Entanglement in Quantum Dots: Insights from Dynamic Susceptibility and Quantum Fisher Information

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Abstract

This study investigates the entanglement properties of quantum dots (QDs) under a universal Hamiltonian where the Coulomb interaction between particles (electrons or holes) decouples into a charging energy and an exchange coupling term. While this formalism typically decouples the charge and spin components, the confinement-induced energy splitting can induce unexpected entanglement in the system. By analyzing the dynamic susceptibility and quantum Fisher information (QFI), we uncover intriguing behaviors influenced by exchange constants, temperature variations, and confinement effects. In Ising QDs, far below the Stoner instability point where the QD is in a disordered paramagnetic phase, temperature reductions unexpectedly lead to decreased entanglement, challenging conventional expectations. Conversely, anisotropic Heisenberg models exhibit enhanced entanglement near isotropic points. Our findings highlight the intricate interplay between exchange interactions and entanglement in QDs, laying the groundwork for future studies on topological entanglement and the influence of entanglement on material properties. Overall, this work contributes to advancing our understanding of entanglement in QDs and its potential applications in quantum technologies.

Keywords: Quantum dots, Etranglement, Exchange interactions, Quantum Fisher Information, Stoner Instability

Introduction

Quantum entanglement, a fundamental concept in quantum physics, serves as a key to unlocking the mysteries of correlations that extend beyond classical physics, propelling advancements in quantum technologies. Exploring entanglement not only enriches our understanding of quantum many-body physics but also drives the development of experimental protocols aimed at detecting and quantifying entanglement in many-body quantum states. Such endeavors are crucial for leveraging entanglement as a resource in quantum computing, communication, and metrology.

Recent studies have made significant strides by introducing a method to measure multipartite entanglement in thermal ensembles through dynamic susceptibility. This technique finds immediate

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applicability with existing experimental setups in cold atomic gases and condensed matter systems.^{2,3} By enabling direct measurement of multipartite entanglement, essential for advancing quantum technologies, this approach also establishes a vital link between multipartite entanglement and observable many-body correlations in experiments.

The elusive nature of entanglement in large quantum systems poses a challenge for its direct measurement, particularly in characterizing quantum phases and transitions. Traditional methods like tomography face limitations due to exponential resource scaling, making them impractical for systems beyond a few qubits.^{4,5} In contrast, quantum Fisher information (QFI), a concept from quantum metrology, can serve as an indicator for multipartite entanglement. QFI, which quantifies the maximum precision achievable in parameter estimation, becomes directly measurable through dynamic susceptibilities.^{5,6} This approach enables the direct measurement of multipartite entanglement in quantum systems.

Quantum dots (QDs), with their discrete energy levels and tunable properties, offer a fertile ground for exploring quantum phenomena at the nanoscale. Among these phenomena, the interplay between quantum entanglement and exchange interactions, both anisotropic Ising and isotropic Heisenberg, presents a rich area of study, especially concerning the Stoner instability (SI). Understanding entanglement between spin and charge degrees of freedom in QDs is vital for advancing quantum information technologies.

The exchange interactions within QDs exhibit notable distinctions in the presence and absence of SOC, with Ising exchange predominating in the presence of SOC and Heisenberg exchange prevailing in its absence.⁷ Additionally, it is essential to recognize that holes and electrons behave distinctively in QDs due to their differing interactions with SOC: holes are typically influenced by SOC, whereas electrons are not. This differentiation underscores the significance of comprehending the unique behaviors of electrons and holes in QDs, thereby enriching our insight into spin-charge entanglement within these systems.

Entanglement, a fundamental concept in quantum physics whereby particles become interconnected to the extent that the overall state of the system cannot be expressed as the product of individual particle states, serves as an essential resource for quantum computation, communication, and metrology. Notably, the capability to generate and manipulate entangled states among various types of qubits, such as spin and charge qubits, paves the way for novel quantum operations and augments the functionality of quantum devices.

The Stoner instability, a quantum phase transition from a paramagnetic to a ferromagnetic state, is a cornerstone concept for understanding magnetism in QDs, with practical implications for their electronic and magnetic properties. The tunneling density of states in QDs and nanoparticles has been reevaluated with the extension of the universal Hamiltonian to include uniaxial anisotropic exchange, shedding light on the systems' behavior near the SI point. The significance of the ground state's total spin in determining the tunneling density of states was highlighted, revealing a single maximum in its energy dependence, which contrasts with previous perturbative analyses predicting multiple extrema.⁷

Dynamic (or AC) susceptibility measurements offer a non-invasive means to probe the dynamic response of a system to an external alternating magnetic field, providing insights into intrinsic properties like magnetic susceptibility and dielectric permittivity. In the context of quantum dots, dynamic susceptibility serves as a proxy to analyze the intricate dynamics of spin and charge entanglement.^{8,9} By examining the response of quantum dots to varying frequencies and amplitudes

of an AC field, researchers can infer the degree of entanglement and coherence between qubits, essential for implementing quantum gates and algorithms.

This paper highlights the methodological importance within QD systems, holding significant experimental implications. It provides a comprehensive analysis of spin-exchange interactions within QDs, contributing to a broader understanding of charge and spin entanglement in these systems. Various exchange interactions, including Ising and Heisenberg models, are considered, covering both isotropic and anisotropic scenarios, with analyses spanning both high and low-temperature extremes. Such comparisons deepen understanding of the mesoscopic SI and the effects of exchange interactions on the quantum states of QDs, particularly in the context of entanglement, where exchange interactions play a crucial role in entangling the spins of confined electrons or holes within QDs. The distinction between Heisenberg and Ising exchange interactions introduces complexity in understanding the quantum behavior of QDs, especially with the presence of SOC, which can significantly affect the quantum states and entanglement properties of the system.

