Dihedral-product groups of nonabelian simple groups

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Abstract

Given a finite group G, a group X is called a *dihedral-product group* of G if $X = GD$ where D is a finite dihedral group. In particular, it is called a *dihedral-skew product group* of G if $G \cap D = 1$ and D has a trivial core in X. In this paper, dihedral-skew product groups of all nonabelian simple groups are classified and dihedral-product groups of all nonabelian simple groups are characterized.

1 Introduction

Throughout the paper all groups are finite. Given a finite group G, a group X is called a *dihedral-product group* of G if $X = GD$ where D is a finite dihedral group. In particular, a dihedral-product group $X = GD$ of G is called a *dihedral-skew product group* of G if $G \cap D = 1$ and D has a trivial core in X. The investigation of dihedral-skew product groups is at least related to the following three topics.

(1) *Factorizations of groups:* A group X is said to be properly *factorizable* if $X = GH$ for two proper subgroups G and H of X, while the expression $X = GH$ is called a *factorization* of X. Furthermore, if $G \cap H = 1$, then we say that X has an *exact factorization*. Hence, a dihedral-skew product group X of a group G means that X has a factorization $X = GD$ such that D is a dihedral group, and a dihedral-skew product group X of a group G means that X has an exact factorization $X = GD$ such that D is a dihedral group with a trivial core in X. Similarly, for any given group G, a group X is called a *skew product group* of G if $X = GC$ where $G \cap C = 1$ and C is a cyclic group with a trivial core in X.

Factorizations of groups naturally arise from the well-known Frattini's argument, including its version in permutation groups. One of the most famous results about factorized groups might be one of theorems of Itô, saying that any group is metabelian whenever it is the product of two abelian subgroups (see [8]). Later, Wielandt and Kegel showed that the product of two nilpotent subgroups must be soluble (see [22] and [10]). Douglas showed that the product of two cyclic groups must be super-solvable (see [6]). The factorizations of the finite almost simple groups were determined in [16] and the factorizations

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of almost simple groups with a solvable factor were determined in [15]. There are many other papers related to factorizations, for instance, finite products of soluble groups, factorizations with one nilpotent factor and so on. Here we are not able to list all references and the readers may refer to a survey paper [1].

(2) *Permutation groups:* For a factorization $X = GH$, let $\Omega = [X : G]$ be the set of right cosets of G in X. If G is core free, then X acts faithfully on Ω. Hence, X is a permutation group with degree $|\Omega|$ and contains a transitive subgroup H. Clearly, a dihedral-skew product group $X = GD$ of G is a permutation group acting on $[X : D]$ and containing a regular subgroup G. The study of a permutation group containing a transitive subgroup played an important role in the history of permutation group theory. A transitive permutation group X is called *primitive* if Ω has no nontrivial X-invariant partition. Burnside and Schur proved that primitive permutation group containing a regualr cyclic group of order p^m and composite order is 2-transitive (see [3, 19]). In 1949, Wielandt showed that such group containing a regualr dihedral group is 2-transitive [23], and in 1957, Scott proved that such group containing a regualr generalized quaternion group is 2-transitive [20]. A permutation group is called *quasiprimitive* if each of its non-trivial normal subgroups is transitive. Recently, Li, Pan and Xia characterized quasiprimitive permutation group containing a regualr metacyclic group [14].

(3) *Cayley maps:* A map $\mathcal{M}$ is said to be *orientable regular* or *regular* if its automorphism group $\mathrm{Aut}\,(\mathcal{M})$ is regular on arcs or flags, see [12] for the definitions. Richter et al. show that a map $\mathcal{M}$ is a *Cayley map* of a group G if and only if the group $\mathrm{Aut}\,(\mathcal{M})$ contains a regular subgroup that is isomorphic to G (see [18]). Jajcay and Širáň showed that for a rotary Cayley map CM of G, the automorphism group Aut (CM) is a skew product group of G (see [9]). Orientable regular Cayley maps of cyclic groups, dihedral groups and elementary abelian p-groups are classified in [5, 11, 7], separately. And orientable regular Cayley maps of finite nonabelian characteristically simple groups are characterized in [4]. Recently, Kwak and Kwon showed that for an unoriented regular Cayley map CM of G, the automorphism group Aut (CM) is a dihedral-skew product group of G (see [12]). Thus the problem of determining all regular Cayley maps of G is suffice to determine all dihedral-skew product groups of G. Since orientable regular Cayley maps of nonabelian simple groups are characterized in [4], we shall consider their unoriented regular Cayley maps. Hence, we need to determine dihedral-skew product groups of all nonabelian simple groups.

Theorem 1.1 *Let $X = GD$ be a dihedral-skew product group of a nonabelian simple group G. Then either $G \lhd X$ or the triple (X, G, D) is given in Table 1.*

Recall that a permutation group is called quasiprimitive if each of its non-trivial normal subgroups is transitive. Then a permutation group is called *non-quasiprimitive* if there exists a non-trivial normal subgroup that is non-transitive. In other words, a non-quasiprimitive permutation group means that there exists a non-trivial normal subgroup

Table 1: Dihedral-skew product groups of G

Row	X	G	D
1	$\mathrm{AGL}(3,2)$	$GL(3,2)$	D_8
2	M_{12}	M_{11}	D_{12}
3	M_{24}	M_{23}	D_{24}
4	$A_{4m}, m \geq 2$	A_{4m-1}	D_{4m}
5	$PGL(2,11)$	A_5	D_{22}
6	$A_{2m+3} \rtimes \mathbb{Z}_2, m \geq 2$	A_{2m+2}	$D_{2(2m+3)}$
7	$\mathrm{Aut}\,(M_{12})$	M_{11}	D_{24}
8	$A_{4m} \rtimes \mathbb{Z}_2, m \geq 2$	A_{4m-1}	D_{8m}

which induces proper blocks. In fact, a permutation group is either quasiprimitive or non-quasiprimitive.

