AN ALGEBRAIC-GEOMETRIC CONSTRUCTION OF "LUMP" SOLUTIONS OF THE KP1 EQUATION

JOHN B. LITTLE

Abstract. In this note, we show how certain everywhere-regular real rational function solutions of the KP1 equation ("multi-lumps") can be constructed via the polynomial analogs of theta functions from singular rational curves with cusps. The method we use can be understood as producing a degeneration of the well-understood soliton solutions from nodal singular curves. Hence it can be seen as a variation on the *long-wave limit* technique of Ablowitz and Satsuma from [1, 12], as developed by Zhang, Yang, Li, Guo, and Stepanyants, [13]. We present an explicit example of a three-lump solution constructed via the polynomial analog of the theta function from a rational curve with two cuspidal singular points, each with semigroup $\langle 2,5\rangle$. (In the theory of curve singularities, these are known as A_4 double points.) We conjecture that these ideas will generalize to give similar M-lump solutions with $M = \frac{N(N+1)}{2}$ for $N > 2$ starting from rational curves with two singular points with semigroup $\langle 2, 2N+1\rangle$ (A_{2N} double points). Similar solutions have been constructed by other methods previously; our contribution is to show how they arise from the algebraic-geometric setting by considering singular curves with several cusps, as in [2].

1. Introduction

We will consider the Kadomtsev-Petviashvili (KP) equation for $u = u(x, y, t)$ in the form

$$(-4u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0. \tag{1}$$

Solutions of these PDEs are of considerable interest in physics since they model several different sorts of wave phenomena in two space dimensions and time. In the PDE literature, taking the $+$ sign on the final term gives what is called the KP2 equation, while the $-$ sign gives the KP1 equation. The differences between these cases are often not emphasized by authors discussing the construction of solutions via the algebraic-geometric techniques studied here. This is no doubt true because solutions of one form of the equation can be taken to solutions of the other by a complex rescaling of the independent variables taking $y \mapsto i \cdot y$. However, the behavior of the solutions for real values of the space variables in the two cases is quite distinct. In particular, the regular real rational function solutions that we are interested in arise only in the KP1 case.

The work of Mikio Sato and his school on construction of solutions via τ-functions corresponding to points of the infinite-dimensional Sato Grassmannian, [11], has illuminated the structure of solutions and shown close connections with combinatorial constructions such as Young diagrams for partitions and Schur polynomials.

Various classes of solutions to KP1 and KP2 (and other related soliton equations) using techniques from algebraic geometry have been known since the late 1970's, based on the so-called Krichever construction, which shows how to produce points of the

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Sato Grassmannian starting from data from an algebraic curve. As a result, the KP equation has generated a great deal of interest in algebraic geometry. In the most spectacular connection, KP solutions play a key role in the results of Shiota, Mulase, and many others on the Schottky problem of characterizing the Jacobian varieties of smooth curves among all principally polarized abelian varieties. In the algebraic-geometric context, the solutions produced from smooth curves via theta functions on their Jacobians have received the greatest amount of attention. In the PDE context, these are the so-called quasiperiodic solutions.

However, it has long been understood–and the details have become increasingly clear–that other classes of KP solutions have connections with other classes of singular curves in a very parallel manner. For instance, one of the classes of solutions that will be very important for us here are the *soliton* solutions. These arise in the algebraic-geometric context by considering the limit of the theta function as a family of smooth curves of genus g degenerates to an irreducible rational curve with g ordinary double points ("nodes"). The connections here were glimpsed very early and exploited by Mumford for the construction of KdV and KP solitons in [10]. They have been studied in much more detail recently in the works of many authors. The most relevant for our purposes are the papers by Agostini, Fevola, Mandelshtam, and Sturmfels, [3], and the related work of Fevola and Mandelshtam, [7]. The fact that *all* real regular KP soliton solutions corresponding to τ-functions from points of the totally nonnegative Grassmannian can be expressed by the theta functions on nodal singular curves has recently been established by Kodama in [8].

