FACE EMBEDDINGS OF ARCHIMEDEAN SOLIDS

TOMMY MURPHY AND DAVID WEED

Abstract. We characterize the Archimedean solids among the convex uniform polyhedra via face-embeddings into a regular Tetrahedron. This result has been listed without proof in the literature. As an application, we obtain a lattice packing of $\mathbb{R}^3$ by the truncated Icosahedron whose packing density is notably close to the optimal value obtained in [1].

To understand any class of objects in mathematics one has to describe how one object of the class "sits inside" another. The present work is concerned with the class of convex uniform polyhedra. The simplest example of such an object is the tetrahedron T, a Platonic/regular solid. In this vein, our motivating question is to determine how convex uniform polyhedra can be naturally placed inside T. We begin with a definition:

Definition 0.1. Let S_1 and S_2 be two distinct convex uniform polyhedra. Then S_1 admits a "*k-face embedding*" in S_2, written $S_1 \subset_k S_2$, if S_1 can be circumscribed by S_2 so that each point of intersection lies on one of k fixed faces of S_1, and extends so that each of these k faces is a subset of a face of S_2.

The reason we are interested in this definition is as follows.

Main Theorem. Let S be a convex uniform polyhedron. Then $S \subset_4 T$ if, and only if, S is the Icosahedron I, the Octahedron O, or an Archimedean solid.

Our interest in this condition was piqued by reading [7], which lists this condition as a characterization of Archimedean solids. No proof is provided, but Pugh is adamant that this is a fundamental fact, going as far as to use this property to actually define the Archimedean solids. At various other places in the literature, this result is mentioned [3, 5] but no details have been provided.

The proof is more important than the theorem. Classically the Archimedean solids are constructed by starting with the Platonic solids and applying various truncations and strictly more complicated operations such as snubification and expansion. We refer the reader to [2, 6, 7, 9]. Moreover, topological arguments are employed as part of the classifications because these operations result in faces where the edge lengths can vary. A cumbersome rescaling of certain faces is then required to produce a uniform solid (c.f. the truncated Cuboctahedron and truncated Icosidodecahedron). This issue was apparent even to Kepler [6]. Kappraff [5] mentions that it is possible to construct all the Archimedean solids in an entirely metric fashion by slicing the Platonic solids with judiciously chosen planes, but does not give precise details. We explicitly describe how to do this and thus eliminate these ambiguous rescaling issues. Throughout we will use Conway's polyhedra notation [4].

To establish the main theorem we need to explicitly construct a four-face embedding of each Archimedean solid. The starting point is the classically known fact that O and I admit a four-face embedding into T [5]. We will build almost all the Archimedean solids from this embedding of O or I using various truncations that do not affect the four faces coincident with the faces of the tetrahedron. The only exception to this approach is tT, the truncated Tetrahedron, where the 4-face embedding is immediate.

Taking two Archimedean solids S_1 and S_2, it is natural to see if the 4-face embeddings $S_1 \subset_4 T$ and $S_2 \subset_4 T$ induce a 4-face embedding $S_1 \subset_4 S_2$. In particular, 4-face embedding S_1 inside the truncated Octahedron tO is desirable as tO tessellates space. This leads naturally to the following corollary.

Corollary 0.2. *There is a lattice packing of the truncated Icosahedron with density* ≈ 0.77.

This packing is interesting as the density is very near the optimal density (determined in [1]) ≈ 0.785. However it is a different lattice.