Remark: X acts quasiprimitively on $[X:G]$ in Rows 1-4 of Table 1 and non-quasiprimitively in Rows 5-8.

Next, dihedral-skew product groups of nonabelian simple groups are characterized. For two groups G and H, let $G.H$ denote an extension of G by H.

Theorem 1.2 *Let $Y = GD$ be a dihedral-product group of a nonabelian simple group G and set $D_Y = \cap_{y\in Y} D^y$, the core of D in Y. Then either $G \lhd Y$ or one of the following holds:*

(1) *$G \cap D = 1$, and one of the following holds:*

(1.1) *$D_Y = 1$ and Y is a dihedral-skew product group of G;*

(1.2) *$D_Y \neq 1$ is cyclic and $Y = D_Y.X = D_Y \rtimes X$, where X is Rows 5 and 6 in Table 1; or*

(1.3) *$D_Y \cong \mathbb{Z}_2$ and $Y = D_Y.X$, where X is one of Table 1.*

(2) *$G \cap D = 2$, and $(Y, G, D) = (A_{m+1}, A_m, D_{2(m+1)})$ or $(A_{m+1} \times \mathbb{Z}_2, A_m, D_{4(m+1)})$, with $\frac{m}{2} \geq 3$ even.*

2 Preliminaries

Notation and terminologies used in the paper are standard and can be found in [17]. In particular, $H_G = \cap_{g\in G} H^g$, the core of the subgroup H in G.

Proposition 2.1 *[13, Theorem 1.5] Let X be a quasiprimitive permutation group of degree n. Then, X contains a dihedral regular subgroup G if and only if X is 2-transitive, and one of the following holds, where $\omega \in \Omega$:*

(i) $(X,G) = (A_4, D_4), (S_4, D_4), (\mathrm{AGL}(3,2), D_8), (\mathrm{AGL}(4,2), D_{16}), (\mathbb{Z}_2^4 \rtimes A_6, D_{16})$, *or* $(\mathbb{Z}_2^4 \rtimes A_7, D_{16})$;

(ii) $(X, G, X_\omega) = (M_{12}, D_{12}, M_{11}), (M_{22}.2, D_{22}, \mathrm{PSL}(3,4).2)$, *or* (M_{24}, D_{24}, M_{23});

(iii) $(X, G, X_\omega) = (S_{2m}, D_{2m}, S_{2m-1})$ *or* $(A_{4m}, D_{4m}, A_{4m-1})$;

(iv) $X = \mathrm{PSL}(2, p^e).O, G = D_{p^e+1}$, *and* $X_\omega \trianglerighteq \mathbb{Z}_p^e \rtimes \mathbb{Z}_{\frac{p^e-1}{2}}.O$, *where* $p^e \equiv 3 \pmod 4$, *and* $O \leq \mathrm{Out}(\mathrm{PSL}(2, p^e)) \cong \mathbb{Z}_2 \times \mathbb{Z}_e$;

(v) $X = \mathrm{PGL}(2, p^e)\mathbb{Z}_f, G = D_{p^e+1}$, *and* $X_\omega = \mathbb{Z}_p^e \rtimes \mathbb{Z}_{p^e-1}$, *where* $p^e \equiv 1 \pmod 4$, *and* $f \mid e$.

Picking up all point stabilizers which are simple groups in Proposition 2.1 and using Magma [2], we immediately get the following corollary.

Corollary 2.2 *Let $X = GD$ be a dihedral-skew product group of a nonabelian simple group G. Suppose that X acts faithfully and quasiprimitively on $[X : G]$. Then $(X, G, D) = (\mathrm{AGL}(3,2), GL(3,2), D_8)$, (M_{12}, M_{11}, D_{12}), (M_{24}, M_{23}, D_{24}), or $(A_{4m}, A_{4m-1}, D_{4m})$ with $m \geq 2$.*

Proposition 2.3 *[4, Theorem 1.1] Let $X = GC$ be a skew group of a nonabelian simple group G. Then one of following holds:*

(1) *$X = G \rtimes C$ and $C \leq \mathrm{Aut}\,(G)$; or*

(2) *$(X, G) = (\mathrm{PSL}(2,11), A_5), (M_{23}, M_{22})$, or (A_{m+1}, A_m) with $m \geq 6$ even.*

For a group H, consider an extension of K by H, that is $G = K.H$. In particular, G is a *proper central extension* of K by H if $K \leq Z(G) \cap G'$ [21]. Then a central extension G is either a proper central extension or $K \times H$.

Proposition 2.4 *[4, Proposition 2.3] For the simple groups $\mathrm{PSL}(2,11), M_{23}$ and A_n for $5 \leqslant n \neq 6$, we have the following results:*

(1) *there exists no nontrivial proper central extension of $\mathrm{PSL}(2,11)$ (A_n) containing a subgroup A_5 (A_{n-1}), separately; and*

(2) *there exists no nontrivial proper central extension of M_{23}.*

Using Magma [2], we get the following facts.

