

THE LOG-OPEN CORRESPONDENCE FOR TWO-COMPONENT LOOIJENGA PAIRS

YANNIK SCHULER

ABSTRACT. A two-component Looijenga pair is a rational smooth projective surface with an anticanonical divisor consisting of two transversally intersecting curves. We establish an all-genus correspondence between the logarithmic Gromov–Witten theory of a two-component Looijenga pair and open Gromov–Witten theory of a toric Calabi–Yau threefold geometrically engineered from the surface geometry. This settles a conjecture of Bousseau, Brini and van Garrel in the case of two boundary components. We also explain how the correspondence implies BPS integrality for the logarithmic invariants and provides a new means for computing them via the topological vertex method.

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INTRODUCTION

In [BBvG24] Bousseau, Brini and van Garrel discovered a surprising relationship between two at first sight quite different curve counting theories:

- **Logarithmic Gromov–Witten theory** of a Looijenga pair

$$(S|D_1 + \dots + D_l)$$

which is a rational smooth projective surfaces S together with an anticanonical singular nodal curve $D_1 + \dots + D_l$. To this datum one can associate a Gromov–Witten invariant

$$\text{LGW}_{g,\hat{c},\beta}(S|D_1 + \dots + D_l) \in \mathbb{Q}$$

enumerating genus g , class β stable logarithmic maps to $(S|D_1 + \dots + D_l)$ with maximum tangency along each irreducible component D_i . Additionally, there are $l - 1$ interior markings with a point condition and an insertion of the top Chern class of the Hodge bundle. The tangency order of the markings along $D_1, \dots, D_l$ is recorded in a matrix $\hat{c}$.

- **Open Gromov–Witten theory** of a toric triple

$$(X, (L_i, f_i)_{i=1}^k)$$

consisting of the toric Calabi–Yau threefold X and a collection of framed Aganagic–Vafa Lagrangian submanifolds (L_i, f_i) . To this data one can associate open Gromov–Witten invariants

$$\text{OGW}_{g,((w_1),\dots,(w_k)),\beta'}(X, (L_i, f_i)_{i=1}^k) \in \mathbb{Q}$$

enumerating genus g , class β' stable maps to X from Riemann surfaces with k boundaries, each wrapping one of the Lagrangian submanifolds L_i exactly w_i times.

It was conjectured in [BBvG24] that starting from a Looijenga pair one can geometrically engineer a toric triple so that the above two curve counts turn out to be essentially equal.

Conjecture A. [BBvG24, Conjecture 1.3] *Let $(S|D_1 + \dots + D_l)$ be a Looijenga pair and β an effective curve class in S satisfying some technical conditions. Then there exists a toric triple $(X, (L_i, f_i)_{i=1}^{l-1})$ and an effective curve class β' in X such that*

$$(1) \quad \sum_{g \geq 0} \hbar^{2g-2} \text{OGW}_{g, \mathbf{c}, \beta'}(X, (L_i, f_i)_{i=1}^{l-1}) = \left(\prod_{i=1}^{l-1} \frac{(-1)^{D_i \cdot \beta}}{D_i \cdot \beta} \right) \frac{(-1)^{D_l \cdot \beta + 1}}{2 \sin \frac{(D_l \cdot \beta) \hbar}{2}} \sum_{g \geq 0} \hbar^{2g-1} \text{LGW}_{g, \hat{\mathbf{c}}, \beta}(S|D_1 + \dots + D_l)$$

where $\mathbf{c}$ is obtained from the contact datum $\hat{\mathbf{c}}$ by deleting the $l-1$ interior markings and the marking with tangency along D_l .

For $l = 1$ the right-hand side of (1) reduces to the generating series of ordinary Gromov–Witten invariants of X in which case the conjecture has already been proven in [vGNS23, Theorem D]. In this paper we investigate the case of two-component Looijenga pairs $(S|D_1 + D_2)$ and as another application of the main result in [vGNS23] we will establish Conjecture A for these geometries.

0.1. Logarithmic-Open Correspondence. Consider a two-component Looijenga pair $(S|D_1 + D_2)$ together with an effective curve class β in S . We will impose

Assumption $\star$. $(S|D_1 + D_2)$ and β satisfy

- $D_i \cdot \beta > 0$, $i \in \{1, 2\}$,
- $D_2 \cdot D_2 \geq 0$,
- $(S|D_1)$ deforms into a pair $(S'|D'_1)$ with S' a smooth projective toric surface and D'_1 a toric hypersurface.¹

Moreover, we denote by $\hat{\mathbf{c}}$ the following contact datum along $D_1 + D_2$

$$\hat{\mathbf{c}} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}}_{m \text{ times}} \overbrace{\begin{pmatrix} c_1 & \cdots & c_n \\ 0 & \cdots & 0 \end{pmatrix}}^{=\mathbf{c}} \begin{pmatrix} 0 \\ D_2 \cdot \beta \end{pmatrix}$$

where we assume $n \geq m$ and $c_i > 0$ for all $i \in \{1, \dots, n\}$. This means there are m interior markings, n markings tangent to D_1 and an additional marking with maximum tangency along D_2 . We set

$$(2) \quad \text{LGW}_{g, \hat{\mathbf{c}}, \beta}(S|D_1 + D_2) := \int_{[\overline{M}_{g, \hat{\mathbf{c}}, \beta}(S|D_1 + D_2)]^{\text{virt}}} (-1)^g \lambda_g \prod_{i=1}^n \text{ev}_i^*(\text{pt}).$$

Especially, note that we impose a point condition at all m interior markings. Our main result is the following logarithmic to open correspondence.

Theorem B. (Theorem 1.9) *Assuming $\star$ there is a toric triple (X, L, f) (see Construction 1.8) and a curve class β' in X satisfying*

$$(3) \quad \sum_{g \geq 0} \hbar^{2g-2} \text{OGW}_{g, \mathbf{c}, \beta'}(X, L, f) = \frac{(-1)^{D_1 \cdot \beta}}{m! \prod_{i=0}^{m-1} c_{n-i}} \frac{(-1)^{D_2 \cdot \beta + 1}}{2 \sin \frac{(D_2 \cdot \beta) \hbar}{2}} \sum_{g \geq 0} \hbar^{2g-1} \text{LGW}_{g, \hat{\mathbf{c}}, \beta}(S|D_1 + D_2).$$

The specialisation of the above correspondence to the case $m = n = 1$ immediately gives

Corollary C. *Conjecture A holds for all two-component Looijenga pairs ($l = 2$) satisfying Assumption $\star$.*

¹We expand on the last condition in Remark 1.7.

0.2. Topological vertex. Computing Gromov–Witten invariants of logarithmic Calabi–Yau surfaces is usually a rather tedious endeavour. In higher genus and the presence of a λ_g insertion the only general tools available are scattering diagrams and tropical correspondence theorems [Bou19; Bou20; Bou21; KSK23].

As an application of Theorem B we obtain a new — highly efficient — means for computing logarithmic Gromov Witten invariants of Looijenga pairs. Since open Gromov–Witten invariants can be computed using the topological vertex [AKMV05; LLLZ09], the same method can be used to determine the opposite side of our correspondence theorem.

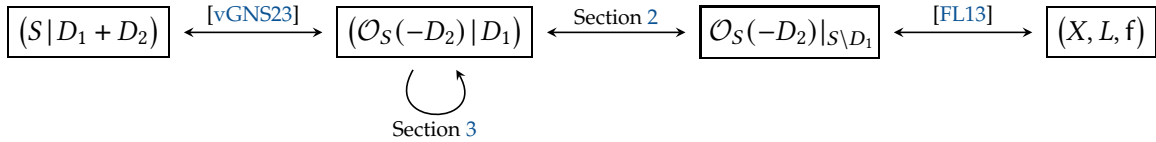
Practical Result D. (Section 4.1) *The logarithmic Gromov–Witten invariants (2) of a two-component Looijenga pair satisfying Assumption $\star$ are computed by the topological vertex.*

0.3. BPS integrality. In general, Gromov–Witten invariants are rational numbers. However, in many cases they are expected to exhibit underlying integral BPS type invariants. For open Gromov–Witten invariants this behaviour was first observed by Labastida–Mariño–Ooguri–Vafa [OV00; LM01; LMV00; MV02], was studied extensively in explicit examples by Luo–Zhu [LZ19; Zhu19; Zhu22] and got recently proven by Yu [Yu23] for general toric targets. Hence, as a corollary of our main result Theorem B we find that logarithmic invariants feature the same property.

Theorem E. (Theorem 4.5) *The logarithmic Gromov–Witten invariants (2) of a two-component Looijenga pair satisfying Assumption $\star$ exhibit BPS integrality.*

In Section 4.2 we spell this statement out in more detail. We conjecture that the above observation holds without imposing Assumption $\star$ as well (Conjecture 4.4).

0.4. Strategy. The proof of Theorem B splits into several steps which partly have already been carried out elsewhere:



In [vGNS23] van Garrel, Nabijou and the author prove a comparison statement between the Gromov–Witten theory of $(S \mid D_1 + D_2)$ and $(\mathcal{O}_S(-D_2) \mid D_1)$. Together with a result of Fang and Liu [FL13] expressing the open Gromov–Witten invariants of a toric triple (X, L, f) as descendant invariants of X it is therefore sufficient to prove a formula for relative Gromov–Witten invariants of $(\mathcal{O}_S(-D_2) \mid D_1)$ in terms of descendant invariants of $X = \text{Tot } \mathcal{O}_S(-D_2)|_{S \setminus D_1}$ in order to establish Theorem B. We will prove such a relation in Section 2 in case there are no interior markings ($m = 0$). Later in Section 3 we will reduce the general case ($m \geq 0$) to the one proven earlier.

0.5. Context & Prospects.

Stable maps to Looijenga pairs. The main motivation for this work clearly comes from the conjectures put forward by Bousseau, Brini and van Garrel. Conjecture A was formulated in [BBvG24] motivated by direct calculations of both sides of the correspondence. Theorem B covers all two-component Looijenga pairs considered in loc. cit. but also previously unknown cases. Most notably, our correspondence theorem also applies to the Looijenga pairs $\text{dP}_1(0, 4)$, $\text{dP}_2(0, 3)$, $\text{dP}_3(0, 2)$ for which Conjecture A was established by proving an intricate identity of q -hypergeometric functions [Kra21; BS23]. We illustrate how our main result can be used to reprove these cases in Section 4.1.2.

There is certainly the question whether the techniques that went into the proof of Theorem B can be generalised to cover Looijenga pairs with $l > 2$ boundary components as well. The most difficult part in this endeavour will most likely be to find an appropriate generalisation of [vGNS23, Theorem A]. Moreover, in the case $l \geq 3$ the construction of the toric triple proposed in [BBvG24, Example 6.5]

involves a flop which numerically does the right job but needs to be understood better geometrically when trying to generalise the techniques of this paper to cover this case as well.

Open/Closed duality of Liu–Yu. Liu and Yu [LY22a; LY22b] prove a correspondence between genus zero open Gromov–Witten invariants of a toric threefold and local Gromov–Witten invariants of some Calabi–Yau fourfold constructed from the threefold. If we combine Theorem B with [vGNS23, Theorem B] we obtain a similar relation between open/local Gromov–Witten invariants of

$$(4) \quad \boxed{(X, L, f)} \longleftrightarrow \boxed{\mathcal{O}_S(-D_1) \oplus \mathcal{O}_S(-D_2)}.$$

It is a combinatorial exercise to check that starting from the toric triple (X, L, f) as we define in Construction 1.8 the fourfold constructed in [LY22b, Section 2.4] is indeed deformation equivalent (in the sense of Remark 1.7) to the right-hand side of (4).² It should, however, be stressed that the correspondence of Liu and Yu holds in a more general context than just local surfaces.