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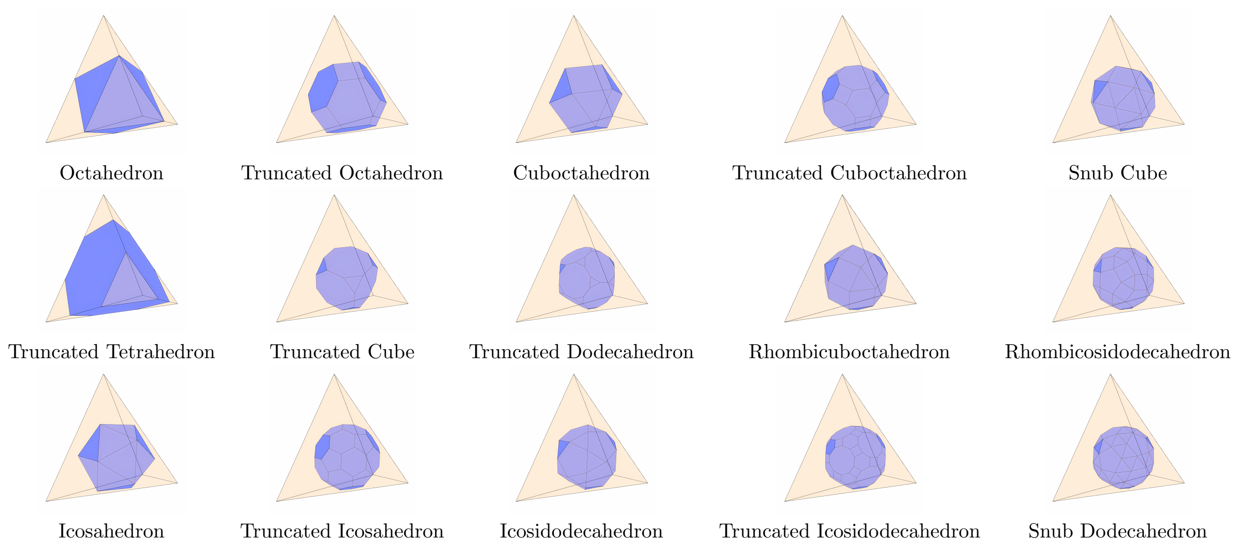


FIGURE 1. Uniform Solids Embedded in a Tetrahedron

1. Vertex Truncation

A vertex truncation is obtained by cutting along planes perpendicular to a vertex. The operation is generally measured by a length along the edge of the solid rather than the actual depth of cut. Without loss of generality, we scale O and I so that the edges have length 1. This is convenient as when we truncate α of the edge length, we can simply say we *vertex truncate by* α, where $\alpha \in (0, 1)$. Vertex truncating to the halfway point of the edge (vertex truncating by 1/2) is also known as *rectification*.

Example 1.1. *Starting from one of its vertices of T, move along the three edges attached by the truncation value. The points on the edges are coplanar and lay on the vertices of an equilateral triangle. More specifically, the plane defined by these points is parallel to the plane tangent to the circumsphere at the original vertex.*

As is well known, starting with the Platonic solids one may vertex truncate by 1/3 to produce three of the Archimedean solids, namely the truncated Tetrahedron (tT), truncated Octahedron (tO), and truncated Icosahedron (tI). A classical fact is that truncation by 1/2 (rectification) produces the Icosidodecahedron (ID) and the Cuboctahedron (CO).

Most authors simply truncate C and D by 1/3 to to produce the truncated Cube (tC) and truncated Dodecahedron (tD) respectively. However, we want to preserve the 4-face embedding for these two solids so we use an alternate construction. Due to the dual nature of O and C the cutting planes for our vertex truncation of O are the faces of C and vice-versa (similarly for I and D). This means we can reach these required solids by truncating past 1/2 for O and I. Explicitly, vertex truncating O by $\frac{2+\sqrt{2}}{3+2\sqrt{2}}$ produces tC and vertex truncating I by $\frac{2+\phi}{3+2\phi}$ where, $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio), produces tD. These values are notable since $\sqrt{2}$ and ϕ are the lengths of the face diagonals of C and D.

2. Vertex and Edge truncation

2.1. **Initial Setup.** Analogous to vertex truncation, edge truncation is defined by cutting by a plane parallel to the tangent plane of the mid-sphere (i.e. the sphere touching each edge midpoint). We measure perpendicular to the edge along the face to determine the depth of cut. This again allows us to abuse notation since the edges have length 1, so we will informally say we edge-truncate by β, where $\beta \in (0, 1)$.

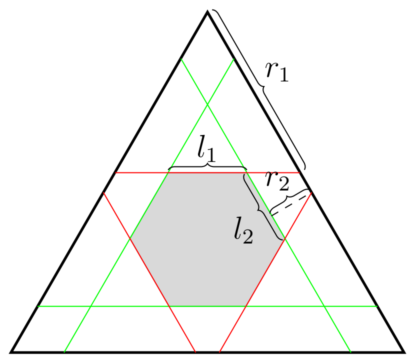
The next solids to construct are the Rhombicuboctahedron (eO), Rhombicosidodecahedron (eI), truncated Cuboctahedron (tCO), and truncated Icosidodecahedron (tID). To achieve this we perform vertex truncation and edge truncation simultaneously. The notations tCO and tID are not wholly accurate since they cannot be produced by simple vertex truncation. As already mentioned, rescaling certain edges is required in this case. The notation eO, eI comes from the fact that these polyhedra are typically constructed via expansion. In fact, Boole-Stott [9] originally produced 11 of the 13 Archimedean solids through vertex, edge, and face expansion. For instance, her construction yields tCO without the issue of having to rescaling certain edges to ensure a uniform solid. However, the price which must be paid for this approach is that a new construction (expansion) is required.