The study employs the universal Hamiltonian approach to simplify the intricate electronelectron interactions within QDs, focusing on zero-mode interactions that encompass charging and spin-exchange terms. While this formalism effectively decouples the charge and spin components, our analysis reveals that the confinement-induced energy splitting unexpectedly induces entanglement in the system. However, it is crucial to acknowledge that strong SOC poses a challenge to the applicability of the universal Hamiltonian to QDs. Under strong SOC conditions, fluctuations in the interaction matrix elements become significant, particularly in the metallic regime where the level spacing (Δ) relative to the Thouless energy (E_{Th}) is very small (i.e., $\Delta/E_{Th} \ll 1$). This complexity suggests that the universal Hamiltonian may not fully capture the behavior of QDs in the presence of strong SOC, indicating the need for a more detailed approach to understand these systems.^{10,11} Furthermore, it is worth noting that this regime is not suitable for a single QD qubit characterized by one or a small number of electrons. Additionally, the paper explores the potential experimental determination of entanglement, a critical parameter for understanding and manipulating QD dynamics.

By comparing Ising and Heisenberg exchange interactions, both anisotropic and isotropic, the study provides a clearer picture of entanglements in each case, advancing theoretical models and suggesting experimental pathways for probing the intricate dynamics of charge and spin within quantum dots. These findings have implications for the development of quantum computing and spintronic devices, where precise control and understanding of spin and charge are essential. The linkage between QFI and dynamic susceptibilities not only facilitates experimental identification of entanglement but also deepens theoretical insight into quantum fluctuations and their influence on the sensitivity of quantum states to external disturbances.

Fundamental Formalism

The anisotropic exchange constant and non-zero dynamical susceptibility are essential parameters in characterizing QD systems. The anisotropic exchange constant describes the interactions among electron spins within QDs, which vary based on their relative orientations. This parameter significantly influences spin dynamics within the QD, playing a key role in quantum information processing applications.

Similarly, the non-zero dynamical susceptibility reveals how the QD responds to an externally applied field, with its sensitivity dependent on the field's frequency. Understanding this parameter is

crucial for devising effective strategies in device engineering and manipulating quantum states within the system.

Magnetic susceptibility, a fundamental property of systems, measures their responsiveness to applied magnetic fields, typically manifested through magnetization fluctuations. This property is intimately linked to spontaneous magnetization within the system.

In paramagnetic systems, devoid of spontaneous magnetization, magnetic susceptibility denotes the magnetization induced by an external magnetic field, expressed as the proportionality constant relating induced magnetization (M) to applied magnetic field strength (H). This relationship is often represented as $M = \chi H$, where χ represents the magnetic susceptibility.

For a time-varying external magnetic field described by $\vec{h}(t) = \vec{h}_0 e^{-i\omega t}$, where h_0 denotes the field magnitude, ω signifies the angular frequency, and 'i' denotes the imaginary unit, the system's response evolves temporally. Consequently, the concept of dynamical susceptibility emerges, capturing the system's response as a function of the frequency (ω) of external fields.

Understanding magnetic susceptibility is crucial in studying quantum systems like QDs, especially concerning gate-defined QD qubits. In these systems, magnetization $(M_{\alpha}(t))$ is fundamental and is expressed as:

$$M_{\alpha}(t) = \langle S_{\alpha}(t) \rangle = \chi_{\alpha\beta} h_{\beta}(t).$$
 (1)

Here, the indices α and β represent the x, y, z dimensions, and magnetization corresponds to the correlation function:

$$M_{\alpha}(t) = -i \int_{-\infty}^{t} dt' \langle [S_{\alpha}(t), V(t')] \rangle.$$
 (2)

In this equation, $V = -\vec{S} \cdot \vec{h}(t)$ represents the Hamiltonian governing the interaction of electron spin with the external magnetic field. By utilizing equations (1) and (2), the magnetic susceptibility can be expressed as:¹²

$$\chi_{\alpha\beta}(\omega) = -i \int_{-\infty}^{t} dt' e^{i\omega(t-t')} \langle [S_{\alpha}(t), S_{\beta}(t')] \rangle. \tag{3}$$

Equation (3) denotes the Fourier transform of a retarded correlation function. Retarded correlation functions can be derived from Matsubara functions via the analytic continuation $i\omega_n \to \omega + i\delta$, where δ is infinitesimal. Expressed in terms of the Matsubara correlation function, the magnetic susceptibility becomes:

$$\chi_{\alpha\beta}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau S_\alpha(t) S_\beta(0) \rangle. \quad (4)$$

Here, $\omega_n = (2n+1)\pi k_B T$, where T represents temperature and constitutes the poles of the fermionic distribution function. T_τ is the time-ordering operator.