The connection between rational function KP solutions, Schur polynomials and polynomial analogs of theta functions from cuspidal singular curves also has a long history. We mention in particular the foundational article of Buchstaber, Leykin, and Enolski, [4]. More recently, this connection been studied in the article [2], which gives more details about the relation between the polynomial analogs of theta functions for cuspidal curves and rational KP solutions. We will make use of several crucial results from that article. We will also follow many of the notational and terminological conventions for singular curves established there, so readers may wish to consult [2] for background material.

Unfortunately, from the point of view of applied PDE, "most of" the rational KP solutions produced from cuspidal singular curves as in [2] are probably of relatively little interest because they tend to be non-regular at some real (x, y) for some or all real t. This is because the denominator of the rational solution

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\ln(\tau(x, y, t)) \tag{2}$$

will "usually" vanish for some real (x, y, t) when $\tau(x, y, t)$ is a polynomial with real coefficients. The same will be true more generally if τ is the product of an exponential factor with exponent linear in x and a polynomial as in Theorem 4.11 from [2]. But in fact we will be somewhat sloppy about the terminology here and essentially ignore any such exponential factor that might be present, calling the polynomial factor itself the τ-function. The reason this is harmless from our point of view is that in passing from $\tau(x, y, t)$ to $u(x, y, t)$ via (2) the exponential factor contributes nothing to the actual KP solution. A simple observation here is that (the polynomial part of) $\tau(x, y, t)$ must have even total degree in (x, y) in order to obtain everywhere-regular real KP solutions.

The complex rescaling $y \mapsto i \cdot y$ to go from a KP2 solution to a KP1 solution can also produce $\tau(x, y, t)$ and $u(x, y, t)$ that take non-real values for some real (x, y, t). These solutions are also of less interest for applications.

From this point of view, possibly the most interesting rational KP solutions are rational "lump" solutions–rational functions $u(x, y, t)$ which are real-valued for all real x, y, t, whose denominators never vanish for real (x, y, t), and which decay to 0 in all directions for all t. Motivated by questions concerning "rogue waves" and other actual physical phenomena, quite a few such solutions have been constructed to date by several authors, and in several different ways.

In this connection, we begin by mentioning the early work of Ablowitz and Satsuma, [1, 12], which showed how to produce regular rational solutions from solitons by what they called the "long-wave limit" process. In our terms, it can be seen easily that their method amounts to taking the theta function from an irreducible rational nodal curve and determining what happens when the nodes degenerate to *ordinary cusps* (i.e. singular points analytically isomorphic to the origin on $y^2 - x^3 = 0$, or so-called A_2 double points). In our terms, the nodes come by identifying pairs of points $\{b, c\}$ on the normalization. We always assume our curves are irreducible and rational, so this is just the complex projective line $\mathbb{P}^1$ (i.e. the Riemann sphere). Then Ablowitz and Satsuma's construction amounts to taking a limit as the pairs $\{b, c\}$ coalesce to single points. Special choices must then be made to ensure that the limit of the soliton solution is regular, essentially by making the limiting polynomial τ-function expressible as a *sum of squares* of real polynomials with a positive nonzero constant term. Those conditions have also been realized by using other methods to find the limiting solutions, most notably the Gram matrix techniques used by Chakravarty and Zowada in [5, 6], and the perturbation processes described by Zhang, et al. in [13].

In this note, we will produce an example of a regular three-lump rational solution of KP1 by using the constructions from [2] and the idea of degenerating a nodal curve to a cuspidal curve (hence degenerating a soliton solution to a rational function solution) in a particular well-chosen way. The idea is to start from a certain soliton solution coming from the theta function of an irreducible nodal curve of arithmetic genus $g = 4$, obtained by identifying *four* pairs of points $\{b_i, c_i\}$, $i = 1, \cdots, 4$, on $\mathbb{P}^1$. By letting $\{b_1, c_1, b_2, c_2\}$ coalesce in a particular way, we produce a singular point with semigroup $\langle 2, 5\rangle$ (an A_4 double point). Simultaneously, $\{b_3, c_3, b_4, c_4\}$ coalesce to a second, distinct but analytically equivalent singular point, also with semigroup $\langle 2, 5\rangle$. From [2], we know that the total degree of the polynomial theta function will be $6 = 3 + 3$ in this case because the Young diagram for each of the singular points is the triangular diagram corresponding to the partition $3 = 2 + 1$. The connection with these particular Young diagrams and the necessity of degenerating to a curve with *two* cuspidal singular points was suggested by the results of [5, 6].