Lemma 2.5 (1) *If $\mathrm{PSL}(2,11) \rtimes \mathbb{Z}_2$ contains a subgroup D_{22}, then $\mathrm{PSL}(2,11) \rtimes \mathbb{Z}_2 = \mathrm{PGL}(2,11) = A_5 D_{22}$;*

(2) *if* $M_{12} \rtimes \mathbb{Z}_2$ *contains a subgroup* D_{24}*, then* $M_{12} \rtimes \mathbb{Z}_2 = \mathrm{Aut}\,(M_{12}) = M_{11}D_{24}$*;*

(3) *there exists no* $M_{23} \rtimes \mathbb{Z}_2$ *(*$M_{24} \rtimes \mathbb{Z}_2$*) containing a subgroup* D_{46} *(*D_{48}*), separately; and*

(4) *there exists no* $\mathrm{AGL}(3,2) \rtimes \mathbb{Z}_2$ *containing a subgroup* D_{16}*.*

3 Proof of Theorem 1.1

Throughout this section, let G be a nonabelian simple group and $X = GD$ a dihedral-skew product group of G, where $D = \langle a\rangle \rtimes \langle b\rangle$.

Since $G_X \trianglelefteq G$ and G is a nonabelian simple group, we get that G_X is either G or 1. Suppose that $G_X = G$. Then $G \lhd X$, that is the former of Theorem 1.1. Since $X = G \rtimes D$ and G is a nonabelian simple group, we know $G\,\mathrm{char}\,X$. So in what follows, we assume that $G_X = 1$.

Lemma 3.1 *Suppose that* $G_X = 1$ *and there exists a maximal subgroup* H *of* X *such that* $G < H$ *and* $H \cap D \le \langle a\rangle$*. Then* $X = (G\langle a\rangle) \rtimes \langle b\rangle$ *and* (X, G, D) *is either* $(\mathrm{PGL}(2,11), A_5, D_{22})$ *or* $(A_{m+1} \rtimes \mathbb{Z}_2, A_m, D_{2(m+1)})$ *with* $m \ge 6$ *even. Moreover,* $G\langle a\rangle\,\mathrm{char}\,X$*.*

Proof Set $H \cap D = \langle a_1\rangle \ne 1$, that is $H = G\langle a_1\rangle$. Then we shall show $\langle a_1\rangle = \langle a\rangle$, that is $H = G\langle a\rangle$. Since

$$a_1 \in \cap_{a^ib\in D} H^{a^ib} = \cap_{ga^ib\in X} H^{ga^ib} = \cap_{x\in X} H^x = H_X, \tag{1}$$

we get $\langle a_1\rangle \le H_X$ which implies $H_X = (H_X \cap G)\langle a_1\rangle$. Noting that G is a nonabelian simple group and $H_X \cap G \trianglelefteq G$, we know that $H_X \cap G$ is either 1 or G. For the contrary, assume that $H_X \cap G = 1$, that is $H_X = \langle a_1\rangle$. Then one can get $H_X = \langle a_1\rangle \le D_X = 1$, a contradiction. Therefore, $H_X \cap G = G$, that is $H_X = G\langle a_1\rangle = H$. Noting that $X/H = \overline{D}$ act regularly and primitively on $[X : H]$, by the definition of H, we get $\langle a_1\rangle = \langle a\rangle$, that is

$$H = G\langle a\rangle \quad \text{and} \quad X = H \rtimes \langle b\rangle.$$

Then $\langle b\rangle_X \le D_X = 1$. Moreover, we get

$$\langle a\rangle_H = \cap_{a^ig\in H} \langle a\rangle^{a^ig} = \cap_{b^ja^ig\in X} \langle a\rangle^{b^ja^ig} = \langle a\rangle_X \le D_X = 1.$$

Noting that $G \cap \langle a\rangle = 1$, we know that H is a skew product group of G. Then by Proposition 2.3, we get that either $H = G \rtimes \langle a\rangle$, or $(H, G) = (\mathrm{PSL}(2,11), A_5)$, (M_{23}, M_{22}) or (A_{m+1}, A_m) with $m \ge 6$ even. For the contrary, assume $H = G \rtimes \langle a\rangle$. Then we get $G\,\mathrm{char}\,H \lhd X$, which implies $G \lhd X$, a contradiction. Hence, $(H, G) = (\mathrm{PSL}(2,11), A_5), (M_{23}, M_{22})$, or (A_{m+1}, A_m).

By $X = H \rtimes \langle b\rangle$, we get that (X, G, D) can only be in the following three cases:

$$(\mathrm{PSL}(2,11) \rtimes \mathbb{Z}_2, A_5, D_{22}), (M_{23} \rtimes \mathbb{Z}_2, M_{22}, D_{46}) \text{ or } (A_{m+1} \rtimes \mathbb{Z}_2, A_m, D_{2(m+1)}).$$

By Lemma 2.5, we get

$$(X, G, D) \in \{(\mathrm{PGL}(2,11), A_5, D_{22}), (A_{m+1} \rtimes \mathbb{Z}_2, A_m, D_{2(m+1)})\}.$$

Next, we shall structure the dihedral-skew product group $A_{m+1} \rtimes \mathbb{Z}_2$ of A_m. Let $D_{2(m+1)} = \langle a, b\rangle \leq S_{m+3}$, where $a = (12 \cdots m+1) \in A_{m+1}$ and $b = (1\, m+1)(2\, m) \cdots (\frac{m}{2}\, \frac{m}{2}+1)(m+2\, m+3)$. Then $A_{m+1} \rtimes \langle b\rangle \cong A_{m+1} \rtimes \mathbb{Z}_2 = A_m D_{2(m+1)}$. Since A_{m+1} is a simple group, we know $D_{2(m+1)_{A_{m+1}}} = 1$, which implies that $A_{m+1} \rtimes \mathbb{Z}_2$ is a dihedral-skew product groups of A_m.

