FRACTAL UNCERTAINTY PRINCIPLE FOR RANDOM CANTOR SETS

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ABSTRACT. We continue our investigation of the fractal uncertainty principle (FUP) for random fractal sets. In the prequel [EH], we considered the Cantor sets in the discrete setting with alphabets randomly chosen from a base of digits so the dimension $\delta \in (0, \frac{2}{3})$. We proved that, with overwhelming probability, the FUP with an exponent $\geq \frac{1}{2} - \frac{3}{4}\delta - \varepsilon$ holds for these discrete Cantor sets with random alphabets.

In this sequel, we construct random Cantor sets with dimension $\delta \in (0, \frac{2}{3})$ in $\mathbb{R}$ via a different random procedure from the one in [EH]. We prove that, with overwhelming probability, the FUP with an exponent $\geq \frac{1}{2} - \frac{3}{4}\delta - \varepsilon$ holds. The proof follows from establishing a Fourier decay estimate of the corresponding random Cantor measures, which is in turn based on a concentration of measure phenomenon in an appropriate probability space for the random Cantor sets.

1. Introduction

We briefly recall the setup of the fractal uncertainty principle (FUP) for random fractal sets and refer to Eswarathasan-Han [EH] for more background. Let $0 < h \leq 1$ be the semiclassical parameter. Define $\mathcal{F}_h$ as the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{ix\cdot\xi}{h}} u(x)\, dx \quad \text{for } u \in C_0^\infty(\mathbb{R}).$$

In the case when $h = 1$, $\mathcal{F} := \mathcal{F}_1$ reduces to the usual Fourier transform. The FUP is formulated in the context of estimating the norm

$$\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}, \tag{1.1}$$

in which the h-dependent sets $X = X(h), Y = Y(h) \subset \mathbb{R}$ are equipped with certain fractal-type structures. Following Dyatlov [Dy, Definition 2.2], the fractal-type structure is characterized by

Definition 1.1 (δ-regular sets). *Let $0 \leq \delta \leq 1$, $R \geq 1$, and $0 \leq \alpha_{\min} \leq \alpha_{\max} \leq \infty$. We say that a non-empty closed set $X \subset \mathbb{R}$ is δ-regular with constant R on scales $\alpha_{\min}$ to $\alpha_{\max}$ if there is a locally finite measure ν_X supported on X such that for every interval I centered at a point in X with $\alpha_{\min} \leq |I| \leq \alpha_{\max}$, we have that*

$$R^{-1}|I|^\delta \leq \nu_X(I) \leq R|I|^\delta.$$

Here, $|E|$ denotes the Lebesgue volume of a measurable set $E \subset \mathbb{R}$.

Since $\mathcal{F}_h$ is unitary in $L^2(\mathbb{R})$, we always have that (1.1)$= O(1)$. We say that X, Y satisfy the FUP with an exponent $\beta \geq 0$ if (1.1) $= O(h^\beta)$ as $h \to 0$. The main objective about the FUP for δ-regular sets is to prove the existence of exponent $\beta > 0$ and to find the sharp one for which the FUP holds. Here, "the sharp exponent", denoted by $\beta^s(X, Y)$, means the largest exponent such that (1.1) $= O(h^\beta)$ holds for all $0 < h < h_0$ with some $h_0 > 0$. It usually depends on the constants δ and R in Definition 1.1 of the sets X, Y in question.

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Firstly, if X and Y are δ-regular on scales h to 1 with $\delta \in [0,1]$, then the FUP (1.1) holds with an exponent

$$\beta_{\mathrm{Vol}} = \max\left\{\frac{1}{2} - \delta, 0\right\}. \tag{1.2}$$

This exponent (1.2) is sometime referred as "the volume bound", because it only takes the volume of X, Y into consideration, that is, $|X|, |Y| \le Ch^{1-\delta}$, see Bourgain-Dyatlov [BD2, Lemma 2.9].

The FUP with improved exponents over the volume bound (1.2) have been proved in various settings by Bourgain-Dyatlov [BD1, BD2], Backus-Leng-Tao [BLT], Cohen [C1, C2], Cladek-Tao [CT], Dyatlov-Jin [DJ1, DJ2], Dyatlov-Zahl [DyZa], Han-Schlag [HS], Jin-Zhang [JZ], and etc. The sets considered in these works are deterministic and the improvement of the exponent is small (and is either implicitly or explicitly dependent on δ and R in Definition 1.1). In particular, the following theorem includes explicit estimates on such improvement, which are exponentially small when $0 < \delta \le \frac{1}{2}$ by Dyatlov-Jin [DJ2, Theorem 1] and super-exponentially small when $\frac{1}{2} < \delta < 1$ by Jin-Zhang [JZ, Theorem 1.2].

Theorem 1.2. *Let $0 < \delta < 1$ and $R \ge 1$. Suppose that X, Y be δ-regular with constant R on scales h to 1. Then the FUP (1.1) holds with an exponent β such that*

$$\beta - \beta_{\mathrm{Vol}} \ge \begin{cases} (5R)^{-\frac{40}{\delta(1-\delta)}} & \text{if } 0 < \delta \le \frac{1}{2}, \\ \exp\left[-\exp\left(C\left(R\delta^{-1}(1-\delta)^{-1}\right)^{C(1-\delta)^{-2}}\right)\right] & \text{if } \frac{1}{2} < \delta < 1. \end{cases}$$

Here, $C > 0$ is an absolute constant.

***Remark*.** The FUP depends on the framework of Fourier transform $\mathcal{F}_h$. In particular, a non-trivial function u and its Fourier transform $\mathcal{F}_h u$ cannot be both compactly supported. However, in the framework of Walsh-Fourier transform, such phenomenon is allowed. In terms of the FUP, there are certain Cantor sets of dimension $\frac{1}{2} \le \delta < 1$ for which the FUP (1.1) does *not* hold with exponents greater than $\beta_{\mathrm{Vol}} = 0$. This is due to Demeter [De].

In an ongoing project, we investigate the FUP (1.1) when the sets X, Y are constructed via certain random procedures, and in these random settings, we aim to prove the FUP with more favorable exponents than the ones in Theorem 1.2 for the deterministic cases. In the paper [EH] by Eswarathasan and the first-named author, we considered the FUP for the random Cantor sets in the discrete setting. That is, let $M, A \in \mathbb{N}$ such that $M \ge 3$ and $A = M^\delta$ with $0 < \delta < 1$. Then the alphabets of cardinality A from the digits $\{0, ..., M-1\}$ form a probability space

$$\mathbb{A}(M, A) = \{\mathcal{A} \subset \{0, ..., M-1\} : \mathrm{Card}(\mathcal{A}) = A\},$$

which is equipped with the uniform counting measure μ_1. Here, $\mathrm{Card}(\mathcal{A})$ is the cardinality of $\mathcal{A}$.

In [EH], an alphabet is chosen at random from $\mathbb{A}(M, A)$; then the discrete Cantor sets are constructed using this alphabet (in each iteration step). Let $\delta \in (0, \frac{2}{3})$. We proved that, with overwhelming probability (that is, except on a subset of $\mathbb{A}(M, A)$ with exponentially small measure depending on M, ε), the Cantor sets with random alphabets satisfy the FUP with an exponent $\beta \ge \frac{1}{2} - \frac{3}{4}\delta - \varepsilon$. It is therefore a significant improvement over the volume bound (1.2) for this random ensemble.

In this paper, we consider the FUP for random Cantor sets in the continuous setting of $\mathbb{R}$. We use a different random procedure with the one in [EH] for the construction. See Section 1.1 for the different random ensembles of Cantor sets, as well as the comparison of approaches and results between the discrete and continuous settings.

Each Cantor set in $\mathbb{R}$ can be built via an iteration process. Let $\mathbb{A}(M,A)$ be as above. Inductively define

$$B_1 = \frac{1}{M}\mathcal{A}, \quad \text{in which } \mathcal{A} \in \mathbb{A}(M,A),$$

and for $j \geq 2$,

$$B_j = \bigcup_{b \in B_{j-1}} \left\{ b + \frac{a}{M^j} : a \in \mathcal{A}(b) \right\},$$

in which $\mathcal{A}(b) \in \mathbb{A}(M,A)$ with $b \in B_{j-1}$ are independent and identical distributed (iid) random variables. In particular, $\mathrm{Card}(B_j) = A^j$ for all $j \in \mathbb{N}$.