From the results of [2], we have a precise recipe for producing the point of the Sato Grassmannian corresponding to the cuspidal curve of arithmetic genus $g = 4$. We also know how the corresponding τ-function for the KP solution is related to the polynomial analog of the theta function for the cuspidal curve. To be clear, we note that the results of [2] are geared toward producing solutions of the KP2 equation. Hence we must apply the complex rescaling $y \mapsto i \cdot y$ to get a KP1 solution. We then conclude by finding a τ-function, hence a solution $u(x, y, t)$, that is real for all real (x, y, t). We also show that our solution is regular for all real (x, y, t) by expressing the τ-function as a sum of squares of real polynomials with a nonzero constant term. Throughout this work, we will report results whose derivations made rather heavy use of symbolic computation. We used the `Maple 2024` computer algebra system, [9], to work with some of the rather complicated formulas involved.

Acknowledgments. This work is essentially a continuation of [2] and I would like to thank Daniele Agostini and Türkü Özlüm Çelik for a number of helpful conversations as this was developing. I would also like to thank Bernd Sturmfels once again for facilitating contact with Daniele and Türkü starting in 2020 and for renewing my interest in algebraic curves and applications to PDE.

2. Nodal curves and KP solitons

We begin by setting up some suitable notation for understanding irreducible rational nodal curves and the corresponding soliton KP solutions. We follow notation from [3, 7]. We start from $\mathbb{P}^1$ and identify g pairs of points to produce an irreducible rational nodal curve of arithmetic genus g. In one of the standard coordinate charts of $\mathbb{P}^1$, say the points identified to give the ith node are $z = b_i, c_i$. Then a basis for the vector space of dualizing, or Rosenlicht, differentials on the nodal curve is given by

$$\omega_i = -\left(\frac{1}{z-b_i} - \frac{1}{z-c_i}\right)\, dz,$$

since the sum of the residues at $z = b_i$ and $z = c_i$ vanishes.

For our purposes, it will be most convenient to change coordinates on $\mathbb{P}^1$, taking $u = \frac{1}{z}$ as the local coordinate at the point at infinity. In terms of u,

$$\omega_i = \left(\frac{1}{1/u - b_i} - \frac{1}{1/u - c_i}\right) \cdot \frac{1}{u^2}\, du \tag{3}$$

$$= \frac{b_i - c_i}{(1-b_i u)(1-c_i u)}\, du. \tag{4}$$

In terms of the coordinate u, these differentials can be expanded via geometric series in the form

$$\omega_i = \big((b_i - c_i) + (b_i^2 - c_i^2)u + (b_i^3 - c_i^3)u^2 + (b_i^4 - c_i^4)u^3 + \cdots\big)\ du \tag{5}$$

It is well-known (see, for instance, [10, 3, 7, 8]) that the Riemann theta functions on the Jacobians of a family of smooth curves degenerating to one of these rational nodal curves have a limit of the form

$$\theta(z_1, \ldots, z_g) = \sum_{m \in \{0,1\}^g} \exp 2\pi i \left(\sum_{1 \le i < j \le g} m_i m_j \Omega_{ij} + \sum_{i=1}^{g} m_i z_i\right)$$

for some constants Ω_{ij}. These are the limits of the off-diagonal terms in the period matrices of the smooth curves, while the diagonal terms do not enter in the limit. It is also possible to determine a shift vector $h = (h_1, \ldots, h_g)$ such that the analog of the theta-divisor, that is, the W_{g-1} subvariety of the generalized Jacobian of the rational nodal curve, is given by the equation

$$\theta(z_1 - h_1, \ldots, z_g - h_g) = 0$$

(the vector h is analogous to the vector of "Riemann constants" which plays the same role for the theta function from a smooth genus g curve).

Soliton solutions of KP1 are then derived from these theta-functions first by substituting

$$z_j = (b_j - c_j)x + (b_j^2 - c_j^2)i \cdot y + (b_j^3 - c_j^3)t + \phi_j,$$

to yield KP1 τ-functions and then applying (2). The ϕ_j are arbitrary constant "phase factors" whose values are then determined to produce real regular solutions.