Hence, $(X, G, D) = (\mathrm{PGL}(2,11), A_5, D_{22})$ or $(A_{m+1} \rtimes \mathbb{Z}_2, A_m, D_{2(m+1)})$ with $m \geq 6$ even. Moreover, one can get $G\langle a\rangle = H \operatorname{char} X$, as desired. $\square$

Lemma 3.2 *Suppose that $G_X = 1$. Then (X, G, D) is given in Table 1, that is the later of Theorem 1.1.*

Proof Let $\Omega = [X : G]$ be the set of right cosets of G in X. Since $G_X = 1$, X acts faithfully on Ω by right multiplication. Suppose that X is quasiprimitive. Then by Corollary 2.2, we get $(X, G, D) = (\mathrm{AGL}(3,2), GL(3,2), D_8), (M_{12}, M_{11}, D_{12}), (M_{24}, M_{23}, D_{24})$ or $(A_{4m}, A_{4m-1}, D_{4m})$ with $m \geq 2$, that is Rows 1-4 of Table 1.

In what follows, we assume that X is non-quasiprimitive, that is there exists a non-transitive normal subgroup N of X that induces proper blocks, which implies $GN < X$. Let H be a maximal subgroup of X containing GN. Clearly, we know $N \leq H_X \neq 1$. By $G < H$, we get $H = G(H \cap D)$ where $H \cap D \neq 1$. If $H \cap D \leq \langle a\rangle$, then noting that $G_X = 1$, by Lemma 3.1, we get

$$(X, G, D) = (\mathrm{PGL}(2,11), A_5, D_{22}), (A_{2m+3} \rtimes \mathbb{Z}_2, A_{2m+2}, D_{2(2m+3)}), \text{ with } m \geq 2,$$

that is Rows 5 and 6 of Table 1. So in what follows, we assume that $H \cap D \nleq \langle a\rangle$. In fact, we can make a more rigorous assumption $\mathfrak{A}$: for any maximal subgroup L of X containing G, if $L_X \neq 1$, then $L \cap D \nleq \langle a\rangle$.

Set $H \cap D = \langle a_1, b\rangle$ for some $\langle a_1\rangle < \langle a\rangle$ and $b \in D \setminus \langle a\rangle$. Then $H = G\langle a_1, b\rangle$. With the same as (1), we get that $\langle a_1\rangle \leq H_X$ and $H_X \cap G$ is either 1 or G. Depending on $H_X \cap G$, we can divide it into the following two cases.

Case 1: $H_X \cap G = 1$.

We claim that there exists the only element $gb \in H_X \setminus \langle a_1\rangle$ with $1 \neq g \in G$. Suppose that there exists no element $gb \in H_X \setminus \langle a_1\rangle$ with $g \in G$. Then $H_X = \langle a_1\rangle \leq D_X = 1$, contradicting to $H_X \neq 1$. Hence, there exists an element $gb \in H_X \setminus \langle a_1\rangle$ with $g \in G$. If $g = 1$, then $\langle a_1, b\rangle \leq H_X$. Since $H_X \cap G = 1$, we know $H_X = \langle a_1, b\rangle \leq D_X = 1$, a contradiction. Hence, $g \neq 1$. If there exists an other element $g_1b \in H_X \setminus \langle a_1\rangle$ with $g \neq g_1 \in G$, then we get $g_1g^{-1} \in H_X \cap G = 1$, a contradiction again. Therefore, there exists the only element $gb \in H_X \setminus \langle a_1\rangle$ with $1 \neq g \in G$.

For the contrary, assume that $(gb)^2 \notin \langle a_1\rangle$. Set $(gb)^2 = g_1a_1^ib^j$ for $g_1 \in G$, $i \in \mathbb{Z}$ and $j \in \{0,1\}$. If $j = 0$, then $g_1 \in H_X$ as $\langle a_1\rangle \leq H_X$, which implies $g_1 = 1$, and so $(gb)^2 = a_1^i$, a contradiction. If $j = 1$, that is $(gb)^2 = g_1a_1^ib = g_1ba_1^{-i}$, then $g_1b \in H_X$, which implies $g_1 = g$, and so $gb = a_1^{-i} \in \langle a_1\rangle$, a contradiction again. Therefore, $(gb)^2 \in \langle a_1\rangle$ and so $H_X = \langle a_1\rangle.\langle gb\rangle$.

