H^1(\Omega; \mathbb{R}) = \operatorname{dom}(\mathfrak{a}_B)_\mathbb{R}$ such that $uv \geq 0$. This shows that condition (ii) of Proposition 3.6 is satisfied, and therefore

$$|e^{-tL_B}f| \leq e^{tL_{-\tilde{B}}}|f| \quad \forall t \geq 0 \tag{3.5}$$

holds for all $f \in L^2(\Omega)$. Inequalities (3.3) and (3.5) together show that e^{-tL_B} is bounded on $L^\infty(\Omega)$ for every $t \geq 0$ and that its norm on this space is bounded for $t \in [0,1]$.

We also need boundedness of e^{-tL_B} on $L^1(\Omega)$. To achieve this, we recall from Remark 3.4 that B^* is also order bounded and moreover it is clear that B^* satisfies (3.1) as well. Hence the preceding constructions apply to B^* in place of B — one easily checks the details using Remark 3.5. Thus there exists a possibly different positive operator $\tilde{B}$ on $L^2(\partial\Omega)$ such that (3.3) and (3.5) hold with B^* in place of B. This shows that $e^{-tL'_B} = (e^{-tL_B})'$ is bounded on $L^\infty(\Omega)$, thus e^{-tL_B} is bounded on $L^1(\Omega)$ for every $t \geq 0$.

Everything is now in place to show that $(e^{-tL_B})_{t\geq 0}$ is ultracontractive. Write $T(t) := e^{-tL_B}$ and $V = H^1(\Omega)$. In dimension $d = 1$, ultracontractivity is obtained immediately from the inclusions

$$T(t)(L^2(\Omega)) \subset D(L_B) \subset V \hookrightarrow L^\infty(\Omega) \quad \forall\, t > 0.$$

The first inclusion comes from the analyticity of the semigroup (thanks to Proposition 2.2), and the embedding $V \hookrightarrow L^\infty(\Omega)$ is elementary (e.g. see [12, Theorem 8.8]).

For higher dimensions, a bit more work is required. Firstly, by Remark 3.8, there is a constant $M > 0$ independent of p such that

$$\sup_{0\leq t\leq 1} \|T(t)\|_p \leq M, \qquad p \in [1,\infty]. \tag{3.6}$$

Thus, as is well-known, $(T(t))_{t\geq 0}$ extrapolates to a consistent family of semigroups on L^p for all $1 \leq p \leq \infty$, and is strongly continuous for $1 < p < \infty$. Moreover, since Ω is a bounded domain and in particular has finite Lebesgue measure, it follows from [6, Theorem 7.2.1] that $(T(t))_{t\geq 0}$ is even strongly continuous on L^1. The above properties together with the analyticity of the semigroup allow us to apply the characterisation [6, Theorem 7.3.2]. In particular, we use the implication (v) $\Rightarrow$ (ii): if the form domain $\operatorname{dom}(\mathfrak{a}_B) = V$ satisfies the embedding $V \hookrightarrow L^{2\mu/(\mu-2)}(\Omega)$ for some $\mu > 2$, and $V \cap L^1(\Omega)$ is dense in $L^1(\Omega)$, then the semigroup $T(t)$ is ultracontractive with the estimate

$$\|T(t)\|_{\mathcal{L}(L^2,L^\infty)} \leq ct^{-\mu/4}, \quad 0 < t \leq 1.$$

In our situation, we have $V = H^1(\Omega) \supset C_c^\infty(\Omega)$, and thus the density condition is clearly satisfied. We conclude using the Sobolev embedding theorems [2, Theorem 4.12, Part I]. If $d \geq 3$, we may choose $\mu = d$ since the embedding $V \hookrightarrow L^{2d/(d-2)}$ is valid in bounded Lipschitz domains. Finally, in dimension $d = 2$, we have $V \hookrightarrow L^q(\Omega)$ for any $2 \leq q < \infty$, so we may choose any $2 < \mu < \infty$. In each case, it follows that $(e^{-tL_B})_{t\geq 0}$ is ultracontractive, and the proof is complete. ☐

Remark 3.9. It is natural to ask if one can obtain Gaussian estimates for the semigroup $(e^{-tL_B})_{t\geq 0}$, which would yield ultracontractivity as an immediate consequence. This was already investigated in [29, 30], where the authors even allow for certain classes of unbounded boundary operators B. However, the assumptions in these articles lead to the domination property

$$|e^{-tL_B} f| \leq e^{t\Delta_N}|f| \qquad t \geq 0, f \in L^2(\Omega)$$

where Δ_N denotes the usual Neumann Laplacian (corresponding to the boundary condition $\partial_\nu u = 0$ on $\partial\Omega$), see in particular [29, Theorem 4.4] and [30, Theorem 3.3]. This is an extremely strong property as the following argument shows:

Suppose we choose B such that the semigroup $(e^{-tB})_{t\geq 0}$ is positive on $L^2(\partial\Omega)$. By Proposition 4.1, this yields the positivity of $(e^{-tL_B})_{t\geq 0}$ on $L^2(\Omega)$. However, Akhlil has shown in [5] that if $0 \leq e^{-tL_B} f \leq e^{t\Delta_N} f$ holds for all $t \geq 0$ and $0 \leq f \in L^2(\Omega)$, then the boundary conditions in L_B are necessarily local. This was later generalised to domination by semigroups associated to general local forms [10, Theorem 3.2]. Thus the problem of Gaussian estimates for 'genuinely non-local'

Robin semigroups cannot be tackled simply using domination by other semigroups associated to local forms.

4. Eventual positivity: the case $\mathrm{s}(-L_B) = 0$

As mentioned in the introduction, positivity of the leading eigenfunction of the differential operator A is an important tool to obtain eventual positivity of the semigroup $(e^{-tA})_{t\geq 0}$. For this reason, we distinguish two cases for the spectral bound of the generalised Robin Laplacian L_B. In this section, we consider $\mathrm{s}(-L_B) = 0$, while Section 5 covers the case $\mathrm{s}(-L_B) > 0$. Actually, in both cases we are able to reformulate the spectral bound condition in terms of the spectrum of the boundary operator B. This is likely to be more practical, since one expects to have more explicit information about the boundary conditions in concrete examples.

We begin with the following characterisation of positivity of the semigroup $(e^{-tL_B})_{t\geq 0}$, which was already observed in the case $L = -\Delta$ in [33, Proposition 11.7.1]. The simple proof carries over to the generalised Laplacians L_B without any difficulty.