In the article [13], Zhang, Yang, Li, Guo, and Stepanyants give examples of degenerations of the b_i and c_i depending on a parameter $\varepsilon \to 0$ such that the lowest nonzero terms in the series expansion of the τ-function in powers of ε give rational functions of x, y, t that yield "lump" solutions. However, they do not make the connection with the cuspidal rational curves that are the limits of the nodal rational curves. Hence the connection between their work and ours is that we will use similar degenerations, but we will show explicitly how the limit curve gives a cuspidal rational curve and how the limit theta function corresponds to the polynomial analog of the theta function on the generalized Jacobian of the cuspidal curve.

But in fact, to do this, it will be most convenient to look at the corresponding family of points in the Sato Grassmannian rather than looking explicitly at the limit of the theta function. (That can also be done, of course, but the computations are more awkward.) We will address these points in the next sections.

3. Degenerating to the bi-cuspidal curve, and the polynomial analog of the theta function

From now on, we will specialize to the case $g = 4$ and a particular degeneration from a rational nodal curve to a cuspidal curve with two singular points. The particular choice we will analyze will be the family of curves constructed from $\mathbb{P}^1$ with the following four pairs of points identified

$$\begin{aligned} b_1 &= 1 + 2\varepsilon \text{ and } c_1 = 1 - 2\varepsilon, \\ b_2 &= 1 + \varepsilon \text{ and } c_2 = 1 - \varepsilon, \\ b_3 &= -1 + 2\varepsilon \text{ and } c_3 = -1 - 2\varepsilon, \\ b_4 &= -1 + \varepsilon \text{ and } c_4 = -1 - \varepsilon. \end{aligned} \tag{6}$$

(These are the z-coordinates of the points, but we will usually pass to the other coordinate $u = \frac{1}{z}$ to work with the dualizing differentials and the abelian integrals.) We assume ε is real and small enough in absolute value that all eight of these points of $\mathbb{P}^1$ are distinct. For $\varepsilon \neq 0$, each pair of points yields a node; in the limit as $\varepsilon \to 0$, two nodes coalesce to a cuspidal singular point at $u = 1$ and the other two nodes coalesce to a cuspidal singular point at $u = -1$. This is a flat family and the total δ-invariant of the singular points (that is, the arithmetic genus of the whole curve) is constant, equal to $g = 4$. Each of the limit singular points is an A_4 double point with semigroup $\langle 2, 5\rangle$. For instance, a local planar model of the degeneration of one pair of nodes to a $\langle 2, 5\rangle$-cusp at $(x, y) = (0, 0)$ is given by the family of parametrizations

$$\begin{aligned} x &= t^2 \\ y &= t^5 - 5\varepsilon^2 t^3 + 4\varepsilon^4 t. \end{aligned} \tag{7}$$

We change basis in the vector space of dualizing differentials on the nodal curves of the family as follows to obtain differentials with "good" limits on the cuspidal curve as

$\varepsilon \to 0$. We consider these linear combinations of the differentials from (3) above:

$$\begin{aligned} \eta_1 &= \frac{1}{\varepsilon}\omega_1 \\ \eta_2 &= \frac{1}{\varepsilon^3}(2\omega_2 - \omega_1), \\ \eta_3 &= \frac{1}{\varepsilon}\omega_3, \text{ and} \\ \eta_4 &= \frac{1}{\varepsilon^3}(2\omega_4 - \omega_3). \end{aligned} \tag{8}$$

It is easy to check that the limits of the differentials from (8) as $\varepsilon \to 0$ exist and give

$$\begin{aligned} \psi_1 &= \lim_{\varepsilon\to 0} \eta_1 = \frac{4}{(u-1)^2}\ du \\ \psi_2 &= \lim_{\varepsilon\to 0} \eta_2 = \frac{12u^2}{(u-1)^4}\ du \\ \psi_3 &= \lim_{\varepsilon\to 0} \eta_3 = \frac{4}{(u+1)^2}\ du \\ \psi_4 &= \lim_{\varepsilon\to 0} \eta_4 = \frac{12u^2}{(u+1)^4}\ du. \end{aligned} \tag{9}$$