Noting $(gb)^2 \in \langle a_1\rangle$, we get $\langle a_1^2\rangle \,\mathrm{char}\, H_X \lhd X$, which implies $\langle a_1^2\rangle \lhd X$. By $D_X = 1$, we know $a_1^2 = 1$. For the contrary, suppose that a_1 is a involution. Then $H_X \cong D_4$ or $\mathbb{Z}_4$, and $H/C_H(H_X) = \overline{G} \leq \mathrm{Aut}\,(H_X)$ as $H = GH_X$ and $H_X \leq C_H(H_X)$. By the definition of H, we know $C_H(H_X) = H$, which implies $H_X \leq Z(H)$. Hence, $\langle a_1\rangle \leq D_X = 1$, a contradiction. Therefore, we know $a_1 = 1$, that is $H_X = \langle gb\rangle \cong \mathbb{Z}_2$ and $H = G\langle b\rangle$. Then $gb \in Z(X)$. If $H \lhd X$, then $G \,\mathrm{char}\, H \lhd X$, which implies $G \lhd X$, a contradiction. Consider $\overline{X} := X/H_X = \overline{G}\langle\overline{a}\rangle$ and then $\overline{G} \ntrianglelefteq \overline{X}$, which implies $\overline{G}_{\overline{X}} = 1$. Set $\langle\overline{a}_2\rangle = \langle\overline{a}\rangle_{X/H_X}$. Since $H \leq G\langle a_2, b\rangle < X$, by the maximality of H, we get $a_2 = 1$, which implies $\langle\overline{a}\rangle_{X/H_X} = 1$. Hence, $\langle\overline{a}\rangle$ is core-free in X/H_X. By Proposition 2.3, we know $(\overline{X}, \overline{G}) = (\mathrm{PSL}(2,11), A_5), (M_{23}, M_{22})$, or (A_{2m+3}, A_{2m+2}) with $m \geq 2$. Hence,

$$(X, G) = (\mathbb{Z}_2.\mathrm{PSL}(2,11), A_5), (\mathbb{Z}_2.M_{23}, M_{22}), \text{ or } (\mathbb{Z}_2.A_{2m+3}, A_{2m+2}).$$

Since $\mathbb{Z}_2.L$ is either a proper central extension or $L\times\mathbb{Z}_2$, where $L \in \{\mathrm{PSL}(2,11), M_{23}, A_{2m+3}\}$, by Lemma 2.5 and Proposition 2.4, we know that there exists no $\mathbb{Z}_2.\mathrm{PSL}(2,11)$ $(\mathbb{Z}_2.M_{23})$ such that $\mathbb{Z}_2.\mathrm{PSL}(2,11) = A_5D_{22}$ $(\mathbb{Z}_2.M_{23} = M_{22}D_{46})$, separately, and if $\mathbb{Z}_2.A_{2m+3} = A_{2m+2}D_{2(2m+3)}$, then $\mathbb{Z}_2.A_{2m+3} = A_{2m+3} \times \mathbb{Z}_2$. Hence,

$$(X, G, D) = (A_{2m+3} \times \mathbb{Z}_2, A_{2m+2}, D_{2(2m+3)}).$$

But $G\langle a\rangle = A_{2m+3}$ is a maximal subgroup of X and $G\langle a\rangle \lhd X$, contradicting to $\mathfrak{A}$.

Case 2: $H_X \cap G = G$.

In this case, H_X is either $G\langle a_1\rangle$ or $G\langle a_1, b\rangle = H$ as $\langle a_1\rangle \leq H_X$. If $H_X = G\langle a_1\rangle$, then $G\langle a\rangle$ is a maximal subgroup of X and $G\langle a\rangle \lhd X$, contradicting to $\mathfrak{A}$ again. Hence, $H_X = G\langle a_1, b\rangle = H$. By $H_X \lhd X$, we know that $H = G\langle a^2, b\rangle$ and $X = H \rtimes \langle ab\rangle$, which implies $4 \mid |D|$. If $G\lhd H$, then $H = G \rtimes \langle a^2, b\rangle$ as $G\cap\langle a^2, b\rangle = 1$, and so $G \,\mathrm{char}\, H \lhd X$ as G is a nonabelian simple group, which implies $G\lhd X$, contradicting with $G_X = 1$. Therefore, $G \ntrianglelefteq H$, which implies $a^2 \neq 1$ and $G_H = 1$. Noting that $\langle a^2, b\rangle_H = 1$, $G \cap \langle a^2, b\rangle = 1$ and $\langle a^2, b\rangle$ is a dihedral group in $H = G\langle a^2, b\rangle$, we know that H is also a dihedral-skew product group of G. By $G_H = 1$, we know that H acts faithfully on $[H : G]$.

Suppose that H is a quasiprimitive group. By Corollary 2.2, we get $(H, G, D) = (\mathrm{AGL}(3,2), GL(3,2), D_8), (M_{12}, M_{11}, D_{12}), (M_{24}, M_{23}, D_{24}), (A_{4m}, A_{4m-1}, D_{4m})$, with $m \geq$

2, which implies $(X,G,D)=(\mathrm{AGL}(3,2)\rtimes\mathbb{Z}_2, GL(3,2), D_{16}), (M_{12}\rtimes\mathbb{Z}_2, M_{11}, D_{24}), (M_{24}\rtimes\mathbb{Z}_2, M_{23}, D_{48}), (A_{4m}\rtimes\mathbb{Z}_2, A_{4m-1}, D_{8m})$. By Lemma 2.5, we get

$$(X,G,D)\in\{(\mathrm{Aut}\,(M_{12}), M_{11}, D_{24}), (A_{4m}\rtimes\mathbb{Z}_2, A_{4m-1}, D_{8m})\}.$$