BPS invariants. Motivated by Theorem E we conjecture BPS integrality for Gromov–Witten invariants of type (2) for two-component logarithmic Calabi–Yau surfaces not necessarily satisfying Assumption ★ (Conjecture 4.4). This predication overlaps with a conjecture of Bousseau [Bou20, Conjecture 8.3] in some special cases. So one may wonder about a simultaneous generalisation of both conjectures. Especially, Bousseau’s conjecture also covers logarithmic Calabi–Yau surfaces with more than two components. A geometric proof of Conjecture A in the case of $l > 2$ components in combination with LMOV integrality [Yu23] might allow to make progress towards this direction.

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1. STATEMENT OF THE MAIN RESULT

1.1. Logarithmic Gromov–Witten theory.

Definition 1.1. An l -component Looijenga pair $(S|D_1 + \dots + D_l)$ is a rational smooth projective surface S together with an anticanonical singular nodal curve $D_1 + \dots + D_l$ with l irreducible components $D_1, \dots, D_l$.

Given a two-component Looijenga pair $(S|D_1 + D_2)$, we denote by

$$\overline{M}_{g, \hat{c}, \beta}(S|D_1 + D_2)$$

the moduli stack of genus g , class β stable logarithmic maps to $(S|D_1 + D_2)$ with markings whose tangency along $D_1 + D_2$ is encoded in $\hat{c}$ [Che14; AC14; GS13]. We fix this tangency data to be

$$\hat{c} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & c_1 & \cdots & c_n & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 \cdot \beta \end{pmatrix}}_{m \text{ times}}$$

²The key observation is that the toric divisor D we are deleting in Construction 1.8 is reinserted in Liu–Yu’s construction [LY22b, Section 2.4] of the fourfold as the divisor associated to the ray $\tilde{b}_{R+1}\mathbb{R}_{\geq 0}$.

for some $c_1, \dots, c_n > 0$ with $\sum_i c_i = D_1 \cdot \beta$. This means there are m interior markings (no tangency), n markings with tangency along D_1 and a marking having maximum tangency with D_2 . The virtual dimension of the moduli stack is $g + m + n$ and so we define

$$\text{LGW}_{g,\hat{c},\beta}(S|D_1 + D_2) := \int_{[\overline{M}_{g,\hat{c},\beta}(S|D_1+D_2)]^{\text{virt}}} (-1)^g \lambda_g \prod_{i=1}^n \text{ev}_i^*(\text{pt})$$

where λ_g is the top Chern class of the Hodge bundle.

The ultimate goal of this section is to relate the above invariants to the ones of an open subset of (a deformation of) the threefold $Y := \text{Tot } \mathcal{O}_S(-D_2)$. Let us recall the main result of [vGNS23] which serves as a first step towards this direction. Writing π for the projection $Y \rightarrow S$, we set $D := \pi^{-1}(D_1)$ and define

$$\text{LGW}_{g,\tilde{c},\beta}(Y|D) := \int_{[\overline{M}_{g,\tilde{c},\beta}(Y|D)]^{\text{virt}}} \prod_{i=1}^n \text{ev}_i^*(\pi^* \text{pt})$$

where $\tilde{c}$ is obtained from $\hat{c}$ by deleting the marking with tangency along D_2 .

Theorem 1.2. [vGNS23, Theorem A] Suppose $D_1 \cdot D_2 \geq 0$ and β is so that $D_1 \cdot \beta \geq 0$ and $D_2 \cdot \beta > 0$. Then

$$(5) \quad \sum_{g \geq 0} \hbar^{2g-2} \text{LGW}_{g,\tilde{c},\beta}(Y|D) = \frac{(-1)^{D_2 \cdot \beta + 1}}{2 \sin \frac{(D_2 \cdot \beta) \hbar}{2}} \sum_{g \geq 0} \hbar^{2g-1} \text{LGW}_{g,\hat{c},\beta}(S|D_1 + D_2).$$

1.2. Open Gromov–Witten theory. Open Gromov–Witten invariants are a delicate topic. We will work in the framework of Fang and Liu [FL13] which builds on ideas of [KL06; LS06]. We keep details to a minimum since at no point we will be working with open Gromov–Witten invariants directly. We refer the interested reader to [FL13; KL06] for more details.

1.2.1. Preliminaries on toric Calabi–Yau threefolds. Let X be a smooth toric Calabi–Yau threefold. We will write $\hat{T} \cong \mathbb{G}_m^3$ for its dense torus and $T \cong \mathbb{G}_m^2$ for its **Calabi–Yau torus**. The latter is defined as the kernel $T = \ker \chi$ of the character χ associated to the induced $\hat{T}$ -action on $K_X \cong \mathcal{O}_X$.

If X can be realised as a symplectic quotient, which is the case when X is semiprojective [HS02], Aganagic and Vafa [AV00] construct a certain class of Lagrangian submanifolds L in X . Relevant for us here is the fact that these Lagrangian submanifolds are invariant under the action of the maximal compact subgroup $T_{\mathbb{R}} \hookrightarrow T$ and are homeomorphic to $\mathbb{R}^2 \times S^1$. Moreover, these Lagrangian submanifolds intersect a unique one dimensional torus orbit closure of X in a circle S^1 . For short we will call a Lagrangian submanifold L of the type considered by Aganagic and Vafa an **outer brane** if it intersects a non-compact one dimensional torus orbit closure.

Let us write μ for the moment map

$$\mu : X \longrightarrow \text{Lie}(T_{\mathbb{R}})^*$$

and fix an isomorphism $\text{Lie}(T_{\mathbb{R}})^* \cong \mathbb{R}^2$. We call the image of the union of all one dimensional torus orbit closures under μ the **toric diagram** of X . It is an embedded trivalent graph since there are always three $\hat{T}$ -preserved one dimensional strata meeting in a torus fixed point (see Figure 1). Now let L be an outer brane in X . By our earlier discussion, its image $\mu(L)$ under the moment must be a point on a non-compact edge ℓ . Let us write q for the unique torus fixed point contained in ℓ and denote by

$$u, v, w \in H_T^2(\text{pt}, \mathbb{Z}) \subset \text{Lie}(T_{\mathbb{R}})^*$$

the weights of the induced T action on the tangent spaces of three torus orbit closures at this point. Here, $u = c_1^T(T_q \ell)$ and v is chosen so that $u \wedge v \geq 0$ as indicated in Figure 1. A **framing** of L is the choice of an element $f \in H_T^2(\text{pt}, \mathbb{Z})$ satisfying³

$$u \wedge f = u \wedge v.$$

³This definition agrees with the one in [FL13, Section 2.5] and differs from [BBvG24, Definition 6.1] by a minus sign.

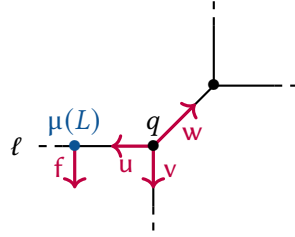


FIGURE 1. The image of the toric skeleton (black) of $\text{Tot } \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with an outer brane L (blue) in framing $f = 0$ (red) under the moment map.

Equivalently, we have

$$(6) \quad f = v - fu$$

for some $f \in \mathbb{Z}$. Moreover, as an element in $H_T^2(\text{pt}, \mathbb{Z})$ one may view f as a character $T \rightarrow \mathbb{G}_m$. This defines a one dimensional subtorus $T_f := \ker f \hookrightarrow T$ which we will refer to as the **framing subtorus**. Finally, let us denote by $|_{f=0}$ the restriction map

$$H_*^T(\text{pt}, \mathbb{Z}) \longrightarrow H_*^{T_f}(\text{pt}, \mathbb{Z})$$

as this morphism essentially sets the weight f equal to zero.

Definition 1.3. A **toric triple** (X, L, f) consists of

- a smooth semiprojective toric Calabi–Yau threefold X ,
- an outer brane L ,
- a framing f .

Remark 1.4. To cover certain interesting cases it will actually be necessary to relax the above definition slightly. Concretely, we would like to include the case where X is not semiprojective in our discussion as well. In this case we replace the second point in the above definition with the choice of some non-compact one dimensional torus orbit closure ℓ .

1.2.2. Open Gromov–Witten invariants. Given a toric triple (X, L, f) and a collection $\mathbf{c} = (c_1, \dots, c_n)$ of positive integers we denote by

$$\overline{M}_{g, \mathbf{c}, \beta'}(X, L)$$

the moduli space parametrising genus g stable maps $f : (C, \partial C) \rightarrow (X, L)$ whose domain C is a Riemann surface with n labelled boundary components $\partial C = \partial C_1 \sqcup \dots \sqcup \partial C_n$ such that for all $i \in \{1, \dots, n\}$

$$f_*[\partial C_i] = c_i[S^1] \in H_1(L, \mathbb{Z}) \quad \text{and} \quad f_*[C] = \beta' + \sum_i c_i[B] \in H_2(X, L, \mathbb{Z}).$$

Here, β' a curve class in X and B is the unique $T_{\mathbb{R}}$ -preserved disk stretching out from q to the circle S^1 in which L and ℓ intersect. The virtual dimension of this moduli problem is zero. We refer to [KL06] for details.

Moduli spaces parametrising stable maps with boundaries are (if they actually exist) usually notoriously hard to work with. In our case there is however a way out. Since the Calabi–Yau torus $T_{\mathbb{R}}$ leaves L invariant the $T_{\mathbb{R}}$ -action on X lifts to an action on $\overline{M}_{g, \mathbf{c}, \beta'}(X, L)$. Then as opposed to its ambient space the (anticipated) fixed locus

$$\overline{M}_{g, \mathbf{c}, \beta'}(X, L)^{T_{\mathbb{R}}}$$

is a compact complex orbifold. So as in [FL13] we define open Gromov–Witten invariants via virtual localisation:

$$(7) \quad \text{OGW}_{g, \mathbf{c}, \beta'}(X, L, f) := \int_{[\overline{M}_{g, \mathbf{c}, \beta'}(X, L)^{T_{\mathbb{R}}}]^{\text{virt}}_{T_{\mathbb{R}}}} \frac{1}{e^{T_{\mathbb{R}}}(N^{\text{virt}})} \Big|_{f=0} \in \mathbb{Q}.$$

The above integral produces a rational function in the two equivariant parameters of homogenous degree zero. Only the restriction to the framing subtorus yields a rational number. This is how definition (7) depends on the choice of framing f .

We recall a result of Fang and Liu which allows us to express open Gromov–Witten invariants of (X, L, f) in terms of descendant invariants of X .

Theorem 1.5. [FL13, Proposition 3.4] *If (X, L, f) is a toric triple then*

$$(8) \quad \text{OGW}_{g,c,\beta'}(X, L, f) = \prod_{i=1}^n (-1)^{f c_i} \frac{\prod_{k=1}^{c_i-1} (f c_i + k)}{u \cdot c_i!} \int_{[\overline{M}_{g,n,\beta'}(X)^T]_T^{\text{virt}}} \frac{1}{e^T(N^{\text{virt}})} \prod_{i=1}^n \frac{\text{ev}_i^* \phi}{(\frac{u}{c_i} - \psi_i)} \Big|_{f=0}$$

where ϕ is the T -equivariant Poincaré dual of the torus fixed point q contained in the one dimensional torus orbit closure ℓ intersected by L .

Remark 1.6. In [LLLZ09] Li, Liu, Liu and Zhou generalise the definition of open Gromov–Witten invariants to the case when X is not semiprojective. (This assumption was necessary for the construction of Aganagic–Vafa Lagrangian submanifolds.) In this case the datum of an outer brane gets replaced by the choice of a non-compact one dimensional torus orbit closure ℓ and open Gromov–Witten invariants are defined via formal relative invariants. It is shown in [FL13, Corollary 3.3] that these invariants still satisfy Theorem 1.5.

1.3. Statement of the main correspondence. Let $(S|D_1+D_2)$ be a two-component Looijenga pair and β an effective curve class in S . We demand that this data satisfies Assumption $\star$ which we recall means that

- $D_i \cdot \beta > 0, i \in \{1, 2\}$,
- $D_2 \cdot D_2 \geq 0$,
- $(S|D_1)$ deforms into a pair $(S'|D'_1)$ with S' a smooth projective toric surface and D'_1 a toric hypersurface.