In terms of the original affine coordinate $z = \frac{1}{u}$, these can be written as

$$\psi_1 = \frac{4}{(z-1)^2}\ dz, \psi_2 = \frac{12}{(z-1)^4}\ dz, \psi_3 = \frac{4}{(z+1)^2}\ dz, \psi_4 = \frac{12}{(z+1)^4}\ dz,$$

which is exactly the expected form for the dualizing differentials on a curve with two $\langle 2,5\rangle$-cusps. (Note that as observed in [2], the exponents after integrating with respect to z would correspond to the "gaps" $1,3$ of this semigroup.)

By the constructions from [2], the $\langle 2,5\rangle$-cusps correspond to the triangular Young diagrams from the partition $3 = 2+1$. Moreover, the total degree of the polynomial analog of the theta function is $3+3=6$ in this case. We can find an implicit equation of the W_3 subvariety of the generalized Jacobian of the cuspidal curve starting from the usual abelian integral parametrization. We take the base point of the abelian integrals as the point $z=0$ or $u=\infty$ on $\mathbb{P}^1$, the normalization of the cuspidal curve:

$$Z_j = \int_\infty^{t_1} \psi_j + \int_\infty^{t_2} \psi_j + \int_\infty^{t_3} \psi_j, \quad j = 1,\ldots,4. \tag{10}$$

Eliminating the t_i via a Gröbner basis calculation yields an implicit equation involving a polynomial analog of the theta function:

(11)

$$\begin{aligned} &Z_1^3Z_3^3 + 24Z_1^3Z_3^2 - 24Z_1^2Z_3^3 + 192Z_1^3Z_3 + 16Z_1^3Z_4 - 540Z_1^2Z_3^2 + 192Z_1Z_3^3 \\ &+ 16Z_2Z_3^3 + 336Z_1^3 - 4032Z_1^2Z_3 - 384Z_1^2Z_4 + 4032Z_1Z_3^2 + 384Z_2Z_3^2 - 336Z_3^3 - 5904Z_1^2 \\ &+ 27936Z_1Z_3 + 3072Z_1Z_4 + 3072Z_2Z_3 + 256Z_2Z_4 - 5904Z_3^2 + 32256Z_1 + 5376Z_2 \\ &- 32256Z_3 - 5376Z_4 = 0. \end{aligned}$$

As expected, the highest-degree term is the $Z_1^3Z_3^3$ of total degree 6.

4. The limiting Sato Grassmannian point and the τ-function

From Theorem 4.11 of [2], to produce a KP2 τ-function from this theta function (up to an exponential factor that does not contribute anything when we apply (2) to produce the actual KP1 solution $u(x, y, t)$), we need to compute a frame for the point of the Sato Grassmannian corresponding to the cuspidal curve. In fact, we have already done the relevant computations needed here, since the crucial part of this comes from the series expansions of the dualizing differentials from (9) above. We have

$$\begin{aligned}
\psi_1 &= (4 + 8u + 12u^2 + 16u^3 + \cdots)\, du \\
\psi_2 &= (-12u^2 - 48u^3 + \cdots)\, du \\
\psi_3 &= (4 - 8u + 12u^2 - 16u^3 + \cdots)\, du \\
\psi_4 &= (-12u^2 + 48u^3 + \cdots)\, du
\end{aligned} \tag{12}$$

Hence, taking

$$\begin{aligned}
Z_1 &= 4x + 8iy + 12t + \phi_1 \\
Z_2 &= -12t + \phi_2 \\
Z_3 &= 4x - 8iy + 12t + \phi_3 \\
Z_4 &= -12t + \phi_4
\end{aligned} \tag{13}$$

and substituting into (11), we obtain what is essentially a τ-function for KP1 solution (note that the complex rescaling of y to $i \cdot y$ has been incorporated here). To clean up the form a bit, we can also divide by the constant factor 4096. The ϕ_j are arbitrary constant parameters (analogous to "phase factors" for soliton solutions). They must be chosen appropriately to obtain a regular real solution. In fact, for "most" values of the ϕ_j the resulting KP1 solutions produced by this recipe will still be non-regular, and they will also take non-real values at some real (x, y, t). We will see how to overcome these difficulties in the next section.

