

Linear-Function Correcting Codes

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Abstract—In this paper, we study linear-function correcting codes, a class of codes designed to protect linear function evaluations of a message against errors. The work "Function-Correcting Codes" by Lenz et al. 2023 provides a graphical representation for the problem of constructing function-correcting codes. We use this graph to get a lower bound on the redundancy required for function correction. By considering the function to be a bijection, such an approach also provides a lower bound on the redundancy required for classical systematic error correcting codes. For linear-function correction, we characterise the spectrum of the adjacency matrix of this graph, which gives rise to lower bounds on redundancy. The work "Function-Correcting Codes" gives an equivalence between function-correcting codes and irregular-distance codes. We identify a structure imposed by linearity on the distance requirement of the equivalent irregular-distance code which provides a simplified Plotkin-like bound. We propose a version of the sphere packing bound for linear-function correcting codes. We identify a class of linear functions for which an upper bound proposed by Lenz et al., is tight. We also identify a class of functions for which coset-wise coding is equivalent to a lower dimensional classical error correction problem.

I. INTRODUCTION

In a typical communication system, the sender wishes to transmit a message to the receiver through an erroneous channel. All symbols of the message are considered equally important and the objective is to create an error correcting code (ECC) with a decoder that can recover the entire message accurately. However, in certain scenarios, the receiver may only be interested in a specific function of the message. In such cases, if the sender knows the function, the message can be encoded so as to ensure that the desired attribute is protected against errors. This approach gives rise to a new class of codes known as function-correcting codes (FCCs) as introduced in [1].

The paradigm of FCCs was introduced in [1], which considers systematic codes that protect a general function evaluation of the message from errors. The channel considered is the substitution channel, in which the errors are symbol substitutions on the transmitted vector. They proposed a function-dependent graph for the problem of constructing FCCs. The vertex set of the graph represents possible codewords of the FCC and two vertices are connected if and only if they can be contained together in an FCC. Thus, independent sets of the graph of sufficient cardinality form FCCs. It is difficult to characterise the independent sets of this graph for a general function. An equivalence between FCCs and irregular-distance codes is shown in [1], where the distance requirements of the irregular-distance code is dictated by the function of interest. Since the problem heavily depends on the function, simplified sub optimal bounds on redundancy were proposed that are easier to evaluate. A fundamental result from classical coding theory is the Singleton bound, which says that to correct t errors, at least $2t$ redundant symbols have to be added. It is shown in [1] that

this lower bound on redundancy holds for FCCs as well. It is also known that there are no non-trivial binary classical ECCs that achieve this lower bound. But in the case of FCCs, a class of functions called locally-binary functions have been proposed for which there exists encoding and decoding schemes so that t errors can be corrected using $2t$ redundant symbols, for a particular range of t . They also compare schemes for certain functions with the corresponding classical ECCs and show that function correction can be done with a lower redundancy length than what is required for classical error correction. FCCs for symbol-pair read channels were explored in [2]. Bounds on the optimal redundancy is provided and a counterpart of locally binary functions is proposed - pair-locally binary functions.

An application that motivates the use of systematic FCCs is archival data storage. Consider a message stored in a noisy storage medium after being encoded using an ECC. Say an attribute of the stored data is to be stored in the medium with an even higher error correcting capability. The classical approach is to use an ECC of higher capability. But in [1], it is proposed to store the parity obtained by using a systematic FCC.

A. Contributions and Organisation

In this work, we study FCCs for the case when the function evaluated is linear. We show that the bounds provided in [1] can be simplified using the structure imposed by linearity. We also propose additional bounds on the redundancy and propose classes of functions where FCCs can achieve lower redundancy lengths than classical codes. The technical contributions of this paper are summarised:

- A graph based representation of the problem is proposed in [1]. Independent sets of this graph, of sufficient cardinality can be chosen to be an FCC. An upper bound for the independence number of this graph is proposed which leads to lower bounds on the redundancy of the FCC. We consider a different graph with the same vertex set as the original one but with a maximum independent set which is a superset of that of the original graph. We express this graph as the Cartesian product of two smaller graphs which helps in characterising its independence number (Section III-A: Theorem 4).
- In [1], a Plotkin-like bound is derived for FCCs. The Hamming distance distribution of the message vector space is required for computing this bound. For linear functions it is shown that only the weight distribution of the kernel of the function is needed to compute the bound (Section III-B: Corollary 3).
- A characterization of the graphical representation of FCCs is given for linear functions. We show that its adjacency matrix has a symmetric block circulant structure. (Section III-B: Theorem 5).

- Using this structure, we show that for linear functions, the adjacency matrix is diagonalised by the tensor powers of DFT matrices. As a special case, for linear functions whose domain is a vector space over a field of characteristic 2, we show that Hadamard matrices can diagonalise the adjacency matrix. The obtained spectrum of the graph is used to obtain bounds on redundancy. (Section III-B: Corollary 5). (Section III-B: Corollary 5).
- Coset-wise coding is proposed where all the messages evaluated to the same function value are assigned the same parity. A sphere packing based bound is proposed for such coding schemes (Section III-C: Theorem 8).
- A class of linear functions for which coset-wise coding is optimal is identified and a scheme for encoding such functions is provided. (Section III-C).
- A class of linear functions is identified for which the problem of coset-wise coding is equivalent to a lower dimensional classical error correction problem (Section III-C).

B. Notations

The set of all positive integers less than or equal to N is denoted by $[N]$. The set of all positive integers is denoted by $\mathbb{N}$ and the set of all non-negative integers is denoted by $\mathbb{N}_0$. The set of all complex numbers is denoted by $\mathbb{C}$. The set of all real numbers is denoted by $\mathbb{R}$. A finite field of size q is represented as $\mathbb{F}_q$. A vector, $\mathbf{a} \in \mathbb{F}_q^n$ is represented as $(a_1 a_2 \dots a_n)$. The Hamming weight of $\mathbf{a} \in \mathbb{F}_q^n$ is denoted by $w_H(\mathbf{a})$. For $k \leq l \leq n$, $\mathbf{a}[k : l]$ represents the vector $(a_k a_{k+1} \dots a_l)$. For any set of vectors $X \subseteq \mathbb{F}_q^n$, $w_H(X)$ denotes the Hamming weight of the minimum Hamming weight vector in X . For any pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$, $d_H(\mathbf{x}, \mathbf{y})$ denotes the Hamming distance between $\mathbf{x}$ and $\mathbf{y}$. For an $N \times N$ matrix $\mathbf{D}$, $[\mathbf{D}]_{ij}$ denotes the (i, j) th entry of $\mathbf{D}$ and $\mathbf{D}_{(i)}$ denotes the i th column of $\mathbf{D}$. For sets of positive integers $X, Y \subseteq [N]$, $\mathbf{D}_{X,Y}$ denotes the submatrix of $\mathbf{D}$ whose rows are indexed by X and columns by Y . We use $\mathbf{e}_i \in \mathbb{F}_q^n$ to represent the unit vector with a 1 in the i th coordinate and 0s everywhere else. We use $\mathbf{I}_n$ to denote the identity matrix of order n . For a matrix $\mathbf{X} \in \mathbb{C}^{n \times n}$, $\mathbf{X}^\dagger$ denotes its conjugate transpose. For matrices $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \otimes \mathbf{Y}$ denotes the Tensor product of $\mathbf{X}$ and $\mathbf{Y}$. We use $\mathbf{X}^{\otimes n}$ to denote the n th order tensor power of $\mathbf{X}$ i.e., $\mathbf{X}^{\otimes n} = \otimes_{i=1}^n \mathbf{X}$. The set of all vectors that are at a Hamming distance less than or equal to t from $\mathbf{u} \in \mathbb{F}_q^n$ is denoted by $B_H(\mathbf{u}, t)$. For any $x \in \mathbb{R}$, $[x]^+ := \max(x, 0)$. For any two integers a and n ,

$$\langle a \rangle_n := \begin{cases} a \pmod{n}; & \text{if } a \pmod{n} \neq 0 \\ n; & \text{otherwise.} \end{cases}$$

II. PRELIMINARIES

A. Graphs

A graph $\mathcal{G}$ is a pair (V, E) where V is the vertex set and $E \subseteq V \times V$ is the edge set. We say that vertices v_1 and v_2 are connected if $(v_1, v_2) \in E$. In this paper we consider simple graphs, i.e., there are no loops or multiple edges.

An independent set of $\mathcal{G}$ is a set $S \subseteq V$ such that $v_1, v_2 \in S \implies (v_1, v_2) \notin E$. The independence number α is the cardinality of the maximum independent set. A maximum independent set of $\mathcal{G}$ is called an α -set of $\mathcal{G}$.