Remark 1.7. By the last condition we mean that there is a logarithmically smooth morphism of fine saturated logarithmic schemes $S \rightarrow T$ with T irreducible and integral together with a line bundle $\mathcal{L}$ on S . Moreover, there are regular points $t, t' : \text{Spec } \mathbb{k} \rightarrow T$ with fibres

$$S_t = (S|D_1), \quad S_{t'} = (S'|D'_1)$$

on which $\mathcal{L}$ restricts to $\mathcal{L}_t = \mathcal{O}_S(-D_2)$ and $\mathcal{L}_{t'} = \omega_{S'}(D'_1)$.

Given $(S'|D'_1)$ as above we write $Y := \text{Tot } \omega_{S'}(D'_1)$ and write D for the preimage of D'_1 under the projection $\pi : Y \rightarrow S'$. We remark that all compact one dimensional torus orbit closures in Y come from the toric boundary of S' embedded via the zero section $S' \hookrightarrow Y$. This way we may especially view D'_1 as a toric curve in Y .

Construction 1.8. To $(S|D_1+D_2)$ as above we associate the toric triple (X, L, f) where

$$X := Y \setminus D$$

and $L \subset X \subset Y$ is an outer brane intersecting a compact one dimensional torus orbit closure $\ell \subset Y$ adjacent to D'_1 . The action of the Calabi–Yau torus T of X naturally extends to Y . Writing F for the fibre $\pi^{-1}(p)$ over the point $p = \ell \cap D'_1$ we set $f := c_1^T(F)$ to be the weight of the T -action on F .

We are now able to state our main result. For this we fix $c_1, \dots, c_n > 0$ with $\sum_i c_i = D_1 \cdot \beta$ and denote by $\hat{c}$ the contact datum

$$(9) \quad \hat{c} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & c_1 & \cdots & c_n & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 \cdot \beta \end{pmatrix}}_{m \text{ times}}$$

along $D_1 + D_2$. We assume $m \leq n$.

Theorem 1.9. (Theorem B) Under Assumption $\star$ we have

$$(10) \quad \sum_{g \geq 0} \hbar^{2g-2} \text{OGW}_{g,c,\beta'}(X, L, f) = \frac{(-1)^{D_1 \cdot \beta}}{m! \prod_{i=0}^{m-1} c_{n-i}} \frac{(-1)^{D_2 \cdot \beta+1}}{2 \sin \frac{(D_2 \cdot \beta) \hbar}{2}} \sum_{g \geq 0} \hbar^{2g-1} \text{LGW}_{g,\hat{c},\beta}(S|D_1 + D_2)$$

where $\beta' = \iota^*(\beta - (D_1 \cdot \beta)[\ell])$ and ι is the open inclusion $X \hookrightarrow Y$.

See Section 4.1.2 for an extended example illustrating Construction 1.8 and an application of the above theorem.

Remark 1.10. Note that Construction 1.8 depends on the choice which of the two torus orbit closures adjacent to D_1 the Lagrangian submanifold L is intersecting. Hence, a priori, one should expect the right-hand side of (11) to depend on this choice as well and it should come as a surprise that Theorem 1.9 actually demonstrates an independence.

Remark 1.11. If X as constructed in 1.8 turns out to be non-semiprojective the left-hand side of equation (10) can still be defined as described in Remark 1.6.

Remark 1.12. Let us quickly comment on the importance of Assumption $\star$. The first two points guarantee that the moduli stack of relative stable maps to $(Y|D)$ is proper which is required in Theorem 1.2. The last point in Assumption $\star$ is necessary for X to be toric which is in turn required in the definition of open Gromov–Witten invariants we use. The condition is rather technical but allows us treat interesting cases such as the example we will discuss in Section 4.1.2.

We observe that in the light of Theorem 1.2 and Theorem 1.5 it suffices to prove the following proposition in order to deduce Theorem 1.9.

Proposition 1.13. Under Assumption $\star$ we have

$$(11) \quad \begin{aligned} & \text{LGW}_{g,\hat{c},\beta}(Y|D) \\ &= (-1)^{D_1 \cdot \beta} m! \left(\prod_{i=0}^{m-1} c_{n-i} \right) \prod_{i=1}^n (-1)^{f_{c_i}} \frac{\prod_{k=1}^{c_i-1} (f_{c_i} + k)}{u \cdot c_i!} \int_{[\overline{M}_{g,n,\beta'}(X)^T]_T^{\text{virt}}} \frac{1}{e^T(N^{\text{virt}})} \prod_{i=1}^n \frac{\text{ev}_i^* \phi}{\frac{u}{c_i} - \psi_i} \Big|_{f=0} \end{aligned}$$

where $\beta' = \iota^*(\beta - (D_1 \cdot \beta)[\ell])$ and ϕ is the T -equivariant Poincaré dual of the torus fixed point q in $\ell \cap X$.

We will first prove the proposition in the special case $m = 0$ in Section 2 and then deduce the general case from a reduction argument proven in Section 3. But before we embark on proving Proposition 1.13, let us first argue that it indeed implies Theorem 1.9.

Proof of Theorem 1.9. Comparing formula (8) and (11) we deduce

$$\text{OGW}_{g,c,\beta'}(X, L, f) = \frac{(-1)^{D_1 \cdot \beta}}{m! \prod_{i=0}^{m-1} c_{n-i}} \cdot \text{LGW}_{g,\hat{c},\beta}(Y|D).$$

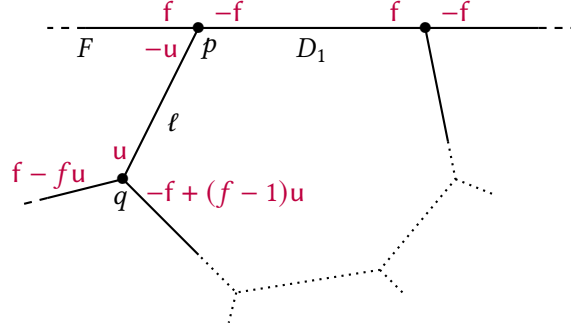
Now by deformation invariance [GS13, Theorem 0.3] (see also [MR20, Appendix A]) the Gromov–Witten invariants of $(S|D_1)$ and $(S'|D'_1)$ agree. In combination with Theorem 1.2 this therefore gives the identity stated in Theorem 1.9. $\square$

2. STEP I: NO INTERIOR MARKINGS

This section is devoted to the proof of a special instance of Proposition 1.13.

Proposition 2.1. The statement of Proposition 1.13 holds if there are no interior markings ($m = 0$).

We will show this by computing the left-hand side of equation (11) via virtual localisation [GP99]. We start with a careful analysis of the target geometry $(Y|D)$ where we adopt the notation introduced in Section 1.3. To unload notation we will assume that S is already toric and D_1 is a toric hypersurface throughout this section which means we can take $(S'|D'_1) = (S|D_1)$.

FIGURE 2. The image of the toric skeleton of Y under the moment map.

2.1. The geometric setup. The Calabi–Yau torus T of X also naturally acts on Y as both varieties share the same dense open torus. The compact one dimensional torus orbit closures of Y are given by the irreducible components of the toric boundary of S embedded into Y by the zero section. We write $\ell \hookrightarrow Y$ for the component intersected by L . By construction ℓ intersects $D_1 \hookrightarrow Y$ transversely in a torus fixed point which we denote p . We write q for the second torus fixed point of ℓ .

Figure 2 shows the image of one dimensional torus orbit closures of Y under the moment map locally around D_1 . In this sketch dashed lines correspond to non-compact orbit closures while the dotted lines indicate that the toric boundary of $S \hookrightarrow Y$ forms a circle. Moreover, the red labels represent the weights with which the torus T is acting on the tangent spaces of the respective torus orbit closures at a fixed point. The observation (see [vGNS23, Lemma 2.7]) that

$$(12) \quad c_1^T(T_p F) = -c_1^T(T_p D_1) = f$$

will be crucial in our localisation calculation later. Moreover, we also note that the framing factor f introduced in (6) can be identified with the degrees

$$f = \deg \mathcal{O}_S(-D_2)|_\ell = -\deg N_\ell Y - 1.$$

The toric diagram of $X = Y \setminus D$ is obtained from the one displayed in Figure 2 by erasing the top vertical line. In this process the divisor ℓ gets decompactified and so L is indeed an outer brane.

2.2. Initialising the localisation. Construction 1.8 involved the choice of a stratum ℓ the outer brane is intersecting. We now do an analogous choice by supporting the cycle whose pushforward gives the Gromov–Witten invariant $\text{LGW}_{g,c,\beta}(Y|D)$ on a substack with evaluations constrained to the fibre $F = \pi^{-1}(p)$ where π is the projection $Y \rightarrow S$:

$$(13) \quad \begin{array}{ccc} \overline{M}_{g,c,\beta}(Y|D)|_F & \hookrightarrow & \overline{M}_{g,c,\beta}(Y|D) \\ \downarrow & \square & \downarrow \\ F^n & \hookrightarrow & D^n. \end{array}$$

Pulling back the obstruction theory along the horizontal arrow we obtain

$$\text{LGW}_{g,c,\beta}(Y|D) = \deg[\overline{M}_{g,c,\beta}(Y|D)|_F]^{\text{virt}}.$$

This choice will turn out rather useful when we will calculate the invariant via virtual localisation. Regarding this we remark that the T -action on Y lifts to an action on $\overline{M}_{g,c,\beta}(Y|D)|_F$ since D and F are both preserved by T . Especially, this also provides us with an action of the framing subtorus T_f on the moduli stack.

Now equation (12) tells us that the divisor D is pointwise fixed under T_f since the restriction $T_f \hookrightarrow T$ effectively sets the weight f equal to zero. This means we are in a situation where the analysis of Graber and Vakil [GV05] of virtual localisation in the context of stable maps relative a smooth

divisor applies. Since their analysis was carried out in the setting of relative stable maps to expanded degenerations as introduced by Li [Li01; Li02] we quietly pass to this modification of $\overline{M}_{g,c,\beta}(Y|D)$ until the end of this section without actually changing our notation. Since eventually we push all cycles forward to a point it does not matter which birational model we choose for our moduli stack [AMW14]. See also [MR19] for an analysis of torus localisation in the context of Kim's logarithmic stable maps to expanded degenerations [Kim10].

Following Graber and Vakil [GV05] we split the T_f -fixed locus $\overline{M}_{g,c,\beta}(Y|D)^{T_f}$ into two distinguished (not necessarily connected or irreducible) components. First, there is the simple fixed locus

$$\overline{M}_{g,c,\beta}(Y|D)^{\text{sim}}$$

whose general points are stable maps with target Y . The complement of this is called composite fixed locus and consequently its general points correspond to maps to non-trivial expanded degenerations of $(Y|D)$. We will denote this component by

$$\overline{M}_{g,c,\beta}(Y|D)^{\text{com}}.$$

As we did in (13) we form fibre products

$$\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{sim}} := \overline{M}_{g,c,\beta}(Y|D)^{\text{sim}} \times_{D^n} F^n, \quad \overline{M}_{g,c,\beta}(Y|D)|_F^{\text{com}} := \overline{M}_{g,c,\beta}(Y|D)^{\text{com}} \times_{D^n} F^n.$$

Then virtual localisation [GP99] gives

$$(14) \quad \begin{aligned} \text{LGW}_{g,c,\beta}(Y|D) &= \deg_{T_f} \frac{1}{e_{T_f}(N_{\text{sim}}^{\text{virt}})} \cap [\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{sim}}]^{\text{virt}} \\ &+ \deg_{T_f} \frac{1}{e_{T_f}(N_{\text{com}}^{\text{virt}})} \cap [\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{com}}]^{\text{virt}}. \end{aligned}$$

We will inspect the contribution of the simple and composite fixed locus to the Gromov–Witten invariant individually in the next two subsections.