5. Finding a real regular solution

We begin with a simple observation related to the form of (11) and (13). As long as x, t are taken to be real, the only possible terms contributing non-real values in the substituted theta function are those containing odd powers of y–that is, y, y^3, and y^5–together with arbitrary powers of x, t giving a total degree at most 6. The coefficient of y^5 yields the terms

$$(-24\phi_1 + 24\phi_3 + 384)i$$

Hence, if $\phi_1 - \phi_3 = 16$, the coefficient of y^5 will be zero. When $\phi_1 = 16 + \phi_3$ is substituted into the coefficient of y^3, a rather surprising amount of cancellation happens and the only remaining terms are

$$(44 - 2\phi_4 + 2\phi_2)i.$$

If also $\phi_2 - \phi_4 = -22$, then the coefficient of y^3 is zero. Moreover, these choices also make the coefficient of y equal to zero, so the KP1 solution from (2) takes only real values for real (x, y, t). The "phase factors" ϕ_3, ϕ_4 are still arbitrary, so to produce an explicit solution, we take $\phi_3 = \phi_4 = 0$. The following is essentially a τ-function for the solution we are considering:

(14)

$$
\begin{aligned}
&198x + (3033/2)x^2t + (6615/2)x^2t^2 + (8883/2)xt^2 + 1494xt + 2592y^2xt^2 + x^6 + 6561xt^3 \\
&+ 306x^2y^2 + 576y^4t + 2970y^2t^2 + 3240x^2t^3 + 180x^4t + 96x^3y^2 + 1080x^3t^2 \\
&+ 741x^3t + 2592y^2t^3 + 1710ty^2 + 4860t^4x + 522xy^2 + 192xy^4 + 1458xt^5 + 432y^4t^2 \\
&+ 972y^2t^4 + 135x^4t^2 + 540x^3t^3 + 18x^5t + 1215x^2t^4 + 48x^2y^4 + 12x^4y^2 + 864x^2y^2t \\
&+ 144x^3ty^2 + 648x^2t^2y^2 + 1908xy^2t + 1296xy^2t^3 + 288xy^4t + 729t^6 + 64y^6 + 405y^2 \\
&+ 2916t^5 + 228y^4 + 12x^5 + 261x^2 + 2142t^2 + 513/8 + (8667/2)t^3 + (345/2)x^3 \\
&+ (1125/2)t + (249/4)x^4 + (19521/4)t^4.
\end{aligned}
$$

(We say "essentially" because as always the actual τ-function also includes an exponential factor that is linear in x, hence does not contribute when (2) is applied.)

Theorem 1. *The KP1 solution from the polynomial in* (14) *is real and regular for all real* (x, y, t).

Proof. That the $u(x, y, t)$ produced by (2) takes only real values for real (x, y, t) is a consequence of the determination of the ϕ_j described above and is also clearly visible from the form of (14). To show that this is a regular solution (a "multi-lump") we will show that the polynomial in (14) is a sum of squares of real polynomials with a nonzero constant term. This will complete the proof. To begin, we note that the polynomial analog of the theta function from (11) above can actually be rewritten in the form

$$
\begin{aligned}
(15)\qquad &(Z_1^3 - 24Z_1^2 + 192Z_1 + 16Z_2 - 336)(Z_3^3 + 24Z_3^2 + 192Z_3 + 16Z_4 + 336) \\
&+ 36(Z_1Z_3 + 10Z_1 - 6Z_3 - 56)(Z_1Z_3 + 6Z_1 - 10Z_3 - 56).
\end{aligned}
$$