Proposition 4.1. *Let A satisfy Assumptions (A1) and (A2). The following assertions are equivalent:*

(i) *The semigroup $(e^{-tL_B})_{t\geq 0}$ on $L^2(\Omega)$ is positive.*
(ii) *The semigroup $(e^{-tB})_{t\geq 0}$ on $L^2(\partial\Omega)$ is positive.*

Proof. We recall the Beurling-Deny criterion [42, Theorem 2.6], which states that the semigroup generated by $-L_B$ is positive if and only if $\mathfrak{a}_B[u^+, u^-] \leq 0$ for all $u \in H^1(\Omega; \mathbb{R})$. Thanks again to [35, Lemma 7.6], we obtain $\int_\Omega A\nabla u^+ \cdot \nabla u^- \, dx = 0$, and therefore

$$\mathfrak{a}_B[u^+, u^-] = \langle B\gamma(u^+), \gamma(u^-)\rangle_{L^2(\partial\Omega)} = \mathfrak{b}[\gamma(u^+), \gamma(u^-)], \tag{4.1}$$

where $\mathfrak{b}$ is defined by (2.3). The equivalence is now proved using (4.1) and by applying the Beurling-Deny criterion to $\mathfrak{b}$. $\square$

Remark 4.2. (i) Proposition 4.1 applies in particular to the classical case of local boundary conditions, since obviously the semigroup $(e^{-tB})_{t\geq 0}$ is positive when B acts as multiplication by a function $\beta \in L^\infty(\partial\Omega)$. In this classical setting, positivity of the semigroup $(e^{-tL_\beta})_{t\geq 0}$ is already well-known, see e.g. [8, Theorem 4.9] and [13, Proposition 8.1].

(ii) In the situation where Ω is an open interval $(a, b) \subseteq \mathbb{R}$, then $\partial\Omega = \{a, b\}$ and $L^2(\partial\Omega)$ can be identified with $\mathbb{C}^2$, so that B has a matrix representation

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad b_i \in \mathbb{R}, i \in \{1, 2, 3, 4\}.$$

In this case, the implication (ii) $\Rightarrow$ (i) can be strengthened: if the matrix semigroup $(e^{-tB})_{t\geq 0}$ is *eventually positive*, then $(e^{-tL_B})_{t\geq 0}$ is positive. This follows immediately from [17, Proposition 6.2], where it was proved that for 2×2 real matrix semigroups, eventual positivity implies positivity.

We now obtain a result concerning the triviality of the peripheral spectrum of the non-local Robin Laplacian.

Theorem 4.3. *Let A satisfy Assumptions (A1) and (A2). If $B + B^*$ is positive semi-definite on $L^2(\partial\Omega)$, then $\mathrm{s}(-L_B) \leq 0$ and*

$$\sigma(-L_B) \cap i\mathbb{R} \subseteq \{0\}.$$

In particular, if $\sigma(-L_B) \cap i\mathbb{R} \neq \emptyset$, then $\sigma(-L_B) \cap i\mathbb{R} = \{0\}$, any 0-eigenfunction is constant, and consequently $\dim \ker(-L_B) = 1$. *In other words, the eigenvalue 0 is* geometrically simple. *Moreover, it holds in this case that* $\mathrm{s}(-L_B) = 0$.

Proof. The assertion $\mathrm{s}(-L_B) \le 0$ is a consequence of Proposition 3.2: since $B+B^*$ is positive semi-definite, it follows that the semigroup $(e^{-tL_B})_{t\ge 0}$ is contractive, and then by a standard result in semigroup theory [23, Proposition IV.2.2] we conclude

$$\mathrm{s}(-L_B) \le \omega_0(-L_B) \le 0.$$

From previous discussions, we know that $-L_B$ has compact resolvent, and hence $\sigma(-L_B)$ consists only of eigenvalues. If $\sigma(-L_B)\cap i\mathbb{R} = \emptyset$, there is nothing more to do, hence we assume $-L_B$ has an eigenvector $v \in \mathrm{dom}(-L_B)$ such that $-Lv = i\omega v$ for some $\omega \in \mathbb{R}$ and $\|v\|_2 = 1$. Using the form associated to $-L_B$, we find $\mathfrak{a}_B[v] = \langle -Lv, v\rangle = i\omega$ and then

$$\int_\Omega A\nabla v \cdot \overline{\nabla v}\, dx = i\omega - \langle B\gamma(v), \gamma(v)\rangle_{L^2(\partial\Omega)}\,.$$

Therefore, upon taking the real part of the above equation and using the ellipticity of L, we obtain

$$0 \le \alpha\|\nabla v\|_2^2 \le \int_\Omega A\nabla v\cdot\overline{\nabla v}\, dx = -\operatorname{Re}\langle B\gamma(v),\gamma(v)\rangle_{L^2(\partial\Omega)} \le 0$$

from the assumption on B. Since $\alpha > 0$, it follows that $\nabla v = 0$, so v is a non-zero constant function, and we conclude $Lv = 0$. This shows that $\omega = 0$. Finally, since the semigroup $(e^{-tL_B})_{t\ge0}$ is analytic (recall Proposition 2.2(ii)), it is known that $\mathrm{s}(-L_B) = \omega_0(-L_B)$, see [23, Corollary IV.3.12]. However, $0 \in \sigma(-L_B)$ also implies $\mathrm{s}(-L_B) \ge 0$. We conclude $\mathrm{s}(-L_B) = 0$, and the proof is complete. ☐

Remark 4.4. (i) Assume that $\sigma(-L_B)\cap i\mathbb{R} \ne \emptyset$. The previous theorem then shows that $\mathrm{s}(-L_B) = 0$, and $\mathbf{1}$ is an associated eigenfunction. If we apply the boundary conditions, we deduce that $B\mathbf{1}_{\partial\Omega} = 0$, where $\mathbf{1}_{\partial\Omega}$ is the constant 1 function on the boundary. Conversely, if $B\mathbf{1}_{\partial\Omega} = 0$, then

$$\mathfrak{a}_B[\mathbf{1},\phi] = \langle B\gamma(\mathbf{1}),\gamma(\phi)\rangle_{L^2(\partial\Omega)} = 0 = \langle 0,\phi\rangle_{L^2(\Omega)}$$

for all $\phi \in H^1(\Omega)$. This proves that $\mathbf{1} \in \mathrm{dom}(L_B)$ and $0 \in \sigma(-L_B)$, and the associated eigenspace is spanned by the constant function $\mathbf{1}$. Hence $0\in\sigma(-L_B)$ if and only if $B\mathbf{1}_{\partial\Omega} = 0$.