Write

$$\mathcal{C}_j = \bigcup_{b \in B_j} \left[b, b + \frac{1}{M^j} \right], \quad \text{and} \quad \mathcal{C} = \bigcap_{j=1}^{\infty} \mathcal{C}_j \subset [0,1]. \tag{1.3}$$

Hence, $\mathcal{C}_j$ is a union of A^j closed intervals, each of which has Lebesgue volume of M^{-j}. Define also the Borel measure ν_j whose density function is given by

$$\rho_j(x) = \frac{M^j}{A^j} \mathbb{1}_{\mathcal{C}_j}(x) \quad \text{for } x \in \mathbb{R}.$$

Finally, the weak limit ν of ν_j as $j \to \infty$ defines the Cantor measure which is supported on $\mathcal{C}$.

We next set up the appropriate probability space for the random Cantor set $\mathcal{C}$ and the random Cantor measure ν. In the first iteration, the probability space is given by $\mathbb{A}(M,A)$; in the j-th iteration for $j \geq 2$, since $\mathrm{Card}(B_{j-1}) = A^{j-1}$, the probability space is given by $\mathbb{A}(M,A)^{A^{j-1}}$ equipped with the uniform counting measure μ_j. Write

$$\mathbb{A}^\infty := \prod_{j=1}^{\infty} \mathbb{A}(M,A)^{A^{j-1}} \text{ equipped with the probability measure } \mu^\infty := \prod_{j=1}^{\infty} \mu_j. \tag{1.4}$$

Then $\mathcal{C}$ is a random Cantor set and ν is a random Cantor measure in $\mathbb{R}$ with respect to the probability space $\mathbb{A}^\infty$.

It is obvious that each Cantor set $\mathcal{C}$ constructed in this way has Hausdorff dimension

$$\delta = \frac{\log A}{\log M} \in (0,1).$$

Moreover, each Cantor set $\mathcal{C}$ is δ-regular with constant $R = 2$ on scales 0 to 1, for which one can simply choose the Cantor measure ν in Definition 1.1, see Dyatlov [Dy, Example 2.6] for the example of the middle-third Cantor set (that is, $M = 3$ and $\mathcal{A} = \{0,2\}$).

Consider the h-neighborhoods $\mathcal{C}(h)$ of $\mathcal{C}$. Then $\mathcal{C}(h)$ is δ-regular with constant $R = 8$ on scales h to 1, see Bourgain-Dyatlov [BD2, Lemma 2.3]. Therefore, for *each* Cantor set $\mathcal{C}$, the FUP (1.1) holds for $X = Y = \mathcal{C}(h)$ with an exponent β explicitly given in Theorem 1.2.

However, in the case when $0 < \delta < \frac{2}{3}$, we prove that, with overwhelming probability (that is, except on a subset of $\mathbb{A}^\infty$ with exponentially small measure), the FUP holds for the random Cantor sets with much better exponents than the ones in Theorem 1.2:

Theorem 1.3 (FUP for random Cantor sets)**.** *Let $0 < \delta < \frac{2}{3}$. Suppose that $M, A \in \mathbb{N}$ such that $M \geq e^{4\delta^{-1}}$ and $A = M^\delta$. Then for $0 < \varepsilon < \frac{\delta}{2}$, there exists $\mathbb{G} \subset \mathbb{A}^\infty$ with*

$$\mu^\infty\left(\mathbb{A}^\infty \setminus \mathbb{G}\right) \leq C_1 e^{-\frac{M^\varepsilon}{C}}$$

such that each Cantor set $\mathcal{C}$ from $\mathbb{G}$ satisfies that

$$\left\| \mathbb{1}_{\mathcal{C}(h)} \mathcal{F}_h \mathbb{1}_{\mathcal{C}(h)} \right\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C h^{\frac{1}{2} - \frac{3}{4}\delta - \varepsilon} \quad \text{for all } 0 < h < M^{-8}.$$

Here, $C > 0$ is an absolute constant and $C_1 = C_1(\varepsilon) > 0$ depends on ε.

We also prove an FUP for the random Cantor measures ν:

Theorem 1.4 (FUP for random Cantor measures). *Under the same conditions as Theorem 1.3, the corresponding random Cantor measure ν satisfies that*

$$\left\| \int_{\mathbb{R}} e^{-\frac{ix\cdot\xi}{h}} u(x)\, d\nu(x) \right\|_{L^2_\nu(\mathbb{R})} \le C h^{\frac{\delta}{4}-\varepsilon} \|u\|_{L^2_\nu(\mathbb{R})} \quad \text{for all } u \in L^2_\nu(\mathbb{R}) \text{ and } 0 < h < M^{-8}.$$

The proofs of the FUP in Theorem 1.3 and 1.4 are based on the following Fourier decay estimate of the random Cantor measures:

Theorem 1.5 (Fourier decay of random Cantor measures). *Let $0 < \delta < 1$. Suppose that $M, A \in \mathbb{N}$ such that $M \ge e^{4\delta^{-1}}$ and $A = M^\delta$. Then for $0 < \varepsilon < \min\{\frac{\delta}{2}, \frac{1}{3}\}$, there exists $\mathbb{G} \subset \mathbb{A}^\infty$ with*

$$\mu^\infty (\mathbb{A}^\infty \setminus \mathbb{G}) \le C_1 e^{-\frac{M^\varepsilon}{C}}$$

such that each Cantor measure ν from $\mathbb{G}$ satisfies that

$$|\mathcal{F}\nu(\xi)| \le C|\xi|^{-\frac{\delta-\varepsilon}{2}} \quad \text{for all } |\xi| \ge M^4.$$

Here, $C > 0$ is an absolute constant and $C_1 = C_1(\varepsilon) > 0$ depends on ε.

Remark (Fourier dimension and Salem sets). The Fourier dimension $\dim_{\mathrm{F}} E$ of a Borel set $E \subset \mathbb{R}$ is the largest number s such that there is a finite Borel measure ν supported on E which satisfies that $\mathcal{F}\nu(\xi) = O(|\xi|^{-\frac{s}{2}})$. We always have that $\dim_{\mathrm{F}} E \le \dim_{\mathrm{H}} E$, the Hausdorff dimension of E, because the strongest Fourier decay estimate for a measure supported on E is $O(|\xi|^{-\frac{\dim_{\mathrm{H}} E}{2}})$. We say that E is a Salem set if $\dim_{\mathrm{F}} F = \dim_{\mathrm{H}} E$. Salem [S] constructed the first example of such sets, which is random in nature and is Cantor-like. See Mattila [M, Chapter 12] for more details.

The random Cantor measures in Theorem 1.5 satisfy the (almost) strongest Fourier decay estimates. Notice also that it is valid for the full range of $\delta \in (0,1)$. However, only in the smaller range of $\delta \in (0, \frac{2}{3})$ does it imply the FUP in Theorems 1.3 and 1.4. Our construction of the random Cantor sets is largely inspired by Łaba-Pramanik [LP, Section 6], which provides examples of Salem sets via a different random procedure with Salem's original approach [S].

1.1. **Random ensembles in the discrete setting and the continuous setting.** Continue with our notations of $\mathbb{A}(M, A)$ with $M, A \in \mathbb{N}$ and $A = M^\delta$, $0 < \delta < 1$. We discuss the FUP for different ensembles of Cantor sets. The boundary points of the Cantor set in the continuous setting of $\mathbb{R}$ at each iteration $j \in \mathbb{N}$ naturally define the Cantor set in the discrete setting of

$$\mathbb{Z}_N := \frac{1}{N}\mathbb{Z}/N\mathbb{Z} \quad \text{with } N = M^j.$$

This connection allows us to compare the approaches and results on the FUP in the discrete setting and in the continuous setting. (We shall remark that for direct comparison between the two settings, the discrete set $\mathbb{Z}_N$ here differs with the one in [DJ1, EH] by a scaling factor of N.)

Ensemble I. Let $\mathcal{A} \in \mathbb{A}(M, A)$ be randomly chosen. For $j \in \mathbb{N}$, define the discrete Cantor set of order j as

$$B_j = \left\{ \sum_{l=1}^{j} \frac{a_l}{M^l} : a_l \in \mathcal{A} \text{ for } l = 1, ..., j \right\},$$

for which the probability space is $\mathbb{A}(M, A)$.

Ensemble II. For $j \in \mathbb{N}$, let $\mathcal{A}_k \in \mathbb{A}(M, A)$ for $l = 1, ..., j$ be iid random variables. Define the discrete Cantor set of order j as

$$B_j = \left\{ \sum_{l=1}^{j} \frac{a_l}{M^l} : a_l \in \mathcal{A}_l \text{ for } l = 1, ..., j \right\},$$

for which the probability space is $\prod_{l=1}^{j} \mathbb{A}(M, A)$.