The Cartesian product $\mathcal{G} \square \mathcal{H}$ of two graphs $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V', E')$ is a graph with its vertex set given by $V \times V'$ and two vertices (u, v) and (u', v') are connected in $\mathcal{G} \square \mathcal{H}$ iff

- 1) $u = u'$ and $(v, v') \in E'$ or,
- 2) $(u, u') \in E$ and $v = v'$.

B. Circulant Matrices

An $n \times n$ matrix $\mathbf{A}$ is said to be a circulant matrix if $[\mathbf{A}]_{ij} = [\mathbf{A}]_{\langle i+1 \rangle_n \langle j+1 \rangle_n}$ for all $i, j \in [n]$. An $nm \times nm$ matrix $\mathbf{B}$ is said to be a block circulant matrix if it is of the following block matrix form:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} & \dots & \mathbf{B}_{1n} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} & \dots & \mathbf{B}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n1} & \mathbf{B}_{n2} & \mathbf{B}_{n3} & \dots & \mathbf{B}_{nn} \end{bmatrix},$$

where each $\mathbf{B}_{ij}$ is an $m \times m$ matrix and $\mathbf{B}_{ij} = \mathbf{B}_{\langle i+1 \rangle_n \langle j+1 \rangle_n}$ for all $i, j \in [n]$.

It is known that an $n \times n$ circulant matrix is diagonalized by the n th order Discrete Fourier Transform (DFT) matrix given by

$$\mathbf{W}_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^8 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix},$$

where ω is the n th root of unity.

C. Hadamard Matrices

A Hadamard matrix of order n is a real $n \times n$ matrix $\mathbf{H}_n$ taking values from the set $\{1, -1\}$ such that $\mathbf{H}_n^T \mathbf{H}_n = n \mathbf{I}_n$. The Sylvester construction [3] gives Hadamard matrices when n is a power of 2 as follows:

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } \mathbf{H}_n = \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & -\mathbf{H}_{n-1} \end{bmatrix}.$$

Note that $\mathbf{H}_2$ is equal to the DFT matrix of order 2, $\mathbf{W}_2$.

D. Classical Error Correcting Codes

An $(n, M, d)_q$ code $\mathcal{C}$ is a set of M vectors of length n with elements from some finite set (alphabet) of cardinality q (say $\mathbb{F}_q$), such that d is the minimum Hamming distance between any pair of vectors (codewords) in $\mathcal{C}$. A code of alphabet size q is said to be a q -ary code.

A code of minimum distance d can correct any error of length $\lfloor \frac{d-1}{2} \rfloor$.

The maximum number of codewords in any q -ary code of length n and minimum distance d between codewords is denoted by $A_q(n, d)$.

E. Linear Functions

A function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ is said to be linear if it satisfies the following condition:

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^k \text{ and } \alpha, \beta \in \mathbb{F}_q$$

It can be expressed as a matrix operation

$$f(\mathbf{x}) = \mathbf{F}\mathbf{x}, \text{ for some } \mathbf{F} \in \mathbb{F}_q^{l \times k}.$$

The kernel of f or the null space of $\mathbf{F}$ is denoted by $\ker(f)$. For the rest of the paper, the considered function is such that $l \leq k$ and $\mathbf{F}$ is full rank, i.e., $\text{rank}_{\mathbb{F}_q}(\mathbf{F}) = l$. We denote the range of f as $\text{Im}(f) = \mathbb{F}_q^l \triangleq \{f_0, f_1, \dots, f_{q^l-1}\}$.

Definition 1. Consider a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$. We define $|\ker(f)|_d$ to be the number of vectors in $\ker(f)$ of Hamming weight d .

Definition 2. Consider a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$. We define $\mathbb{F}_q^k / \ker(f)$ to be the set of all cosets of $\ker(f)$ in $\mathbb{F}_q^k$.

Remark 1. Consider a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$. The cosets in $\mathbb{F}_q^k / \ker(f)$ partition $\mathbb{F}_q^k$, where the elements of each coset are assigned to a unique function value, i.e. there is an induced isomorphism $\bar{f} : \mathbb{F}_q^k / \ker(f) \rightarrow \mathbb{F}_q^l$ such that $\mathbf{u} + \ker(f) \mapsto f(\mathbf{u})$. We use $\bar{f}_i$ to denote the coset mapped to $f_i \in \mathbb{F}_q^l$ and $\bar{f}_0$ to denote $\ker(f)$.

Due to linearity, the distance distribution of $\bar{f}_i \forall i \in \mathbb{F}_q^l$ is the same and is equal to the weight distribution of $\ker(f)$.

F. Function-Correcting Codes

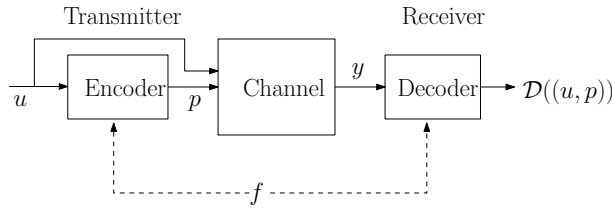


Fig. 1: The function Correction Setting

The system model illustrated in Fig. 1 consists of a transmitter that wants to send a vector $\mathbf{u} \in \mathbb{F}_q^k$. The receiver wants to evaluate a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ at $\mathbf{u}$. The transmitter encodes the data using a systematic encoding $\mathcal{E} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^{k+r}$ such that $\mathcal{E}(\mathbf{u}) = (\mathbf{u}, \mathbf{p})$.

The encoded data $\mathcal{E}(\mathbf{u})$ is sent over a channel that can introduce upto t symbol errors. The receiver receives $\mathbf{y} = \mathcal{E}(\mathbf{u}) + \mathbf{e} \in \mathbb{F}_q^{k+r}$, $w_H(\mathbf{e}) \leq t$. The receiver has a decoding function $\mathcal{D} : \mathbb{F}_q^{k+r} \rightarrow \mathbb{F}_q^k$, which satisfies $\mathcal{D}(\mathbf{y}) = f(\mathbf{u}) \forall \mathbf{u} \in \mathbb{F}_q^k$.

The definition for an FCC was given in [1].

Definition 3. [1] An encoding $\mathcal{E} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^{k+r}$ is said to be an (f, t) -FCC for a function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ if for all $\mathbf{u}_i, \mathbf{u}_j \in \mathbb{F}_q^k$ with $f(\mathbf{u}_i) \neq f(\mathbf{u}_j)$,

$$d_H(\mathcal{E}(\mathbf{u}_i), \mathcal{E}(\mathbf{u}_j)) \geq 2t + 1. \quad (1)$$

Remark 2. When the function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ is a bijection, an (f, t) -FCC is equivalent to a systematic $(n, q^k, 2t + 1)$ code.

Definition 4. [1] The optimal redundancy $r_f(k, t)$ is defined as the smallest r possible such that there exists an (f, t) -FCC with an encoding function $\mathcal{E} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^{k+r}$.

The distance requirement matrix was introduced in [1].

Definition 5. [1] Let, $\mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^k-1} \in \mathbb{F}_q^k$. The distance requirement matrix $\mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^k-1})$ for an (f, t) -FCC is a $q^k \times q^k$ matrix with entries

$$[\mathbf{D}_f(t, \mathbf{u}_0, \dots, \mathbf{u}_{q^k-1})]_{ij} = \begin{cases} [2t + 1 - d_H(\mathbf{u}_i, \mathbf{u}_j)]^+; & \text{if } f(\mathbf{u}_i) \neq f(\mathbf{u}_j) \\ 0; & \text{otherwise.} \end{cases}$$

For linear functions, we use the shorthand $\mathbf{D}_f(t) := \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^k-1})$.

Remark 3. For a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$, $\ker(f)$ has a dimension of $k - l$ over $\mathbb{F}_q$. So, each column of $\mathbf{D}_f(t)$ will have at least q^{k-l} number of 0s, since the cosets in $\mathbb{F}_q^k / \ker(f)$ are of the same cardinality and all the elements of a particular coset are mapped to the same element in $\mathbb{F}_q^l$.

Remark 4. Since $\mathbb{F}_q^k$ is a vector space over $\mathbb{F}_q$, the distance distribution is same as the weight distribution. As a result, the columns (and rows) of $\mathbf{D}_f(t)$ are permutations of each other.

Irregular-distance codes of constraint $\mathbf{D}$ were defined in [1] as follows:

Definition 6. [1] Let $\mathbf{D} \in \mathbb{N}_0^{M \times M}$. Then $P = \{\mathbf{p}_i : i \in [M]\}$ is said to be an irregular-distance code of constraint $\mathbf{D}$ or a $\mathbf{D}$ -code if there is an ordering of P such that $d_H(\mathbf{p}_i, \mathbf{p}_j) \geq [\mathbf{D}]_{ij} \forall i, j \in [M]$.