(ii) The conclusion of Theorem 4.3 also holds for $-(L_B)'$ (recall Remark 3.5 for the precise description of the dual generator). Obviously the positive definiteness of $B + B^*$ is unchanged if B is replaced by B^*.

In the one-dimensional case $\Omega = (a, b)$ (see Remark 4.2), we obtain a perhaps surprising corollary for positivity of the semigroup.

Corollary 4.5. *In dimension $n = 1$, if $B + B^*$ is positive semi-definite (on $\mathbb{C}^2$) and $(1\quad 1)^\top \in \ker B$, then $(e^{-tL_B})_{t\ge0}$ is positive.*

Proof. By Remark 4.4(i), the condition $(1\quad 1)^\top \in \ker B$ is equivalent to $0 \in \sigma(-L_B)$. Hence Theorem 4.3 implies that $\sigma(-L_B)\cap i\mathbb{R} = \{0\}$, $\mathrm{s}(-L_B) = 0$, and any eigenvector v corresponding to 0 is a non-zero constant function on Ω. The positive semi-definite property of $B + B^*$ implies that the diagonal entries of B are non-negative, while the condition $(1\quad 1)^\top \in \ker B$ implies that the off-diagonal entries of B are non-positive. It follows that the matrix semigroup $(e^{-tB})_{t\ge0}$ is positive, and hence $(e^{-tL_B})_{t\ge0}$ is a positive semigroup by Proposition 4.1. ☐

In higher dimensions, we can only expect eventual positivity of the semigroup; however, we do obtain a rather strong form of eventual positivity. This is possible due to the criterion from [19, Theorem 3.1] and our earlier result on ultracontivity, Theorem 3.3.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded Lipschitz domain. Assume that L_B satisfies Assumption 3.1, and $B \in \mathcal{L}(L^2(\partial\Omega))$ also satisfies (3.1). If in addition $B + B^*$ is positive semi-definite and $B\mathbf{1}_{\partial\Omega} = 0$, then the semigroup $(e^{-tL_B})_{t\geq 0}$ has the following property: there exists $t_0 \geq 0$ and a constant $\delta > 0$ such that*

$$e^{-tL_B} f \geq \delta \left(\int_\Omega f \, dx \right) \mathbf{1} \qquad \forall t \geq t_0$$

for all $0 \leq f \in L^2(\Omega)$. We say that the semigroup is uniformly eventually strongly positive *with respect to* $\mathbf{1}$.

In the proof, we use the facts on duality and adjoints presented in the introductory Section 2.4.

Proof of Theorem 4.6. Throughout the proof, we denote $E = L^2(\Omega)$, and identify E' with E via the Riesz isomorphism, and likewise for the space $L^2(\partial\Omega)$. We also use the notation $v \gg_u 0$ for positive elements u, v in a Banach lattice to mean that $v \geq cu$ for some constant $c > 0$. We will show that the semigroup $(e^{-tL_B})_{t\geq 0}$ satisfies all the assumptions of [19, Theorem 3.1].

Since B and B^* are real operators, so are the generators $-L_B$ and $-L_B'$, and hence $(e^{-tL_B})_{t\geq 0}$ and $(e^{-tL_B'})_{t\geq 0}$ are real semigroups. Now let $v := \mathbf{1}$, which is clearly a quasi-interior point of E. Via the previously mentioned identification $E = E' = L^2(\Omega)$, we also consider $\psi := \mathbf{1}$ as the linear functional $\psi(f) = \int_\Omega f \, dx$. Due to Theorem 4.3 and Remark 4.4(i), we know that $0 \in \sigma(-L_B)$ and v is an eigenfunction for the geometrically simple and dominant eigenvalue $0 = \mathrm{s}(-L_B)$. By Remark 4.4(ii), the same assertions are true for ψ and the dual operator $-L_B'$. With slight abuse of notation, we set $u = \varphi = \mathbf{1}$ and then, tautologically, we have that $v \gg_u 0$ and $\psi \gg_\varphi 0$. Thus condition (b) of [19, Theorem 3.1] is satisfied.

From the remarks at the beginning of this proof, we see in addition that B^* is order bounded. Hence by Theorem 3.3 we obtain that $e^{-tL_B}(E) \subset E_u$ and $e^{-tL_B'}(E) \subset E_\varphi$ for all $t > 0$. This shows that condition (a) of [19, Theorem 3.1] holds, and thus the proof is complete. ☐

We now give some explicit examples of boundary operators B that satisfy the assumptions of Theorem 4.6 but for which the semigroup $(e^{-tB})_{t\geq 0}$ is *not* positive.

Example 4.7. (a) Let $0 \neq v \in L^\infty(\partial\Omega)$ be a real-valued function satisfying $\int_{\partial\Omega} v \, d\sigma = 0$, and consider the rank-1 operator defined by

$$Bf := (v \otimes v)(f) = \left(\int_{\partial\Omega} v f \, d\sigma \right) v, \qquad f \in L^2(\partial\Omega).$$

This is clearly a bounded operator on $L^2(\partial\Omega)$, and extrapolates to a bounded operator on $L^1(\partial\Omega)$ and $L^\infty(\partial\Omega)$. By construction it holds that $B\mathbf{1} = 0$. Since there exists a constant $c > 0$ such that $v \leq c\mathbf{1}$, we see that $B = c^2 \mathbf{1} \otimes \mathbf{1} - (c^2 \mathbf{1} \otimes \mathbf{1} - v \otimes v)$ is the difference of two positive operators, and is hence order bounded. As v is real, B is then self-adjoint, and moreover

$$\langle (v \otimes v) f, f \rangle_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} v f \, d\sigma \right) \left(\int_{\partial\Omega} v \overline{f} \, d\sigma \right) = \left| \int_{\partial\Omega} v f \, d\sigma \right|^2 \geq 0$$

for all $f \in L^2(\partial\Omega)$. Thus $B + B^* = 2B$ is positive semi-definite. To show that $-B$ does not generate a positive semigroup, we test the form $\mathfrak{b}$ with v itself and obtain

$$\mathfrak{b}[v^+, v^-] = \langle Bv^+, v^- \rangle_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |v^+|^2 \, d\sigma \right) \left(\int_{\partial\Omega} |v^-|^2 \, d\sigma \right) > 0,$$

which shows that form $\mathfrak{b}$ violates the Beurling-Deny criterion (see the proof of Proposition 4.1). Hence $(e^{-tB})_{t\geq 0}$ is not positive.