2.3. The simple fixed locus. Closely following [GV05], we start with an analysis of the simple fixed locus. By definition the image of a relative stable map $f : C \rightarrow Y$ parametrised by

$$\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{sim}}$$

intersects D transversely, ie. $f^*D = \sum_{i=1}^n c_i x_i$ as Cartier divisors where $x_1, \dots, x_n$ denote the markings. If we write

$$C_1, \dots, C_n$$

for the irreducible components containing these markings then since $f : C \rightarrow Y$ lies in the T_f -fixed locus and by assumption none of the components C_i get contracted there must be a (non-trivial) lift of the action T_f on C_i (possibly after passing to some cover of T_f) making $f|_{C_i}$ T_f -equivariant. As a consequence, for all $i \in \{1, \dots, n\}$ both C_i and its image under f must be rational curves. Moreover, writing q_i for the second T_f -fixed point of the image rational curve, $f|_{C_i}$ is the unique degree c_i cover maximally ramified over $f(x_i)$ and q_i .

Now since we confined the evaluation morphisms to the fibre $F \subset Y$, we deduce that the image of each component C_i is contained in $\pi^{-1}(\ell)$ where π is the projection $Y \rightarrow S$ and ℓ is as in Figure 2. However, since the pullback of the line bundle $\mathcal{O}_S(-D_2)$ under $\pi \circ f$ is negative we deduce that f has to factor through the zero section $S \hookrightarrow Y$. Hence, the image of each irreducible component C_i has to be the curve ℓ already. This also means that $q_i = q$ for all $i \in \{1, \dots, n\}$.

Now if we write $C' \sqcup \bigsqcup_{i=1}^n C_i$ for the connected components of the partial normalisation of C along the nodes $y_i \in C_i$ mapping to q we observe that $f|_{C'}$ is a genus g , class

$$\beta' = \iota^*(\beta - (D_1 \cdot \beta)[\ell])$$

$T_{\mathbb{F}}$ -fixed stable map to $X = Y \setminus D$ with markings $y_1, \dots, y_n$ confined to q . Hence, forming the fibre square

$$\begin{array}{ccc} \overline{M}_{g,n,\beta'}(X)|_q & \hookrightarrow & \overline{M}_{g,n,\beta'}(X) \\ \downarrow & \square & \downarrow \Pi_i \text{ev}_{y_i} \\ q & \hookrightarrow & X^n \end{array}$$

we can summarise the discussion up to now as follows.

Lemma 2.2. *We have*

$$(15) \quad \overline{M}_{g,c,\beta}(Y|D)|_F^{\text{sim}} = \left[\overline{M}_{g,n,\beta'}(X)|_q^{T_{\mathbb{F}}} / \prod_{i=1}^n \mathbb{Z}_{c_i} \right].$$

Using this description of the simple fixed locus we are now able to determine its contribution to the Gromov–Witten invariant.

Proposition 2.3. *We have*

$$(16) \quad \deg_{T_{\mathbb{F}}} \frac{1}{e^{T_{\mathbb{F}}}(N_{\text{sim}}^{\text{virt}})} \cap [\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{sim}}]_{T_{\mathbb{F}}}^{\text{virt}} = (-1)^{D_1 \cdot \beta} \prod_{i=1}^n (-1)^{f c_i} \frac{\prod_{k=1}^{c_i-1} (f c_i + k)}{u \cdot c_i!} \\ \times \int_{[\overline{M}_{g,n,\beta'}(X)]_{T_{\mathbb{F}}}^{T_{\mathbb{F}}}} \frac{1}{e^{T_{\mathbb{F}}}(N^{\text{virt}})} \prod_{i=1}^n \frac{\text{ev}_i^* \phi}{\frac{u}{c_i} - \psi_i}.$$

where ϕ is the $T_{\mathbb{F}}$ -equivariant Poincaré dual of q .

Proof. With our characterisation (15) of the simple fixed locus it remains to analyse the difference between $N_{\text{sim}}^{\text{virt}}$ and the virtual normal bundle N^{virt} of the fixed locus

$$\overline{M}_{g,n,\beta'}(X)^{T_{\mathbb{F}}} \hookrightarrow \overline{M}_{g,n,\beta'}(X).$$

First of all note that we get an overall factor

$$\prod_{i=1}^n \frac{1}{c_i}$$

from the quotient. Moreover, partially normalising along the nodes $y_1, \dots, y_n$ connecting C' with $C_1, \dots, C_n$ we see that each degree c_i cover

$$f_i := f|_{C_i} : C_i \rightarrow \ell$$

contributes with factors

$$\begin{aligned} \frac{1}{e^{T_{\mathbb{F}}}(\mathbf{H}^\bullet(C_i, f_i^* T_{\ell|p}^{\log})^{\text{mov}})} &= \frac{1}{c_i!} \left(\frac{u}{c_i} \right)^{-c_i} \\ \frac{1}{e^{T_{\mathbb{F}}}(\mathbf{H}^\bullet(C_i, f_i^* N_{\ell S})^{\text{mov}})} &= \frac{1}{(-(f+1)c_i)!} \left(\frac{u}{c_i} \right)^{(f+1)c_i} \\ \frac{1}{e^{T_{\mathbb{F}}}(\mathbf{H}^\bullet(C_i, f_i^* \mathcal{O}_S(-D_2))^{\text{mov}})} &= (-1)^{f c_i + 1} (-f c_i - 1)! \left(\frac{u}{c_i} \right)^{-f c_i - 1} \end{aligned}$$

where $f = \deg \mathcal{O}_S(-D_2)|_{\ell} = -\deg N_{\ell} Y - 1 < 0$. Note that if we multiply all these factors together we indeed obtain the top line on the right-hand side of (16). Regarding the second line let us only remark that the insertions

$$\frac{u}{c_i} - \psi_i$$

for $i \in \{1, \dots, n\}$ come from smoothings of the nodes $y_1, \dots, y_n$. □

2.4. Vanishing of composite contributions. We observe that the integral on the right-hand side of equation (11) is the result of computing the one in (16) via T -virtual localisation. Hence, to prove Proposition 2.1 it suffices to show

Proposition 2.4. *We have*

$$\deg_{T_f} \frac{1}{e_{T_f}(N_{\text{com}}^{\text{virt}})} \cap [\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{com}}]^{\text{virt}} = 0.$$

The proof of this proposition necessitates a careful analysis of the composite fixed locus for which we will closely follow [GV05, Section 3].

2.4.1. The setup. We first set the notation required in this section. Denote by

$$(17) \quad Z := \mathbb{P}_D(\mathcal{O}_D \oplus N_D Y)$$

the projective completion of $N_D Y$ and write D_0 and D_∞ for its zero and infinity section. Note that Z is a line bundle over

$$(18) \quad P := \mathbb{P}_{D_1}(\mathcal{O}_{D_1} \oplus N_{D_1} S).$$

More precisely, we have $Z \cong \mathcal{O}_P(-2f)$ where f is a fibre of $P \rightarrow D_1$.

By definition, the composite fixed locus parametrises relative stable maps whose target is an expanded degeneration of $(Y|D)$ which is the result of gluing $(Y|D)$ and multiple copies of $(Z|D_0 + D_\infty)$ to form an accordion. The composite locus decomposes into a disjoint union of components labelled by certain decorated graphs indicating how a relative stable map distributes over the expanded target.

Definition 2.5. A **splitting type** Γ is the data of

- (i) a bipartite graph with vertices $V(\Gamma)$, edges $E(\Gamma)$ and legs $L(\Gamma)$;
- (ii) an assignment of a curve class β_v to every vertex $v \in V(\Gamma)$;
- (iii) a genus label $g_v \in \mathbb{Z}_{\geq 0}$ for every vertex $v \in V(\Gamma)$;
- (iv) a contact order $d_e \in \mathbb{Z}_{>0}$ assigned to every edge $e \in E(\Gamma)$;
- (v) a labelling $\{1, \dots, n\} \cong L(\Gamma)$.

This data has to be compatible with (g, c, β) in the usual way. Vertices are separated into Y - and Z -vertices and we will write Γ_Y and Γ_Z for the type of relative stable maps (with possibly disconnected domain) to $(Y|D)$, respectively $(Z|D_0 + D_\infty)$, obtained by cutting Γ along its edges. All legs are adjacent to Z -vertices.

With this notation the composite locus can be written as a union of disconnected components labelled by splitting types:

$$\overline{M}_{g,c,\beta}(Y|D)^{\text{com}} = \bigsqcup_{\Gamma} \overline{F}_{\Gamma}.$$

To characterise the component labelled by a fixed splitting type Γ , let us denote by

$$\overline{M}_{\Gamma_Y}(Y|D)^{\text{sim}}$$

the simple T_f -fixed locus of relative stable maps of type Γ_Y and write

$$\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^{\sim}$$

for the moduli stack of type Γ_Z relative stable maps to non-rigid $(Z|D_0 + D_\infty)$ and form a substack with constrained evaluations at marking legs

$$\begin{array}{ccc} \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} & \hookrightarrow & \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^\sim \\ \downarrow & \square & \downarrow \Pi_i \text{ev}_i \\ F^n & \hookrightarrow & D_\infty^n. \end{array}$$

The two moduli stacks support evaluation morphisms to $D \cong D_0$ associated to the edges of Γ . Let Δ denote the diagonal $D^{E(\Gamma)} \hookrightarrow D^{E(\Gamma)} \times D^{E(\Gamma)}$ and form the fibre product

$$(19) \quad \begin{array}{ccc} \overline{M}_\Gamma & \hookrightarrow & \overline{M}_{\Gamma_Y}(Y|D)^{\text{sim}} \times \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} \\ \downarrow & \square & \downarrow \\ D^{E(\Gamma)} & \xrightarrow{\Delta} & D^{E(\Gamma)} \times D^{E(\Gamma)}. \end{array}$$

Then the connected component $\overline{F}_\Gamma$ of the composite fixed locus is the quotient of $\overline{M}_\Gamma$ by $\text{Aut}(d_e)$ and moreover, by [GV05, Lemma 3.2], the Gysin pullback

$$[\overline{M}_\Gamma]^{\text{virt}} := \Delta^!([\overline{M}_{\Gamma_Y}(Y|D)^{\text{sim}}]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}}]^{\text{virt}})$$

pushes forward to $[\overline{F}_\Gamma]^{\text{virt}}$ under the quotient morphism. This means we get

$$(20) \quad \deg_{T_f} \frac{1}{e_{T_f}(N_{\text{com}}^{\text{virt}})} \cap [\overline{M}_{g,c,\beta}(Y|D)|_F^{\text{com}}]_{T_f}^{\text{virt}} = \sum_{\Gamma} \frac{1}{|\text{Aut}(d_e)|} \deg_{T_f} \frac{1}{e_{T_f}(N_{\Gamma}^{\text{virt}})} \cap [\overline{M}_\Gamma]^{\text{virt}}.$$

2.4.2. First vanishing: restricted curve classes. To prove Proposition 2.4 we will show that each term in the sum (20) vanishes in several steps. We first recall a lemma from [vGNS23].

Lemma 2.6. [vGNS23, Section 2.3.3] *If there is a Z -vertex $v \in V(\Gamma)$ such that β_v is not a multiple of a fibre of $P \rightarrow D_1$, then*

$$(21) \quad \deg_{T_f} \frac{1}{e_{T_f}(N_{\Gamma}^{\text{virt}})} \cap [\overline{M}_\Gamma]^{\text{virt}} = 0.$$

Remark 2.7. The above lemma was proven in [vGNS23] under the assumption that $D_1 \cdot \beta = 0$ but it is straight forward to generalise the proof to the case at hand where $D_1 \cdot \beta > 0$. The key observation is that after localising the left-hand side of (21) with respect to T one can identify a trivial factor in the obstruction bundle guaranteeing the vanishing.

2.4.3. Second vanishing: star shaped graphs. From now on we may assume that every Z -vertex v is decorated with curve class β_v which is a multiple of a fibre of $P \rightarrow D_1$.

The following results are inspired by vanishing arguments appearing in the localisation calculations [LY22a; LLLZ09; LLZ03]. We will present more geometric versions of these arguments here.