Ensemble III. Let $B_1 = \frac{1}{M}\mathcal{A}$ with $\mathcal{A} \in \mathbb{A}(M, A)$. For $j \geq 2$, let $\mathcal{A}(b) \in \mathbb{A}(M, A)$ for $b \in B_{j-1}$ be iid random variables, and define the discrete Cantor set of order j as

$$B_j = \bigcup_{b \in B_{j-1}} \left\{ b + \frac{a}{M^j} : a \in \mathcal{A}(b) \right\},$$

for which the probability space is $\prod_{k=1}^{j} \mathbb{A}(M, A)^{A^{k-1}}$.

In each of the ensembles above, the Cantor sets $\mathcal{C}$ in $\mathbb{R}$ are defined by (1.3). The following three figures demonstrate (the initial iterations of) examples of the Cantor sets with $M = 3$ and $A = 2$.

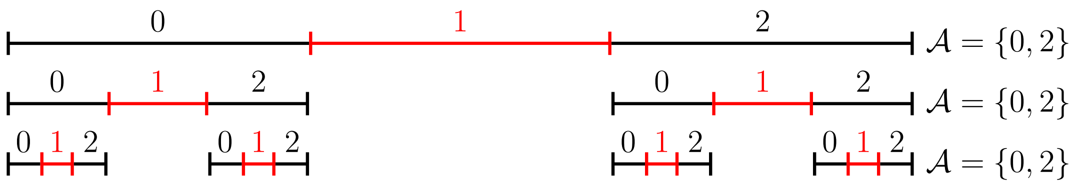

FIGURE 1. The initial three iterations and the alphabet used for a Cantor set in Ensemble I. (The intervals colored red are removed in the iteration process.)

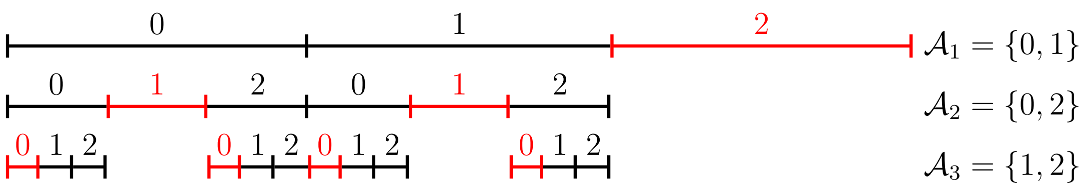

FIGURE 2. The initial three iterations and the alphabets used for a Cantor set in Ensemble II.

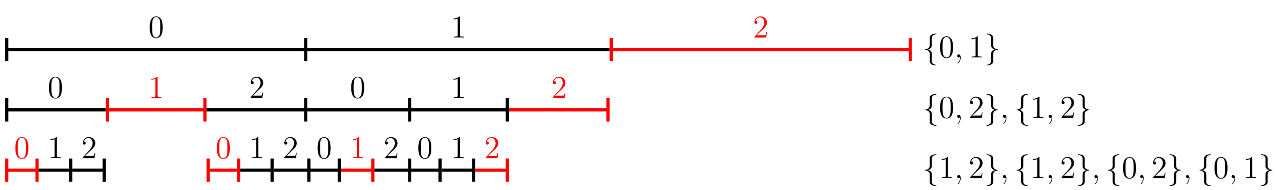

FIGURE 3. The initial three iterations and the alphabets used for a Cantor set in Ensemble III.

Remark (The FUP for Cantor sets in the discrete setting of $\mathbb{Z}_N$). Let $\mathcal{F}_N$ be the discrete Fourier transform which is unitary on $l^2(\mathbb{Z}_N)$. For each $j \in \mathbb{N}$, the discrete Cantor set $B_j \in \mathbb{Z}_N$. The FUP for Cantor sets in the discrete setting is concerned with the estimate of the form

$$\left\| \mathbb{1}_{B_j} \mathcal{F}_N \mathbb{1}_{B_j} \right\|_{l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)} = O\left(N^{-\beta}\right) \quad \text{as } j \to \infty, \tag{1.5}$$

in which $N^{-1} = M^{-j} \to 0$ plays the role of the semiclassical parameter.

For the deterministic Cantor sets in Ensemble I, Dyatlov-Jin [DJ1] introduced a FUP theory and proved the FUP (1.5) with exponent $\beta = \beta(M, \mathcal{A}) > \beta_{\mathrm{Vol}}$ for all $M, \mathcal{A}$. They also extensively studied the sharp exponent $\beta_s(M, \mathcal{A})$ in the FUP for different $M, \mathcal{A}$. On one hand, they found examples of $M, \mathcal{A}$ for which the FUP holds with the exponent $\frac{1-\delta}{2}$, which is the best possible one. On the other hand, they also found examples of $M, \mathcal{A}$ for which the sharp exponent $\beta_s(M, \mathcal{A}) = \beta_{\mathrm{Vol}} + o_M(1)$.

For the random Cantor sets in Ensemble I, Eswarathasan-Han [EH] introduced a probabilistic approach to the FUP and proved that for $0 < \delta < \frac{2}{3}$, $\beta_s(M, \mathcal{A}) \geq \frac{1}{2} - \frac{3}{4}\delta - \varepsilon$ with overwhelming probability. The proof is similar to the one used in this paper, that is, it is based on establishing a Fourier decay estimate of the corresponding discrete Cantor measure $\tilde{\nu}_j = \sum_{b \in B_j} \delta_b$. However, the proof is simplified because of the submultiplicativity property in the discrete setting: If $j = j_1 + j_2$, then

$$\left\|\mathbb{1}_{B_j} \mathcal{F}_N \mathbb{1}_{B_j}\right\|_{l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)} \leq \left\|\mathbb{1}_{B_{j_1}} \mathcal{F}_{N_1} \mathbb{1}_{B_{j_1}}\right\|_{l^2(\mathbb{Z}_{N_1}) \to l^2(\mathbb{Z}_{N_1})} \left\|\mathbb{1}_{B_{j_2}} \mathcal{F}_{N_2} \mathbb{1}_{B_{j_2}}\right\|_{l^2(\mathbb{Z}_{N_2}) \to l^2(\mathbb{Z}_{N_2})}$$

in which $N = N_1 N_2$ with $N_1 = M^{j_1}$ and $N_2 = M^{j_2}$. Hence, the estimate of the FUP at the initial iteration $j = 1$ implies one for all iterations $j \in \mathbb{N}$. Indeed, the Fourier transform of the discrete Cantor measure at the first iteration,

$$\mathcal{F}\tilde{\nu}_1(m), \quad \text{in which } \tilde{\mu}_1 = \sum_{b \in B_1} \delta_b = \sum_{a \in \mathcal{A}} \delta_{\frac{a}{M}},$$

has decay for $m \in \mathbb{Z}_M \setminus \{0\}$ for randomly chosen $\mathcal{A} \in \mathbb{A}(M, A)$, which is sufficient to imply the FUP in the discrete setting.

For the deterministic Cantor sets in Ensembles II and III, one can prove an FUP with exponents in Theorem 1.2, following an approach of Dyatlov-Jin [DJ2] to reduce the FUP for the discrete Fourier transform $\mathcal{F}_N$ to one for $\mathcal{F}_h$ with respect to the discrete measures $\tilde{\nu}_j$. Moreover, Dyatlov-Jin [DJ1] still supplies examples of Cantor sets for which the FUP holds with the best possible exponent (by simply choosing the same alphabet at each iteration).

However, much less is known for the random Cantor sets in Ensembles II and III than Ensemble I. In particular, the submultiplicativity property is not necessarily true, since different alphabets can be used in different iterations. So one has to consider higher iterations when proving the FUP via the probabilistic approach. – This is open.

Remark (The FUP for Cantor sets in the continuous setting of $\mathbb{R}$). All Cantor sets $\mathcal{C}$ in each ensemble above are δ-regular with an absolute constant R on scales 0 to 1 in Definition 1.1. Their h-neighborhood $\mathcal{C}(h)$ are δ-regular on scales h to 1. As a consequence, the h-neighborhoods $\mathcal{C}(h)$ satisfy the FUP in Theorem 1.2.

Our main results in Theorems 1.3 and 1.4 are concerned with the random Cantor sets in Ensemble III, which state that if $0 < \delta < \frac{2}{3}$, then the sharp constant $\beta_s \geq \frac{1}{2} - \frac{3}{4}\delta - \varepsilon$ in the FUP with overwhelming probability.

However, for the random Cantor sets in Ensembles I and II, much less is known. In particular, the Cantor measures in these ensembles do not necessarily have Fourier decay, that is, the Fourier dimension of some Cantor sets can be zero, see again Mattila [M, Chapter 12]. The probabilistic approach to the FUP in these ensembles is open.

Moreover, for the deterministic case in all the ensembles above, no examples of Cantor sets are known to satisfy the FUP with the best possible exponent $\frac{1-\delta}{2}$ in the continuous setting of $\mathbb{R}$ to the authors' knowledge.