(b) Let us also give an example where B is not a kernel operator. Consider $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ the unit disk in $\mathbb{R}^2$ with boundary $\Gamma = \{x \in \mathbb{R}^2 : |x| = 1\}$, and let R be the operator of anticlockwise rotation by $\frac{\pi}{2}$. This operator is unitary, and its adjoint $R^* = R^{-1}$ is the clockwise rotation by $\frac{\pi}{2}$. With slight abuse of notation, we write the action of R on functions $f \in L^2(\Gamma)$ as $(R \cdot f)(x) := f(Rx)$, and define $Bf := (R - R^*) \cdot f$. Then $B + B^* = 0$ and hence is positive semi-definite, albeit in a trivial way. Since R and R^* are clearly positive operators, we also have that B is order bounded by construction, and in addition extrapolates to a bounded operator on $L^1(\Gamma)$ and $L^\infty(\Gamma)$ (since the same is true for R and R^*). The constant function $\mathbf{1}_\Gamma$ is rotationally invariant, so $B\mathbf{1}_\Gamma = 0$ is also satisfied. Again, let us show that $(e^{-tB})_{t\geq 0}$ is not positive using the Beurling-Deny criterion. For all $f, g \in L^2(\Gamma; \mathbb{R})$, we have

$$\mathfrak{b}[f, g] = \langle (R^* - R) \cdot f, g\rangle_{L^2(\Gamma)} = \int_\Gamma [f(R^*x) - f(Rx)]g(x)\, dx.$$

Let the four (open) quadrants of $\mathbb{R}^2$ be denoted by

$$Q_1 = \{x_1 > 0, x_2 > 0\}, \quad Q_2 = \{x_1 < 0, x_2 > 0\}, \quad Q_3 = -Q_1, \quad Q_4 = -Q_2,$$

and set

$$\Gamma_j := \Gamma \cap Q_j, \qquad j \in \{1, 2, 3, 4\}.$$

We construct the function

$$f := \mathbf{1}_{\Gamma_4} - \mathbf{1}_{\Gamma_1\cup\Gamma_2}, \quad \text{with } f^+ = \mathbf{1}_{\Gamma_4}, f^- = \mathbf{1}_{\Gamma_1\cup\Gamma_2}.$$

Elementary computations then yield

$$f^+(R^*x) = \mathbf{1}_{\Gamma_1}(x), f^+(Rx) = \mathbf{1}_{\Gamma_3}(x)$$

and

$$\begin{aligned}\mathfrak{b}[f^+, f^-] &= \int_\Gamma [f^+(R^*x) - f^+(Rx)]f^-(x)\, dx \\ &= \int_\Gamma (\mathbf{1}_{\Gamma_1}(x) - \mathbf{1}_{\Gamma_3}(x))\mathbf{1}_{\Gamma_1\cup\Gamma_2}(x)\, dx = \int_\Gamma \mathbf{1}_{\Gamma_1}(x)\, dx = \frac{\pi}{2} > 0,\end{aligned}$$

hence the Beurling-Deny criterion is violated by f.

Note that for the operators B in Example 4.7, ultracontractivity of $(e^{-tL_B})_{t\geq 0}$ — which is essential to get eventual positivity — cannot be shown by means of the domination result in [30, Theorem 3.3]. Indeed, for this one would need a negative operator on $L^2(\partial\Omega)$, called Θ_1 in [30], such that

$$|e^{-tL_B} f| \leq e^{-tL_{\Theta_1}}|f| \leq e^{t\Delta_N}|f|$$

for all $f \in L^2(\Omega)$ and $t \geq 0$, where Δ_N denotes the Neumann Laplacian. But as explained in Remark 3.9, this implies that the form $\mathfrak{a}_{\Theta_1}$ is local and hence it follows from Proposition 3.6 that $\mathfrak{a}_B$ is local too. Yet, this is clearly not true for the operators B in Example 4.7.

5. Eventual positivity: the case $\mathrm{s}(-L_B) > 0$

In Theorem 4.3, the spectral condition $\mathrm{s}(L_B) = 0$ conveniently led to a positive constant eigenfunction, which in turn yielded the eventual strong positivity in Theorem 4.6, but the arguments cannot be adapted to the case $\mathrm{s}(-L_B) > 0$. Thus, instead of developing a general theory, we change the perspective of our analysis in this section and will show how symmetry conditions on the domain Ω and the coefficients of the differential operator can yield a positive leading eigenfunction of the non-local Robin Laplacian. The interaction between symmetry and spectral theory is a classical area of study. Indeed, quite general properties of eigenfunctions of the Dirichlet Laplacian on symmetric domains were extensively investigated by Pereira [43].

5.1. **Some notation and terminology.** We begin with some very general observations. Let G be a group which acts on $\mathbb{R}^n$ and preserves the Lebesgue measure. We say that a domain $\Omega \subseteq \mathbb{R}^n$ is *G-invariant* if $g(\overline{\Omega}) = \overline{\Omega}$ for all $g \in G$. For such a domain, the group G has a natural action on functions $u \in H := L^2(\Omega)$, given by 'left translation' operators

$$(L_g u)(x) := u(g^{-1}x).$$

In fact, it is easy to verify that the map $G \ni g \mapsto L_g \in \mathcal{L}(H)$ is a representation of G, and this is the reason for using g^{-1} instead of g in the definition. In Pereira's terminology, this is called the *quasi-regular representation* of G.

Given an element $u \in H$, the *orbit* of u under the action of G is given by

$$G(u) := \{L_g u : g \in G\}.$$

We say that u is G-invariant if $G(u) = \{u\}$. The *symmetric subspace* is defined to be the closed subspace

$$H_G := \{u \in H : L_g u = u \quad \forall\, g \in G\}$$

consisting of G-invariant functions. In other words, H_G is precisely the fixed space of the set $\{L_g : g \in G\}$. The orthogonal complement $H_G^\perp$ of H_G is therefore called the *anti-symmetric* subspace, and H admits the decomposition $H = H_G \oplus H_G^\perp$.

A closed operator $A : D(A) \subseteq H \to H$ is called *G-equivariant* if for all $u \in D(A)$ and $g \in G$ it holds that

$$L_g u \in D(A) \quad \text{and} \quad A(L_g u) = L_g(Au). \tag{5.1}$$

In regards to spectral theory, the following observation is crucial. If A is a G-equivariant operator and u is an eigenfunction with eigenvalue λ, then we observe

$$A(L_g u) = L_g(Au) = \lambda L_g u \qquad \forall\, g \in G,$$

which shows that $L_g u$ is also an eigenfunction corresponding to λ. Hence the λ-eigenspace V_λ is invariant under all the operators L_g and defines a sub-representation of the quasi-regular representation. Moreover, the dimension of V_λ is at least the dimension of the span of the orbit $G(u)$. We remark that an eigenvalue is called *G-simple* if the equality

$$V_\lambda = \operatorname{span} G(u)$$

holds, i.e. the action of the group on a single eigenfunction generates the entire eigenspace. See [43] for further investigations on G-simplicity.