Lemma 2.8. *Unless every vertex in Γ_Z carries exactly one leg we have*

$$[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}}]^{\text{virt}} = 0.$$

Proof. Clearly, every Z -vertex v has to have at least one leg for the moduli stack to be non-empty since $D_\infty \cdot \beta_v > 0$. So suppose there is some vertex with at least two legs. Since the curve class carried by this vertex is a fibre class and the restriction of the line bundle $Z \rightarrow P$ to a fibre trivialises, the product

of the evaluation morphisms of these two markings factors through the diagonal $D_\infty \rightarrow D_\infty^2$. Hence, we have the following commuting diagram with Cartesian squares:

$$\begin{array}{ccc}
\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} & \hookrightarrow & \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^\sim \\
\downarrow g & \square & \downarrow \\
F^{n-1} & \xrightarrow{\quad \bar{\eta} \quad} & D_\infty^{n-1} \\
\downarrow f & \square & \downarrow \\
F^n & \xrightarrow{\quad \eta \quad} & D_\infty^n.
\end{array}$$

Note that the excess intersection bundle

$$E = f^* N_{F^n} D_\infty^n / N_{F^{n-1}} D_\infty^{n-1}$$

is trivial as is every vector bundle on $F^{n-1} \cong \mathbb{A}^{n-1}$. Moreover, note that since T_f is acting trivially on D_∞ and F , this bundle is also T_f -equivariantly trivial. Therefore, by the excess intersection formula [Ful98, Theorem 6.3] we have

$$[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}}]^{\text{virt}} = \eta^! [\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^\sim]^{\text{virt}} = c_1^{T_f}(g^* E) \cap \bar{\eta}^! [\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^\sim]^{\text{virt}} = 0. \quad \square$$

As another consequence of our assumption that all curve classes of Z -vertices are multiples of fibre classes the evaluation morphism associated to edges

$$\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} \longrightarrow D_0^{E(\Gamma)}$$

factors through $F^{E(\Gamma)} \hookrightarrow D_0^{E(\Gamma)}$. If we combine this with the fact that the evaluation morphism

$$\overline{M}_{\Gamma_Y}(Y|D)^{\text{sim}} \rightarrow D^{E(\Gamma)}$$

factors through $D_1^{E(\Gamma)} \hookrightarrow D^{E(\Gamma)}$ we may restrict the evaluations to the intersection point $p = F \cap D_1$. We get the following commuting diagram with Cartesian squares:

$$\begin{array}{ccc}
\overline{M}_{\Gamma_Y}(Y|D)|_F^{\text{sim}} \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0 + \infty)^\sim & \longrightarrow & \overline{M}_{\Gamma_Y}(Y|D)^{\text{sim}} \times \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} \\
\downarrow & \square & \downarrow \\
p & \xrightarrow{\quad \quad \quad} & (D_1)^{E(\Gamma)} \times F^{E(\Gamma)} \\
\downarrow & \square & \downarrow \\
D^{E(\Gamma)} & \xrightarrow{\quad \Delta \quad} & D^{E(\Gamma)} \times D^{E(\Gamma)}.
\end{array}
\tag{22}$$

Especially, note that we have a fibre square

$$\begin{array}{ccc}
\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0 + \infty)^\sim & \hookrightarrow & \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\tilde{F}} \\
\downarrow & \square & \downarrow \\
p & \xrightarrow{\quad \xi \quad} & F^{E(\Gamma)}.
\end{array}
\tag{23}$$

First, we record the following vanishing.

Lemma 2.9. *We have $[\overline{M}_\Gamma]^{\text{virt}} = 0$ unless every Z -vertex has a single adjacent edge.*

Proof. The lemma can be proven with an argument similar to the one in the proof of Lemma 2.8 using diagram (22). We leave this to the reader. $\square$

Moreover, comparing our definition of $\overline{M}_\Gamma$ in (19) with diagram (22) we learn that

$$\overline{M}_\Gamma \cong \overline{M}_{\Gamma_Y}(Y|D)|_F^{\text{sim}} \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim.$$

On the level of virtual fundamental classes we find the following identity.

Lemma 2.10. *If there is a Z -vertex v with $g_v > 0$ the cycle $[\overline{M}_\Gamma]^\text{virt}$ vanishes. Otherwise, the cycle equates to*

$$[\overline{M}_{\Gamma_Y}(Y|D)|_F^{\text{sim}}]^\text{virt} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim]^\text{virt}.$$

Proof. Let ξ be as in (23). Then since all squares in (22) are Cartesian we have

$$\begin{aligned} [\overline{M}_\Gamma]^\text{virt} &= \Delta^!([\overline{M}_{\Gamma_Y}(Y|D)|_F^{\text{sim}}]^\text{virt} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim]^\text{virt}) \\ &= [\overline{M}_{\Gamma_Y}(Y|D)|_F^{\text{sim}}]^\text{virt} \times \xi^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim]^\text{virt} \end{aligned}$$

by [Ful98, Theorem 6.2 (c)]. Comparing the usual perfect obstruction theory of $\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim$ with the one pulled back from $\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim$ we find

$$\xi^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim]^\text{virt} = \lambda_{\text{top}}^2 \cap [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim]^\text{virt}$$

where λ_{top} is the top Chern class of the Hodge bundle. The statement of the lemma then follows from the fact that $\lambda_{\text{top}}^2 = 0$ if any of the vertices in Γ_Z carries a non-zero genus label and $\lambda_{\text{top}}^2 = 1$ otherwise. $\square$

2.4.4. Final vanishing. Having proven Lemmas 2.8–2.10 we are finally able to deduce Proposition 2.4.

Proof of Proposition 2.4. According to Lemma 2.8 and 2.9 the only splitting types Γ whose associated summand on the left-hand side of (20) may be non-zero consist of Z -vertices with exactly one adjacent edge and leg. Moreover, by Lemma 2.10 every such vertex has to carry a vanishing genus label. Therefore, the virtual dimension of $\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim$ is

$$\sum_{v \in V(\Gamma_Z)} (2g_v - 2) + |E(\Gamma)| + |L(\Gamma)| - 1 = -1$$

and so the cycle $[\overline{M}_\Gamma]^\text{virt}$ vanishes by Lemma 2.10. $\square$

3. STEP II: REDUCING INTERIOR TO CONTACT INSERTIONS

3.1. The comparison statement. Throughout this section we will assume that $(S|D_1+D_2)$ is a logarithmic Calabi–Yau surface with boundary the union of two transversally intersecting smooth irreducible curves D_1, D_2 . Moreover, we fix an effective curve class β in S and assume

- $D_i \cdot \beta > 0, i \in \{1, 2\},$
- $D_2 \cdot D_2 \geq 0.$

Especially, in this section we do not necessarily assume $(S|D_1)$ to deform to a toric pair. We set $Y := \text{Tot } \mathcal{O}_S(-D_2)$ and write D for the preimage of D_1 under the projection morphism.

As before we fix a contact datum $\mathbf{c} = (c_1, \dots, c_n)$ with D where $c_i > 0$ for all $i \in \{1, \dots, n\}$. We set

$$(24) \quad \text{LGW}_{g,\mathbf{c},\beta}(Y|D) = \int_{[\overline{M}_{g,\mathbf{c},\beta}(Y|D)]^\text{virt}} \prod_{i=1}^n \text{ev}_i^*(\pi^* \text{pt}).$$

Now denote by

$$\tilde{\mathbf{c}} = (0, \dots, 0, c_1, \dots, c_n)$$

the result of adding $m \leq n$ interior markings. Similarly to (24) we define $\text{LGW}_{g,\tilde{\mathbf{c}},\beta}(Y|D) \in \mathbb{Q}$ by imposing point conditions at the first n markings. Note that this especially means that there is a point

condition at all m interior markings. We have the following comparison statement which reduces computations to the case without any interior markings.

Proposition 3.1. *We have*

$$(25) \quad \text{LGW}_{g,\tilde{c},\beta}(Y|D) = m! \left(\prod_{i=0}^{m-1} c_{n-i} \right) \cdot \text{LGW}_{g,c,\beta}(Y|D).$$

Together with the special case $m = 0$ of Proposition 1.13, which we have already proven in the last section, the above reduction result implies the general case $m \geq 0$ of Proposition 1.13 as well. The rest of this section is hence dedicated to the proof of Proposition 3.1.

3.2. Degeneration. To prove the main result of this section we will use a degeneration argument similar to [vGGR19]. Since by now this approach has become a well-established technique [TY20; BNTY23; vGNS23; RMS24; BFGW21] we economise on details.

Recall the definition of the line bundle $Z \rightarrow P$ from equation (17) and (18) (where we identify $S = S'$). We defined Z to be the projective completion of $N_D Y$ whose zero and infinity section we denote by D_0 and D_∞ respectively. Let us consider the degeneration to the normal cone of D in Y :

$$\mathcal{Y} \longrightarrow \mathbb{A}^1.$$

$\mathcal{Y}$ is a line bundle over the degeneration to the normal cone of D_1 in S . While the general fibre of $\mathcal{Y}$ is Y , the special fibre $\mathcal{Y}_0 = Y \sqcup_D Z$ is the result of gluing Y along D to Z along the zero section D_0 . Now let us write $\mathcal{D}$ for the closure of $D \times (\mathbb{A}^1 - \{0\})$ in $\mathcal{Y}$ and equip $\mathcal{Y}$ with the divisorial logarithmic structure with respect to $Y + Z + \mathcal{D}$. The resulting logarithmic scheme has general fibre $(Y|D)$ and its special fibre is obtained by gluing

$$(Y|D) \quad \text{and} \quad (Z|D_0 + D_\infty).$$

Finally, for F a fibre of $D \rightarrow D_1$ we denote by $\mathcal{F}$ the closure of $F \times (\mathbb{A}^1 - \{0\})$ in $\mathcal{Y}$. The intersection $\mathcal{F}_0$ of $\mathcal{F}$ with the special fibre $\mathcal{Y}_0$ is a fibre of $D_\infty \rightarrow D_1$.

3.3. Decomposition. Composition with the blowup morphism $\mathcal{Y} \rightarrow Y \times \mathbb{A}^1$ produces a pushforward

$$\rho : \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0) \longrightarrow \overline{M}_{g,\tilde{c},\beta}(Y|D)$$

and by the conservation of number principle we have

$$(26) \quad [\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\text{virt}} = (\rho_{\mathcal{F}_0})_* [\overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\text{virt}}$$

where we introduced the fibre products

$$\begin{array}{ccc} \overline{M}_{g,\tilde{c},\beta}(Y|D)|_F & \hookrightarrow & \overline{M}_{g,\tilde{c},\beta}(Y|D) \\ \downarrow & \square & \downarrow \Pi_{i=1}^n \text{ev}_i \\ F^n & \hookrightarrow & Y^n \end{array} \quad \begin{array}{ccc} \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0} & \hookrightarrow & \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0) \\ \downarrow & \square & \downarrow \Pi_{i=1}^n \text{ev}_i \\ \mathcal{F}_0^n & \hookrightarrow & \mathcal{Y}_0^n. \end{array}$$

and pulled back the respective virtual fundamental classes along horizontal arrows. The pushforward of the left-hand side of (26) to $\overline{M}_{g,\tilde{c},\beta}(Y|D)$ gives

$$\Pi_{i=1}^n \text{ev}_i^*(\pi^* \text{pt}) \cap [\overline{M}_{g,\tilde{c},\beta}(Y|D)]^{\text{virt}}$$

and so pushing forward to a point returns the Gromov–Witten invariant we are after:

$$\text{LGW}_{g,\tilde{c},\beta}(Y|D).$$

We combine this with the decomposition formula [ACGS20, Theorem 5.4]

$$(27) \quad [\overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\text{virt}} = \sum_{\tau} \frac{m_{\tau}}{|\text{Aut } \tau|} \iota_* [\overline{M}_{\tau}(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\text{virt}}$$

where ι denotes the inclusion

$$\overline{M}_{\tau}(\mathcal{Y}_0)|_{\mathcal{F}_0} \hookrightarrow \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}$$

and the sum runs over all rigid tropical types τ of tropical maps to the polyhedral complex

$$\Sigma : \quad \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \rightarrow$$

$\sigma_Y \qquad \qquad \sigma_Z$

which is the fibre over $1 \in \mathbb{R}_{\geq 0}$ of the tropicalisation $\Sigma(\mathcal{Y}) \rightarrow \Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}$.