The claim is that performing the following vertex and edge truncations of O and I produces eO, eI, tCO, and tID while preserving the four-face embedding. As before, $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

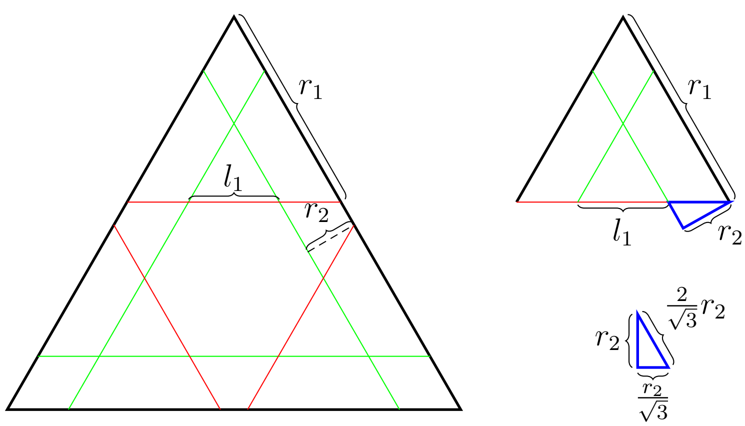
Seed	Solid	Vertex	Edge
O	eO	$\frac{2}{3+\sqrt{2}}$	$\frac{\sqrt{3}}{6+2\sqrt{2}}$
	tCO	$\frac{2+\sqrt{2}}{3+3\sqrt{2}}$	$\frac{\sqrt{3}}{6+6\sqrt{2}}$
I	eI	$\frac{2}{3+\phi}$	$\frac{\sqrt{3}}{6+2\phi}$
	tID	$\frac{2+\phi}{3+3\phi}$	$\frac{\sqrt{3}}{6+6\phi}$

Table 1. Parameters for simultaneous vertex and edge truncation.

2.2. **Explicit Computations.** Here we present the details of the construction of tCO by applying vertex and edge truncations to O using the values given in the table. For the remaining solids mentioned, the proof is entirely analogous, and details are left to the reader. Starting with $O \subset_4 T$, the first step is to mark the cutting lines on the triangular faces. The red lines represent the planes for vertex truncation by $r_1 \in (0,1)$ and the green lines denotes edge truncation by $r_2 \in (0,1)$. In the center the shaded hexagon is a new face of tCO. The four faces of O which are subsets of the faces of the circumscribing T will give us the 4-face embedding desired.



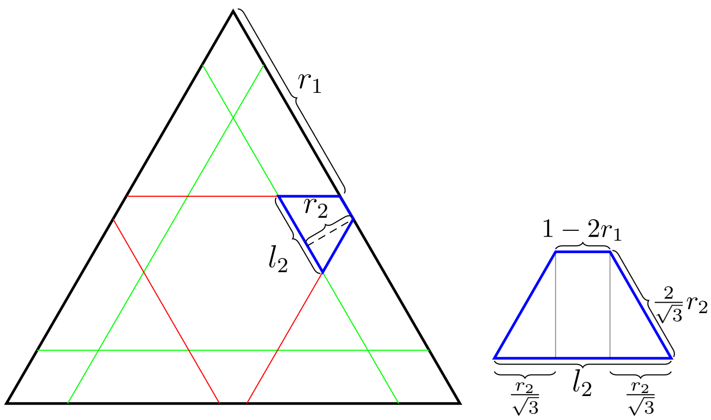
Firstly we parameterize l_1 and l_2 in terms of r_1 and r_2. We obtain l_1 by observing the top triangle with side labeled r_1 is equilateral.