The main example of a G-equivariant operator for our purposes is, of course, the Laplacian.

Example 5.1. (a) It is straightforward to check that the Laplacian on $L^2(\mathbb{R}^n)$, with natural domain $H^2(\mathbb{R}^n)$, is G-equivariant for any subgroup G of the orthogonal group $O(n)$. If we consider a bounded G-invariant domain Ω, then the Laplacian with boundary conditions remains G-equivariant provided that the boundary operator enjoys the same property. (More precisely, we consider the action of G on the boundary $\partial\Omega$, and require that the boundary operator commutes with the group action in the sense of (5.1) for all $u \in L^2(\partial\Omega)$). Thus, Δ equipped with homogeneous Dirichlet or Neumann conditions is G-equivariant. The generalised Robin Laplacian (2.2) is also G-equivariant whenever the operator $B \in \mathcal{L}(L^2(\partial\Omega))$ commutes with the group action, i.e. $B(L_g f) = L_g(Bf)$ for all $g \in G$ and $f \in L^2(\partial\Omega)$.

(b) We can also consider a non-divergence form operator

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \operatorname{tr}(A(x)D^2 u)$$

where the coefficient matrix $A = (a_{ij})$ satisfies Assumptions (A1) and (A2), and $D^2 u$ denotes the Hessian matrix of u. For sufficiently smooth functions u and all $g \in G$, we calculate

$$L[u(gx)] = \operatorname{tr}(A(x)g^\top (D^2u)(gx)g) = \operatorname{tr}(gA(x)g^\top (D^2u)(gx)).$$

On the other hand, $(Lu)(gx) = \operatorname{tr}(A(gx)(D^2u)(gx))$. Hence, L is G-equivariant if

$$gA(x)g^\top = A(gx) \qquad \forall\, g \in G, x \in \Omega.$$

Now let G be a subgroup of $O(n)$, and assume that the coefficients of A are G-invariant functions, i.e. $a_{ij}(gx) = a_{ij}(x)$ for all $1 \le i, j \le n$, $g \in G$, and $x \in \Omega$. Then the above condition reduces to the simple commutation relation

$$gA(x) = A(x)g \qquad \forall\, g \in G, x \in \Omega.$$

For divergence form operators $L = -\operatorname{div}(A\nabla\cdot)$, we can replace (5.1) by a condition more suitable for operators arising from a sesquilinear form. Moreover, for our purposes, it will be sufficient to specialise G to a group of rotations. In this case, the left translation operators satisfy $(L_g)^* = L_{g^{-1}}$. Assuming that $AL_g = L_g A$ as in (5.1), a formal calculation yields

$$\langle AL_g u, v\rangle = \langle L_g Au, v\rangle = \langle Au, (L_g)^* v\rangle = \langle Au, L_{g^{-1}} v\rangle\,,$$

which motivates the following definition.

Definition 5.2. Let G be a subgroup of $O(n)$. Suppose $A : D(A) \subseteq H \to H$ is a closed operator arising from a sesquilinear form $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \subset H \times H \to \mathbb{C}$. Then $\mathfrak{a}$ is called G-equivariant if for all $u \in D(\mathfrak{a})$ and $g \in G$, it holds that

$$L_g u \in D(\mathfrak{a}) \quad \text{and} \quad \mathfrak{a}[L_g u, v] = \mathfrak{a}[u, L_{g^{-1}} v]$$

for all $v \in D(\mathfrak{a})$. In this case, we also call A a G-equivariant operator.

5.2. **Symmetry and the Robin Laplacian.** We turn our attention to the analysis of the generalised Robin Laplacian L_B in the presence of symmetry. Our assumptions here are quite different from the previous sections, so we highlight them separately.

Assumption 5.3. Let G be a subgroup of $O(n)$, and $\Omega \subseteq \mathbb{R}^n$ a bounded, Lipschitz, G-invariant domain. As before, let L_B denote the generalised Laplacian with coefficient matrix $A = (a_{ij})$ and boundary operator $B \in \mathcal{L}(L^2(\partial\Omega))$. We assume the following.

(B1) A continues to satisfy Assumptions (A1) and (A2) (i.e. uniformly elliptic with bounded coefficients). In addition, we assume that each a_{ij} is a G-invariant function, $a_{ij} = a_{ji}$, and

$$gA(x) = A(x)g \qquad \forall\, g \in G, x \in \Omega. \tag{5.2}$$

(B2) B is a real, self-adjoint, and G-equivariant operator.

Remark 5.4. The commutation relation (5.2) is quite a strong assumption, especially in the case when $G = O(n)$. Indeed, if a matrix $h \in \mathbb{R}^{n\times n}$ commutes with all $g \in O(n)$, then h commutes with all real $n \times n$ matrices, and hence h is a multiple of the identity matrix. Let us briefly describe the proof.

Write $\mathcal{M}$ for the algebra of real $n \times n$ matrices, and denote by e the unit element of $\mathcal{M}$ (i.e. the identity matrix). If h commutes with all $g \in O(n)$, then h also commutes with all g in the sub-algebra of $\mathcal{M}$ generated by $O(n)$, which we call

$\mathcal{A}$. We will show that $\mathcal{A} = \mathcal{M}$. If k is skew-symmetric, then $\exp(tk) \in O(n)$ for all $t \geq 0$. Hence $k \in \mathcal{A}$ since it is obtained as the limit as $t \downarrow 0$ of the elements $t^{-1}(\exp(tk) - e) \in O(n)$.

Now suppose $k \in \mathcal{M}$ is symmetric. After rescaling, we may assume that the eigenvalues of k (which are all real) lie in the interval $[-1, 1]$. By the spectral theorem for symmetric matrices, there exists $g \in O(n)$ and a diagonal matrix d with entries in $[-1, 1]$ such that $k = gdg^\top$. The matrix d can then be expressed as a convex combination of diagonal matrices with entries in $\{-1, 1\}$, and thus $k \in \mathcal{A}$. Hence $\mathcal{A}$ contains all symmetric matrices too.