By the discussion in [ACGS20, Section 5.1] τ can only be rigid if all vertices of the underlying graph map to either one of the vertices σ_X or σ_Z . Hence, τ is uniquely reconstructed from the data of its underlying decorated bipartite graph which is exactly the data of a splitting type as introduced in Definition 2.5. Hence, to be closer to the notation used in Section 2.4 let us write Γ for the splitting type underlying τ and Γ_Y, Γ_Z for the type of stable logarithmic maps to $(Y|D)$, respectively $(Z|D_0+D_\infty)$, obtained by cutting Γ along edges.

We remark that only such rigid tropical types can contribute non-trivially to the sum (27) for which all marking legs are attached to a Z -vertex. Indeed, contact markings necessarily have to be adjacent to a Z -vertex and if an interior marking is carried by a Y -vertex then $\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0}$ is empty (where we crucially use our assumption that $m \leq n$). Moreover, the factor m_τ in the decomposition formula (27) equates to $\text{lcm}(d_e)$ where d_e denotes contact order of an edge e .

Since we are in the special setting where Σ is of dimension one the discussion in [Gro23, Section 7] applies. There is a morphism v with target the fibre product

$$(28) \quad \begin{array}{ccccc} \overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0} & \xrightarrow{v} & \overline{M}_\Gamma & \longrightarrow & \overline{M}_{\Gamma_Y}(Y|D) \times \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \\ & & \downarrow & \square & \downarrow \\ & & D_0^{E(\Gamma)} & \xrightarrow{\Delta} & D_0^{E(\Gamma)} \times D_0^{E(\Gamma)} \end{array}$$

which is finite and satisfies

$$(29) \quad v_*[\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\text{virt}} = \frac{\Pi_e d_e}{\text{lcm}(d_e)} \Delta^!([\overline{M}_{\Gamma_Y}(Y|D)]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\text{virt}}).$$

There is a gluing morphism $\theta : \overline{M}_\Gamma \rightarrow \overline{M}_{g,m+n,\beta}(Y)$ such that

$$(30) \quad \begin{array}{ccc} \overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0} & \xrightarrow{v} & \overline{M}_\Gamma \\ \downarrow \iota & & \downarrow \theta \\ \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0} & \xrightarrow{\zeta \circ \rho_{\mathcal{F}_0}} & \overline{M}_{g,m+n,\beta}(Y) \end{array}$$

commutes where ζ denotes the forgetful morphism

$$\zeta : \overline{M}_{g,\tilde{c},\beta}(Y|D)|_F \longrightarrow \overline{M}_{g,m+n,\beta}(Y).$$

To summarize the discussion so far we combine equation (27) and (29) and use the fact that diagram (30) commutes to deduce

$$(31) \quad \zeta_*[\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\text{virt}} = \sum_{\Gamma} \frac{\Pi_e d_e}{|\text{Aut } \Gamma|} \theta_*[\overline{M}_\Gamma]^{\text{virt}}$$

where we introduced the notation

$$[\overline{M}_\Gamma]^{\text{virt}} := \Delta^!([\overline{M}_{\Gamma_Y}(Y|D)]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\text{virt}}).$$

3.4. Vanishing. In this section we will argue that for most splitting types Γ the cycle $[\overline{M}_\Gamma]^{\text{virt}}$ vanishes. We note that our setup is almost the same as in Section 2.4. The only notable differences are as follows.

- $\overline{M}_{\Gamma_Y}(Y|D)$ parametrise logarithmic stable maps to $(Y|D)$. In Section 2.4 we considered the simple fixed locus of this moduli stack under some torus action.
- $\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)$ parametrises logarithmic stable maps to $(Z|D_0 + D_\infty)$ while in Section 2.4 the target was non-rigid $(Z|D_0 + D_\infty)$.
- Γ_Z contains interior markings.

We note that all arguments in Section 2.4 are essentially insensitive to first two differences which allows us to prove the vanishing of $[\overline{M}_\Gamma]^{\text{virt}}$ in similar steps. Only the third point requires some attention.

3.4.1. First vanishing: fibre classes. We start with the analogue of Lemma 2.6.

Lemma 3.2. *We have $[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\mathcal{F}_0}]^{\text{virt}} = 0$ unless Γ_Z only consists of vertices decorated with a curve class which is a multiple of a fibre class.*

Proof. Let us write β_Z for the overall curve class carried by Γ_Z and suppose its projection $p_*\beta_Z$ along the morphism $p : P \rightarrow D_1$ is non-zero. In other words we assume $p_*\beta_Z = d[D_1]$ for some $d > 0$. We claim that in this case

$$[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)]^{\text{virt}} = 0.$$

To prove this we use an argument from [vGGR19, Section 4]. Set $k = m + n + |E(\Gamma)|$ and consider the morphism

$$\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty) \longrightarrow \overline{M}_{g_Z, k, d[D_1]}(D_0)$$

obtained by forgetting the logarithmic structure and composing with the projection $Z \rightarrow D_0$. Write $H_2(P)^+$ and $H_2(D_1)^+$ for the semigroups of effective curve classes in P , respectively D_1 and denote by $\mathfrak{M}_{g, n, H}^{(\log)}$ the Artin stack of prestable (logarithmic) curves with irreducible components weighted by an element in a semigroup H . We factor the projection morphism through

$$\begin{array}{ccccc} \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty) & \xrightarrow{u} & \widetilde{M}_{\Gamma_Z} & \longrightarrow & \overline{M}_{g_Z, k, d[D_1]}(D_0) \\ & \searrow & \downarrow & \square & \downarrow \\ & & \mathfrak{M}_{g_Z, k, H_2(P)^+}^{\log} & \xrightarrow{v} & \mathfrak{M}_{g_Z, k, H_2(D_1)^+} \end{array}$$

Then by [vGGR19, Theorem 4.1] we have⁴

$$[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)]^{\text{virt}} = u^!_{\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)/\widetilde{M}_{\Gamma_Z}} v^! [\overline{M}_{g_Z, k, d[D_1]}(D_0)]^{\text{virt}}.$$

Our claim that $[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)]^{\text{virt}} = 0$ directly follows from the fact that

$$[\overline{M}_{g_Z, k, d[D_1]}(D_0)]^{\text{virt}} = 0$$

since $D_0 \cong \text{Tot } \mathcal{O}_{\mathbb{P}^1}(-2)$ is a (non-compact) K3 surface and $d > 0$. □

⁴In the statement of [vGGR19, Theorem 4.1] van Garrel, Graber and Ruddat assume a positivity property for the logarithmic tangent sheaf of the target which guarantees that u is smooth. In our case at hand this positivity condition is in general not satisfied and u is only virtually smooth in the sense of [Man12b, Definition 3.4]. The latter is however sufficient for our purposes thanks to [Man12a, Corollary 4.9].

3.4.2. *Second vanishing: star shaped graphs.* In this section we apply the same arguments we used in Section 2.4.3 in order to constrain the shape of Γ to star shaped graphs.

Notation 3.3. We introduce the following notation for labelling sets:

- Interior markings with a point condition $I_{\text{pt}} := \{1, \dots, m\}$;
- Contact markings with a point condition $J_{\text{pt}} := \{m+1, \dots, n\}$;
- Contact markings without an insertion $J_{\perp} := \{n+1, \dots, m+n\}$.

We may assume that all Z -vertices of Γ are decorated with a curve class which is a multiple of a fibre class. This has the following consequence.

Lemma 3.4. *We have $[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\mathcal{F}_0}]^{\text{virt}} = 0$ unless every Z -vertex*

- *either carries two legs, one labelled in I_{pt} and one in $J_{\perp}$;*
- *or carries a single leg with label in J_{pt} .*

Proof. It suffices to show that every Z -vertex is carrying at most one leg with label in $I_{\text{pt}} \cup J_{\text{pt}}$. Indeed, in this case the number of vertices in Γ_Z is bounded from below by $|I_{\text{pt}}| + |J_{\text{pt}}| = n$. On the other hand it is also bounded from above by the number of contact markings n . We deduce that there have to be exactly n vertices each carrying one of the legs labelled by $J_{\perp} \cup J_{\text{pt}}$. Since we assumed that every vertex carries at most one leg with label in $I_{\text{pt}} \cup J_{\text{pt}}$, the m interior markings are necessarily distributed among the vertices carrying a leg labelled in $J_{\perp}$.

So suppose there is a Z -vertex with more than one leg with label in $I_{\text{pt}} \cup J_{\text{pt}}$. Since the evaluation morphism of an interior marking has target Z let us first define a substack with confined evaluations:

$$\overline{N}_{\Gamma_Z} := \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty) \times_{Z^n} D_\infty^n.$$

As every Z -vertex is decorated with a curve class which is a multiple of a fibre class and we assume there is a vertex with two legs labelled in $I_{\text{pt}} \cup J_{\text{pt}}$ the resulting evaluation morphism

$$\overline{N}_{\Gamma_Z} \longrightarrow D_\infty^n$$

factors through $D_\infty^{n-1} \rightarrow D_\infty^n$. We obtain the following commuting diagram with Cartesian squares:

$$\begin{array}{ccccc} \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\mathcal{F}_0} & \longrightarrow & \overline{N}_{\Gamma_Z} & \longrightarrow & \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty) \\ \downarrow & & \downarrow & & \downarrow \Pi_{i=1}^n \text{ev}_i \\ \mathcal{F}_0^{n-1} & \longrightarrow & D_\infty^{n-1} & \longrightarrow & D_\infty^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_0^n & \longrightarrow & D_\infty^n & \longrightarrow & Z^n \end{array}$$

The vanishing of $[\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\mathcal{F}_0}]^{\text{virt}}$ then immediately follows from an excess bundle calculation as the one we did in Lemma 2.8. $\square$

Moreover, by our assumption that all Z -vertices carry curve classes which are multiples of a fibre class the evaluation morphism associated to edges

$$\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_{\mathcal{F}_0} \longrightarrow D_0^{E(\Gamma)}$$

factors through $F^{E(\Gamma)} \hookrightarrow D_0^{E(\Gamma)}$ where we write F for the image of $\mathcal{F}_0$ under $Z \rightarrow D_0$. We combine this observation with the fact that stable maps to Y factor through the zero section $S \hookrightarrow Y$ in order to constrain the image of Z -vertices to the fibre of $Z \rightarrow D_0$ over the point $p = F \cap D_1$. Indeed, as we just argued the evaluation morphism

$$\overline{M}_{\Gamma_Y}(Y|D) \longrightarrow D^{E(\Gamma)}$$

factors through $D_1^{E(\Gamma)} \hookrightarrow D^{E(\Gamma)}$. Writing $\overline{M}_{\Gamma_Y}(Y|D)|_F$ for the fibre product $\overline{M}_{\Gamma_Y}(Y|D) \times_{D^{E(\Gamma)}} F^{E(\Gamma)}$ we find that

$$(32) \quad \begin{array}{ccc} \overline{M}_{\Gamma_Y}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty & \longrightarrow & \overline{M}_{\Gamma_Y}(Y|D) \times \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \\ \downarrow & \square & \downarrow \\ p & \longrightarrow & D_1^{E(\Gamma)} \times F^{E(\Gamma)} \\ \downarrow & \square & \downarrow \\ D_0^{E(\Gamma)} & \xrightarrow{\Delta} & D_0^{E(\Gamma)} \times D_0^{E(\Gamma)}. \end{array}$$

commutes and both squares are Cartesian. An excess intersection argument as in Lemma 2.9 gives

Lemma 3.5. *We have $[\overline{M}_\Gamma]^{\text{virt}} = 0$ if there is a Z-vertex with more than one edge.* $\square$

Since all squares in (32) are Cartesian, a comparison with (28) gives

$$\overline{M}_\Gamma \cong \overline{M}_{\Gamma_Y}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty.$$

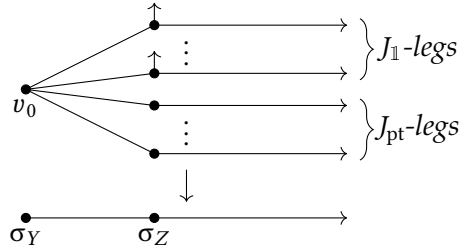
The following lemma comparing the virtual fundamental classes of the above moduli stacks can be proven with the same arguments we used to show Lemma 2.10.