Next, we shall structure the dihedral-skew product group $A_{4m}\rtimes\mathbb{Z}_2$ of A_{4m-1}. Let $D_{8m}=\langle a,b\rangle\le S_{4m+2}$, where $a=(1\,2\,\cdots 4m)(4m{+}1\; 4m{+}2)$ and $b=(1\,4m)(2\,4m{-}1)\cdots(2n\;2m{+}1)\in A_{4m}$. Then

$$A_{4m}\rtimes\langle ab\rangle = A_{4m}\times\langle(4m+1\;4m+2)\rangle\cong A_{4m}\times\mathbb{Z}_2=A_{4m-1}D_{8m}.$$

Since A_{4m} is a simple group, we know $D_{8m\,A_{4m}}=1$, which implies that $A_{4m}\rtimes\mathbb{Z}_2$ is a dihedral-skew product group of A_m. Hence,

$$(X,G,D)=(\mathrm{Aut}\,(M_{12}), M_{11}, D_{24})\,\text{or}\,(A_{4m}\rtimes\mathbb{Z}_2, A_{4m-1}, D_{8m}),$$

that is Rows 7 and 8 in Table 1.

In what follows, we shall show that H may not be a non-quasiprimitive. For the contrary, assume that H is a non-quasiprimitive. Then there exists a maximal subgroup $K=G\langle a_1,b^j\rangle$ of H such that $K_X\ne 1$, where $a_1\in\langle a^2\rangle$ and $j\in\{0,1\}$. Suppose that $j=0$, that is $K=G\langle a_1\rangle$. Then using Lemma 3.1 for H, we get $K=G\langle a^2\rangle\,\mathrm{char}\,H\lhd X$, which implies $K\lhd X$. Hence, $G\langle a\rangle\le X$, contradicting to $\mathfrak{A}$. Therefore, $j=1$, that is $K=G\langle a_1,b\rangle$, and furthermore, by the above argument, we know that $G\langle a^2\rangle$ can not be a subgroup of H. Clearly, $K_H\cap G$ is either 1 or G. If $K_H\cap G=1$, then with the same argument as Case 1, we know $G\langle a^2\rangle\,\mathrm{char}\,H$, a contradiction. Thus, $K_H\cap G=G$ and so K_H is either $G\langle a_1\rangle$ or K. For the former, that is $K_H=G\langle a_1\rangle$, we get $G\langle a^2\rangle\le H$, a contradiction. For the later, that is $K_H=K$, we get $\langle a_1\rangle=\langle a^4\rangle$ and $4\mid o(a)$, which implies $K=K_H=G\langle a^4,b\rangle$. Clearly, $\langle a^4\rangle\le K_X$ and $b\notin K_X$. Since $|X:K|=4$ and $X/K_X\le S_4$, we know $G\le K_X$, which implies $K_X=G\langle a^4\rangle$. Then we also get $G\langle a^2\rangle\le H$, a contradiction again. □

4 Proof of Theorem 1.2

Let G be a nonabelian simple group and $Y=GD$ a dihedral-product group of G, where $D=\langle a\rangle\rtimes\langle b\rangle$. Set $D_Y=\cap_{y\in Y}D^y$, the core of D in Y.

Since $G_Y\trianglelefteq G$ and G is a nonabelian simple group, we get that G_Y is either G or 1. Suppose that $G_Y=G$. Then $G\lhd Y$, that is the part (i) of Theorem 1.2. So in what follows, we assume that $G_Y=1$.

Generally, for a dihedral-product group $Y=GD$ of any group G (maybe not a nonabelian simple group) with a trivial core in Y, we have the following result.

Lemma 4.1 *Let H be a group and $Y=HD$ a dihedral-product group of H. Set $D=\langle a\rangle\rtimes\langle b\rangle$. Suppose that H is core-free. Then $H\cap D$ is either 1 or $\langle b_1\rangle$ with $b_1\in D\setminus\langle a\rangle$.*

Proof Since H is core-free, Y acts faithfully on $\Omega := [Y : H]$. Set $H \cap \langle a \rangle = \langle a_1 \rangle$. Then

$$\langle a_1 \rangle \leq \cap_{a^ib^j \in D} H^{a^ib^j} = \cap_{h \in H, a^ib^j \in D} H^{ha^ib^j} = \cap_{y \in Y} H^y = H_Y = 1,$$

which implies $H \cap \langle a \rangle = 1$. Hence, $H \cap D$ is either 1 or $\langle b_1 \rangle$ with $b_1 \in D \setminus \langle a \rangle$, as desired. $\square$

By Lemma 4.1, we get $|G \cap D| \leq 2$ for $Y = GD$. Depending on $|G \cap D|$, we get Lemmas 4.2 and 4.3.

Lemma 4.2 *Let G be a nonabelian simple group and $Y = GD$ a dihedral-product group of G. Set $D = \langle a \rangle \rtimes \langle b \rangle$. Suppose that $G \cap D = 1$ and G is core fore. Then $D_Y < \langle a \rangle$ the following holds:*

(i) *$|D_Y| \leq 2$ and $Y = D_Y.X$, where X is one of Table 1; or*

(ii) *$D_Y \neq 1$ and $Y = D_Y.X = D_Y \rtimes X$, where X is Rows 5 and 6 in Table 1.*

Proof Let G be a nonabelian simple group and $Y = GD$ a dihedral-product group of G such that $G \cap D = 1$ and G is core free. Set $D = \langle a \rangle \rtimes \langle b \rangle$ and $D_Y = \cap_{y \in Y} D^y$, the core of D in Y.