Finally, any $h \in \mathcal{M}$ can be decomposed into a symmetric and skew-symmetric part simply by $h = \frac{1}{2}(h + h^\top) + \frac{1}{2}(h - h^\top)$. Thus $\mathcal{A} = \mathcal{M}$ as claimed.

Under Assumption 5.3, it is straightforward to verify that the form $\mathfrak{a}_B$ is symmetric and G-equivariant in the sense of Definition 5.2. Hence L_B is self-adjoint and G-equivariant. In contrast to the results in Section 4, we no longer require the 'accretivity' assumption that $B + B^*$ is positive semi-definite, but we require L_B to be self-adjoint in order to employ a variational principle in the following result.

Theorem 5.5. *Let Assumption 5.3 be satisfied, and write $L^2_0(\partial\Omega)$ for the subspace of $L^2(\partial\Omega)$ consisting of mean-zero functions, i.e. $f \in L^2(\partial\Omega)$ such that $\int_{\partial\Omega} f \, d\sigma = 0$. Let B_0 denote the restriction of B to $L^2_0(\partial\Omega)$. If*

$$\langle B\mathbf{1}_{\partial\Omega}, \mathbf{1}_{\partial\Omega}\rangle_{L^2(\partial\Omega)} < 0 \tag{5.3}$$

then $\mathrm{s}(-L_B) > 0$. Moreover, there exists a constant $\beta > 0$ (depending only on the domain and $-L_B$) such that if

$$\|B_0\|_{\mathcal{L}(L^2_0(\partial\Omega), L^2(\partial\Omega))} \leq \beta^{-1}$$

then every eigenfunction associated to $\mathrm{s}(-L_B)$ is G-symmetric, i.e. belongs to the space $H_G \cap \mathrm{dom}(L_B)$.

Proof. As above, we write H_G for the symmetric subspace of $L^2(\Omega)$. For brevity, we also write $F := L^2(\partial\Omega)$ and $F_0 := L^2_0(\partial\Omega)$. Clearly $\mathbf{1} \in H_G \cap H^1(\Omega)$, so if u is an anti-symmetric function in $H^1(\Omega)$, the Poincaré inequality [38, Theorem 13.27] implies

$$\|u\|_2^2 \leq c_0 \|\nabla u\|_2^2 \qquad \forall\, u \in H_G^\perp \cap H^1(\Omega)$$

with a constant $c_0 > 0$ depending only on Ω. We combine this result with the standard trace inequality $\|\gamma(u)\|^2_{2,\partial\Omega} \leq c_1 \|u\|^2_{H^1(\Omega)}$ to obtain

$$\|\gamma(u)\|_F^2 \leq c_1 \|u\|^2_{H^1(\Omega)} \leq (c_0 + 1) c_1 \|\nabla u\|_2^2$$

for all $u \in H_G^\perp \cap H^1(\Omega)$. Now we set $C := (c_0 + 1)c_1$ and $\beta := C\alpha^{-1}$, and use the uniform ellipticity of the coefficient matrix A to get

$$\mathfrak{a}_B[u] \geq \alpha \|\nabla u\|_2^2 + \langle B\gamma(u), \gamma(u)\rangle_F \geq \big(\alpha C^{-1} \|\gamma(u)\|_F - \|B\gamma(u)\|_F\big) \|\gamma(u)\|_F$$

for all $u \in H_G^\perp \cap H^1(\Omega)$. Since $\gamma(u) \in F_0$, the assumption $\|B\|_{\mathcal{L}(F_0, F)} \leq \beta^{-1} = \alpha C^{-1}$ and the above inequality imply that

$$\mathfrak{a}_B[u] \geq 0 \qquad \text{for all } u \in H_G^\perp \cap H^1(\Omega). \tag{5.4}$$

Since B is self-adjoint, the form $\mathfrak{a}_B$ is symmetric. Thus, by a standard variational principle the smallest eigenvalues of LB is given by

$$-\mathrm{s}(-L_B) = \min_{0 \neq u \in H^1(\Omega)} \frac{\mathfrak{a}_B[u]}{\|u\|_2^2}.$$

We claim that condition (5.3) implies $\mathrm{s}(-L_B) > 0$. Indeed, if we use the constant function $\mathbf{1}$ in the form $\mathfrak{a}_B$ associated to the operator L_B, we obtain

$$-\mathrm{s}(-L_B) \leq \frac{\mathfrak{a}_B[\mathbf{1}]}{\|\mathbf{1}\|_2^2} = \frac{1}{|\Omega|^2} \langle B\gamma(\mathbf{1})), \gamma(\mathbf{1})\rangle_{L^2(\partial\Omega)} < 0,$$

or equivalently $\mathrm{s}(-L_B) > 0$ as asserted. However, (5.4) shows that $\mathfrak{a}_B[u] \geq 0$ for all anti-symmetric functions u in $H^1(\Omega)$. Hence, the normalised eigenfunction associated to $\mathrm{s}(-L_B)$ which attains the minimum of the Rayleigh quotient must be a symmetric function. This completes the proof. □

Recall that an action of a group G on a (non-empty) set X is called *transitive* if for every pair of distinct $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$. If $f : X \to \mathbb{C}$ is a G-invariant function and G acts transitively on X, then f must be constant. Indeed, for $x \neq y \in X$, we choose $g \in G$ such that $y = g \cdot x$, and then $f(y) = f(g \cdot x) = f(x)$ by the invariance of f. With this in mind, and together with some PDE arguments, we can say much more about the eigenfunctions for $\mathrm{s}(-L_B)$ in the case that Ω is a ball.

Theorem 5.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a ball, let Assumption 5.3 be satisfied, and suppose that G acts transitively on the unit sphere $\partial\Omega = \mathbb{S}^{n-1}$. If B satisfies $\langle B\mathbf{1}_{\partial\Omega}, \mathbf{1}_{\partial\Omega}\rangle_{L^2(\partial\Omega)} < 0$ and in addition*

$$\|B_0\|_{\mathcal{L}(L^2_0(\partial\Omega), L^2(\partial\Omega))} \leq \beta^{-1}, \tag{5.5}$$

using the notation of Theorem 5.5, then $\mathrm{s}(-L_B)$ admits a rotationally symmetric eigenfunction $\varphi \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that $\varphi(x) \geq \delta$ for all $x \in \overline{\Omega}$, for some $\delta > 0$. Consequently $\mathrm{s}(-L_B)$ is a simple eigenvalue, and it is the only positive eigenvalue.