Lemma 3.6. *If Γ contains a Z-vertex v with $g_v > 0$ the cycle $[\overline{M}_\Gamma]^{\text{virt}}$ vanishes. Otherwise, it equates to*

$$[\overline{M}_{\Gamma_Y}(Y|D)|_F]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\text{virt}}. \quad \square$$

Let us summarise the results of this section.

Proposition 3.7. *The cycle $[\overline{M}_\Gamma]^{\text{virt}}$ vanishes unless the rigid tropical type τ associated to Γ is of the form*



where v_0 is decorated with $g_{v_0} = g$, $\beta_{v_0} = \beta$ and all other vertices are decorated with multiples of fibre classes. Moreover, in this case

$$[\overline{M}_\Gamma]^{\text{virt}} = [\overline{M}_{\Gamma_Y}(Y|D)|_F]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\text{virt}}. \quad \square$$

3.5. Analysis of remaining terms. Now our discussion will deviate from the analysis in Section 2.4 because as opposed to Section 2.4.4 we will show that all remaining contributions characterised in Proposition 3.7 do contribute to the Gromov–Witten invariant and will exactly yield the right-hand side of equation (25).

Let Γ be as in Proposition 3.7. We fix a labelling of the markings of $\overline{M}_{\Gamma_Y}(Y|D)|_F$ by fixing a bijection $\{1, \dots, n\} \cong E(\Gamma)$ where we impose that the i th edge is adjacent to the Z-vertex carrying the $(m+i)$ th leg. With this assignment we can naturally identify Γ_Y with the data (g, c, β) .

Our first goal is to simplify

$$\theta_*[\overline{M}_\Gamma]^{\text{virt}} = \theta_*([\overline{M}_{g,c,\beta}(Y|D)|_F]^{\text{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\text{virt}}).$$

Under θ all images of stable maps which lie in a fibre of $P \rightarrow D_1$ get contracted to a point. Hence, the morphism factors into

$$\begin{array}{ccc} \overline{M}_{g,c,\beta}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty & \xrightarrow{\text{Id} \times \pi} & \overline{M}_{g,c,\beta}(Y|D)|_F \times (\overline{M}_{0,3})^m \xrightarrow{\phi} \overline{M}_{g,m+n,\beta}(Y) \\ & \searrow \theta & \end{array}$$

where π is the forgetful morphism which only remembers the stabilised domain curve and ϕ is a morphism which forgets the logarithmic structure and attaches a contracted rational component at the last m markings.

Consequently, a Z -vertex carrying a leg labelled with $i + m \in J_{\mathbb{I}}$ contributes with a factor

$$\deg[\overline{M}_{0,((c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)|_\infty]^{\text{virt}} = \deg[\overline{M}_{0,((c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)]^{\text{virt}} = \frac{1}{c_i}$$

where the denominator is due to the degree c_i automorphism group acting on the unique degree c_i cover $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ fully ramified over 0 and infinity. Similarly, a Z -vertex with leg labelled by $i + m \in J_{\text{pt}}$ contributes

$$\deg[\overline{M}_{0,(0,(c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)|_\infty]^{\text{virt}} = \deg(\text{ev}_1^*(\text{pt}) \cap [\overline{M}_{0,(0,(c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)]^{\text{virt}}) = 1$$

because the additional marking kills the automorphism group acting on the domain. We obtain

$$(33) \quad \theta_*[\overline{M}_\Gamma]^{\text{virt}} = \left(\prod_{i=1}^{n-m} \frac{1}{c_i} \right) \cdot \phi_* \left([\overline{M}_{g,c,\beta}(Y|D)|_F]^{\text{virt}} \times [\overline{M}_{0,3}]^{\times m} \right).$$

Note that a splitting type Γ as in Proposition 3.7 is uniquely determined by the bijection $I_{\text{pt}} \cong J_{\mathbb{I}}$ it induces. This means there are exactly $m!$ such splitting types each one contributing with (33) to the overall Gromov–Witten invariant. Thus, plugging the last equation into (31) we find

$$\zeta_*[\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\text{virt}} = m! \left(\prod_{i=0}^{m-1} c_{n-i} \right) \cdot \phi_* \left([\overline{M}_{g,c,\beta}(Y|D)|_F]^{\text{virt}} \times [\overline{M}_{0,3}]^{\times m} \right).$$

Pushing the identity forward to a point yields formula (25) which closes the proof of Proposition 3.1.

4. APPLICATIONS

4.1. Topological Vertex. Open Gromov–Witten invariants can be computed efficiently using the topological vertex method [AKMV05; LLLZ09]. Hence, formula (10) equating logarithmic Gromov–Witten invariants — which are usually rather cumbersome to compute — against open ones presents a new venue for calculating the former.

4.1.1. The general recipe. We refer to [Yu23, Section 2] or [BBvG24, Section 6] for a general summary of the topological vertex method which allows to determine the open Gromov–Witten invariants of a toric triple just from the information of its toric diagram. Let (X, L, f) be the outcome of Construction 1.8 for some two-component Looijenga pair $(Y|D_1 + D_2)$. We will write

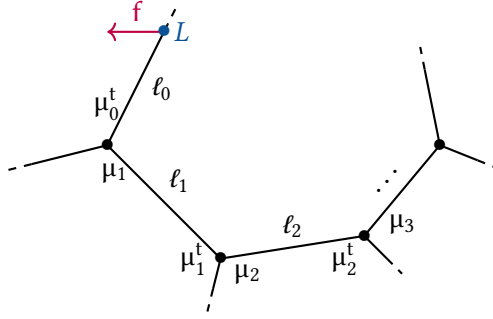
$$(34) \quad \mathcal{W}_\mu(X, L, f)(q, Q)$$

for the generating series of disconnected open Gromov–Witten invariants of (X, L, f) in representation basis where the winding around L is encoded in a partition μ . In the special case of a single boundary wrapping the Lagrangian submanifold L we can extract the connected invariant defined in (7) via [BBvG24, Equation (6.16)]⁵

$$(35) \quad \sum_{\beta'} \sum_{g \geq 0} \text{OGW}_{g,(c),\beta'}(X, L, f) \hbar^{2g-1} Q^{\beta'} = -i \sum_{k=0}^c \frac{(-1)^k}{c} \frac{\mathcal{W}_{(c-k,1^k)}(X, L, f)}{\mathcal{W}_\emptyset(X, L, f)}$$

under the change of variables $q = e^{i\hbar}$ where we fixed a square root i of -1 .

⁵We corrected [BBvG24, Equation (6.16)] by an overall factor $-i$. See [Yu23, Appendix A] for a discussion of this factor.

FIGURE 3. Toric diagram of (X, L, f) decorated with partitions.

As we discussed in Section 2.1, the toric diagram of (X, L, f) will have a shape as displayed in Figure 3 where we labelled the compact one dimensional torus orbit closures by $\ell_1, \dots, \ell_k$ starting from the stratum ℓ_0 intersected by L and assigned partitions $\mu_1, \dots, \mu_k$ to all edges. Then every edge ℓ_i , $i \in \{0, \dots, k\}$, contributes with a factor

$$(-1)^{(\deg N_{\ell_i} S') \cdot |\mu_i|} q^{(\deg N_{\ell_i} S' + 1) \cdot \kappa_{\mu_i} / 2}$$

to (34) where $\kappa_v := \sum_j v_j(v_j - 2j + j)$. Each trivalent vertex corresponding to a torus fixed point $\ell_i \cap \ell_{i+1}$ contributes

$$\mathcal{W}_{\mu_{i+1}, \mu_i^t, \emptyset} := s_{\mu_{i+1}}(q^{p+\mu_i}) s_{\mu_i^t}(q^p) = q^{-\kappa_{\mu_i} / 2} s_{\mu_{i+1}}(q^p) s_{\mu_i}(q^{p+\mu_{i+1}}) = q^{\kappa_{\mu_{i+1}} / 2} \sum_v s_{\frac{\mu_i}{v}}(q^p) s_{\frac{\mu_{i+1}}{v}}(q^p)$$

where by convention $\mu_0 = \mu$ and $\mu_{k+1} = \emptyset$ and $s_{\alpha/\beta}(q^{p+\gamma})$ denotes the skew Schur function $s_{\alpha/\beta}(x_1, x_2, \dots)$ evaluated at $x_j = -j + \frac{1}{2} - \gamma_j$. As an application of [LLLZ09, Proposition 7.4] we obtain

Lemma 4.1. *For (X, L, f) the outcome of Construction 1.8 we have*

$$\mathcal{W}_\mu(X, L, f)(q, Q) = \sum_{\mu_1, \dots, \mu_k} Q^{\sum_{i=1}^k |\mu_i| \cdot [\ell_i]} \prod_{i=0}^k (-1)^{(\deg N_{\ell_i} S') \cdot |\mu_i|} q^{(\deg N_{\ell_i} S' + 1) \cdot \kappa_{\mu_i} / 2} \mathcal{W}_{\mu_{i+1}, \mu_i^t, \emptyset}.$$

In case the toric triple is a strip geometry the above sum of partitions can be carried out explicitly to yield a closed form solution for $\mathcal{W}_\mu(X, L, f)$ [IK06]. Plugged into (35) and (10) this can be used to give a closed form solution for the logarithmic Gromov–Witten invariants of $(S|D_1 + D_2)$ and will be illustrated in a concrete example in the following.

4.1.2. Extended example: $dP_3(0, 2)$. Let us consider $(dP_3|D_1 + D_2)$ where dP_3 is a del Pezzo surface which is the result of blowing up three points in $\mathbb{P}^2$. This Looijenga pair was denoted by $dP_3(0, 2)$ in [BBvG24]. If we write E_1, E_2, E_3 for exceptional divisors we take $D_1 \in |H - E_1|$ and $D_2 \in |2H - E_2 - E_3|$ smooth with transverse intersections. We use the following notation for a curve class in dP_3 :

$$\beta = d_0(H - E_1 - E_2 - E_3) + d_1E_1 + d_2E_2 + d_3E_3.$$

It was conjectured in [BBvG24] and later proven in [BS23] that the maximum contact logarithmic Gromov–Witten invariants of this geometry have the following closed form solution.

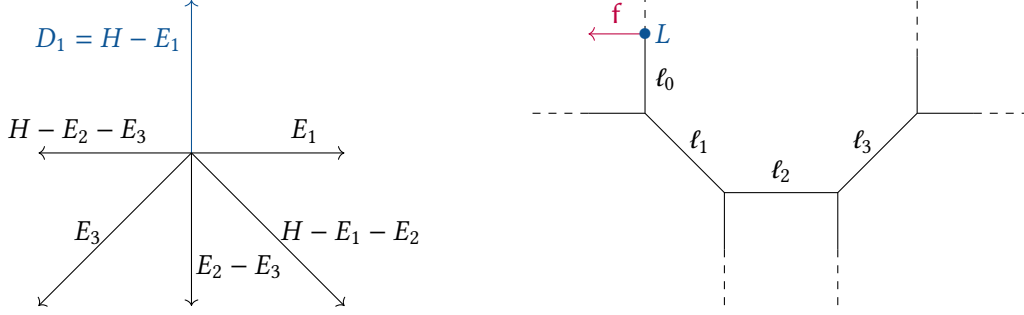


FIGURE 4. The fan of dP'_3 on the left and the toric diagram of (X, L, f) on the right [BBvG24, Figure 6.5].