Further, $N_q(\mathbf{D})$ is defined as the smallest integer r such that there exists a $\mathbf{D}$ -code of length r over $\mathbb{F}_q$.

We can thus see that an (f, t) -FCC is an irregular-distance code with $\mathbf{D} = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^k-1})$.

It is shown in [1] that the problem of constructing an FCC can be formulated as that of finding sufficiently large independent sets of the following graph.

Definition 7. [1] We define $\mathcal{G}_f(t, k, r)$ as the graph with vertex set $V = \mathbb{F}_q^k \times \mathbb{F}_q^r$ such that any two vertices $\mathbf{v}_i = (\mathbf{u}_i, \mathbf{r}_i)$ and $\mathbf{v}_j = (\mathbf{u}_j, \mathbf{r}_j)$ are connected if and only if

- $\mathbf{u}_i = \mathbf{u}_j$, or
- $f(\mathbf{u}_i) \neq f(\mathbf{u}_j)$ and $d_H(\mathbf{v}_i, \mathbf{v}_j) \leq 2t + 1$.

That is, if two vertices $(\mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{u}_j, \mathbf{v}_j)$ are connected in $\mathcal{G}_f(t, k, r)$, then an (f, t) -FCC cannot contain both $(\mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{u}_j, \mathbf{v}_j)$ as codewords. So if we can find an independent set of the graph of size q^k , it can be used as an (f, t) -FCC.

Definition 8. Let the vertex set of $\mathcal{G}_f(t, k, r)$ be $V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{q^{k+r}}\}$. The adjacency matrix $\mathbf{G}$ of graph $\mathcal{G}_f(t, k, r)$ is a $q^{k+r} \times q^{k+r}$ matrix such that

$$[\mathbf{G}]_{i,j} = \begin{cases} 1; & \text{if } \mathbf{v}_i \text{ and } \mathbf{v}_j \text{ are connected} \\ 0; & \text{otherwise.} \end{cases}$$

We use $\gamma_f(k, t)$ to denote the smallest integer r such that there exists an independent set of size q^k in $\mathcal{G}_f(t, k, r)$.

Example 1. Consider the function $f : \mathbb{F}_2^2 \rightarrow \{0, 1\}$ defined as $f((u_1 u_2)) = u_1 \cup u_2$, where $\cup$ denotes the logical OR operator. The graph $\mathcal{G}_f(t, k, r)$ for $t=1$ and $r = 2$ is shown in Fig. 2.

The following theorem from [1] shows the connection between the redundancy of optimal FCCs, irregular-distance codes and independent sets of $\mathcal{G}_f(t, k, r)$.

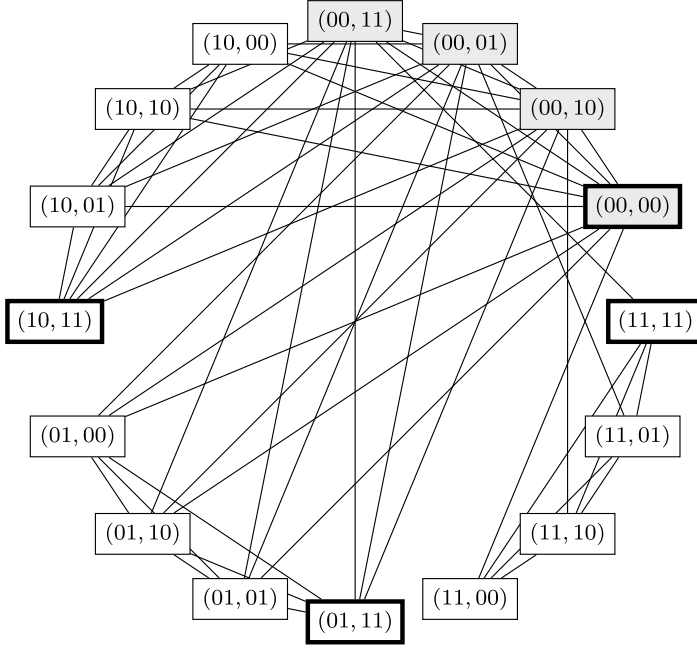


Fig. 2: [1] Graph $\mathcal{G}_f(t, k, r)$ for $t = 1, k = 2$ and $r = 2$ and the function $f((u_1 u_2)) = u_1 \cup u_2$. An independent set of size 4 is highlighted in bold.

Theorem 1. [1] For any function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$,

$$r_f(k, t) = \gamma_f(k, t) = N(\mathbf{D}_f(t)) \quad (2)$$

We can thus see that an (f, t) -FCC is an irregular-distance code with $\mathbf{D} = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^t-1})$.

The definitions for function distance and function distance matrix as in [1] are given below:

Definition 9. [1] For a function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$, the distance between $f_i, f_j \in \text{Im}(f) = \mathbb{F}_q^l$ is defined as

$$d_f(f_i, f_j) \triangleq \min_{\mathbf{u}_i, \mathbf{u}_j \in \mathbb{F}_q^k} d_H(\mathbf{u}_i, \mathbf{u}_j), \text{ s.t. } f(\mathbf{u}_i) = f_i, f(\mathbf{u}_j) = f_j. \quad (3)$$

Definition 10. [1] The function distance matrix of a function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ is a $q^l \times q^l$ matrix with its entries given by

$$[\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1})]_{ij} = \begin{cases} [2t + 1 - d_f(f_i, f_j)]^+; & \text{if } i \neq j \\ 0; & \text{otherwise.} \end{cases} \quad (4)$$

Remark 5. It follows from Remarks 1 and 3 that the problem of finding function distances becomes that of finding inter-coset distances. We also have,

$$d_f(f_i, 0) = w_H(\bar{f}_i) = \min_{\mathbf{u} \in \bar{f}_i} w_H(\mathbf{u}).$$

Because of the property of cosets, the inter-coset distance distributions are uniform, i.e. the column/row entries of $\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1})$ are from the set $\{d_f(f_i, 0); f_i \in \mathbb{F}_q^l\}$, i.e. the columns (and rows) are permutations of each other.

Remark 6. Consider functions f and g such that $f(\mathbf{x}) = \mathbf{F}\mathbf{x}$ and $g(\mathbf{x}) = \mathbf{G}\mathbf{x}$. Then the distance requirement matrix (and function distance matrix) for f and g are the same if $\mathbf{F}$ can

be obtained from $\mathbf{G}$ using only elementary row operations and column permutations.

The following lower bound on redundancy is given in [1].

Theorem 2. [1] For any function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ and $\{\mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_M\} \subseteq \mathbb{F}_q^k$

$$r_f(k, t) \geq N_q(\mathbf{D}_f(t, \mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_M)) \quad (5)$$

and for $|\text{Im}(f)| \geq 1$

$$r_f(k, t) \geq 2t. \quad (6)$$

As shown in [1], an existential bound on redundancy is obtained on doing coset-wise coding, i.e., when the same parity vector is assigned to all the messages in one coset.

Theorem 3. [1] For any function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$,

$$r_f(k, t) \leq N(\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1})) \quad (7)$$

where $\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1})$ is the function distance matrix.

The bound in Theorem 3 can be tight as shown in the following corollary which follows from Theorems 2 and 3.

Corollary 1. If there exists a set of representative information vectors $\{\mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^t-1}\}$ with $\{f(\mathbf{u}_0) \dots f(\mathbf{u}_{q^t-1})\} = \text{Im}(f)$ and

$$\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1}) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^t-1}), \quad (8)$$

then

$$r_f(k, t) = N(\mathbf{D}_f(t, f_0, f_1 \dots f_{q^t-1})).$$

We propose a class of linear functions for which such a selection of representative vectors exists in Section III-C.

A generalized version of the Plotkin bound for binary FCCs is provided in [1].