1.2. **Organization of the paper.** In Section 2, we prepare the probabilisitic tools from concentration of measure theory for our proof of the main theorems. In Section 3, we prove the Fourier decay of the random Cantor measures. In Section 4, we use the Fourier decay to establish the FUP for the random Cantor measures and random Cantor sets.

2. Probabilistic estimates

Recall that $M, A \in \mathbb{N}$ with $M \geq 3$ and $A = M^\delta$ with $0 < \delta < 1$, and that $\mathbb{A}(M, A)$ (equipped with the uniform counting probability measure μ_1) is the probability space of alphabets of cardinality A from the digits $\{0, ..., M-1\}$.

In this section, we establish the probabilistic estimates, which are used to prove the Fourier decay estimate of random Cantor measures, see Section 3. The following function (2.1) appears naturally in such an estimate.

Suppose that B is a finite set and $\mathcal{A}(b) \in \mathbb{A}(M, A)$ for $b \in B$ are independent and identical random variables. For $N \in \mathbb{N}$ and $\eta \in \mathbb{R} \setminus \{0\}$, define

$$\frac{1}{\operatorname{Card}(B)} \sum_{b \in B} e^{iN\eta b} F_\eta\left(\mathcal{A}(b)\right), \tag{2.1}$$

in which

$$F_\eta(\mathcal{A}) = \frac{1}{A} \sum_{a \in \mathcal{A}} e^{-i\eta a} - \frac{1}{M} \sum_{a=0}^{M-1} e^{-i\eta a}. \tag{2.2}$$

Hence, (2.1) is a random variable with respect to the probability space $\mathbb{A}(M, A)^{\operatorname{Card}(B)}$ equipped with the uniform counting measure μ. The main estimate in this section is

Theorem 2.1. *Let $N \in \mathbb{N}$ and $\eta \in \mathbb{R} \setminus \{0\}$. Then*

$$\mu\left(\left|\frac{1}{\operatorname{Card}(B)} \sum_{b \in B} e^{iN\eta b} F_\eta\left(\mathcal{A}(b)\right)\right| \geq t\right) \leq 2 \exp\left(-\frac{\operatorname{Card}(B) A t^2}{64 + \frac{4}{3} A t}\right) \quad \text{for all } t > 0.$$

We estimate $F_\eta(\mathcal{A})$ for $\mathcal{A} \in \mathbb{A}(M, A)$ and first observe that

$$|F_\eta(\mathcal{A})| \leq 2 \quad \text{for all } \mathcal{A} \in \mathbb{A}(M, A). \tag{2.3}$$

We next show that the random variable $F_\eta(\mathcal{A})$ for $\mathcal{A} \in \mathbb{A}(M, A)$ is concentrated near its expectation at an exponential rate, which therefore satisfies a much better estimate than the above trivial bound with overwhelming probability. It is a consequence of the concentration of measure theory in the metric space $\mathbb{A}(M, A)$ established by Eswarathasan-Han [EH, Section 2].

Set the metric in $\mathbb{A}(M, A)$ by

$$d(\mathcal{A}_1, \mathcal{A}_2) = \operatorname{Card}\left(\mathcal{A}_1 \triangle \mathcal{A}_2\right) = \operatorname{Card}(\mathcal{A}_1 \setminus \mathcal{A}_2) + \operatorname{Card}(\mathcal{A}_2 \setminus \mathcal{A}_1) \quad \text{for } \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}(M, A). \tag{2.4}$$

Here, $\mathcal{A}_1 \triangle \mathcal{A}_2$ denotes the symmetric difference. For a function $F : \mathbb{A}(M, A) \to \mathbb{C}$, its Lipschitz norm is defined by

$$\|F\|_{\mathrm{Lip}} = \max_{\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}(M,A), \mathcal{A}_1 \neq \mathcal{A}_2} \frac{|F(\mathcal{A}_1) - F(\mathcal{A}_2)|}{d(\mathcal{A}_1, \mathcal{A}_2)}.$$

Under this setup, we have Eswarathasan-Han [EH, Theorem 2.1]:

Theorem 2.2 (Concentration of measure in the space of alphabets)**.** *Let $F : \mathbb{A}(M, A) \to \mathbb{C}$ and $t > 0$. Then*

$$\mu_1\left(\{\mathcal{A} \in \mathbb{A}(M, A) : |F(\mathcal{A}) - \mathbb{E}[F]| \geq t\}\right) \leq 2 \exp\left(-\frac{t^2}{16 A \|F\|_{\mathrm{Lip}}^2}\right),$$

in which $\mathbb{E}[F]$ *is the expectation of* F *with respect to* μ_1*.*

To apply Theorem 2.2 to F_η in (2.2), we compute that

$$\begin{aligned}
\mathbb{E}\left[F_\eta\right] &= \frac{1}{\operatorname{Card}\left(\mathbb{A}(M,A)\right)}\sum_{\mathcal{A}\in\mathbb{A}(M,A)} F_\eta(\mathcal{A}) \\
&= \binom{M}{A}^{-1}\sum_{\mathcal{A}\in\mathbb{A}(M,A)}\left(\frac{1}{A}\sum_{a\in\mathcal{A}(b)} e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\eta a}\right) \\
&= \frac{1}{A}\binom{M}{A}^{-1}\sum_{a=0}^{M-1}\sum_{\mathcal{A}\in\mathbb{A}(M,A),\mathcal{A}\ni a} e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\eta a} \\
&= \frac{1}{A}\binom{M}{A}^{-1}\binom{M-1}{A-1}\sum_{a=0}^{M-1}e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\eta a} \\
&= \frac{1}{M}\sum_{a=0}^{M-1}e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\eta a} \\
&= 0.
\end{aligned}$$

Here, we used the fact that for any fixed digit $a=0,...,M-1$, the number of alphabets $\mathcal{A}$ in $\mathbb{A}(M,A)$ which contain a is $\binom{M-1}{A-1}$.

To estimate the Lipschitz norm of F_η in (2.2), let $\mathcal{A}_1,\mathcal{A}_2\in\mathbb{A}(M,A)$. Then

$$\begin{aligned}
\left|F_\eta\left(\mathcal{A}_1\right)-F_\eta\left(\mathcal{A}_2\right)\right| &= \frac{1}{A}\left|\sum_{a\in\mathcal{A}_1}e^{-i\eta a}-\sum_{a\in\mathcal{A}_2}e^{-i\eta a}\right| \\
&= \frac{1}{A}\left|\sum_{a\in\mathcal{A}_1\triangle\mathcal{A}_2}e^{-i\eta a}\right| \\
&\le \frac{1}{A}\cdot\operatorname{Card}(\mathcal{A}_1\triangle\mathcal{A}_2) \\
&= \frac{1}{A}\cdot d(\mathcal{A}_1,\mathcal{A}_2),
\end{aligned}$$

in the view of the metric (2.4). Hence, $\|F\|_{\mathrm{Lip}}\le\frac{1}{A}$. By Theorem 2.2, we therefore have the following proposition.

Proposition 2.3. *Let* $\eta\in\mathbb{R}\setminus\{0\}$ *and* $t>0$*. Then*

$$\mu_1\left(\{\mathcal{A}\in\mathbb{A}(M,A):|F_\eta(\mathcal{A})|\ge t\}\right)\le 2\exp\left(-\frac{At^2}{16}\right).$$

An immediate consequence is an estimate on the variance $\operatorname{Var}[F_\eta]$ of F_η:

Corollary 2.4. *Let* $\eta\in\mathbb{R}\setminus\{0\}$*. Then*

$$\operatorname{Var}\left[F_\eta\right]\le\frac{32}{A}.$$

Proof. Since $\mathbb{E}[F_\eta]=0$, compute that

$$\operatorname{Var}\left[F_\eta\right] = \mathbb{E}\left[\left|F_\eta\right|^2\right]$$

$$\begin{aligned} &= 2\int_0^\infty t\cdot \mu_1\left(\{\mathcal{A}\in\mathbb{A}(M,A) : |F_\eta(\mathcal{A})|\geq t\}\right)\,dt \\ &\leq 4\int_0^\infty te^{-\frac{At^2}{16}}\,dt \\ &= \frac{32}{A}. \end{aligned}$$

□

We now prove Theorem 2.1 and need the following version of Bernstein inequality from Bennett [B].