Furthermore, the correlation function can be evaluated by defining the wave-vector dependent susceptibility: 12

$$\chi_{\alpha\beta}(\vec{q}, i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau S_\alpha(\vec{q}, \tau) S_\beta(-\vec{q}, 0) \rangle. \tag{5}$$

The spin operators are typically expressed in terms of electron creation and annihilation operators:

$$S_{\alpha}(\vec{q}) = \frac{1}{2} \sum_{\vec{p}\sigma\sigma}, a^{\dagger}_{\vec{p}+\vec{q}\sigma} \sigma^{\alpha}_{\sigma\sigma}, a_{\vec{p}\sigma}, . \quad (6)$$

Here, σ^{α} (α =x,y,z) represents the Pauli spin matrices. The dynamic longitudinal and transverse susceptibilities can be calculated as follows:¹²

$$\begin{cases} \chi_{zz}(\vec{q}, i\omega_n) = \frac{1}{4} \sum_{\vec{p}\vec{k}ss'} ss' \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau a^{\dagger}_{\vec{p}+\vec{q}s}(\tau) a_{\vec{p}s}(\tau) a^{\dagger}_{\vec{k}-\vec{q}s'}(0) a_{\vec{k}s'}(0) \rangle, \\ \chi_{+-}(\vec{q}, i\omega_n) = \sum_{\vec{p}\vec{k}} \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau a^{\dagger}_{\vec{p}+\vec{q}\uparrow}(\tau) a_{\vec{p}\downarrow}(\tau) a^{\dagger}_{\vec{k}-\vec{q}\downarrow}(0) a_{\vec{k}\uparrow}(0) \rangle, \\ \chi_{-+}(\vec{q}, i\omega_n) = \sum_{\vec{p}\vec{k}} \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau a^{\dagger}_{\vec{p}+\vec{q}\downarrow}(\tau) a_{\vec{p}\uparrow}(\tau) a^{\dagger}_{\vec{k}-\vec{q}\uparrow}(0) a_{\vec{k}\downarrow}(0) \rangle. \end{cases}$$
(7)

The crucial step in calculating the magnetic susceptibility is to derive an expression for the bracket in equation (7). For a noninteracting free electron gas, these operators can be paired using Wick's theorem. To obtain both the dynamic and static susceptibilities, a single-particle Green's function is defined as:

$$G_s^{(0)}(\vec{p},\tau) = -\langle T_\tau a_{\vec{p}s}(\tau) a_{\vec{p}s}^\dagger(0) \rangle, \quad (8)$$

where $s = \uparrow, \downarrow$. This function can be expanded into a Fourier series:

$$G_s^{(0)}(\vec{p},\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} G_s^{(0)}(\vec{p},i\omega_n).$$
 (9)

Here, $G_s^{(0)}(\vec{p}, i\omega_n) = (i\omega_n - \xi_{\vec{p}})^{-1}$ with $\xi_{\vec{p}} = \varepsilon_{\vec{p}} - \mu$, where $\varepsilon_{\vec{p}}$ is the single-particle energy and μ is the chemical potential. In terms of single-particle Green's function the transverse dynamic susceptibility of a noninteracting electron gas is given by:¹²

$$\chi_{+-}^{(0)}(\vec{q}, i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \sum_{\vec{k}} G_{\uparrow}^{(0)}(\vec{k} + \vec{q}, -\tau) G_{\downarrow}^{(0)}(\vec{k}, \tau) = \frac{1}{N} \sum_{\vec{k}} \frac{n_F(\xi_{\vec{k}}) - n_F(\xi_{\vec{k}+\vec{q}})}{i\omega_n + \xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}, \quad (10)$$

where $n_{\rm F}$ is the Fermi distribution function, and the longitudinal susceptibility is given in terms of transverse as $\chi_{+-}^{(0)}(\vec{q},i\omega_n)=2\chi_{zz}^{(0)}(\vec{q},i\omega_n)$.

The static susceptibility is given by $\chi = \lim_{\vec{q} \to 0} \chi_{zz}(\vec{q}, i\omega_n = 0)$. For noninteracting systems, the static susceptibility reduces to $\chi^{(0)} = \mu_B^2 N_F$, where μ_B is the Bohr magneton and N_F is the electron density of state at the Fermi energy.

Electron-Electron Interaction and Universal Hamiltonian in QDs

In the metallic regime characterized by high Thouless conductance ($g_T \gg 1$), the behavior of QDs is fully described by a universal Hamiltonian. This Hamiltonian encapsulates the electron-electron interaction within the QD and comprises three spatially independent components: the charging energy, the ferromagnetic exchange coupling, and the Cooper channel interaction.¹³ In the presence of the first two terms, the universal Hamiltonian for QDs is expressed as follows:

$$H = H_0 + H_C + H_S,$$

$$H_0 = \sum_{\alpha\sigma} \epsilon_{\alpha} a_{\alpha\sigma}^{\dagger} a_{\alpha\sigma}, \ H_C = E_C (\hat{n} - N_0)^2, \ H_S = -J_{\perp} (S_x^2 + S_y^2) - J_z S_z^2.$$
 (11)

Here, H_0 represents the non-interacting Hamiltonian, where $a_{\alpha\sigma}^{\dagger}$ is the electron creation operator for an orbital level α and spin $\frac{\sigma}{2}(\sigma=\pm1)$, and ϵ_{α} denotes spin-degenerate single-particle levels. H_C stands for the charging interaction, with $\hat{n}=\sum_{\alpha}\hat{n}_{\alpha}=\sum_{\alpha,\,\sigma=\uparrow,\downarrow}a_{\alpha\sigma}^{\dagger}a_{\alpha\sigma}$ representing the number operator. E_C represents the charging energy, and N_0 represents a positive background

charge. The last term in equation (11), where $S_{i(=x,y,z)} = \sum_{\alpha} S_{i\alpha}$, $S_{i\alpha} = \frac{1}{2} \sum_{\sigma\sigma'} a_{\alpha\sigma'}^{\dagger} a_{\alpha\sigma'}$, encapsulates exchange interactions within the QD, with σ representing the Pauli vector matrix, and J_z and J_{\perp} denoting the exchange interaction constants.