Lemma 1. For any distance requirement matrix $\mathbf{D} \in \mathbb{N}_0^{M \times M}$,

$$N_2(\mathbf{D}) \geq \begin{cases} \frac{4}{M^2} \sum_{i,j:i < j} [\mathbf{D}]_{ij}, & \text{if } M \text{ is even} \\ \frac{4}{M^2-1} \sum_{i,j:i < j} [\mathbf{D}]_{ij}, & \text{if } M \text{ is odd.} \end{cases}$$

Example 2. Consider a function $f : \mathbb{F}_2^4 \rightarrow \mathbb{F}_2^2$ with the mapping

$$\mathbf{x} \mapsto \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

The coset decomposition $\mathbb{F}_2^4 / \ker(f)$ and the induced mapping is:

$$\begin{aligned} \{0000, 0001, 0110, 0111\} &\mapsto 00 \\ \{0010, 0011, 0100, 0101\} &\mapsto 11 \\ \{1000, 1001, 1110, 1111\} &\mapsto 10 \\ \{1100, 1101, 1010, 1011\} &\mapsto 01. \end{aligned}$$

The distance requirement matrix for an (f, t) -FCC is

$$\mathbf{D}_f(t, u_0, u_1 \dots u_{15}) = (2t+1)\mathbf{I}_{16} -$$

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 \\ 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 1 & 3 & 2 & 3 & 2 & 4 & 3 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 1 & 2 & 3 & 4 & 2 & 3 \\ 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 3 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 3 & 4 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 2 & 4 & 3 & 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 3 & 4 & 2 & 3 & 2 & 3 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 4 & 3 & 3 & 2 & 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 2 & 3 & 2 & 4 & 3 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 2 & 3 & 4 & 2 & 3 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 3 & 3 & 4 & 1 & 2 & 2 & 3 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 \\ 3 & 2 & 4 & 3 & 2 & 1 & 3 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 3 & 4 & 2 & 3 & 2 & 3 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 4 & 3 & 3 & 2 & 3 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

The function distance matrix is given by

$$\mathbf{D}_f(t, f_0, f_1, f_2, f_3) = (2t+1)\mathbf{I}_4 - \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

If we choose a set of representative vectors $\{u_0, u_1, u_2, u_3\} = \{0000, 0100, 1000, 1100\}$, we can see that $\mathbf{D}_f(t, f_0, f_1, f_2, f_3) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. So, by Corollary 1, $N_2(\mathbf{D}_f(t, f_0, f_1, f_2, f_3)) = N_2(\mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{15}))$. That is, the simplified problem of coset-wise coding in 2 dimensions can attain the optimal redundancy for the original problem of dimension 4.

III. MAIN RESULTS

In this section, we provide our main results. We propose a lower bound on the redundancy of an FCC. When the function considered is a bijection, we obtain a bound for classical systematic ECCs. For linear functions, we show that the bound provided in Lemma 1 simplifies and can be obtained in terms of the weight distribution of the kernel of the function. We show that the adjacency matrix of the graph $\mathcal{G}_f(t, k, r)$ has a recursive block circulant structure for linear functions. For functions whose domain is a vector space over a finite field of characteristic 2, we characterise the spectrum of the adjacency matrix, leading to lower bounds on redundancy. We consider the case of coset-wise coding and identify functions for which coset-wise coding is optimal.

A. A Lower Bound on Redundancy

We provide a lower bound on the redundancy required for the case when $2t+1 \leq k$. We consider a graph $G'_f(t, k, r)$ with the same vertex set as of $\mathcal{G}_f(t, k, r)$ such that its maximum independent set contains the maximum independent set of $\mathcal{G}_f(t, k, r)$. This leads to an upper bound on $\alpha(\mathcal{G}_f(t, k, r))$, which gives a lower bound on the redundancy required.

Theorem 4. *For a function $f : \mathbb{F}_q^k \rightarrow \text{Im}(f)$ and $t \in \mathbb{N}$ such that $2t+1 \geq k$, there exists an (f, t) -FCC of redundancy length r only if*

$$q^r \geq \frac{q^k}{\alpha(\mathcal{G}_f(t, k, 0))}.$$

Proof: Consider the graph $\mathcal{G}_f(t, k, 0)$ with vertex set $\mathbb{F}_q^k$. Two vertices $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_q^k$ are connected if and only if $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and $d_H(\mathbf{x}_1, \mathbf{x}_2) < 2t+1$. Let $\mathcal{R}$ be a graph with vertex

set $\mathbb{F}_q^r$ such that any two distinct vertices are connected. Note that $\alpha(\mathcal{R}) = 1$. Let $G'_f(t, k, r)$ be the Cartesian product of the graphs $\mathcal{G}_f(t, k, 0)$ and $\mathcal{R}$, i.e.,

$$G'_f(t, k, r) = \mathcal{G}_f(t, k, 0) \square \mathcal{R}.$$

An illustration is provided in Example 3. The vertex set of $\mathcal{G}_f(t, k, r)$ is the same as that of $G'_f(t, k, r)$.

Now, we show that if two vertices are unconnected in $\mathcal{G}_f(t, k, r)$, then they are unconnected in $G'_f(t, k, r)$. Let, $(\mathbf{x}_i, \mathbf{v}_i)$ and $(\mathbf{x}_j, \mathbf{v}_j) \in \mathbb{F}_q^{k+r}$ be connected in $G'_f(t, k, r)$. This happens if and only if

- (i) $\mathbf{x}_i = \mathbf{x}_j$, or
- (ii) $d_H(\mathbf{x}_i, \mathbf{x}_j) < 2t+1$ and $\mathbf{v}_i = \mathbf{v}_j$.

Consider condition (i). If $\mathbf{x}_i = \mathbf{x}_j$, then $(\mathbf{x}_i, \mathbf{v}_i)$ and $(\mathbf{x}_j, \mathbf{v}_j)$ are connected in $\mathcal{G}_f(t, k, r)$ by definition. Consider condition (ii). If $d_H(\mathbf{x}_i, \mathbf{x}_j) < 2t+1$ and $\mathbf{v}_i = \mathbf{v}_j$, then $d_H((\mathbf{x}_i, \mathbf{v}_i), (\mathbf{x}_j, \mathbf{v}_j)) < 2t+1$, which implies that $(\mathbf{x}_i, \mathbf{v}_i)$ and $(\mathbf{x}_j, \mathbf{v}_j)$ are connected in $\mathcal{G}_f(t, k, r)$. Hence, any α -set of $\mathcal{G}_f(t, k, r)$ is a subset of some independent set of $G'_f(t, k, r)$, which implies that

$$\alpha(\mathcal{G}_f(t, k, r)) \leq \alpha(G'_f(t, k, r)). \quad (9)$$

Now, we bound the independence number of $G'_f(t, k, r)$. Since $G'_f(t, k, r) = \mathcal{G}_f(t, k, 0) \square \mathcal{R}$, from [4] and [5], we can use the upper bound on the independence number of a graph Cartesian product:

$$\begin{aligned} \alpha(G'_f(t, k, r)) &\leq \min\{q^k \cdot \alpha(\mathcal{R}), q^r \cdot \alpha(\mathcal{G}_f(t, k, 0))\} \\ &= \min\{q^k, q^r \cdot \alpha(\mathcal{G}_f(t, k, 0))\}. \end{aligned} \quad (10)$$

For an (f, t) -FCC of length r to exist, $\alpha(\mathcal{G}_f(t, k, r))$ should be q^k . So, from (9) and (10), we can say that if an (f, t) -FCC of length r exists, then

$$q^r \cdot \alpha(\mathcal{G}_f(t, k, 0)) \geq q^k,$$

which gives the required result. $\blacksquare$

The result of Theorem 4 can be used to get a lower bound on the parity required for classical systematic ECCs by considering the function f to be a bijection.

Corollary 2. *There exists a (q^{k+r}, q^k, d) systematic block code where $\lfloor \frac{d-1}{2} \rfloor \leq k$ only if*

$$q^r \geq \frac{q^k}{A_q(k, d)},$$

where $A_q(k, d)$ is the maximum number of codewords in any q -ary code of length k and minimum distance d .