Theorem 2.5 (Bernstein inequality)**.** *Let $X_1,...,X_n$ be independent and identical random variables with expectation zero. Suppose that for some $C>0$, $|X_j|\leq C$ for all $j=1,...,n$. Then for $t>0$,*

$$P\left(\left|\frac{1}{n}\sum_{j=1}^n X_j\right|\geq t\right)\leq 2\exp\left(-\frac{nt^2}{\frac{2}{n}\sum_{j=1}^n \mathrm{Var}\left[X_j^2\right]+\frac{2}{3}Ct}\right).$$

Proof of Theorem 2.1. We apply Bernstein inequality with $n=\mathrm{Card}(B)$ and

$$X(b)=e^{iN\eta b}F_\eta\left(\mathcal{A}(b)\right),$$

for which $X(b)\leq 2$ from (2.3) for all $b\in B$. Moreover,

$$\mathbb{E}[X(b)]=0 \quad\text{and}\quad \mathrm{Var}[X(b)]\leq\frac{32}{A}.$$

Hence,

$$\mu\left(\left|\frac{1}{\mathrm{Card}(B)}\sum_{b\in B}e^{i\eta b}F_\eta\left(\mathcal{A}(b)\right)\right|\geq t\right)\leq 2\exp\left(-\frac{\mathrm{Card}(B)t^2}{\frac{64}{A}+\frac{4}{3}t}\right)=2\exp\left(-\frac{\mathrm{Card}(B)At^2}{64+\frac{4}{3}At}\right).$$

□

3. Fourier decay of random Cantor measures

We derive the Fourier transform of the Cantor measure ν supported on the Cantor set $\mathcal{C}$, see (1.3). Let $\xi\in\mathbb{R}\setminus\{0\}$. Then

$$\mathcal{F}\nu(\xi)=\lim_{j\to\infty}\mathcal{F}\nu_j(\xi)=\frac{1}{\sqrt{2\pi}}\lim_{j\to\infty}\int_{\mathbb{R}}e^{-ix\xi}\rho_j(x)\,dx.$$

Here, the density function ρ_j of the measure ν_j assigns a measure of $\frac{M^j}{A^j}$ to each of the A^j intervals which defines $\mathcal{C}_j$. Hence,

$$\begin{aligned} \mathcal{F}\nu_j(\xi) &= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_j}\frac{M^j}{A^j}\int_b^{b+\frac{1}{M^j}}e^{-ix\xi}\,dx \\ &= \frac{M^j}{\sqrt{2\pi}A^j}\sum_{b\in B_j}\frac{e^{-i\left(b+\frac{1}{M^j}\right)\xi}-e^{-ib\xi}}{-i\xi} \\ &= \frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j}\cdot\left(\frac{1}{A^j}\sum_{b\in B_j}e^{-i\xi b}\right). \end{aligned}$$

Since $\mathrm{Card}(B_j) = A^j$,

$$|\mathcal{F}\nu_j(\xi)| \leq \left| \frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j} \cdot \left(\frac{1}{A^j}\sum_{b\in B_j} e^{-i\xi b}\right)\right| \leq \left|\frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j}\right| \leq C\cdot\min\left\{1, \frac{M^j}{|\xi|}\right\}.$$

in which $C>0$ is an absolute constant, that is, it is independent of M, A, ξ, j. Here, we used the fact that

$$\frac{(e^{-i\eta}-1)}{\eta} = O(1) \text{ as } \eta\to 0, \quad \text{and} \quad \frac{(e^{-i\eta}-1)}{\eta} = O\left(|\eta|^{-1}\right) \text{ as } |\eta|\to\infty.$$

This estimate of $\mathcal{F}\nu_j$ is not strong enough to derive a Fourier decay of ν. To do so, we next compare the Fourier transforms of $\mathcal{C}_j$ for consecutive j's, which requires a different way of computing $\mathcal{F}\nu_j$ as above. Let $j\geq 2$. Then

$$\begin{aligned}
\mathcal{F}\nu_{j-1}(\xi) &= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j-1}}\frac{M^{j-1}}{A^{j-1}}\int_b^{b+\frac{1}{M^{j-1}}} e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j-1}}\frac{M^{j-1}}{A^{j-1}}\sum_{a=0}^{M-1}\int_{b+\frac{a}{M^j}}^{b+\frac{a+1}{M^j}} e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j-1}}\frac{M^{j-1}}{A^{j-1}}\sum_{a=0}^{M-1}\frac{e^{-i\left(b+\frac{a+1}{M^j}\right)\xi}-e^{-i\left(b+\frac{a}{M^j}\right)\xi}}{-i\xi}\\
&= \frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j}\cdot\left(\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}}e^{-i\xi b}\cdot\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\xi a/M^j}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{F}\nu_{j}(\xi) &= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j}}\frac{M^{j}}{A^{j}}\int_b^{b+\frac{1}{M^{j}}} e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j-1}}\frac{M^{j}}{A^{j}}\sum_{a\in\mathcal{A}(b)}\int_{b+\frac{a}{M^j}}^{b+\frac{a+1}{M^j}} e^{-ix\xi}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\sum_{b\in B_{j-1}}\frac{M^{j}}{A^{j}}\sum_{a\in\mathcal{A}(b)}\frac{e^{-i\left(b+\frac{a+1}{M^j}\right)\xi}-e^{-i\left(b+\frac{a}{M^j}\right)\xi}}{-i\xi}\\
&= \frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j}\cdot\left(\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}}e^{-i\xi b}\cdot\frac{1}{A}\sum_{a\in\mathcal{A}(b)}e^{-i\xi a/M^j}\right).
\end{aligned}$$

Denoting $\eta = \xi/M^j$, we have that

$$\begin{aligned}
&\mathcal{F}\nu_j(\xi) - \mathcal{F}\nu_{j-1}(\xi)\\
= &\frac{i\left(e^{-i\xi/M^j}-1\right)}{\sqrt{2\pi}\xi/M^j}\cdot\left[\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}}e^{-i\xi b}\left(\frac{1}{A}\sum_{a\in\mathcal{A}(b)}e^{-i\xi a/M^j}-\frac{1}{M}\sum_{a=0}^{M-1}e^{-i\xi a/M^j}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta}\cdot\left[\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}\left(\frac{1}{A}\sum_{a\in\mathcal{A}(b)} e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1} e^{-i\eta a}\right)\right]\\
&= \frac{i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta}\cdot\left[\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}F_\eta\left(\mathcal{A}(b)\right)\right]\\
&:= G(\eta).
\end{aligned}
$$

We use Theorem 2.1 to estimate $G(\eta)$. Firstly, compute that

$$
\begin{aligned}
G'(\eta) &= \frac{\eta e^{-i\eta}-i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta^2}\cdot\left[\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}F_\eta\left(\mathcal{A}(b)\right)\right]\\
&\quad+\frac{i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta}\cdot\left[\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}\left(-iM^jbF_\eta\left(\mathcal{A}(b)\right)+\partial_\eta F_\eta\left(\mathcal{A}(b)\right)\right)\right].
\end{aligned}
$$

Since $a\in\{0,1,...,M-1\}$,

$$
\left|\partial_\eta F_\eta\left(\mathcal{A}(b)\right)\right| = \left|\frac{1}{A}\sum_{a\in\mathcal{A}(b)}\left(-iae^{-i\eta a}\right)-\frac{1}{M}\sum_{a=0}^{M-1}\left(-iae^{-i\eta a}\right)\right|\le M.
$$

Since $|F_\eta(\mathcal{A})|\le 2$ by (2.3) and $B_{j-1}\in[0,1]$ with $\mathrm{Card}(B_{j-1})=A^{j-1}$, we have that

$$
\begin{aligned}
|G'(\eta)| &\le \left|\frac{\eta e^{-i\eta}-i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta^2}\right|\cdot\left|\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}F_\eta\left(\mathcal{A}(b)\right)\right|\\
&\quad+\left|\frac{i\left(e^{-i\eta}-1\right)}{\sqrt{2\pi}\eta}\right|\cdot\left|\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}\left(-iM^jbF_\eta\left(\mathcal{A}(b)\right)+\partial_\eta F_\eta\left(\mathcal{A}(b)\right)\right)\right|\\
&\le C\cdot\min\left\{1,|\eta|^{-1}\right\}+C\min\left\{1,|\eta|^{-1}\right\}\cdot\left(M^j+M\right)\\
&\le C\cdot\min\left\{1,|\eta|^{-1}\right\}\cdot M^j,
\end{aligned}
$$

in which $C>0$ is an absolute constant. Here, we used the fact that

$$
\frac{\eta e^{-i\eta}-i\left(e^{-i\eta}-1\right)}{\eta^2}=O(1)\text{ as }\eta\to 0,\quad\text{and}\quad\frac{\eta e^{-i\eta}-i\left(e^{-i\eta}-1\right)}{\eta^2}=O\left(|\eta|^{-1}\right)\text{ as }|\eta|\to\infty.
$$