It is well-recognized that in the presence of ferromagnetic exchange interaction, bulk systems may undergo a Stoner transition from a paramagnetic to a ferromagnetic phase.¹³ However, in finite systems like QDs, an intermediate phase may emerge depending on the strength of the exchange constants, sometimes between the paramagnetic and ferromagnetic phases. In QDs with isotropic Heisenberg exchange constant (i.e., when $J_z = J_{\perp} = J$), three different regimes are observed:¹³

- 1. At small values of $J \ll \Delta$, satisfying $g_T = E_T/\Delta \gg 1$, the total spin of the QD is zero, and the QD is in the disordered paramagnetic phase.
- 2. As J increases, the ferromagnetic exchange coupling attempts to align the spins of electrons, and the competition between the exchange coupling and the kinetic energy leads to an intermediate partially polarized phase in which the total spin of the QD becomes nonzero. This intermediate regime occurs in the interval $\Delta/2 \leq J < \Delta$, and the total spin of the QD increases monotonically with J as $S = J/[2(\Delta J)]$. This intermediate regime is referred to as the mesoscopic Stoner regime, which disappears as $\Delta \to 0$.
- 3. By further increasing J towards Δ , the system approaches the thermodynamic SI point, and the QD undergoes a second-order phase transition to a spin-polarized ferromagnetic phase. In the mesoscopic Stoner regime, the total spin is finite but not dependent on the volume of the system. However, in the ferromagnetic phase, all electrons become fully spin-polarized, and the magnetization is proportional to the volume of the QD.

For QDs with anisotropic exchange Hamiltonian $(J_z \neq J_\perp)$, the mesoscopic Stoner regime emerges in the interval $\Delta - J_z \ll J_\perp < J_z$. In this regime, the total spin is approximately given by ¹⁶: $S \cong (J_\perp + J_z - \Delta)/[2(\Delta - J_z)]$

However, for the case of an Ising exchange Hamiltonian, the mesoscopic Stoner regime vanishes.¹³ In this scenario, if $J_z < \Delta$, the QD resides in the disordered paramagnetic phase with S = 0. Conversely, as J_z increases towards Δ , the paramagnetic phase undergoes an abrupt transition to the fully polarized ferromagnetic phase at $J = \Delta$.

Dynamic Transverse Susceptibility Analysis in QDs with Ising Exchange Hamiltonian

An exact solution for QDs with Ising exchange constant $-J_zS_z^2$, $(J_\perp=0)$ has been presented in [14] in the Coulomb blockade (CB) regime. In their work, I. S. Burmistrov et al. considered the spin-disordered regime below the SI, wherein the parameters of the Hamiltonian (11) satisfy $J_z < \Delta \ll T \ll E_c$. The condition $E_c \gg \Delta$ is met in relatively large QDs, as it scales with size L as $1/L^2$, while the charging energy scales as $E_c \sim 1/L$. For a two-dimensional QD, $E_c \gg \Delta$ is equivalent to $L \gg a_B$, where a_B represents the effective Bohr radius for electrons in the dot.

In the Ising model, the absence of spin-flip processes results in the dynamic longitudinal susceptibility vanishing, leaving only the static component non-zero. To calculate the dynamic transverse susceptibility, as outlined in equation (7), we must evaluate the average

 $\langle T_{\tau} a_{\vec{p}+\vec{q}\uparrow}^{\dagger}(\tau) a_{\vec{p}\downarrow}(\tau) a_{\vec{k}-\vec{q}\downarrow}^{\dagger}(0) a_{\vec{k}\uparrow}(0) \rangle$ using the Hamiltonian defined by H. In the CB regime the dynamic transverse susceptibility in terms of imaginary time is:¹⁵

$$\chi^{+-}(\tau) = \frac{\beta e^{J_z \tau}}{\tilde{Z}(\mu)} \sum_{N} e^{-\beta E_c (N - \tilde{N}_0)^2} \sum_{M = -N}^{N} \{ e^{-\frac{\beta}{4} (\Delta - J_z) M^2} e^{J_z \tau M} \sum_{\alpha} [1 - n_{\alpha}(\bar{\mu}_{\uparrow})] n_{\alpha}(\bar{\mu}_{\downarrow}) \}$$
(12)

where $\beta = \frac{1}{T}$ ($k_{\rm B} = 1$) is temperature inverse. In this equation, $\bar{\mu}_{\sigma} \equiv N_{\sigma} \Delta$, and N_{σ} is the total number of electrons with spin $\sigma = \uparrow, \downarrow$. $\tilde{N}_0 \equiv N_0 + \frac{\mu}{2E_c}$, and $\tilde{Z}(\mu)$ denotes the grand canonical partition function. The double summation above arises from replacing the grand canonical partition function in terms of the sum over canonical ones. The summation parameters are the electrons number, N and the total spin of the dot (in units of $\hbar/2$), M. Below the SI point, in the disordered regime no symmetry breaking happens in the system to distinguish opposite spin polarization. In this regime, in the high temperatures limit (i.e., when the QD contains many electrons, the following conditions; $N\Delta \gg T$ and $(N-|M|)\Delta \gg T$ are satisfied, and therefore the sum over M in Eq. Error! Reference source not found. can be replaced by an integral from $-\infty$ to $+\infty$. After Fourier-transforming $\chi^{+-}(\tau)$ to Matsubara frequencies and performing analytic continuation, the imaginary part of the physical response function $\chi^{+-}(\omega)$ is acquired as:

Im
$$\chi^{+-}(\omega) = \frac{\sqrt{\pi\beta(\Delta - J_z)}}{2J_z} e^{\frac{\beta}{4} \left[\Delta + J_z - (\Delta - J_z) \left(\frac{\omega}{J_z}\right)^2\right]} (1 + \omega/J_z) \frac{\sinh\left[\beta\omega/2\right]}{\sinh\left[\frac{\beta\Delta}{2}(1 + \omega/J_z)\right]}$$
. (13)

The imaginary component of $\chi^{+-}(\omega)$ signifies the system's capacity to absorb and dissipate magnetic energy at a nonzero frequency, offering a vital understanding of the system's dynamics. It's evident that the transverse susceptibility remains unaffected by the charging interaction in the QD. Indeed, under the specified conditions, the charge and spin degrees of freedom are effectively decoupled.