Part of this reflects the general patterns determined in the proof of Theorem 5.3 of [2]. The two factors of degree 3 on the first line are actually the implicit equations of the theta-divisors for the two partial normalizations of the bicuspidal curve, where one of the cusps is smoothed and the other one remains untouched. The other term is of lower total degree 4 and it happens to factor in this way in this case. When the values from (13) with ϕ_j as above are substituted into this polynomial, the two factors of total degree 3 in the term on the first line become complex conjugates, and the product has the form $(A+iB)(A-iB) = A^2+B^2$, where A, B are polynomials with real coefficients. The factors on the second line yield

$$36((4x + 12t + 10)^2 + 64y^2 + 4)((4x + 12t + 6)^2 + 64y^2 + 4)$$

which is also a sum of squares of real polynomials. It is easy to see that the value of the whole polynomial when $x = y = t = 0$ is a strictly positive constant. Hence the corresponding KP1 solution is regular for all real (x, y, t). It decays to zero as $(x^2 + y^2)^{-2}$ as $x^2 + y^2 \to \infty$. $\square$

We include some numerically-generated plots of the solution given by (14) to demonstrate how it evolves over time.

From Figures 1 and 3, we see that there are three local maxima (the "lumps") contained in the graph of $u(x, y, t)$ for t outside a central region. In Figure 2, however, we see that the lumps are undergoing an interesting interaction where two lumps have apparently coalesced and exchanged form with the taller single lump. This means that our solution is *not* the same as the traveling wave 3-lump solutions found in [13].

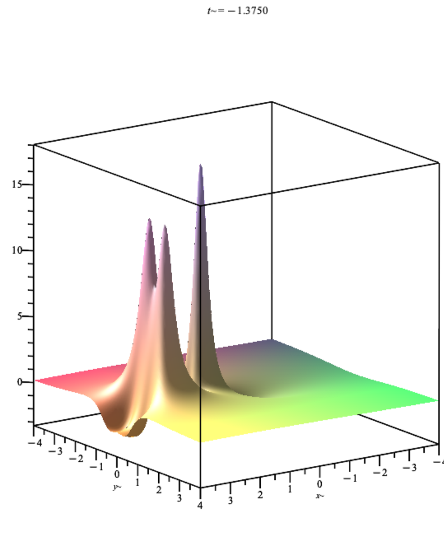

FIGURE 1. The solution (14) at $t = -1.375$.

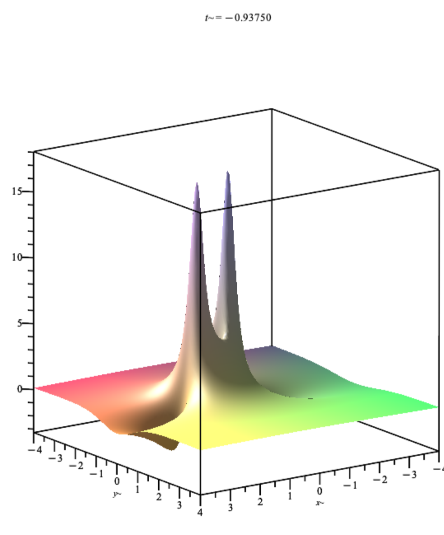

FIGURE 2. The solution (14) at $t = -0.9375$.

The reason is that the authors of [13] actually started from an analog of the Boussinesq equation in which dependence of u on t is omitted and a solution $v(x, y)$ of that equation is then used to generate a KP solution by setting $u(x, y, t) = v(d(x+Vt), y)$ where d, V are constants and V represents a wave speed. Solutions like ours have been produced by other authors by different methods, though. We refer the interested reader to the literature review and the bibliography of [13] for pointers to the relevant articles.

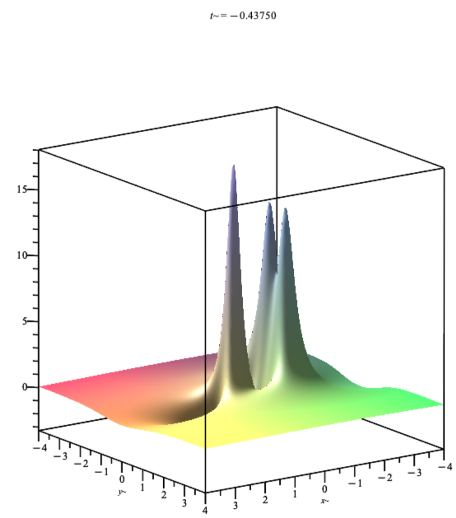

FIGURE 3. The solution (14) at $t = -0.4375$.