For the contrary, assume that $D_Y \not\leq \langle a \rangle$. Then by $D_Y \trianglelefteq D$, we get D_Y is either D or $\langle a^2, b_1 \rangle$ with $b_1 \in D \setminus \langle a \rangle$. If $D_Y = D$, then $Y = D \rtimes G$. Since $Y/C_Y(D) \leq \mathrm{Aut}\,(D)$, $\mathrm{Aut}\,(D)$ is a solvable group and G is an nonabelian group, we get $G \leq C_Y(D)$, which implies $G \lhd Y$, a contradiction. If $D_Y = \langle a^2, b_1 \rangle$ with $b_1 \in D \setminus \langle a \rangle$, then $Y = (D_Y \rtimes G) \rtimes \langle ab \rangle$. With the same argument as the case $D_Y = D$, we get $Y = (G \times \langle a^2, b_1 \rangle) \rtimes \langle ab \rangle$. Consider $H = G \times \langle a^2, b_1 \rangle$. We shall show $G \,\mathrm{char}\, H$. If not, then there exists a subgroup K in H such that $K \neq G$, $K \cong G$ and $K \lhd H$. Since $G \cap K \lhd G$ and G is an nonabelian group, we get $G \cap K = 1$ and so $G \times K \leq H$. Then $\langle K, a^2, b_1 \rangle \leq C_H(G) = \langle a^2, b_1 \rangle$, which implies $K \leq \langle a^2, b_1 \rangle$, a contradiction. Hence, $G \,\mathrm{char}\, H$. By $G \,\mathrm{char}\, G \times \langle a^2, b_1 \rangle \lhd Y$, we know $G \lhd Y$, a contradiction again. Therefore, $D_Y \leq \langle a \rangle$.

Set $D_Y = \langle a_1 \rangle$. Then $Y/C_Y(\langle a_1 \rangle) \leq \mathrm{Aut}\,(\langle a_1 \rangle)$ is abelian, which implies that $GC_Y(\langle a_1 \rangle)/C_Y(\langle a_1 \rangle)$ is abelian. Since $\langle a \rangle \leq C_Y(\langle a_1 \rangle)$ and $C_Y(\langle a_1 \rangle) \cap G \trianglelefteq G$, we get that $C_Y(\langle a_1 \rangle)$ is either Y or $G\langle a \rangle$. Hence, $\langle D_Y, G \rangle = D_Y \times G$. If $D_Y \times G \lhd Y$, then $G \,\mathrm{char}\, D_Y \times G \lhd Y$, which implies $G \lhd Y$, a contradiction. Therefore,

$$Y = (D_Y \times G)D, \text{ where } D_Y \times G \ntrianglelefteq Y.$$

Note that the quotient group $\overline{Y} := Y/D_Y = \overline{GD}$ is a dihedral-skew product group of G as $\overline{G} \cong G$, and $\overline{G}$ is core free.

Suppose that $C_Y(\langle a_1 \rangle) = Y$. Then by the definition of D, we get $\mathrm{o}(a_1) \leq 2$. If $\mathrm{o}(a_1) = 1$, then D is core free and so Y is a dihedral-skew product group of G with $G_Y = 1$, which implies that (Y, G, D) is given in Table 1. If $\mathrm{o}(a_1) = 2$, then by Theorem 1.1, we know that Y is a central extension of $\mathbb{Z}_2$ by X, where X is given in Table 1, noting $\overline{G} \cong G$

in $\overline{Y}$. Hence, $Y = D_Y.X$, where X is given in Table 1, that is the former part of this lemma.

Suppose that $C_Y(\langle a_1\rangle) = G\langle a\rangle$. Then $a_1 \neq 1$ and $\overline{Y} = (\overline{G}\langle\overline{a}\rangle) \rtimes \langle\overline{b}\rangle$. Noting that $\overline{G}_{\overline{Y}} = 1$, by Lemma 3.1, we get $(\overline{Y}, \overline{G}\langle\overline{a}\rangle, \overline{G}, \overline{D})$ is either $(\mathrm{PGL}(2,11), \mathrm{PSL}(2,11), A_5, D_{22})$ or $(A_{m+1} \rtimes \mathbb{Z}_2, A_{m+1}, A_m, D_{2(m+1)})$ with $m \geq 6$ even. Since the pre-image $\overline{G}\langle\overline{a}\rangle$ is $\langle a_1\rangle.(\overline{G}\langle\overline{a}\rangle)$, the central extension of $\langle a_1\rangle$ by $G\langle a\rangle$, by Proposition 2.4, we get $G\langle a\rangle = \langle a_1\rangle \times (G\langle a_2\rangle)$, where $\langle a\rangle = \langle a_1\rangle \times \langle a_2\rangle$ and $G\langle a_2\rangle$ is either $\mathrm{PSL}(2,11)$ or A_{m+1}. By $G\langle a_2\rangle \,\mathrm{char}\, G\langle a\rangle \lhd Y$, we know $G\langle a_2\rangle \lhd Y$. Then

$$Y = (\langle a_1\rangle \times (G\langle a_2\rangle)) \rtimes \langle b\rangle = \langle a_1\rangle \rtimes X,$$

where $X = (G\langle a_2\rangle) \rtimes \langle b\rangle$ is either $\mathrm{PGL}(2,11)$ or $A_{m+1} \rtimes \mathbb{Z}_2$ with $m \geq 6$ even, that is the later part of this lemma. □