Proof. Clearly, $\overline{\Omega}$ is invariant under G. By Theorem 5.5, if φ is an eigenfunction associated to $\lambda := \mathrm{s}(-L_B) > 0$, then φ is G-invariant. The remarks on transitivity preceding the corollary then show that $\varphi|_{\partial\Omega}$ is constant. The standard interior regularity results, e.g. [35, Corollary 8.11], yield $\varphi \in C^\infty(\Omega)$, from which it now follows that $\varphi \in C(\overline{\Omega})$ (because φ is constant on the boundary). Since the equation $(-L - \lambda)\varphi = 0$ holds in Ω with $-\lambda < 0$, the maximum principle for divergence-form elliptic equations [35, Theorem 8.1] implies that $\varphi \geq 0$ in Ω.

We know that φ is constant on $\partial\Omega$. The maximum principle ([35, Corollary 8.2] in particular) also shows that this constant cannot be 0. Hence, after multiplying by a suitable scalar, we may assume $\varphi|_{\partial\Omega} = 1$. The argument in [15, Theorem 3.1], which employs the weak Harnack inequality, now shows that $\varphi(x) > 0$ for all $x \in \Omega$. Combined with the continuity of φ on $\overline{\Omega}$ and $\varphi|_{\partial\Omega} = 1$, we can conclude that there exists $\delta > 0$ such that $\varphi(x) \geq \delta$ for all $x \in \overline{\Omega}$.

The rotational symmetry and strict positivity of φ also imply the simplicity of the eigenvalue $\mathrm{s}(-L_B)$. Finally, if $\mu \in (0, \mathrm{s}(-L_B))$ is another positive eigenvalue, then the μ-eigenspace is orthogonal to φ. However, since $-\mu < 0$, the Rayleigh quotient arguments in the proof of Theorem 5.5 show that a μ-eigenfunction ψ must be symmetric. By the arguments in the preceding paragraphs, we conclude that some scalar multiple $c\psi$ is non-negative in Ω, which contradicts the orthogonality with φ. Hence the proof is concluded. □

The preceding theorem now leads to sufficient conditions for uniform eventual positivity of the semigroup $(e^{-tL_B})_{t\geq 0}$ in a setting different from Section 4.

Corollary 5.7. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a ball, let Assumption 5.3 be satisfied, and suppose that G acts transitively on the unit sphere $\partial\Omega = \mathbb{S}^{n-1}$. If in addition B is order bounded and satisfies $B(L^\infty(\partial\Omega)) \subseteq L^\infty(\partial\Omega)$, then the semigroup $(e^{-tL_B})_{t\geq 0}$*

is uniformly eventually strongly positive with respect to $\mathbf{1}$*; that is, there exists* $t_0 \geq 0$ *and a constant* $\delta > 0$ *such that*

$$e^{-tL_B} f \geq \delta \left(\int_\Omega f \, dx \right) \mathbf{1} \qquad \forall\, t \geq t_0$$

for all $0 \leq f \in L^2(\Omega)$*.*

Proof. The semigroup $(e^{-tL_B})_{t\geq 0}$ is self-adjoint and real. From the assumptions that B is order bounded, self-adjoint on $L^2(\partial\Omega)$, and leaves $L^\infty(\partial\Omega)$ invariant, we deduce from Theorem 3.3 that $e^{-tL_B}(L^2(\Omega)) \subset L^\infty(\Omega)$ for all $t > 0$.

Let $\varphi \gg_{\mathbf{1}} 0$ be the principal eigenfunction (unique after normalisation) associated to $s(-L_B)$ from Theorem 5.6. Since $s(-L_B)$ is a simple eigenvalue, [19, Corollary 3.5] can now be applied to deduce that $(e^{-tL_B})_{t\geq 0}$ is uniformly eventually strongly positive with respect to $\mathbf{1}$. □

Remark 5.8. By Theorem 5.6, there exist constants $c_1, c_2 > 0$ such that $c_1\mathbf{1} \leq \varphi \leq c_2\mathbf{1}$. Remark 3.8 then shows that φ and $\mathbf{1}$ actually generate the same principal ideal, namely $L^\infty(\Omega)$. Thus we have

$$e^{-tL_B}(L^2(\Omega)) \subset E_\varphi \qquad \forall\, t > 0$$

as well, and there exist constants $c_t > 0$ such that $|e^{-tL_B} f| \leq c_t \varphi$ for all $t > 0$. In the terminology introduced by Davies and Simon [21], this shows that the semigroup $(e^{-tL_B})_{t\geq 0}$ is *intrinsically ultracontractive*. This property is well-known for a large variety of second-order elliptic operators with *local* boundary conditions.

Example 5.9. We revisit the setting of the Bose condensation example in [18, Section 6]. Let $\Omega = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ be the unit disk and let $G = SO(2)$. Functions on the boundary $\Gamma = \{x \in \mathbb{R}^2 : |x| = 1\}$ can be identified with 2π-periodic functions defined on $(-\pi, \pi)$. Thus, given a $q \in L^1(\Gamma)$, the convolution operator

$$(Bf)(x) := (q * f)(x) = \int_{-\pi}^{\pi} q(x-y) f(y) \, dy$$

is well-defined and continuous on $L^2(\Gamma)$ by Young's convolution inequality, and clearly also leaves $L^\infty(\Gamma)$ invariant. The decomposition $q = q^+ - q^-$ into a difference of positive functions shows that B is order bounded. To ensure that B is self-adjoint, we also assume that the Fourier coefficients of q, given by

$$q_k := \int_{-\pi}^{\pi} q(x) e^{-ikx} \, dx \qquad (k \in \mathbb{Z}),$$

are real-valued.

The disk Ω is of course invariant under the group G. It is well-known that G is the symmetry group of the circle, and consequently each $g \in G$ can be identified with a point on Γ. For $f \in L^2(\Gamma)$, we therefore have the translation operators $(L_y f)(x) = f(x-y)$ for all $x, y \in (-\pi, \pi)$. With this identification, we obtain

$$L_y(Bf) = (q*f)(\cdot - y) = \int_{-\pi}^{\pi} q(z) f(\cdot - y - z) \, dz = \int_{-\pi}^{\pi} q(z)(L_y f)(\cdot - z) \, dz = B(L_y f)$$

for all $f \in L^2(\Gamma)$, which shows that B is G-equivariant. Finally, to obtain the condition $\langle B\mathbf{1}, \mathbf{1}\rangle_{L^2(\Gamma)} < 0$, one simply requires the 0-th Fourier mode to satisfy $q_0 < 0$.