6. Comments and Generalizations

By analogy with other physical situations, multi-lump solutions of KP1 such as the one from (14) have been called "bound states" in [13] and elsewhere. There are also other configurations of lumps that have been constructed by other methods and we do not understand whether or how the ideas here might be applied to all those other sorts of examples yet.

On a more hopeful note, much of the construction we have presented, starting from the choice of the pairs of points from (6) for the family of nodal curves generalizes immediately to give families degenerating to cuspidal curves with two A_{2N} double points for all $N \geq 1$. The forms of the dualizing differentials on the cuspidal limits will be parallel, and the results of [2] yield a similar factorization of the leading terms of the polynomial analog of the theta function on the cuspidal curve. Theorem 4.11 of [2] applies in general as here to give KP1 solutions.

Conjecture 1. *We conjecture that the methods used to generate this example will generalize to give similar real regular M-lump solutions with $M = \frac{N(N+1)}{2}$ for all $N \geq 1$ starting from rational curves with two cusps with semigroup $\langle 2, 2N+1\rangle$ (A_{2N} double points). We expect these solutions will have triangular configurations of lumps with N rows containing $1, 2, \ldots, N$ lumps respectively.*

In additional support of this conjecture, we include a plot of a similar solution constructed from a bicuspidal curve with two $\langle 2, 7\rangle$ cusps (A_6 double points) following the same plan as that used in the calculations reported above. This comes from a

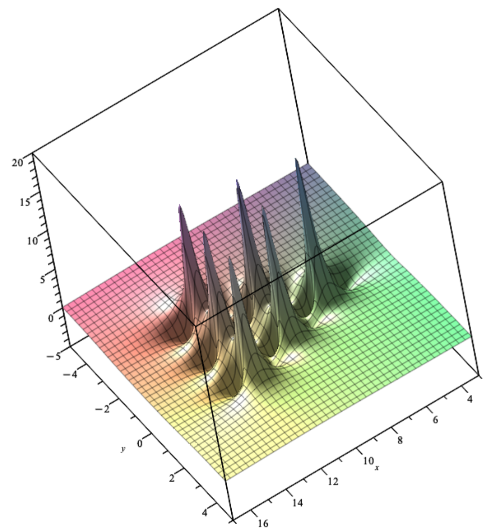

FIGURE 4. The case $N = 3$—another KP1 solution with 6 lumps.

family of rational nodal curves as in (6), but with

$$
\begin{aligned}
b_1 &= 1 + 3\varepsilon \text{ and } c_1 = 1 - 3\varepsilon,\\
b_2 &= 1 + 2\varepsilon \text{ and } c_2 = 1 - 2\varepsilon,\\
b_3 &= 1 + \varepsilon \text{ and } c_3 = 1 - \varepsilon,\\
b_4 &= -1 + 3\varepsilon \text{ and } c_4 = -1 - 3\varepsilon,\\
b_5 &= -1 + 2\varepsilon \text{ and } c_5 = -1 - 2\varepsilon.\\
b_6 &= -1 + \varepsilon \text{ and } c_6 = -1 - \varepsilon,
\end{aligned}
\tag{16}
$$

The polynomial analog of the theta function in this case has degree 12. The polynomials involved are too complicated to be readily understandable, so they are omitted. The pattern established in Theorem 5.3 of [2] is clear, though. The highest degree term in the polynomial analog of the theta function is $Z_1^6 Z_4^6$ since the corresponding partition for each cusp is the triangular $6 = 3 + 2 + 1$.

The plot in Figure 4 shows the surface of $u(x, y, t)$ from above for the one value $t = -3$ so the lump arrangement is visible. The lumps seem to coalesce for t around 0, then emerge in a reflected version of this same pattern as t increases.

The obstacle that we have not overcome as of yet is proving that there will always be choices of the "phase factors" ϕ_j as above that produce real regular KP1 solutions. This will require more detailed information about the form of the terms of lower total degree in the polynomial analog of the theta function on the cuspidal limit curve. This seems somewhat similar to, but more complicated than, the form seen in (15) when $N \geq 3$. We hope to return to this in the future.

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Professor *emeritus*, Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA 01610

Email address: jlittle@holycross.edu