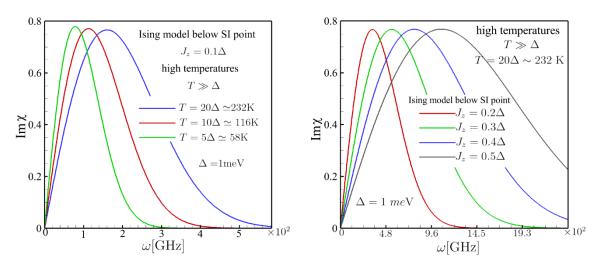


Figure 1: Comparison of the imaginary component of the dynamic transverse susceptibility as a function of frequency (ω): depicting variations across a range of temperatures (left graph) and different values of the exchange constant (right graph).

Figure 1 illustrates the plot of the function $\text{Im}\chi^{+-}$ with respect to frequency (ω) . The left plot corresponds to $J_z = 0.1\Delta$ for various temperature values that satisfy $\frac{\Delta}{T} \ll 1$, while the right plot is configured at $T = 20\Delta$ for different J_z values less than Δ . The characteristic curve of $\text{Im}\chi^{+-}$ starts

from zero at ω =0, increases linearly at small ω values, reaches a maximum at $\omega_0 \approx \sqrt{\frac{2TJ^2}{(\Delta - J)}}$, and eventually decreases for larger ω values.

The location of the maximum primarily depends on the strength of the coupling constant, while its value remains unaffected. Furthermore, the peak value weakly depends on the QD temperature, but the peak location increases with the rising QD temperature. The imaginary part of the susceptibility accurately reflects the QD's capability to absorb and dissipate magnetic energy at a non-zero frequency. Lowering the temperature reduces the maximum magnetic energy absorbed or dissipated by the QD, primarily due to the decrease in thermal fluctuations.

The full width of susceptibility at half-maximum is directly related to the resonance frequency, as demonstrated by the relation $\omega=1.59\omega_0$. The width of the susceptibility is proportional to magnetization fluctuations. As J_z increases towards Δ , approaching the SI point, fluctuations in magnetization increase. Consequently, the susceptibility curve becomes broader. Moreover, decreasing the temperature reduces magnetization fluctuations, resulting in a narrower susceptibility curve.

The resonance frequency is on the order of 100 GHz, which can be challenging to access in experiments. However, by applying an external dc magnetic field, Zeeman splitting is induced in the degenerate single-electron energy levels. In this case, it is possible to observe the aforementioned nontrivial behavior of the dynamic transverse spin susceptibility at lower frequencies, accessible in research laboratories.

Regarding the static susceptibility, only the real part is finite. A straightforward calculation yields:

$$Re\chi^{+-}(0) = \frac{e^{\frac{1}{4T}(\Delta + J_z)}}{\Delta}$$
. (14)

It's worth noting that for $J_z = 0$, we revert to the well-known identity $\chi^{+-}(0) = 2\chi^{zz}$ for static susceptibilities. The real part of χ^{+-} at finite frequencies can be obtained either directly or through the application of Kramers-Kronig relations. However, the result is not discussed here due to its minor physical relevance.

An important point to mention is that the dynamic transverse magnetic susceptibility in Eq. **Error! Reference source not found.** does not depend on the charging energy E_C , and is solely a function of the exchange constant J_z and the mean level spacing Δ . This implies that the chargespin coupling in QD is very weak in the limit $J_z < \Delta \ll T \ll E_c$, under the conditions $N\Delta \gg 1$ and $(N - |M|)\Delta \gg 1$.

If the exchange interaction of electrons in the QD follows the isotropic Heisenberg Hamiltonian, resulting in isotropic exchange interaction $(J_z = J_\perp)$, the dynamical spin response $\chi(\omega \neq 0)$ does not exist unless the dot is connected to reservoirs. Actually, the dynamical susceptibility of QDs with the isotropic Heisenberg model $(J_z = J_\perp)$ is fully captured by magnetization M as $\text{Im}\chi(\omega) = 2\pi M\delta(\omega)$, where $\delta(\omega)$ is the Dirac delta function. Therefore, as long as the magnetization is zero, the susceptibility is also zero. As we discussed, far below the SI point, in the disordered paramagnetic phase, the system exhibits no magnetization in the absence of an external uniform magnetic field. In the presence of an external magnetic field b, the

dynamical susceptibility is given by $2\pi M\delta(\omega - b)$, where nonzero magnetization results in a finite dynamical susceptibility.¹⁶

Spin-Spin Entanglement in QDs Using Quantum Fisher Information

The Quantum Fisher Information (QFI) acts as a quantifiable measure for multipartite entanglement, discerning the entanglement intricacies embedded within the many-body quantum correlations inherent in dynamic susceptibility. This relationship between the QFI and $Im\chi$ is mathematically expressed as:¹

QFI =
$$\frac{4}{\pi} \int_0^\infty d\omega \tanh \left[\frac{\beta \omega}{2} \right] \text{Im} \chi$$
. (15)

By substituting equation **Error! Reference source not found.** into integral (15), we can effectively derive the QFI, thereby illuminating the entanglement originating from dynamic transverse spin correlations within the QD.