Proof: If d is odd, take $t = \frac{d-1}{2}$. When the function f is a bijection, the independent sets of $\mathcal{G}_f(t, k, 0)$ will be the sets of vectors in $\mathbb{F}_q^k$ such that the Hamming distance between them is at least $2t+1$, which implies that the cardinality of the α -set will be $\alpha(\mathcal{G}_f(t, k, 0)) = A_q(k, 2t+1)$. A similar argument can be used for the case when d is even to get the required result. $\blacksquare$

Example 3. Consider the function $f : \mathbb{F}_2^3 \rightarrow \{0, 1\}$ defined as $f((u_1 u_2 u_3)) = u_1 \cup u_2 \cup u_3$. Let $t = 1$. For $r = 1$, consider the graph $\mathcal{R}$ with vertex set $\mathbb{F}_2$ with the two elements in $\mathbb{F}_2$ being connected. Consider the graph $\mathcal{G}_f(t, k, 0)$ with vertex set

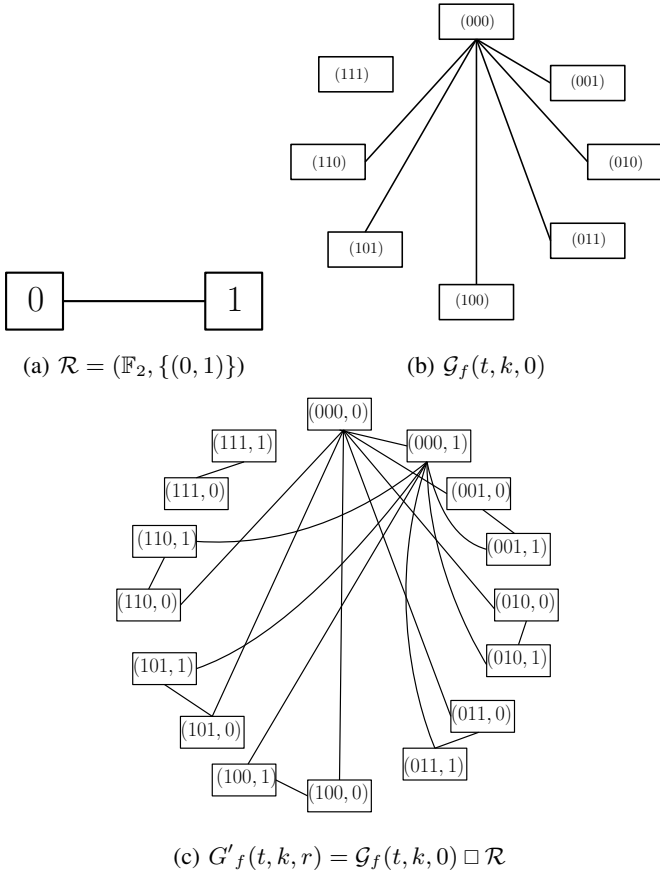


Fig. 3: Graphs $\mathcal{R}$, $\mathcal{G}_f(t, k, 0)$ and $\mathcal{G}'_f(t, k, r)$ for $t = 1$, $r = 1$, $k = 3$, and the function $f : \mathbb{F}_2^3 \rightarrow \{0, 1\}$ defined as $f((u_1 u_2 u_3)) = u_1 \cup u_2 \cup u_3$.

$\mathbb{F}_2^3$ such that any two vertices $\mathbf{x}_i$ and $\mathbf{x}_j$ are connected if and only if $f(\mathbf{x}_i) \neq f(\mathbf{x}_j)$ and $d_H(\mathbf{x}_i, \mathbf{x}_j) < 3$. Their Cartesian Product $\mathcal{G}'_f(t, k, r) = \mathcal{G}_f(t, k, 0) \square \mathcal{R}$ is shown in Fig. 3.

B. Linear-Function Correcting Codes

As mentioned in Remarks 3 and 5, linearity imposes a structure on the distance requirement matrix and function distance matrix of an FCC. This simplifies the bounds proposed in [1] and gives rise to new bounds as shown in this section.

Now, for linear functions, we show that the Plotkin bound for FCCs provided in Lemma 1 can be simplified and can be made more computationally efficient to calculate.

Corollary 3. For a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$, the optimal redundancy of an (f, t) FCC

$$r_f(k, t) \geq \left(\frac{q}{q-1}\right)(2t+1)(1-q^{-l}) - k + \frac{s}{(q-1)(q^k-1)},$$

where $s = \sum_{\mathbf{x} \in \ker(f)} w_H(\mathbf{x})$ i.e., the sum of Hamming weights of the vectors in $\ker(f)$.

Proof: The bound can be obtained by counting $\sum_{i \leq j} d_H(\mathbf{p}_i, \mathbf{p}_j)$ in two different ways. Let $\mathbf{D} = \mathbf{D}_f(t)$ be the distance requirement matrix for the function f .

Let $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{q^k}$ be the codewords of a $\mathbf{D}$ -code of length r . Arrange the codewords as a $q^k \times r$ matrix $\mathbf{P}$. Since each

column of $\mathbf{P}$ can contribute at most $q^{2k}(1 - \frac{1}{q})$ to the sum $\sum_{i,j} d_H(\mathbf{p}_i, \mathbf{p}_j)$, we have

$$\sum_{i,j} d_H(\mathbf{p}_i, \mathbf{p}_j) \leq r q^{2k-1}(q-1). \quad (11)$$

Furthermore, by definition,

$$d_H(\mathbf{p}_i, \mathbf{p}_j) \geq [\mathbf{D}]_{ij},$$

which implies that

$$\sum_{i,j} d_H(\mathbf{p}_i, \mathbf{p}_j) \geq \sum_{i,j} [\mathbf{D}]_{ij}. \quad (12)$$

Note that $\dim_{\mathbb{F}_q}(\ker(f)) = k - l$. Thus, there will be at least q^{k-l} 0s in each column. Because of linearity, the sum of non-zero entries of each column will be the same. Thus,

$$\sum_{i,j} [\mathbf{D}]_{ij} = (\text{no. of columns}) \times (\text{sum of one column}).$$

To find the sum of one column, consider the first column of $\mathbf{D}$. Let I be the row indices of the non-zero entries in the first column of $\mathbf{D}$. Let the first column correspond to the all zero vector $\mathbf{0}$. From the definition of $\mathbf{D}_f(t)$ and Remark 5, we have

$$\begin{aligned} [\mathbf{D}]_{i1} &\geq 2t + 1 - d_H(\mathbf{u}_i, \mathbf{0}) \\ &= 2t + 1 - w_H(\mathbf{u}_i). \end{aligned}$$

This implies that

$$\sum_i [\mathbf{D}]_{i1} \geq (2t+1)(q^k - q^{k-l}) - \sum_{i \in I} w_H(\mathbf{u}_i).$$

The sum of weights of all the vectors in $\mathbb{F}_q^k$ can be found to be $k(q-1)q^{k-1}$. Let the sum of weights of the vectors in $\ker(f)$ be s , which gives

$$\sum_i [\mathbf{D}]_{i1} \geq (2t+1)(q^k - q^{k-l}) - k(q-1)q^{k-1} + s.$$

Thus,

$$\begin{aligned} \sum_{i \leq j} [\mathbf{D}]_{ij} &\geq q^k \times \sum_i [\mathbf{D}]_{i1} \\ &= q^k((2t+1)(q^k - q^{k-l}) - k(q-1)q^{k-1} + s). \end{aligned} \quad (13)$$

From (11), (12) and (13), we get the required result. $\blacksquare$

The Plotkin bound provided in [1] requires the sum of all $[\mathbf{D}]_{ij}$ to calculate the bound. For linear functions, we can see that we require only the sum of the Hamming weights of the vectors in $\ker(f)$.

We now consider the structure linearity imposes on the graph $\mathcal{G}_f(t, k, r)$. We show that its adjacency matrix $\mathbf{G}$ has a recursive block circulant structure.

Definition 11. For some $\mathbf{u} = (u_1 u_2 \dots u_m) \in \mathbb{F}_q^m$ and $m \in [n] \cup \{0\}$, define $U_m^n \triangleq \{(x_1 x_2 \dots x_n) \in \mathbb{F}_q^n : (x_1 x_2 \dots x_m) = (u_1 u_2 \dots u_m)\}$ if $m \neq 0$ and $U_m^n = \mathbb{F}_q^n$ if $m = 0$.

That is, U_m^n is the set of all n length vectors in $\mathbb{F}_q$ such that its first m indices is equal to $\mathbf{u}$.

Theorem 5. For a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$, let $\mathbf{G} \in \{0, 1\}^{q^n \times q^n}$, where $n := k + r$, be the adjacency matrix of the graph $\mathcal{G}_f(t, k, r)$. Let the rows and columns of the matrix $\mathbf{G}$ be indexed by the vectors in $\mathbb{F}_q^n$. Then, $\mathbf{G}$ satisfies the following condition:

C1 For all $\mathbf{u}_m, \mathbf{v}_m \in \mathbb{F}_q^m$ and for all $m \in [n] \cup \{0\}$, for some ordering of U_m^n and V_m^n , if the submatrix $\mathbf{S} := \mathbf{G}_{U_m^n, V_m^n}$ is expressed in block matrix form as

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{1,1} & \mathbf{S}_{1,2} & \cdots & \mathbf{S}_{1,q} \\ \mathbf{S}_{2,1} & \mathbf{S}_{2,2} & \cdots & \mathbf{S}_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{q,1} & \mathbf{S}_{q,2} & \cdots & \mathbf{S}_{q,q} \end{bmatrix},$$

where $\mathbf{S}_{i,j}$ is of order q^{n-m-1} , then for any $i, j \in [q]$,

$$\mathbf{S}_{i,j} = \mathbf{S}_{<i+1>_q, <j+1>_q} = \forall i, j \in [q].$$

Proof: Consider the case when q is a prime number p . For any non-negative integer $i \leq p$, we use $\mathbf{i}_p^n$ to denote the n length p -ary representation of i . We have to show that for all $m \in [n] \cup \{0\}$ and for all $\mathbf{u}_m, \mathbf{v}_m \in \mathbb{F}_p^m$, the submatrix $\mathbf{G}_{U_m^n, V_m^n}$ satisfies condition C1. Assume that U_m^n and V_m^n are ordered lexicographically. For an ordered set U , let $U(i)$ represent the i th element in this ordering.