Next we move on to estimating the following term in $G(\eta)$:

$$
\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}F_\eta\left(\mathcal{A}(b)\right)=\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}\left(\frac{1}{A}\sum_{a\in\mathcal{A}(b)} e^{-i\eta a}-\frac{1}{M}\sum_{a=0}^{M-1} e^{-i\eta a}\right),
$$

which has a period of 2π. Thus, it suffices to consider this term only in the case when $\eta\in(0,2\pi]$. Let

$$
0<L\le A^{\frac{j}{2}-1}.
$$

Set K as the smallest integer such that

$$
l:=\frac{2\pi}{K}\le\frac{\sqrt{2\pi}L}{A^{\frac{j}{2}}M^j}.
$$

Then

$$\left\lfloor \frac{\sqrt{2\pi}A^{\frac{j}{2}}M^j}{L} \right\rfloor \leq K \leq \left\lfloor \frac{\sqrt{2\pi}A^{\frac{j}{2}}M^j}{L} \right\rfloor + 1.$$

Divide $(0, 2\pi]$ into sub-intervals of equal length l. Denote the boundary points of these sub-intervals (except 0) by

$$\eta_k = kl, \quad \text{in which } k = 1, 2, ..., K.$$

We are now ready to apply Theorem 2.1 to η_k. Here, the probability space $\mathbb{A}(M,A)^{A^{j-1}}$ for $\mathcal{C}_j$ is equipped with the probability measure μ_j. Take $t = LA^{-\frac{j}{2}}$. Then there is $\Omega_k \subset \mathbb{A}(M,A)^{A^{j-1}}$ (depending on η_k) with

$$\mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \Omega_k\right) \leq 2\exp\left(-\frac{A^{j-1}\cdot A\left(LA^{-\frac{j}{2}}\right)^2}{64+\frac{4}{3}A\left(LA^{-\frac{j}{2}}\right)}\right) \leq 2\exp\left(-\frac{L^2}{66}\right)$$

such that for all $\mathcal{C}_j$ in Ω_k,

$$\left|\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta_k b}F_\eta\left(\mathcal{A}(b)\right)\right| \leq LA^{-\frac{j}{2}}.$$

Write

$$\mathbb{G}_j = \bigcup_{k=1}^{K}\Omega_k. \tag{3.1}$$

Then for some absolute constant $C > 0$,

$$\begin{aligned}
\mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \mathbb{G}_j\right) &\leq \sum_{k=1}^{K}\mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \Omega_k\right)\\
&\leq 2K\exp\left(-\frac{L^2}{66}\right)\\
&\leq C\cdot\frac{A^{\frac{j}{2}}M^j}{L}\cdot\exp\left(-\frac{L^2}{C}\right),
\end{aligned}$$

and for all $\mathcal{C}_j$ in $\mathbb{G}_j$,

$$\left|\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta_k b}F_\eta\left(\mathcal{A}(b)\right)\right| \leq LA^{-\frac{j}{2}} \quad \text{for all } k = 1, ..., K.$$

Hence, under the same conditions,

$$|G(\eta_k)| = \left|\frac{i\left(e^{-i\eta_k}-1\right)}{\sqrt{2\pi}\eta_k}\right|\cdot\left|\frac{1}{A^{j-1}}\sum_{b\in B_{j-1}} e^{-iM^j\eta b}F_\eta\left(\mathcal{A}(b)\right)\right| \leq C\cdot\min\left\{1, |\eta_k|^{-1}\right\}\cdot LA^{-\frac{j}{2}}.$$

To pass the estimate for η_k, $k = 1, ..., K$, to the one for all $\eta \in (0, 2\pi]$, we use the mean value theorem of $G(\eta)$. That is, for each $\eta \in (0, 2\pi]$, there is η_k such that $\eta_{k-1} = \eta_k - l < \eta \leq \eta_k$. Thus,

$$|G(\eta_k) - G(\eta)| \leq \sup_{\zeta\in(\eta,\eta_k)}|G'(\zeta)|\cdot l$$

$$\begin{aligned} &\leq C \cdot \min\left\{1, |\eta|^{-1}\right\} \cdot M^j \cdot \frac{\sqrt{2\pi}L}{A^{\frac{j}{2}}M^j} \\ &\leq C \cdot \min\left\{1, |\eta|^{-1}\right\} \cdot LA^{-\frac{j}{2}}. \end{aligned}$$

Finally,

$$|G(\eta)| \leq |G(\eta_k) - G(\eta)| + |G(\eta_k)| \leq C \cdot \min\left\{1, |\eta|^{-1}\right\} \cdot LA^{-\frac{j}{2}}.$$

Recall that $\eta = \xi/M^j$. We have established that

Proposition 3.1. *Let $j \geq 2$ and $0 < L \leq A^{\frac{j}{2}-1}$. Then there is $\mathbb{G}_j \subset \mathbb{A}(M,A)^{A^{j-1}}$ with*

$$\mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \mathbb{G}_j\right) \leq C \cdot \frac{A^{\frac{j}{2}}M^j}{L} \cdot \exp\left(-\frac{L^2}{C}\right)$$

such that for all $\mathcal{C}_j$ in $\mathbb{G}_j$ (that is, $\{\mathcal{A}(b), b \in B_{j-1}\}$ are from $\mathbb{G}_j$),

$$|\mathcal{F}\nu_j(\xi) - \mathcal{F}\nu_{j-1}(\xi)| \leq C \cdot \min\left\{1, \frac{M^j}{|\xi|}\right\} \cdot LA^{-\frac{j}{2}} \quad \text{for all } \xi \in \mathbb{R}\setminus\{0\}. \tag{3.2}$$

Here, $C > 0$ is an absolute constant.

Let

$$0 < \varepsilon \leq \frac{1}{3}.$$

For $j \geq 3$, set

$$L = L_j = M^{\frac{j\varepsilon}{2}}.$$

Since $\frac{j\varepsilon}{2} \leq \frac{j}{2} - 1$ for all $j \geq 3$, we apply the proposition above. Then there is $\mathbb{G}_j \in \mathbb{A}(M,A)^{A^{j-1}}$ with

$$\mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \mathbb{G}_j\right) \leq C \cdot \frac{A^{\frac{j}{2}}M^j}{L_j} \cdot \exp\left(-\frac{L_j^2}{C}\right)$$

such that for all $\mathcal{C}_j$ in $\mathbb{G}_j$, (3.2) holds.

Set $\mathbb{G}_1 = \mathbb{A}(M,A)$, $\mathbb{G}_2 = \mathbb{A}(M,A)^A$, and $\mathbb{G}_j$ for $j \geq 3$ be given in (3.1). Let

$$\mathbb{G} = \prod_{j=1}^{\infty} \mathbb{G}_j \subset \mathbb{A}^\infty = \prod_{j=1}^{\infty} \mathbb{A}(M,A)^{A^{j-1}}.$$

Recall that $A = M^\delta$ with $0 < \delta < 1$. Hence,

$$\begin{aligned} \mu^\infty\left(\mathbb{A}^\infty \setminus \mathbb{G}\right) &\leq \sum_{j=1}^{\infty} \mu_j\left(\mathbb{A}(M,A)^{A^{j-1}} \setminus \mathbb{G}_j\right) \\ &\leq \sum_{j=3}^{\infty} C \cdot \frac{A^{\frac{j}{2}}M^j}{L_j} \cdot \exp\left(-\frac{L_j^2}{C}\right) \\ &\leq \sum_{j=3}^{\infty} C \cdot M^{\frac{3j}{2}} \cdot \exp\left(-\frac{M^{j\varepsilon}}{C}\right) \\ &\leq C\exp\left(-\frac{M^\varepsilon}{C}\right)\sum_{j=3}^{\infty} M^{\frac{3j}{2}} \cdot \exp\left(-\frac{M^{(j-1)\varepsilon}}{C}\right) \\ &\leq C_1\exp\left(-\frac{M^\varepsilon}{C}\right). \end{aligned}$$

Here, $C > 0$ is an absolute constant and $C_1 = C_1(\varepsilon) > 0$ depends on ε.