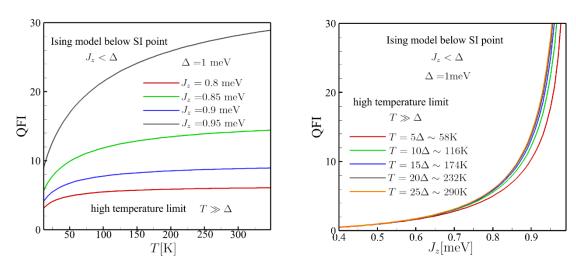


Figure 2: Quantum Fisher Information (QFI) in a QD with Ising exchange Hamiltonian plotted as a function of temperature for various exchange constant strengths, situated far from the SI point (left), and as a function of the exchange interaction constant J_z across different temperatures (right).

Figure 2 depicts the QFI as a function of temperature for various strengths of the exchange constant (on the left) and as a function of J_z for different temperatures (on the right). At elevated temperatures, the QFI exhibits significant values, indicating the presence of robust entanglement among Ising spins within the QD.

As the temperature decreases, the QFI diminishes, eventually reaching zero at sufficiently low temperatures. This behavior is intriguing since, typically in a quantum spin system, lowering the temperature fosters entanglement between spins. However, in the case of the QD, we observe an inverse effect.

Furthermore, our QD employs the classical Ising model to represent spin interactions. In a many-body classical Ising model, both the ground state and all excited states are product states with zero entanglement. Nevertheless, in our QD scenario, despite the Ising nature of the spin interaction, the spin entanglement persists due to the influence of other factors introduced in Hamiltonian (11). Far below the SI point, i.e., when $J_z \ll \Delta$, the spin-charge coupling is exceedingly weak, and the QFI encoded in the dynamical susceptibility is null. By increasing the exchange constant towards the $J_z/\Delta = 1$ point, we progressively approach the SI point. In this scenario, there's an escalation

in dynamic transverse fluctuations of magnetization, consequently leading to an augmentation in the dynamical susceptibility and thus an increase in the entanglement within the QD. As noted earlier under the condition of $N\Delta\gg 1$, where the QD contains a considerable number of electrons, the charge and spin degrees of freedom become decoupled, and the QFI derived from the susceptibility in Eq. **Error! Reference source not found.** mirrors the spin-spin entanglement in the QD.

Dynamic transverse Susceptibility Analysis in QDs with Anisotropic Exchange Hamiltonian

In this section, we explore the entanglement characteristics of a QD featuring an anisotropic exchange Hamiltonian $(J_z \neq J_\perp)$. At elevated temperatures, when $\Delta \ll T$, employing the saddle point approximation facilitates the factorization of the QD's grand partition function into two distinct components: $Z = Z_C Z_S$. Here, Z_C and Z_S respectively denote the contributions of the charging energy and exchange coupling to the grand partition function.

Similar to the Ising case, the z-component of the spin operator exhibits no dynamics. However, the dynamics of the transverse components are governed by the anisotropic exchange Hamiltonian. The imaginary component of the dynamic transverse spin susceptibility is derived from Z_S using the following expression:¹⁶

$$\operatorname{Im}\chi_{+-}(\omega) = -\frac{\pi}{2|J_z - J_\perp|Z_S} \sum_{\sigma = \pm} \left(n + \sigma T \frac{\partial}{\partial J_\perp} \right) Z_S(n)|_{n = -\overline{\omega} + \sigma/2} , \quad (16)$$

where $\overline{\omega} = \omega/[2(J_z - J_\perp)]$. For the two easy-axis $(J_z \ge J_\perp)$ and easy-plane $(J_z < J_\perp)$ scenarios, under the condition $T \gg \Delta > J_\perp - J_z$, the partition function Z_S takes the forms:¹⁶

$$Z_{S} = \begin{cases} \sqrt{\frac{\Delta}{\Delta - J_{Z}}} e^{\frac{\beta J_{\perp}^{2}}{4(\Delta - J_{\perp})}} F_{1}\left(\frac{\Delta}{\Delta - J_{\perp}}, \sqrt{\beta J_{*}}\right), & J_{Z} \geq J_{\perp} \\ \sqrt{\frac{\Delta}{\Delta - J_{Z}}} e^{\frac{\beta J_{\perp}^{2}}{4(\Delta - J_{\perp})}} F_{2}\left(\frac{\Delta}{\Delta - J_{\perp}}, \sqrt{\beta |J_{*}|}\right), & J_{Z} < J_{\perp}, \end{cases}$$

$$(17)$$

where, $J_* = \frac{(\Delta - J_\perp)(J_z - J_\perp)}{\Delta - J_z}$ represents the energy scale specific to the anisotropic problem, interpolating between 0 (for $J_z = J_\perp$) and $\frac{\Delta J_z}{\Delta - J_z}$ (for $J_\perp = 0$). The functions F_1 and F_2 are defined as follows:

$$F_1(x,y) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{\pi}} \frac{\sinh[xyt]}{\sinh[yt]} e^{-t^2}, \ F_2(x,y) = \int_{-\frac{\pi}{2y}}^{+\frac{\pi}{2y}} \frac{dt}{\sqrt{\pi}} \frac{\sin[xyt]}{\sin[yt]} e^{-t^2}.$$
 (18)