Consider the matrix $\mathbf{G}' \in \mathbb{N}_0^{p^n \times p^n}$ defined as $[\mathbf{G}']_{ij} = d_H(\mathbf{i}_p^n, \mathbf{j}_p^n)$. We can express $\mathbf{S}' = \mathbf{G}'_{U_m^n, V_m^n}$ as

$$\mathbf{S}' = \begin{bmatrix} \mathbf{S}'_{1,1} & \mathbf{S}'_{1,2} & \cdots & \mathbf{S}'_{1,p} \\ \mathbf{S}'_{2,1} & \mathbf{S}'_{2,2} & \cdots & \mathbf{S}'_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}'_{p,1} & \mathbf{S}'_{p,2} & \cdots & \mathbf{S}'_{p,p} \end{bmatrix},$$

where $\mathbf{S}'_{i,j} \in \{0, 1\}^{p^{n-m-1} \times p^{n-m-1}} \forall i, j \in [2]$. It is evident that

$$[\mathbf{S}'_{k,l}]_{ij} = \begin{cases} d_H(U_m^n(i), V_m^n(j)); & k = l \\ d_H(U_m^n(i), V_m^n(j)) + 1; & k \neq l. \end{cases}$$

Thus, $\mathbf{G}'_n$ satisfies condition C1.

Now, consider the matrix $\mathbf{G}''$ defined as

$$[\mathbf{G}'']_{ij} = \begin{cases} 1; & \text{if } [\mathbf{G}']_{ij} \leq 2t + 1 \\ 0; & \text{otherwise.} \end{cases}$$

We can see that $\mathbf{G}''$ will also satisfy condition C1 as it is obtained by thresholding the entries of $\mathbf{G}'$. The adjacency matrix $\mathbf{G}$ can be obtained from $\mathbf{G}''$ as

$$[\mathbf{G}]_{\mathbf{i}_p^n, \mathbf{j}_p^n} = \begin{cases} 0; & \text{if } \mathbf{i}_p^n[1:k] - \mathbf{j}_p^n[1:k] \in \ker(f) \\ [\mathbf{G}'']_{ij}; & \text{otherwise.} \end{cases}$$

To show that $\mathbf{G}$ satisfies condition C1, it is now enough to show that for all $m \in [n] \cup \{0\}$ and for all $\mathbf{u}_m, \mathbf{v}_m \in \mathbb{F}_p^m$, the submatrix $\mathbf{G}_{U_m^n, V_m^n}$ satisfies the following condition:

$$[\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{i}_p^n, \mathbf{j}_p^n} = 0 \iff [\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{g}_p^n, \mathbf{h}_p^n} = 0$$

where $\mathbf{g}_p^n = \mathbf{i}_p^n + a \cdot \mathbf{e}_{m+1 \pmod{(n-m)}}$ and $\mathbf{h}_p^n = \mathbf{j}_p^n + a \cdot \mathbf{e}_{m+1 \pmod{(n-m)}}$, where $a \in \mathbb{F}_p$.

From the definition of $\mathbf{G}$, we know that $[\mathbf{G}]_{\mathbf{i}_p^n, \mathbf{j}_p^n} = 0$ if and only if

- 1) $f(\mathbf{i}_p^n[1:k]) \neq f(\mathbf{j}_p^n[1:k])$ and $d_H(\mathbf{i}_p^n, \mathbf{j}_p^n) \geq 2t + 1$, or
- 2) $\mathbf{i}_p^n[1:k] \neq \mathbf{j}_p^n[1:k]$ and $f(\mathbf{i}_p^n[1:k]) = f(\mathbf{j}_p^n[1:k])$.

Since $\mathbf{G}$ is obtained from $\mathbf{G}''$, for all $\mathbf{i}_p^n$ and $\mathbf{j}_p^n$ satisfying the first condition, $[\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{i}_p^n, \mathbf{j}_p^n} = 0 \iff [\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{g}_p^n, \mathbf{h}_p^n} = 0$, as $\mathbf{g}_p^n[1:k]$ and $\mathbf{h}_p^n[1:k]$ lie in different cosets of $\mathbb{F}_p^k / \ker(f)$ and $d_H(\mathbf{g}_p^n, \mathbf{h}_p^n) = d_H(\mathbf{i}_p^n, \mathbf{j}_p^n)$. Now, consider $\mathbf{i}_p^n$ and $\mathbf{j}_p^n$ satisfying the second condition, i.e., $\mathbf{i}_p^n[1:k] \neq \mathbf{j}_p^n[1:k]$ and $\mathbf{i}_p^n[1:k] - \mathbf{j}_p^n[1:k] \in \ker(f)$. Then, from the property of cosets, $\mathbf{g}_p^n[1:k] - \mathbf{h}_p^n[1:k] \in \ker(f)$. That is for such $\mathbf{i}_p^n$ and $\mathbf{j}_p^n$, $[\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{i}_p^n, \mathbf{j}_p^n} = 0 \iff [\mathbf{G}_{U_m^n, V_m^n}]_{\mathbf{g}_p^n, \mathbf{h}_p^n} = 0$. Thus, $\mathbf{G}$ satisfies condition C1.

When q is a prime power, i.e., $q = p^m$, for some prime p , $\mathbb{F}_q^k$ is isomorphic to the vector space $\mathbb{F}_p^{mk}$. So the same arguments can be considered where the matrix $\mathbf{G}$ is indexed by the vectors in $\mathbb{F}_p^{mk}$. ■

Remark 7. Equivalently the structure of $\mathbf{G}$ can be seen as follows: express $\mathbf{G}$ in block matrix form as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \cdots & \mathbf{G}_{1,q} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \cdots & \mathbf{G}_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{q,1} & \mathbf{G}_{q,2} & \cdots & \mathbf{G}_{q,q} \end{bmatrix}.$$

Let $\mathbf{G}_i := \mathbf{G}_{1,i}$. Then $\mathbf{G}$ will be of the form

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 & \cdots & \mathbf{G}_{q-1} & \mathbf{G}_q \\ \mathbf{G}_q & \mathbf{G}_1 & \mathbf{G}_2 & \cdots & \mathbf{G}_{q-2} & \mathbf{G}_{q-1} \\ \mathbf{G}_{q-1} & \mathbf{G}_q & \mathbf{G}_1 & \cdots & \mathbf{G}_{q-3} & \mathbf{G}_{q-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{G}_3 & \mathbf{G}_4 & \mathbf{G}_5 & \cdots & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2 & \mathbf{G}_3 & \mathbf{G}_4 & \cdots & \mathbf{G}_q & \mathbf{G}_1 \end{bmatrix}.$$

Note that the matrix $\mathbf{G}$ is symmetric as it is an adjacency matrix. Thus, $\mathbf{G}$ is a symmetric block circulant matrix. The submatrices $\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_q$ can further be divided into their submatrices recursively and the block circulant structure will hold.

Corollary 4. Consider the adjacency matrix $\mathbf{G} \in \{0, 1\}^{2^{nm} \times 2^{nm}}$ corresponding to a function $f : \mathbb{F}_{2^m}^k \rightarrow \mathbb{F}_{2^m}^l$. Let the rows and columns of the matrix $\mathbf{G}$ be indexed by the set of vectors $\mathbb{F}_{2^m}^{nm}$ in lexicographic order. Then $\mathbf{G}$ satisfies the following condition:

C1 For all $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_2^i$ and for all $i \in [nm] \cup \{0\}$, if the submatrix $\mathbf{S} := \mathbf{G}_{U_i^{nm}, V_i^{nm}}$ is expressed in block matrix form as

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{1,1} & \mathbf{S}_{1,2} \\ \mathbf{S}_{2,1} & \mathbf{S}_{2,2} \end{bmatrix},$$

where $\mathbf{S}_{i,j} \in \{0, 1\}^{2^{n-m-1} \times 2^{n-m-1}} \forall i, j \in [2]$, then $\mathbf{S}_{1,1} = \mathbf{S}_{2,2}$ and $\mathbf{S}_{1,2} = \mathbf{S}_{2,1}$.