To summarise, an operator $B \in \mathcal{L}(L^2(\Gamma))$ given by convolution with a function $q \in L^1(\Gamma)$ satisfies all the assumptions of Corollary 5.7 if q is real-valued and $q_0 < 0$. Thus we obtain quite a general family of convolution operators for which we have uniform eventual positivity for the heat equation with non-local Robin boundary conditions on the unit disk.

We will not investigate whether the smallness condition (5.5) is optimal. However, the following computations show that Theorem 5.6 is not true without some upper bound on the norm of B.

Example 5.10. In this example, we consider the 1-dimensional 'ball' $\Omega = (-1, 1)$. The group $G = O(1)$ of order 2 acts on Ω by reflection, and clearly Ω is G-invariant. The space $L^2(\partial\Omega)$ may be identified with $\mathbb{C}^2$, and G acts on $\mathbb{C}^2$ by permuting the coordinates. Define the boundary operator

$$B = b \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

with a parameter $b > 0$. One easily verifies that B satisfies (B2) — in particular it commutes with the permutation matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it is G-equivariant. Our objective is to show explicitly that if $b > 0$ is sufficiently large, then the conclusions of Theorem 5.6 fail to hold.

For $\lambda > 0$, we solve the eigenvalue problem

$$u'' = \lambda u \quad \text{in } \Omega, \qquad \begin{pmatrix} -u'(-1) \\ u'(1) \end{pmatrix} + B \begin{pmatrix} u(-1) \\ u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A fundamental solution to the eigenvalue equation is given by

$$u(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x), \qquad \mu := \sqrt{\lambda}.$$

It is helpful to observe that

$$\begin{pmatrix} -u'(-1) \\ u'(1) \end{pmatrix} = \begin{pmatrix} \mu \sinh \mu & -\mu \cosh \mu \\ \mu \sinh \mu & \mu \cosh \mu \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{and}$$
$$\begin{pmatrix} u(-1) \\ u(1) \end{pmatrix} = \begin{pmatrix} \cosh \mu & -\sinh \mu \\ \cosh \mu & \sinh \mu \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

After some simple computations, the boundary conditions can be rewritten as

$$\begin{pmatrix} \mu \sinh \mu - b \cosh \mu & 3b \sinh \mu - \mu \cosh \mu \\ \mu \sinh \mu - b \cosh \mu & \mu \cosh \mu - 3b \sinh \mu \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

Therefore, the condition for λ to be an eigenvalue of $-L_B$ is

$$\det \begin{pmatrix} \mu \sinh \mu - b \cosh \mu & 3b \sinh \mu - \mu \cosh \mu \\ \mu \sinh \mu - b \cosh \mu & \mu \cosh \mu - 3b \sinh \mu \end{pmatrix}$$
$$= 2(\mu \sinh \mu - b \cosh \mu)(\mu \cosh \mu - 3b \sinh \mu) = 0.$$

Consider the two functions given by

$$f_1(\mu) = \mu \tanh \mu \quad \text{and} \quad f_2(\mu) = \frac{1}{3} \mu \coth \mu \tag{5.7}$$

for $\mu \geq 0$, which are plotted in Figure 1. Then $\lambda = \mu^2$ is an eigenvalue of $-L_B$ if and only if $f_1(\mu) = b$ or $f_2(\mu) = b$.

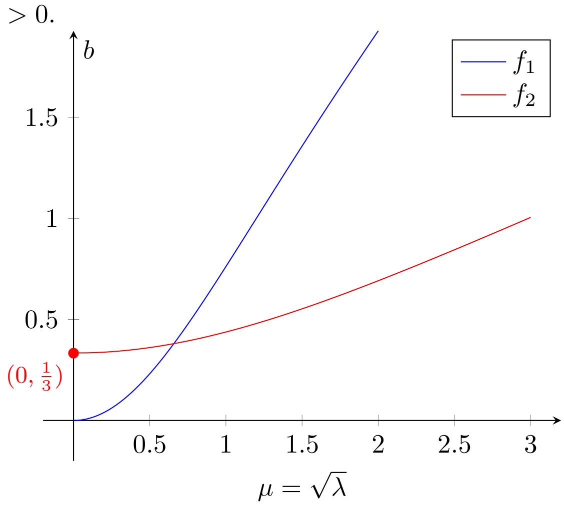
Observe that $f_2(0) = \frac{1}{3}$, and when $b < \frac{1}{3}$, there is precisely one positive eigenvalue; namely $\lambda = \mu^2$, where μ is the unique positive solution to the equation $b = f_1(\mu) = \mu \tanh \mu$. In that case, the first column of the matrix in equation (5.6) consists of zeroes, and thus we can take $c_2 = 0$ and $c_1 > 0$ arbitrary to obtain the eigenfunction

$$\varphi(x) = c_1 \cosh(\mu x), \quad \mu = f_1^{-1}(b).$$

Evidently φ is strictly positive and symmetric on $[-1, 1]$.

When $b = \frac{1}{3}$, a second solution $\mu = 0$ (yielding the eigenvalue $\lambda = 0$) appears. By simple calculations, one checks readily that the 0-eigenspace is spanned by the

FIGURE 1. The graphs of f_1, f_2 encode the positive eigenvalues for a given $b > 0$.



function $\psi(x) = x$, which is notably not positive on $[-1, 1]$. The curves f_1, f_2 cross at the value

$$\mu^* = \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.658479,$$

in which case the corresponding eigenspace is two-dimensional and spanned by $\{\cosh(\mu^* x), \sinh(\mu^* x)\}$. As b increases further, the double eigenvalue splits. However, the leading eigenvalue now arises from f_2, and one can verify using (5.6) that the corresponding eigenspace is spanned by $\sinh(\mu x)$. Thus the positivity of the leading eigenfunction is lost, and it is the smaller positive eigenvalue arising from f_1 that yields a positive eigenfunction.

Open problem. Consider again the simple case $L = -\Delta$. As far as spectral theory is concerned, the results of this article are complementary to the analysis in [18], in the sense that we have developed a general theory under the condition $s(\Delta_B) \geq 0$, whereas the examples considered in Theorems 6.11 and 6.13 in the aforementioned paper satisfy $s(\Delta_B) < 0$. Eventual strong positivity of the semigroup $(e^{t\Delta_B})_{t\geq 0}$ was proved by showing strong positivity of the resolvent operator $(\lambda - \Delta_B)^{-1}$ at $\lambda = 0$, and this was achieved by explicit computations. At the time of writing, it is not clear to us how to obtain general results in the case of a negative spectral bound.

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