Equating sides of the upper triangle yields $r_1 = l_1 + 2\left(\frac{2}{\sqrt{3}}r_2\right)$, or equivalently

$$l_1 = r_1 - \frac{4}{\sqrt{3}}r_2 \tag{2.1}$$

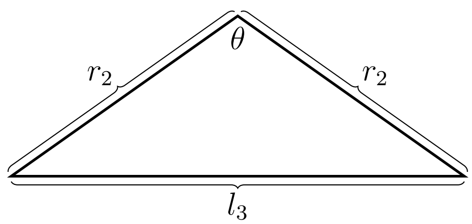
To determine l_2 we focus on the following blue trapezoid:



The top value of $1-2r_1$ comes from the fact that O has edge lengths of 1 and we vertex truncate by r_1. Clearly

$$l_2 = 1 - 2r_1 + \frac{2}{\sqrt{3}}r_2. \tag{2.2}$$

On the other hand, l_2 is an edge of our new solid and is also be the edge of the adjacent rectangular face. This new rectangular face is going to be coincident to the edge truncation plane and we can determine its other side length l_3 using the following diagram.



This diagram is a cross section of O perpendicular to the edge, where the top vertex would be the midpoint of the edge. To determine the length l_3 of the new edge created after edge truncating by r_2, an easy application of the Law of Cosines yields

$$l_3 = r_2\sqrt{2}\sqrt{1-\cos\theta} \tag{2.3}$$

where θ is the dihedral angle of the seed solid. Here the angle θ is the dihedral angle of O, $\arccos(-1/3)$. Substituting this in yields,

$$l_3 = \frac{2\sqrt{2}}{\sqrt{3}}r_2. \tag{2.4}$$

Equations (2.1), (2.2) and (2.4) express l_1, l_2 and l_3 as functions of r_1 and r_2. Now to construct tCO, the requirement that the hexagon is regular forces $l_1 = l_2$, which from Equations (2.1) and (2.2) results in $r_1 = \frac{1}{3} + \frac{2}{\sqrt{3}}r_2$. Similarly looking at the rectangular face adjacent to the top edge of the hexagon, we must have $l_1 = l_3$ to ensure that the resulting solid is Archimedean. From Equations (2.1) and (2.4) this forces $1 - 2r_1 + \frac{2r_2}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}}r_2$. Solving this system of equations results in $r_1 = \frac{2+\sqrt{2}}{3+3\sqrt{2}}$ and $r_2 = \frac{\sqrt{3}}{6+6\sqrt{2}}$.

As already mentioned these simultaneous truncations also produce a rectangular edge with side lengths l_2 and l_3. A direct computation, which we leave to the reader, shows that simultaneously vertex and edge truncating by the given values for r_1 and r_2 forces $l_2 = l_3$, and we conclude we have consistency and thus have constructed tCO as claimed.

2.3. **Skew Truncation.** The final two Archimedean Solids to construct are the Snub Cube (sC) and the Snub Dodecahedron (sD). Either of the chiral forms can be produced by the following approach. The desired embedding follows from a construction due to Rotgé [8] (see also [5]) which we call skew truncation. Starting with O or I we subdivide each edge into two parts that have ratios as in Table 1. The point where the edge is divided is connected to the opposite vertex. This will yield three lines on the face of the original solid intersecting at three points of a regular triangle that has been rotated on the face. Our cutting plane is defined by selecting one edge of this new triangle and the point of the new triangle opposite the original edge. This describes a unique plane for each edge of the original solid. We also define a cutting plane by selecting the vertex of the rotated triangle and the vertices of the rotated triangles around the adjacent original vertex. The key point is that this operation preserves a section of the faces of the original solids, namely the rotated triangle. This means our four-face embedding is preserved throughout this operation.

Solid	Exact Ratio	Approximate
sC	$\frac{1}{3}(1+\sqrt[3]{19-3\sqrt{33}}+\sqrt[3]{19+3\sqrt{33}})$	1.839286755...
sD	$\frac{1}{3}+\frac{2^{5/3}(1+i\sqrt{3}))}{3\sqrt[3]{-49-27\sqrt{5}+3\sqrt{3(186+98\sqrt{5})}}}+\frac{(1-i\sqrt{3})\sqrt[3]{-49-27\sqrt{5}+3\sqrt{3(186+98\sqrt{5})}}}{6\cdot 2^{2/3}}$	1.943151259...