In Eq. (16), the function $Z_s(n)$ is given by:¹⁶

$$Z_{S}(n) = \sqrt{\frac{\beta \Delta}{\pi}} e^{\beta (J_{z} - J_{\perp})n^{2}} \left[\sum_{m=|n|} e^{-\beta (\Delta - J_{\perp})m^{2} + \beta J_{\perp}m} - \sum_{m=|n|+1} e^{-\beta (\Delta - J_{\perp})m^{2} - \beta J_{\perp}m} \right].$$
(19)

In the case where $\beta(\Delta - J_{\perp})|n| \ll 1$, $Z_{S}(n)$ simplifies to:¹⁶

$$Z_{S}(n) = \frac{1}{2} \sqrt{\frac{\Delta}{\Delta - J_{\perp}}} e^{\frac{\beta J_{\perp}^{2}}{4(\Delta - J_{\perp})}} e^{\beta (J_{z} - J_{\perp})n^{2}} \sum_{s=\pm} \operatorname{erf} \left[\sqrt{\beta (\Delta - J_{\perp})} (s|n| + \frac{J_{\perp}}{2(\Delta - J_{\perp})}) \right] + \sqrt{\frac{\beta \Delta}{\pi}} e^{-\beta (\Delta - J_{z})n^{2}} \operatorname{cosh} \left[\beta J_{\perp} |n| \right], \qquad (20)$$

and in the opposite case of $\beta(\Delta - J_{\perp})|n| \gg 1$, $Z_S(n)$ reduces to $\sqrt{\frac{\beta\Delta}{\pi}}e^{-\beta(\Delta - J_Z)n^2 + \beta J_{\perp}|n|}$. Moreover, in the limit of large frequencies, i.e., for $\beta(\Delta - J_{\perp})|\overline{\omega}| \gg 1$, the imaginary part of the transverse spin susceptibility is exponentially small:¹⁶

$$\operatorname{Im}\chi_{+-}(\omega) = -\frac{\overline{\omega}\sqrt{\pi\beta\Delta}}{|J_z - J_\perp|Z_S} e^{-\beta(\Delta - J_z)|\overline{\omega}|(\overline{\omega} + 1) + \beta J_\perp|\overline{\omega}|}.$$
 (21)

In the opposite limit, when $\omega \to 0$, the imaginary part of the transverse spin susceptibility has linear behavior:

$$\operatorname{Im}\chi_{+-}(\omega) = -\frac{\omega\sqrt{\pi\beta\Delta}}{2|J_z - J_\perp|(\Delta - J_\perp)Z_S} \left[\frac{2\Delta - J_\perp}{2(\Delta - J_\perp)} + \frac{\sqrt{\pi}}{2\sqrt{\beta(\Delta - J_\perp)}} g\left(\frac{\beta J_\perp^2}{4(\Delta - J_\perp)} \right) \right], \tag{22}$$

where $g(x) = (1 + 2x)e^x \operatorname{erf}[\sqrt{x}].$

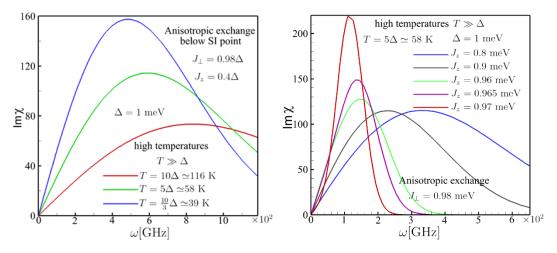


Figure 3: Plots illustrating the variation of the imaginary component of the dynamic transverse susceptibility as a function of frequency (ω) , comparing across a range of temperatures (left graph) and across various values of the exchange constant (right graph).

Figure 3 illustrates the plot of the function $\text{Im}\chi_{+-}$ as a function of frequency (ω) . The left plot depicts results obtained at $J_z = 0.4\Delta$ and $J_{\perp} = 0.98\Delta$ for various temperature values satisfying $\Delta/T \ll 1$. On the other hand, the right plot shows data at $T = 5\Delta$ with $J_{\perp} = 0.98\Delta$ for different J_z values less than Δ and J_{\perp} . The curve of $\text{Im}\chi_{+-}$ initiates from zero at $\omega=0$ and gradually increases (in a linear or nonlinear manner for \tilde{J} values very close to 1) at lower ω values, eventually reaching a maximum at a certain frequency before diminishing for larger ω values.

In contrast to the QD with Ising exchange coupling, both the location and maximum value of the magnetic susceptibility in this case rely on the strengths of the coupling constants. As J_z increases, approaching the isotropic point $J_z \sim J_\perp$, the susceptibility curve becomes narrower, with a larger peak occurring at lower frequencies. Additionally, both the peak value and its location are influenced by temperature; decreasing the temperature results in a narrower susceptibility curve with a higher peak value, but the peak occurs at higher frequencies. Furthermore, it is important to note that the susceptibility becomes zero precisely at the isotropic point $J_z = J_\perp$.

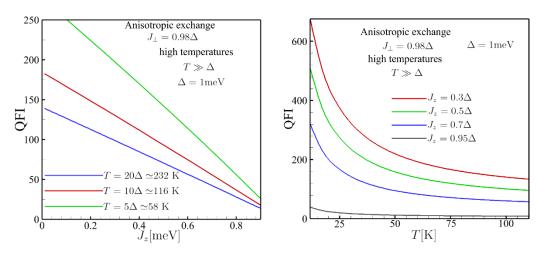


Figure 4: Quantum Fisher information as a function of J_z for different temperatures (left), and as a function of temperature for $J_z < J_\perp$ (right).