Remark 8. Equivalently the structure of $\mathbf{G}$ can be seen as follows: express $\mathbf{G}$ in block matrix form as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} \end{bmatrix}.$$

Then $\mathbf{G}_{1,1} = \mathbf{G}_{2,2}$ and $\mathbf{G}_{1,2} = \mathbf{G}_{2,1}$. The submatrices $\mathbf{G}_{1,1}, \mathbf{G}_{1,2}, \mathbf{G}_{2,1}$ and $\mathbf{G}_{2,2}$ can further be divided into their submatrices recursively and the above condition will hold. The upshot is that $\mathbf{G}$ can be fully specified using just its first row.

We now characterise the spectrum of the adjacency matrix of $\mathcal{G}_f(t, k, r)$.

The necessary condition for t -error function correction is that $B_H(f_i, t)$ should be disjoint for all $f_i \in \mathbb{F}_q^l$. We have,

$$\bigcup_{f_i \in \mathbb{F}_q^l} B_H(f_i, t) \subseteq \mathbb{F}_q^n,$$

which implies,

$$\sum_{f_i \in \mathbb{F}_q^l} |B_H(f_i, t)| \leq q^n. \quad (15)$$

Consider the kernel $\bar{f}_0$. Because of linearity, it is enough to find a bound on $|B_H(f_0, t)|$ as $|B_H(f_i, t)| = |B_H(f_0, t)| \forall f_i \in \mathbb{F}_q^l$. Define, $W_i \triangleq \{\mathbf{v} \in \mathbb{F}_q^n : w_H(\mathbf{v}) = i\}$ and

$$S_i \triangleq \{\mathbf{v} \in \mathbb{F}_q^n : d_H(\mathbf{v}, \mathbf{c}) \geq i \forall \mathbf{c} \in \bar{f}_0 \text{ and } d_H(\mathbf{u}, \mathbf{c}) = i \text{ for some } \mathbf{c} \in \bar{f}_0\}.$$

Let $\delta = \lfloor \frac{d_t-1}{2} \rfloor$. Note that $B_H(f_i, t) = S_0 \cup S_1 \dots \cup S_t$ where $S_0, S_1, \dots, S_\delta$ are disjoint. It is straightforward to see that

$$|S_0 \cup S_1 \dots \cup S_\delta| = q^{k-l} \left(\sum_{i=0}^{\delta} \binom{n}{i} (q-1)^i \right).$$

We estimate $S_{\delta+1}, S_{\delta+2}, \dots, S_t$ to get a stronger bound on $B_H(f_i, t)$.

Assume $\mathbf{0}$ (all zero vector) is a codeword. The number of codewords that are at a distance $\delta + i$ from $\mathbf{0}$ and at a distance $\delta + i$ or greater from the other codewords in $\mathcal{C}(\bar{f}_0)$ is given by

$$\begin{aligned} |W_{\delta+i} \cap S_{\delta+i}| &= |W_{\delta+i}| - \sum_{j=1-i}^{i-1} |W_{\delta+i} \cap S_{\delta+j}| \\ &\geq \binom{n}{\delta+i} (q-1)^{\delta+i} - \sum_{j=1}^{2i-1} a_{ij} |ker(f)|_{(2\delta+j)}, \end{aligned}$$

where

$$a_{ij} = \sum_{l=j-i}^{\lfloor \frac{i-1}{2} \rfloor} \binom{2\delta+j}{\delta+l} \binom{n-2\delta-j}{i-j+l} (q-1)^{\delta+2l+i-j}.$$

We use a_{ij} to denote the number of $\delta + i$ weight vectors that are at a distance less than $\delta + i$ from a particular $2\delta + j$ weight codeword and recall that $|ker(f)|_{(2\delta+j)}$ represents the number of vectors of weight $2\delta + j$ in $\bar{f}_0$.

A vector R in $W_{\delta+i} \cap S_{\delta+i}$ is at a distance of $\delta + i$ from say N codewords. It can be verified that these codewords have to be of even weight. Thus, we can say that N is upper bounded by

$$N \leq 1 + \sum_{\substack{j=2 \\ j \text{ is even}}}^{2(i-\delta)} |ker(f)|_{(2\delta+j)}.$$

Considering all the codewords corresponding to the vectors in $ker(f)$, we get

$$\begin{aligned} |B_H(f_i, t)| &\geq q^{k-l} \left[\sum_{i=0}^{\delta} \binom{n}{i} (q-1)^i \right. \\ &\quad \left. + \sum_{i=\delta+1}^t \frac{\binom{n}{i} (q-1)^i - \sum_{j=1}^{2(i-\delta)-1} a_{ij} |ker(f)|_{(2\delta+j)}}{1 + \sum_{j=2}^{2(i-\delta)} |ker(f)|_{(2\delta+j)}} \right]. \end{aligned}$$

Substituting in (15) gives (14). $\blacksquare$

Now, we provide a class of functions for which coset-wise coding is optimal. From Corollaries 1 and 5, coset-wise coding is optimal for a function f , if there is a selection of minimum weight representative vectors from each coset in $\mathbb{F}_q^k / ker(f)$ such that it satisfies (8). Now, we give a class of functions for which such a selection of minimum weight vectors is possible.

Definition 12. Consider a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$. Then

$$\mathcal{S} := \{ \arg \min_{\mathbf{u} \in \bar{f}} w_H(\mathbf{u}) : \bar{f} \in \mathbb{F}_q^k / ker(f) \},$$

i.e., $\mathcal{S}$ is a selection of minimum weight vectors from each coset in $\mathbb{F}_q^k / ker(f)$. Note that $\mathcal{S}$ is not unique.

Definition 13. We define $C_f(i)$ to be the number of cosets in $\mathbb{F}_q^k / ker(f)$ whose minimum weight vector has 1s in i indices and 0s in all the other indices, i.e.,

$$C_f(i) = \left| \left\{ \mathbf{u} \in \mathcal{S} : \mathbf{u} = \sum_{j \in I} \mathbf{e}_j \text{ for some } I \subseteq [k], |I| = i \right\} \right|.$$

Lemma 2. For a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$, at least $rank(\mathbf{F})$ cosets in $\mathbb{F}_q^k / ker(f)$ should have a unit vector as its minimum weight vector, i.e., $C_f(1) \geq rank(\mathbf{F})$.

Proof: Consider the linear function $f(\mathbf{x}) = \mathbf{F}\mathbf{x}$. We have, $C_f(1) = |\{\mathbf{u} \in \mathcal{S} : \mathbf{u} = \mathbf{e}_j \text{ for some } j \in [k]\}|$. The function f maps all the vectors in a particular coset $\bar{f}_i \in \mathbb{F}_q^k / ker(f)$ to the same image $f_i \in \mathbb{F}_q^l$ and vectors in distinct cosets will not have the same image. This implies that for some $i, j \in [k]$, unit vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ belong to the same coset if and only if the columns $\mathbf{F}_{(i)}$ and $\mathbf{F}_{(j)}$ are equal. This implies that the number of distinct non-zero columns of $\mathbf{F}$ is equal to $C_f(1)$. Since $rank(\mathbf{F})$ cannot be more than the number of distinct non-zero columns of $\mathbf{F}$, $C_f(1) \geq rank(\mathbf{F})$. $\blacksquare$

Lemma 3. Consider a linear function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$. There exists a selection $\mathcal{S} := \{\arg \min_{\mathbf{u} \in \bar{f}} w_H(\mathbf{u}) : \bar{f} \in \mathbb{F}_q^k / ker(f)\}$ that forms a subspace of $\mathbb{F}_q^k$ if and only if exactly l cosets of $\mathbb{F}_q^k / ker(f)$ have a unit vector as its minimum weight vector, i.e., $C_f(1) = l$.

Proof: For the forward implication, consider a selection $\mathcal{S}$ which is a subspace of $\mathbb{F}_q^k$. Note that $\dim_q \mathcal{S} = l$. From Lemma 2, $C_f(1) = l' \geq l$. For the sake of contradiction, let $l' > l$. Then, there exists a set of l' unit vectors in $\mathcal{S}$, which are independent. This is a contradiction as $\dim_q \mathcal{S} = l$.

For the reverse implication, let $C_f(1) = l$. Let $E = \{\mathbf{u} \in \mathcal{S} : \mathbf{u} = \mathbf{e}_j \text{ for some } j \in [k]\}$. We now show that vectors in $span(E)$ occur in distinct cosets. Assume WLOG that $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_l\}$. Since E is a set of independent vectors, $\mathbf{F}E := \{\mathbf{F}\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3, \dots, \mathbf{F}\mathbf{e}_l\}$ is a basis of $\mathbb{F}_q^l$, the image of f .