Now suppose that $\mathcal{C}$ is in $\mathbb{G}$, that is, $\mathcal{C}_j$ are in $\mathbb{G}_j$ for all $j \in \mathbb{N}$, see (1.4). Then

$$\begin{aligned} |\mathcal{F}\nu(\xi)| &\leq |\mathcal{F}\nu_1(\xi)| + |\mathcal{F}\nu_2(\xi)| + \sum_{j=2}^{\infty} |\mathcal{F}\nu_j(\xi) - \mathcal{F}\nu_{j-1}(\xi)| \\ &\leq C \cdot \min\left\{1, \frac{M}{|\xi|}\right\} + C \cdot \min\left\{1, \frac{M^2}{|\xi|}\right\} + \sum_{j=3}^{\infty} C \cdot \min\left\{1, \frac{M^j}{|\xi|}\right\} \cdot L_j A^{-\frac{j}{2}}. \end{aligned}$$

Divide the summation into the cases when $j < J$ and $j \geq J$, in which

$$J = \left\lfloor \frac{\log|\xi|}{M} \right\rfloor.$$

Assume further that

$$0 < \varepsilon \leq \min\left\{\frac{\delta}{2}, \frac{1}{3}\right\}.$$

Then

$$\begin{aligned} & \sum_{j=3}^{\infty} C \cdot \min\left\{1, \frac{M^j}{|\xi|}\right\} \cdot L_j A^{-\frac{j}{2}} \\ = \ & \sum_{j=3}^{J-1} C \cdot \min\left\{1, \frac{M^j}{|\xi|}\right\} \cdot L_j A^{-\frac{j}{2}} + \sum_{j=J}^{\infty} C \cdot \min\left\{1, \frac{M^j}{|\xi|}\right\} \cdot L_j A^{-\frac{j}{2}} \\ \leq \ & C \sum_{j=3}^{J-1} \frac{M^j}{|\xi|} \cdot M^{\frac{j\varepsilon}{2}} A^{-\frac{j}{2}} + C \sum_{j=J}^{\infty} M^{\frac{j\varepsilon}{2}} A^{-\frac{j}{2}} \\ \leq \ & C|\xi|^{-1} \sum_{j=3}^{J-1} M^{\left(1-\frac{\delta-\varepsilon}{2}\right)j} + C \sum_{j=J}^{\infty} M^{-\frac{(\delta-\varepsilon)j}{2}} \\ \leq \ & C|\xi|^{-1} \cdot \frac{M^{\left(1-\frac{\delta-\varepsilon}{2}\right)J}}{M^{1-\frac{(\delta-\varepsilon)}{2}} - 1} + C \cdot \frac{M^{-\frac{(\delta-\varepsilon)J}{2}}}{1 - M^{-\frac{(\delta-\varepsilon)}{2}}} \\ \leq \ & C \cdot \frac{M^{-\frac{(\delta-\varepsilon)J}{2}}}{M^{\frac{1}{2}} - 1} + C \cdot \frac{M^{-\frac{(\delta-\varepsilon)J}{2}}}{1 - M^{-\frac{\delta-\varepsilon}{2}}} \\ \leq \ & \frac{C}{1 - M^{-\frac{\delta}{4}}} \cdot |\xi|^{-\frac{\delta-\varepsilon}{2}} \\ \leq \ & C|\xi|^{-\frac{\delta-\varepsilon}{2}}, \end{aligned}$$

for some absolute constant $C > 0$, provided that $M^{-\frac{\delta}{4}} < \frac{1}{2}$. This can be guaranteed by

$$M \geq e^{-4\delta^{-1}}.$$

In this case, if $|\xi| \geq M^4$, then

$$\begin{aligned} |\mathcal{F}\nu(\xi)| &\leq |\mathcal{F}\nu_1(\xi)| + |\mathcal{F}\nu_2(\xi)| + \sum_{j=2}^{\infty} |\mathcal{F}\nu_j(\xi) - \mathcal{F}\nu_{j-1}(\xi)| \\ &\leq CM^2|\xi|^{-1} + C|\xi|^{-\frac{\delta-\varepsilon}{2}} \\ &\leq C|\xi|^{-\frac{1}{2}} + C|\xi|^{-\frac{\delta-\varepsilon}{2}} \end{aligned}$$

$$\leq \quad C|\xi|^{-\frac{\delta-\varepsilon}{2}},$$

because $0 < \delta < 1$.

Remark (Fourier decay estimates of ν_j). Notice that for each $j \in \mathbb{N}$,

$$|\mathcal{F}\nu_j(\xi)| \leq |\mathcal{F}\nu_1(\xi)| + |\mathcal{F}\nu_2(\xi)| + \sum_{k=3}^{\infty} |\mathcal{F}\nu_k(\xi) - \mathcal{F}\nu_{k-1}(\xi)|.$$

The Fourier decay estimate of the Cantor measure ν in Theorem 1.5 also applies to the Cantor measure ν_j at each iteration $j \in \mathbb{N}$.

Recall that the density function ρ_j of the Cantor measure ν_j is given by

$$\rho_j(x) = \frac{M^j}{A^j}\mathbb{1}_{\mathcal{C}_j}(x) \quad \text{for } x \in \mathbb{R}.$$

It then follows that

$$\mathcal{F}\mathbb{1}_{\mathcal{C}_j}(\xi) = \frac{A^j}{M^j}\mathcal{F}\nu_j(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

Hence, with the same conditions as Theorem 1.5,

$$\left|\mathcal{F}\mathbb{1}_{\mathcal{C}_j}(\xi)\right| \leq C\frac{A^j}{M^j}|\xi|^{-\frac{\delta-\varepsilon}{2}} \quad \text{for all } j \in \mathbb{N} \text{ and } |\xi| \geq M^4. \tag{3.3}$$

4. From Fourier decay to the FUP

In this section, we prove the FUP in Theorems 1.3 and 1.4, following the approach to the FUP by the Fourier decay suggested by Dyatlov [Dy, Section V]. See also Bourgain-Dyatlov [BD1, Section 4], in which they used the (generalized) Fourier decay estimate of the Patterson-Sullivan measures to establish an FUP for the related fractal sets (that is, the limit sets of Schottky groups).

We first prove the FUP in Theorems 1.4 for the random Cantor measures ν:

$$\|T\|_{L^2_\nu(\mathbb{R})\to L^2_\nu(\mathbb{R})} \leq Ch^{\frac{\delta}{4}-\varepsilon},$$

in which

$$Tu(\xi) = \int_{\mathbb{R}} e^{-\frac{ix\cdot\xi}{h}}u(x)\,d\nu(x) \quad \text{for } u \in L^2_\nu(\mathbb{R}).$$

Here, the Cantor set $\mathcal{C}$ is chosen from $\mathbb{G} \subset \mathbb{A}^\infty$ in Theorem 1.5 so that the corresponding Cantor measure ν satisfies the Fourier decay estimate:

$$|\mathcal{F}\nu(\xi)| \leq C|\xi|^{-\frac{\delta-\varepsilon}{2}} \quad \text{for all } |\xi| \geq M^4.$$

Notice that

$$\|T\|_{L^2_\nu(\mathbb{R})\to L^2_\nu(\mathbb{R})} = \|T^\star T\|_{L^2_\nu(\mathbb{R})\to L^2_\nu(\mathbb{R})},$$

in which $T^\star T$ is an integral operator with kernel

$$\mathcal{K}_\nu(\xi,\eta) = \int_{\mathbb{R}} e^{\frac{i(\xi-\eta)x}{h}}\,d\nu(x) = \sqrt{2\pi}\cdot\mathcal{F}\nu\left(\frac{\eta-\xi}{h}\right).$$

The Fourier decay estimate of ν in Theorem 1.5 implies that if $|\frac{\eta-\xi}{h}| \geq M^4$, then

$$|\mathcal{K}_\nu(\xi,\eta)| = \sqrt{2\pi}\cdot\left|\mathcal{F}\nu\left(\frac{\eta-\xi}{h}\right)\right| \leq C\left|\frac{\eta-\xi}{h}\right|^{-\frac{\delta-\varepsilon}{2}} = Ch^{\frac{\delta-\varepsilon}{2}}|\eta-\xi|^{-\frac{\delta-\varepsilon}{2}}.$$

Here, $C > 0$ is an absolute constant. Whereas if $|\frac{\eta-\xi}{h}| \le M^4$, then the trivial estimate holds:

$$|\mathcal{K}_\nu(\xi,\eta)| = \sqrt{2\pi}\cdot\left|\mathcal{F}\nu\left(\frac{\eta-\xi}{h}\right)\right| \le C.$$

By Schur's test,

$$\begin{aligned}
&\|T^\star T\|_{L^2_\nu(\mathbb{R})\to L^2_\nu(\mathbb{R})}\\
\le\ & \sqrt{\sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\eta)\cdot\sup_{\eta\in\mathbb{R}}\int_{\mathbb{R}}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\xi)}\\
=\ & \sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\eta).
\end{aligned}$$

Since ν is supported on $[0,1]$, divide the integral to the ones on dyadic intervals of the form

$$\left[\xi+2^{-j},\xi+2^{-j+1}\right]\quad\text{with } j=1,...,J,$$

in which J is the largest integer such that $2^{-J}\ge M^4h$. Then

$$\left\lfloor\frac{|\log(M^4h)|}{\log 2}\right\rfloor \le J \le \left\lfloor\frac{|\log(M^4h)|}{\log 2}\right\rfloor + 1.$$