Proposition 4.2. [BS23, Proposition 1.4] For all effective curve classes β in dP_3 with $D_i \cdot \beta > 0$, $i \in \{1, 2\}$, and contact data $\hat{\mathbf{c}} = \begin{pmatrix} 0 & D_1 \cdot \beta & 0 \\ 0 & 0 & D_2 \cdot \beta \end{pmatrix}$ we have

$$(36) \quad \sum_{g \geq 0} \hbar^{2g-1} \mathrm{LGW}_{g, \hat{\mathbf{c}}, \beta}(\mathrm{dP}_3 | D_1 + D_2) = \frac{[d_1]_q [d_2 + d_3]_q}{[d_0]_q [d_1 + d_2 + d_3 - d_0]_q} \begin{bmatrix} d_1 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_1 + d_2 + d_3 - d_0 \\ d_1 \end{bmatrix}_q$$

as formal Laurent series in $\hbar$ under the identification $q = e^{\hbar}$.

In this proposition we use the notation

$$[n]_q := q^{n/2} - q^{-n/2}, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \prod_{k=1}^m \frac{[n-m+k]_q}{[k]_q} & 0 \leq m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The statement was proven in [BS23] by extracting the Gromov–Witten invariants from a quantum scattering diagram of a toric model of $(\mathrm{dP}_3 | D_1 + D_2)$. The outcome of this calculation is a convoluted sum of products of q -binomial coefficients which with a lot of effort can be simplified to yield the above formula. We will give a new proof of this identity using the topological vertex method.

Proof of Proposition 4.2. We start by applying Construction 1.8 to $(\mathrm{dP}_3 | D_1 + D_2)$. For this let us write p_1, p_2, p_3 for the three points in $\mathbb{P}^2$ whose blow up gives dP_3 . Then D_1 is the strict transform of a line through p_1 and D_2 the strict transform of a conic through p_2 and p_3 . We note that we can always choose the $\mathbb{G}_m^2$ -action on $\mathbb{P}^2$ in such a way that D_1 is a toric hypersurface and p_1 and p_2 are torus fixed points. Now with this choice of $\mathbb{G}_m^2$ -action the blowups at p_1 and p_2 are toric as opposed to the one at p_3 since all points are assumed to be in general position. However, there is a deformation of dP_3 to the blow up of $\mathbb{P}^2$ in p_1, p_2 and a torus fixed point of E_2 leaving D_1 invariant. We denote this deformation by dP'_3 which is indeed toric as required. We display its fan in Figure 4 where we labelled each ray with its curve class. Since dP'_3 is toric with D_1 a toric hypersurface, we can choose

$$X = \mathrm{Tot} \mathcal{O}_{\mathrm{dP}'_3}(-D_2)|_{\mathrm{dP}'_3 \setminus D_1}.$$

The toric diagram of (X, L, f) is displayed in Figure 4 where we chose L to intersect the torus invariant curve in class $H - E_2 - E_3$. We note that this particular toric triple has already been studied by Bousseau, Brini and van Garrel in [BBvG24, Section 6.3.1]. Hence, from here on we basically follow their calculation.

Observe that each compact edge in the toric diagram corresponds to a ray in the fan. Hence, we can read off the degree of the normal bundles of $\ell_0, \dots, \ell_3$ from Figure 4 to be $-1, -1, -2$ and -1 respectively.

Thus, by Lemma 4.1 we have

$$\mathcal{W}_\mu(X, L, f) = \sum_{\substack{\mu_1, \mu_2, \mu_3 \\ \nu_1, \nu_2}} (-1)^{|\mu|+|\mu_1|+|\mu_3|} s_{\mu^t}(q^\rho) s_{\mu_1^t}(q^{\rho+\mu}) s_{\underline{\mu}_1}(q^\rho) s_{\underline{\mu}_2}(q^\rho) s_{\underline{\mu}_2}(q^\rho) s_{\underline{\mu}_2}(q^\rho) s_{\underline{\mu}_3}(q^\rho) s_{\mu_3^t}(q^\rho) Q^{\sum_{i=1}^3 |\mu_i| \cdot [\ell_i]}.$$

This can now be plugged into (35) and simplified using the techniques developed in [IK06]. These steps have already been carried out in [BBvG24, Equation (6.18–6.23)] and result in

$$\begin{aligned} & \sum_{\beta'} \sum_{g \geq 0} \text{OGW}_{g, (c), \beta'}(X, L, f) \hbar^{2g-1} Q^{\beta'} \\ &= \sum_{c_1, c_2, c_3} i(-1)^{c+c_1+c_3+1} \frac{1}{c[c_1]_q} \begin{bmatrix} c \\ c_1 - c_2 \end{bmatrix}_q \begin{bmatrix} c \\ c_2 - c_3 \end{bmatrix}_q \begin{bmatrix} c + c_3 - 1 \\ c_3 \end{bmatrix}_q \begin{bmatrix} c_1 \\ c \end{bmatrix}_q Q^{\sum_{i=1}^3 c_i [\ell_i]}. \end{aligned}$$

Extracting the coefficient of $Q^{i^*(\beta-d_1[\ell_0])}$ from the above formula we get

$$\begin{aligned} & \sum_{g \geq 0} \hbar^{2g-1} \text{OGW}_{g, (d_1), i^*(\beta-d_1[\ell_0])}(X, L, f) \\ &= \frac{i(-1)^{d_1+d_2+d_3+1} [d_1]_q}{d_1[d_0]_q [d_1+d_2+d_3-d_0]_q} \begin{bmatrix} d_1 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_1 + d_2 + d_3 - d_0 \\ d_1 \end{bmatrix}_q \end{aligned}$$

which can be plugged into (10) to yield the formula stated in Proposition 4.2. $\square$

We stress that the above calculation is substantially easier than the one in [BS23] using scattering diagrams. This hopefully illustrates the usefulness of our main result Theorem 1.9.

4.2. BPS integrality. It is a general feature rather than a coincidence that the all-genus generating series of logarithmic Gromov–Witten invariants lifts to a rational function in $e^{\hbar}$ as we saw for instance in formula (36). This property is usually called BPS integrality.

4.2.1. LMOV invariants. In the context of open Gromov–Witten invariants of toric Calabi–Yau threefolds BPS integrality was first systematically studied by Labastida–Marino–Ooguri–Vafa [OV00; LM01; LMV00; MV02]. Just recently Yu gave a proof of this phenomenon by carefully analysing the combinatorics of the topological vertex [Yu23].

To review Yu’s result, let (X, L, f) be a toric triple. For a curve class β' in X and a winding profile $\mathbf{c} = (c_1, \dots, c_n)$ along L we will write

$$k | (\mathbf{c}, \beta')$$

for $k \in \mathbb{N}$ if k divides c_i for all $i \in \{1, \dots, n\}$ and there is a curve class β'' satisfying $k\beta'' = \beta'$.

Now for every tuple $(\mathbf{c}, \beta')$ we define

$$\text{LMOV}_{\mathbf{c}, \beta'}(\hbar) \in \mathbb{Q}[[\hbar^2]]$$

by demanding that these formal power series satisfy

$$\sum_{g \geq 0} \hbar^{2g-2+n} \text{OGW}_{g, \mathbf{c}, \beta'}(X, L, f) =: \prod_{i=1}^n \frac{2 \sin \frac{\hbar c_i}{2}}{c_i} \cdot \sum_{k | (\mathbf{c}, \beta')} k^{n-1} \left(2 \sin \frac{\hbar k}{2} \right)^{-2} \text{LMOV}_{\mathbf{c}/k, \beta'/k}(k\hbar).$$

Writing μ for the Möbius function one can check that

$$\text{LMOV}_{\mathbf{c}, \beta'}(\hbar) = \left(2 \sin \frac{\hbar}{2} \right)^2 \prod_{i=1}^n \frac{c_i}{2 \sin \frac{\hbar c_i}{2}} \cdot \sum_{k | (\mathbf{c}, \beta')} \frac{\mu(k)}{k} \sum_{g \geq 0} (k\hbar)^{2g-2+n} \text{OGW}_{g, \mathbf{c}/k, \beta'/k}(X, L, f)$$

is indeed a solution to the above defining equation.

Theorem 4.3 (LMOV integrality). [Yu23, Theorem 1.1] $\text{LMOV}_{\mathbf{c}, \beta'}$ lifts to a Laurent polynomial in $q = e^{\hbar}$ with integer coefficients.

4.2.2. *BPS integrality of logarithmic invariants.* Now let $(S|D_1+D_2)$ be a logarithmic Calabi–Yau surface with boundary the union of two transversally intersecting smooth curves D_1, D_2 . To a tuple $(\hat{c}, \beta)$, where β is an effective curve class in S satisfying $D_1 \cdot \beta \geq 0$ and $D_2 \cdot \beta > 0$ and $\hat{c}$ is a contact condition of the form

$$\hat{c} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & c_1 & \cdots & c_n & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 \cdot \beta \end{pmatrix}}_{m \text{ times}}$$

for some $c_1, \dots, c_n > 0$ and $m \leq n$, we associate a formal power series

$$\text{BPS}_{\hat{c}, \beta}(\hbar) \in \mathbb{Q}[[\hbar^2]]$$

solving the recursive system

$$(37) \quad \begin{aligned} & (-1)^{K_S \cdot \beta + 1} \sum_{g \geq 0} \hbar^{2g-1+n} \text{LGW}_{g, \hat{c}, \beta}(S|D_1 + D_2) \\ &= \left(\prod_{i=1}^{n-m} \frac{2 \sin \frac{\hbar c_i}{2}}{c_i} \right) \left(\frac{1}{m!} \prod_{i=n-m+1}^n 2 \sin \frac{\hbar c_i}{2} \right) \left(2 \sin \frac{\hbar(D_2 \cdot \beta)}{2} \right) \sum_{k | (\hat{c}, \beta)} k^{n-1} \left(2 \sin \frac{\hbar k}{2} \right)^{-2} \text{BPS}_{\hat{c}/k, \beta/k}(k\hbar). \end{aligned}$$

Again, one can check that

$$\begin{aligned} \text{BPS}_{\hat{c}, \beta} &= \left(2 \sin \frac{\hbar}{2} \right)^2 \left(\prod_{i=1}^{n-m} \frac{c_i}{2 \sin \frac{\hbar c_i}{2}} \right) \left(\frac{1}{m!} \prod_{i=n-m+1}^n \frac{1}{2 \sin \frac{\hbar c_i}{2}} \right) \frac{1}{2 \sin \frac{\hbar(D_2 \cdot \beta)}{2}} \\ &\quad \times \sum_{k | (\hat{c}, \beta)} \mu(k) k^{m-1} (-1)^{K_S \cdot \beta/k + 1} \sum_{g \geq 0} (k\hbar)^{2g-1+n} \text{LGW}_{g, \hat{c}/k, \beta/k}(S|D_1 + D_2) \end{aligned}$$

is indeed a solution to this defining equation. Motivated by the last section we expect the following.

Conjecture 4.4. $\text{BPS}_{\hat{c}, \beta}$ lifts to a Laurent polynomial in $q = e^{\hbar}$ with integer coefficients.

Evidence for the conjecture comes from the following immediate corollary of Yu’s LMOV integrality (Theorem 4.3) and our logarithmic-open correspondence (Theorem 1.9).

Theorem 4.5. *Conjecture 4.4 holds if $(S|D_1 + D_2)$ satisfies Assumption $\star$. Especially, if (X, L, f) is the toric triple from Construction 1.8 we have*

$$\text{BPS}_{\hat{c}, \beta} = \text{LMOV}_{c, \beta'}.$$

□

Note that the pattern in which the sine factors enter the right-hand side of equation (37) naturally suggests a generalisation to the case where we permit more than one point of contact with D_2 . It remains to be seen whether BPS integrality persists in this case as well.

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UNIVERSITY OF SHEFFIELD, SCHOOL OF MATHEMATICS AND STATISTICS, HOUNSFIELD ROAD, SHEFFIELD S3 7RH, UNITED KINGDOM.

UNIVERSITY OF CAMBRIDGE, DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UNITED KINGDOM

Email address: ys667@cam.ac.uk