Consider $\mathbf{v}_1, \mathbf{v}_2 \in span(E)$. Let $\mathbf{v}_1 = \sum_{i=1}^l a_i \mathbf{e}_i$ and $\mathbf{v}_2 = \sum_{i=1}^l b_i \mathbf{e}_i$, where $a_i, b_i \in \mathbb{F}_q \forall i \in [l]$. Then,

$\mathbf{v}_1$ and $\mathbf{v}_2$ are in the same coset

$$\iff \mathbf{v}_1 - \mathbf{v}_2 \in ker(f)$$

$$\iff \mathbf{F}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$$

$$\iff \mathbf{F} \left(\sum_{i=1}^l a_i \mathbf{e}_i - \sum_{i=1}^l b_i \mathbf{e}_i \right) = \mathbf{0}$$

$$\begin{aligned} \Leftrightarrow \sum_{i=1}^l a_i \mathbf{F} \mathbf{e}_i - \sum_{i=1}^l b_i \mathbf{F} \mathbf{e}_i &= 0 \\ \Leftrightarrow a_i &= b_i \quad \forall i \in [l] \quad \{\cdot \cdot \mathbf{F} \mathbf{E} \text{ is a basis of } \mathbb{F}_q^l\}. \end{aligned}$$

So, distinct vectors in $\text{span}(E)$ are present in distinct cosets. It is easy to verify that the vectors in $\text{span}(E)$ are minimum weight vectors in the corresponding cosets. Thus, $S = \text{span}(E)$. ■

Thus, for a selection S which is a vector space to exist, exactly l cosets should have unit vectors as minimum weight vectors. So, the matrix representation $\mathbf{F}$ of f must have exactly l distinct non-zero columns. Since $\text{rank}(\mathbf{F}) = l$, these columns have to be independent.

Lemmas 2 and 3 lead to a class of functions for which the representative vectors satisfy (8) which is given in the following theorem.

Theorem 9. *Consider a linear function*

$$\begin{aligned} f : \mathbb{F}_q^k &\rightarrow \mathbb{F}_q^l \\ \mathbf{u} &\mapsto \mathbf{F}\mathbf{u}; \mathbf{F} \in \mathbb{F}_q^{l \times k}. \end{aligned}$$

There exists a selection of representative vectors $\{\mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1}\}$, where $\{f(\mathbf{u}_0), f(\mathbf{u}_1) \dots f(\mathbf{u}_{q^l-1})\} = \text{Im}(f)$ such that $\mathbf{D}_f(t, f_0, f_1 \dots f_{q^l-1}) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1})$ if the number of distinct, non-zero, independent columns of $\mathbf{F}$ is l .

Proof: From Remark 5, we know that for linear functions, the weight distribution of the vectors in $S = \{\mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1}\}$ is the same as $\{d_f(f_i, 0) : f_i \in \mathbb{F}_q^l\}$, the intercoset distance distribution. According to Lemma 3, there should be exactly l unit vectors in S for it to be a vector space and this implies that $\mathbf{F}$ should have exactly l non-zero distinct columns. Since $\text{rank}(\mathbf{F})$ is assumed to be l , the non-zero columns have to be independent. If S is a vector space, then its distance distribution will also be uniform, which implies that $\mathbf{D}_f(t, f_0, f_1 \dots f_{q^l-1}) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1})$. ■

The function f given in Example 2 has exactly 2 distinct, non-zero columns independent over $\mathbb{F}_2$. We can see that there exists a selection $S = \{0000, 0100, 1000, 1100\}$ which is a subspace of $\mathbb{F}_2^4$ and $\mathbf{D}_f(t, f_0, f_1, f_2, f_3) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where $S = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We now provide a coding scheme for the class of functions specified in Theorem 9. Consider a function $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ with $\mathbf{F} \in \mathbb{F}_q^{l \times k}$ as its matrix representation which has exactly l distinct non-zero independent columns. Choose the set of minimum weight representative vectors $S = \{\mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1}\}$ such that S is a subspace of $\mathbb{F}_q^k$. From Corollary 1, the optimal redundancy is given by $r_f(k, t) = N_2(\mathbf{D}_f(t, f_0, f_1 \dots f_{q^l-1})) = N_2(\mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1}))$. Let E be a spanning set of S . Assume WLOG that $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_l\}$. Then S is of the following form:

$$S = \{(x_1 x_2 \dots x_k) \in \mathbb{F}_q^k : x_i = 0 \quad \forall i \in [l+1 : k]\}.$$

Define,

$$S' = \{(x_1 x_2 \dots x_l) : (x_1 x_2 \dots x_l \dots x_k) \in S\}.$$

Note that $S' \equiv \mathbb{F}_q^l$. Let $S' = \{\mathbf{u}'_0, \mathbf{u}'_1 \dots \mathbf{u}'_{q^l-1}\}$. Note that $\mathbf{D}_f(t, \mathbf{u}'_0, \mathbf{u}'_1 \dots \mathbf{u}'_{q^l-1}) = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1})$.

Note that the distance requirement matrix $\mathbf{D}_f(t, \mathbf{u}'_0, \mathbf{u}'_1 \dots \mathbf{u}'_{q^l-1})$ is the same as the distance requirement matrix for the parity vectors of a classical systematic ECC of cardinality q^l and minimum distance $2t+1$ (The parity vectors of a classical systematic ECC can be considered as the parity vectors of an FCC where the function to be corrected is a bijection). So, the parity bits of a systematic $(n, q^l, 2t+1)$ block code will be a $\mathbf{D}$ -code where $\mathbf{D} = \mathbf{D}_f(t, \mathbf{u}_0, \mathbf{u}_1 \dots \mathbf{u}_{q^l-1})$.

For the function in Example 2, the matrix $\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ satisfies the above condition.

We can choose $S = \{0000, 1000, 0100, 1100\}$ and $S' = \{00, 01, 10, 11\} \equiv \mathbb{F}_2^2$. The function distance matrix is the same as $\mathbf{D}_f(t, 00, 01, 10, 11)$. Say, we need an $(f, 1)$ -FCC. This is equivalent to the distance constraint that should be satisfied by the parity bits of a block code of cardinality 4 and minimum distance 3. From [7], there exists a $(5, 4, 3)$ block code. Using the 3 parity bits of this code for coset-wise coding, we have an FCC of $r = 3$. As it is well known from classical coding theory, that non-trivial binary t -error correcting codes of parity length $2t$ (MDS codes) do not exist, $r_f(k, 1) \geq 3$. Thus, for the function in Example 2, $r_f(k, 1) = 3$.

Now, we consider linear functions where coset-wise coding is equivalent to a reduced dimensional classical error correction problem. Consider a linear function, $f : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^l$ and $\mathbf{u} \mapsto \mathbf{F}\mathbf{u}; \mathbf{F} \in \mathbb{F}_q^{l \times k}$ with $k \geq q^l - 1$. If the number of distinct columns of $\mathbf{F}$ is at least $q^l - 1$, then all the cosets of $\mathbb{F}_q^k / \ker(f)$ will have a unit vector. Then, the function distance matrix will be of the form

$$[\mathbf{D}_f(t, f_0, f_1 \dots f_{q^l-1})]_{ij} = \begin{cases} 2t; & \text{if } i \neq j \\ 0; & \text{else,} \end{cases} \quad (16)$$

i.e, the diagonal entries are 0 and the non-diagonal entries are $2t$.

So, the parity vectors satisfying the distance requirements will be an ECC of cardinality q^l and minimum distance $2t$. Essentially we are reducing the dimension of the error correction problem from $k \geq q^l$ to l and minimum distance requirement from $2t+1$ to $2t$.

IV. DISCUSSION

We proposed a lower bound for the redundancy required for a function-correcting code, which also led to a bound on the redundancy of classical ECCs when the function considered is a bijection. We explored a class of function-correcting codes where the function to be computed at the receiver is linear. We showed that the adjacency matrix of $\mathcal{G}_f(t, k, r)$ has a recursive block circulant structure. When the domain of the function is a vector space over a field of characteristic 2, we characterised the spectrum of the adjacency matrix, which leads to lower bounds on redundancy. Functions that can attain this lower bound are yet to be found. For linear functions, we showed that the generalised Plotkin bound proposed in [1] simplifies for linear functions. We proposed a version of the sphere packing bound for coset-wise coding. We characterised a class of functions for which coset-wise coding is optimal and proposed a coding scheme for such functions. We also characterised a class of functions for which the redundancy to be added for coset-wise coding is equivalent to a lower dimensional classical ECC. Further research directions include

using the the structure of $\mathcal{G}_f(t, k, r)$ to get improved bounds and coding schemes and to identify functions and coding schemes that can attain or come close to the proposed bounds. Linear codes for function correction are yet to be explored.

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