For each $I\subset\mathbb{R}$, $\nu(I)\le C|I|^\delta$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\eta)\\
\le\ & \int_{[\xi,\xi+M^4h]}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\eta) + \sum_{j=1}^{J}\int_{[\xi+2^{-j},\xi+2^{-j+1}]}|\mathcal{K}_\nu(\xi,\eta)|\ d\nu(\eta)\\
\le\ & C\left[\nu\left(\left[\xi,\xi+M^4h\right]\right)+\sum_{j=1}^{J}h^{\frac{\delta-\varepsilon}{2}}\left(2^{-j}\right)^{-\frac{\delta-\varepsilon}{2}}\cdot\nu\left(\left[\xi+2^{-j},\xi+2^{-j+1}\right]\right)\right]\\
\le\ & C\left[\left(M^4h\right)^\delta+\sum_{j=1}^{J}h^{\frac{\delta-\varepsilon}{2}}\left(2^{-j}\right)^{-\frac{\delta-\varepsilon}{2}}\cdot\left(2^{-j}\right)^\delta\right]\\
\le\ & Ch^{\frac{\delta-\varepsilon}{2}}|\log h|\\
\le\ & Ch^{\frac{\delta}{2}-2\varepsilon},
\end{aligned}$$

provided that $h\le M^{-8}$ (so $(M^4h)^\delta\le h^{\frac{\delta}{2}}\le h^{\frac{\delta-\varepsilon}{2}}$).

We next prove the FUP in Theorems 1.4 for the h-neighborhood $\mathcal{C}(h)$ of the random Cantor set $\mathcal{C}$:

$$\left\|\mathbb{1}_{\mathcal{C}(h)}\mathcal{F}_h\mathbb{1}_{\mathcal{C}(h)}\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|\mathcal{F}_h\|_{L^2(\mathcal{C}(h))\to L^2(\mathcal{C}(h))} = \left\|\mathcal{F}_h^\star\mathbb{1}_{\mathcal{C}(h)}\mathcal{F}_h\right\|_{L^2(\mathcal{C}(h))\to L^2(\mathcal{C}(h))},$$

in which $\mathcal{F}_h^\star\mathbb{1}_{\mathcal{C}(h)}\mathcal{F}_h$ is an integral operator with kernel

$$\mathcal{K}_L(\xi,\eta) = \frac{1}{2\pi h}\int_{\mathcal{C}(h)}e^{\frac{i(\xi-\eta)x}{h}}\,dx = \frac{1}{\sqrt{2\pi}h}\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}\left(\frac{\eta-\xi}{h}\right).$$

Here, the Cantor set $\mathcal{C}$ is chosen from $\mathbb{G}\subset\mathbb{A}^\infty$ so that the corresponding Cantor measure ν satisfies the Fourier decay estimate in Theorem 1.5. By (3.3), for the Cantor set $\mathcal{C}_j$ at each iteration $j\in\mathbb{N}$,

$$\left|\mathcal{F}\mathbb{1}_{\mathcal{C}_j}(\xi)\right| \le C\frac{A^j}{M^j}|\xi|^{-\frac{\delta-\varepsilon}{2}}\quad\text{for all } |\xi|\ge M^4.$$

Recall that $\mathcal{C}_j$ at the j-th iteration is a union of A^j intervals of equal length M^{-j}. Let J is the largest integral such that $M^{-J} \geq h$. Then

$$\left\lfloor \frac{|\log h|}{\log M} \right\rfloor \leq J \leq \left\lfloor \frac{|\log h|}{\log M} \right\rfloor + 1.$$

We compare the Fourier transform of $\mathbb{1}_{\mathcal{C}(h)}$ and of $\mathbb{1}_{\mathcal{C}_J}$. Notice that

$$|\mathcal{C}(h) \triangle \mathcal{C}_j| = |\mathcal{C}(h) \setminus \mathcal{C}_J| + |\mathcal{C}_J \setminus \mathcal{C}(h)| \leq CA^J h \leq CM^{\delta J} h \leq Ch^{1-\delta}.$$

Then for all $\eta \in \mathbb{R}$,

$$\left|\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}(\xi) - \mathcal{F}\mathbb{1}_{\mathcal{C}_J}(\xi)\right| \leq C \cdot |\mathcal{C}(h) \triangle \mathcal{C}_j| \leq Ch^{1-\delta}.$$

Therefore, for all $|\xi| \geq M^4$,

$$\begin{aligned} \left|\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}(\xi)\right| &\leq \left|\mathcal{F}\mathbb{1}_{\mathcal{C}_j}(\xi)\right| + \left|\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}(\xi) - \mathcal{F}\mathbb{1}_{\mathcal{C}_J}(\xi)\right| \\ &\leq Ch^{1-\delta}|\xi|^{-\frac{\delta-\varepsilon}{2}} + Ch^{1-\delta} \\ &\leq Ch^{1-\delta}|\xi|^{-\frac{\delta-\varepsilon}{2}} \end{aligned}$$

As a consequence, if $|\eta - \xi| \geq M^4 h$, then

$$|\mathcal{K}_L(\xi, \eta)| = \frac{1}{\sqrt{2\pi} h} \left|\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}\left(\frac{\eta - \xi}{h}\right)\right| \leq Ch^{-\frac{\delta+\varepsilon}{2}} |\eta - \xi|^{-\frac{\delta-\varepsilon}{2}}.$$

Whereas if $|\frac{\eta-\xi}{h}| \leq M^4$, then the trivial estimate holds:

$$|\mathcal{K}_L(\xi, \eta)| = \frac{1}{\sqrt{2\pi} h} \cdot \left|\mathcal{F}\mathbb{1}_{\mathcal{C}(h)}\left(\frac{\eta - \xi}{h}\right)\right| \leq Ch^{-\delta}.$$

Here, we used the fact that $|\mathcal{C}(h)| \leq Ch^{1-\delta}$.

With this finite scale version of the Fourier decay estimate, we repeat the same process as above: By Schur's test,

$$\begin{aligned} &\|\mathcal{F}_h^\star \mathcal{F}_h\|_{L^2(\mathcal{C}(h)) \to L^2(\mathcal{C}(h))} \\ \leq\ & \sqrt{\sup_{\xi \in \mathcal{C}(h)} \int_{\mathcal{C}(h)} |\mathcal{K}_L(\xi, \eta)|\, d\eta \cdot \sup_{\eta \in \mathcal{C}(h)} \int_{\mathcal{C}(h)} |\mathcal{K}_L(\xi, \eta)|\, d\xi} \\ =\ & \sup_{\xi \in \mathcal{C}(h)} \int_{\mathcal{C}(h)} |\mathcal{K}_L(\xi, \eta)|\, d\eta. \end{aligned}$$

Divide the integral to the ones on dyadic intervals of the form

$$\left[\xi + 2^{-j}, \xi + 2^{-j+1}\right] \quad \text{with } j = 1, ..., J,$$

in which J is the largest integral such that $2^{-J} \geq M^4 h$. Then

$$\left\lfloor \frac{\log(M^4 h)}{\log 2} \right\rfloor \leq J \leq \left\lfloor \frac{\log(M^4 h)}{\log 2} \right\rfloor + 1.$$

For each dyadic interval I above, $|\mathcal{C}(h) \cap I| \leq Ch^{1-\delta}|I|^\delta$. Hence,

$$\begin{aligned} &\int_{\mathcal{C}(h)} |\mathcal{K}_h(\xi, \eta)|\, d\eta \\ \leq\ & \int_{\mathcal{C}(h) \cap [\xi, \xi + M^4 h]} |\mathcal{K}_h(\xi, \eta)|\, d\eta + \sum_{j=1}^{J} \int_{\mathcal{C}(h) \cap [\xi + 2^{-j}, \xi + 2^{-j+1}]} |\mathcal{K}_h(\xi, \eta)|\, d\eta \end{aligned}$$

$$\leq \ Ch^{1-\delta}\left[\left(M^4h\right)^{\delta}\cdot h^{-\delta}+\sum_{j=1}^{J}\left(2^{-j}\right)^{\delta}\cdot h^{-\frac{\delta+\varepsilon}{2}}\left(2^{-j}\right)^{-\frac{\delta-\varepsilon}{2}}\right]$$
$$\leq \ Ch^{1-\frac{3\delta}{2}-\frac{\varepsilon}{2}}|\log h|$$
$$\leq \ Ch^{1-\frac{3\delta}{2}-2\varepsilon},$$

provided that $h\leq M^{-8}$ (so $M^{4\delta}\leq h^{-\frac{\delta}{2}}\leq h^{-\frac{\delta+\varepsilon}{2}}$).

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