The approximate values here are pulled from Rotgé and the exact values are the results of solving the following cubic,

$$r^3 - r^2 - r - 1 + 2\cos\alpha = 0$$

for $\alpha = 90°,\ 108°$ as shown in [8]. The value provided for sC is known as the Tribonacci Constant where as the the value for sD does not appear to have an analogous title.

3. Proof of the Main Theorem

As we have explicitly established the 4-face embeddings required, it remains to show the converse statement. Specifically, we claim that the remaining convex uniform polyhedra are not four-face embeddable in T. The other members of this class are the remaining Platonic solids C and D, the prisms P_n, and the antiprisms A_n. Our argument uses the dihedral angles produced by various faces. Observe that when a solid $S \subset_4 T$, there must be four faces of S that have pairwise dihedral angles equal to $\arccos(1/3)$, the dihedral angle of T.

For each of these solids, the possible angles between pairs of faces are as follows.

Solid	Possible Face Angles
Cube	$\pi/2,\ \pi$
Dodecahedron	$\arccos(\pm\sqrt{5}/5),\ \pi$
n-Prism	$(n-2)k\pi/n : 1 \le k \le \lfloor n/2 \rfloor,\ \pi/2,\ \pi$
n-Antiprism	$\arccos(\pm 1/\sqrt{3}\tan(\pi/2n))$

This table shows that none of the attainable angles are equivalent to $\arccos(1/3)$. This is immediate for all the families except for the antiprisms. An antiprism A_n has two n-gonal faces and $2n$ triangular faces.The two possible angles are between the n-gonal face and the edge (or vertex) adjacent triangular face. The only way that the angles coincide is when $n = 3$. Here, A_3 is equivalent to O and we have already covered this case. The only final possibility is if T is not coincident with the n-gonal face and only touches the triangular faces. This implies opposite triangular faces of A_n coincide with T and thus $2\arccos(\pm 1/\sqrt{3}\tan(\pi/2n)) = \arccos(1/3)$ for some n, which is never true.

The main Theorem now follows. $\square$

4. The Packing result

A (Bravais) lattice packing L is one in which the centroids of the non-overlapping particles are located at the points of L, each oriented in the same direction. $\mathbb{R}^3$ can then be geometrically divided into identical regions F called fundamental cells, each of which contains just the centroid of one particle. A periodic packing of particles S is obtained by placing a fixed non-overlapping configuration of N particles (where $N \geq 1$) with arbitrary orientations in each fundamental cell of L. Thus, the packing is still periodic under translations by L, but the N particles can occur anywhere in the chosen cell subject to the non-overlap condition. The density of a periodic packing is $\frac{N\cdot(\mathrm{vol}(S))}{\mathrm{vol}(F)}$.

Now that some basic language has been set up, observe that tO tessellates $\mathbb{R}^3$, forming the bitruncated cubic honeycomb. The centroid of each tO forms a standard body-centered cubic (bcc) lattice. By our main theorem, $tO \subset_4 T$ and $tI \subset_4 T$. Using this orientation we can place a tI at each point in the bcc lattice. This forms a new packing of $\mathbb{R}^3$ that has density $\frac{\mathrm{vol}(tI)}{\mathrm{vol}(tO)} \approx 0.770$. In terms of a Bravais lattice packing tO is our fundamental cell and the other solid is the particle. Curiously, this packing of tI is within 1.25% of the optimal lattice packing (c.f [1], [10]).

To explore the lattice packing structures, a Grasshopper script was used in Rhino3D. Since each solid has a k-face embedding in T we scale the radii of the insphere that is tangent to those 4 faces to all be equal. This means that each solid can be embedded in the same tetrahedron. Then we make use of Grasshopper's boundary representation to determine if one solid is completely contained within another. Normally the orientation would need to be accounted for but the default orientation for each of the Archimedean Solids in Grasshopper is already such that only scaling is required. We then found the ratio in the volume of each solid compared to tO, but all other instances the ratios were lower and consequently not as interesting.

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(T. Murphy) Department of Mathematics, California State University Fullerton, 800 N. State College Blvd., Fullerton 92831 CA, USA

Email address: tmurphy@fullerton.edu

URL: www.fullerton.edu/math/faculty/tmurphy/

(D. Weed) Department of Mathematics, University of California Riverside

Email address: david@davidweed.net

URL: www.davidweed.net