Lemma 4.3 *Let G be a nonabelian simple group and $Y = GD$ a dihedral-product group of G. Suppose that $|G\cap D| = 2$ and G is core fore. Then $(Y, G, D) = (A_{m+1}, A_m, D_{2(m+1)})$ or $(A_{m+1} \times \mathbb{Z}_2, A_m, D_{4(m+1)})$, with $\frac{m}{2} \geq 3$ even.*

Proof Let G be a nonabelian simple group and $Y = GD$ a dihedral-product group of G such that $|G \cap D| = 2$ and G is core free. By Lemma 4.1, set $C = \langle a\rangle$ and $\langle b\rangle \cong \mathbb{Z}_2$ such that $D = C \rtimes \langle b\rangle$, $G \cap D = \langle b\rangle$. Then $Y = GC$ where $G \cap C = 1$. With the same argument as Lemma 4.2, consider C_Y, the core of C in Y.

Suppose that $C_Y = 1$. Then Y is a skew product group of G with $G_Y = 1$. By Proposition 2.3, we get that $(Y, G, C) = (\mathrm{PSL}(2,11), A_5, \mathbb{Z}_{11}), (M_{23}, M_{22}, \mathbb{Z}_{23})$, or $(A_{m+1}, A_m, \mathbb{Z}_{m+1})$ with $m \geq 6$ even. By the definition of Y, we know that there exists an involution $b \in G$ such that $\langle C, b\rangle$ is dihedral. But by Magma [2], we know that there no exists D_{22} (D_{46}) in $\mathrm{PSL}(2,11)$ (M_{23}), separately. Next, consider $(Y, G, C) = (A_{m+1}, A_m, \mathbb{Z}_{m+1})$. We get

$$a = (1\,2\,\cdots\,m\,m+1)\,\text{and}\,C = \langle a\rangle \cong \mathbb{Z}_{m+1}.$$

Clearly, there exists an involution $b = (1\,m+1)(2\,m)\cdots(\frac{m}{2}\,\frac{m}{2}+2) \in S_{m+1}$ such that $\langle a, b\rangle = D_{2(m+1)}$. Moreover, $b \in A_{m+1}$ if $\frac{m}{2}$ is even. Then we get

$$(Y, G, D) = (A_{m+1}, A_m, D_{2(m+1)}),\ \text{with}\ \frac{m}{2} \geq 3\,\text{even},$$

that is the former part of the lemma. So in what follows, we assume that $|C_Y| \neq 1$.

Set $C_Y = \langle a_1\rangle$ with $a_1 \in \langle a\rangle$. By $|C_Y| \neq 1$, we know $a_1 \neq 1$. Then $Y/C_Y(\langle a_1\rangle) \leq \mathrm{Aut}\,(\langle a_1\rangle)$ is abelian, which implies that $GC_Y(\langle a_1\rangle)/C_Y(\langle a_1\rangle)$ is abelian. Since $\langle a\rangle \leq C_Y(\langle a_1\rangle)$ and $C_Y(\langle a_1\rangle) \cap G \unlhd G$, we get that $C_Y(\langle a_1\rangle) = G\langle a\rangle = Y$. By the definition of Y and $a_1 \neq 1$, we get $\mathrm{o}(a_1) = 2$. Consider $\overline{Y} = Y/\langle a_1\rangle = \overline{GD} = \overline{GC}$. Since $\overline{G} \cong G$ and $\overline{C}_{\overline{Y}} = 1$, we know that $\overline{Y}$ is a dihedral-skew product group of G with $\overline{G} \cap \overline{D} = 2$ and $\overline{C}_{\overline{Y}} = 1$. Then with the same as the above, we know $\overline{Y} = A_{m+1}$, with $\frac{m}{2} \geq 3$ even.

Hence, Y is a central extension of $\langle a_1 \rangle$ by X, where X is A_{m+1}, with $\frac{m}{2} \geq 3$ even. Then $(Y, G, D) = (\mathbb{Z}_2.A_{m+1}, A_m, D_{2(m+1)})$ with $\frac{m}{2} \geq 3$ even. By Proposition 2.4, we know

$$(Y, G, D) = (\mathbb{Z}_2.A_{m+1}, A_m, D_{4(m+1)}), \text{ with } \frac{m}{2} \geq 3 \, \text{even},$$

that is the later part of the lemma. □

5 Further research

In this paper, we classify dihedral-skew product groups of all nonabelian simple groups. Since all nonabelian simple groups are finite nonabelian characteristically simple groups and orientable regular Cayley maps of finite nonabelian characteristically simple groups are characterized in [4], we propose the following problem for further research.

Question 1 *Determine dihedral-skew product groups of all finite nonabelian characteristically simple groups and classify regular Cayley maps of unoriented finite nonabelian characteristically simple groups.*

Since dihedral-product groups of all nonabelian simple groups are characterized in Theorem 1.2, we also want to classify them.

Question 2 *Determine dihedral-product groups of all nonabelian simple groups.*

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