We have illustrated in Figure 4 the plot of QFI for QDs with anisotropic exchange coupling, showing its variation as a function of J_z for different temperatures (left), and as a function of temperature for different strengths of J_z (right). The decrease in QFI with increasing the strength of the coupling constant at a fixed temperature is due to the approaching of J_z towards J_\perp , the isotropic point, where the magnetic susceptibility is zero in the absence of an external magnetic field. In Figure 4, we have also depicted the plot of QFI as a function of temperature for different strengths of exchange coupling. At high temperatures, QFI is small, indicating that the electrons in the QD are disentangled due to thermal fluctuations. By lowering the temperature, thermal fluctuations are reduced, and the spins become entangled. This behavior contrasts with the Ising case (see Figure 2-left), where lowering the temperature reduces the entanglement. The reason behind this difference lies in the opposite behaviors of the dynamical susceptibilities as a function of temperature in the two systems. As discussed, in the Ising case, although lowering the temperature narrows the dynamical susceptibility, its peak value remains almost constant. In contrast, in the anisotropic case, the peak value of the susceptibility increases significantly, leading to the enhancement of the QFI with decreasing temperature.

It should be noted that we have neglected the influence of QD's level fluctuations on the behavior of susceptibility and entanglement. We anticipate that the interplay between level fluctuations and exchange interactions will give rise to intriguing physics related to the quantum entanglement of the QD.

Spin-Spin Entanglement in QDs at Low Temperatures

In the low-temperature regime ($T \ll \Delta$), the behavior of QDs with anisotropic spin Hamiltonian can be effectively described using the grand partition function expressed as:¹⁶

$$Z = \sum_{n_{\uparrow} n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta E_{c}(n-N_{0})^{2} + \beta \mu n + \beta J_{\perp} m(m+1)} \operatorname{Sgn}(2m+1) \sum_{l=-|m+1/2|+1/2}^{|m+1/2|-1/2} e^{\beta (J_{z}-J_{\perp})l^{2}}, (23)$$

where $n = n_{\uparrow} + n_{\downarrow}$ and $m = (n_{\uparrow} - n_{\downarrow})/2$. The function Z_n is the canonical partition function of the QD with n number of electrons. The dynamical susceptibility is given by:

$$\chi_{\pm}(\omega) = \frac{1}{Z} \sum_{n_{\uparrow}, n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta E_{c}(n-N_{0})^{2} + \beta \mu n + \beta J_{\perp} m(m+1)} \operatorname{sgn}(2m+1) \sum_{l=-\left|m+\frac{1}{2}\right| + \frac{1}{2}}^{\left|m+\frac{1}{2}\right| - \frac{1}{2}} e^{-\beta E_{c}(n-N_{0})^{2} + \beta \mu n + \beta J_{\perp} m(m+1)} \operatorname{sgn}(2m+1) \sum_{l=-\left|m+\frac{1}{2}\right| + \frac{1}{2}}^{\left|m+\frac{1}{2}\right| - \frac{1}{2}}$$

$$e^{\beta(J_z - J_\perp)l^2} \left[\frac{m(m+1) - l^2 - l}{\omega + (J_\perp - J_z)(2l+1) + i0^+} - \frac{m(m+1) - l^2 + l}{\omega + (J_\perp - J_z)(2l-1) + i0^+} \right]$$
(24)

The imaginary part of susceptibility is:

$$\operatorname{Im}\chi_{\pm}(\omega) = \frac{-\pi}{Z} \sum_{n_{\uparrow}, n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta E_{c}(n-N_{0})^{2} + \beta \mu n + \beta J_{\perp} m(m+1)} \operatorname{sgn}(2m+1) \sum_{l=-\left|m+\frac{1}{2}\right| + \frac{1}{2}}^{\left|m+\frac{1}{2}\right| - \frac{1}{2}} e^{\beta (J_{z} - J_{\perp})l^{2}} \left[(m(m+1) - l^{2} - l)\delta(\omega + (J_{\perp} - J_{z})(2l+1)) - (m(m+1) - l^{2} + l)\delta(\omega + (J_{\perp} - J_{z})(2l-1)) \right]$$
(25)

In the regime of low temperatures ($T \ll \Delta$), where thermal fluctuations are minimal, the canonical partition function can be approximated as $Z_n \sim e^{-\beta \Delta n(n-1)/2}$. In Equation (25), we must consider the contribution where m = S and $l = \pm S$ only, where S represents the total spin of the ground state.

In this limit, the imaginary part of the dynamical susceptibility simplifies to a series of Dirac delta functions, given by:

$$\operatorname{Im}\chi_{+-}(\omega) = \pi S \left[\delta \left(\omega - (J_z - J_\perp)(2S - 1) \right) - \delta \left(\omega + (J_z - J_\perp)(2S - 1) \right) \right] \tag{26}$$

This expression reflects the behavior of the system's susceptibility at low temperatures, indicating the discrete nature of the spin transitions within the QD.

The behavior of Im χ strongly relies on the value of the total spin S. As discussed earlier, within the mesoscopic Stoner regime where the conditions $\Delta - J_z \ll J_{\perp} < J_z$ are satisfied, the total spin of the ground state can be approximated as $S \cong (J_{\perp} + J_z - \Delta)/[2(\Delta - J_z)]$. This expression yields a positive variable for S. In terms of the mean level spacing, the second term in the argument of the delta functions simplifies to $(J_z - J_{\perp}) \left(\frac{J_{\perp}}{1 - J_z} - 2 \right)$, which is always positive within the mesoscopic Stoner regime.

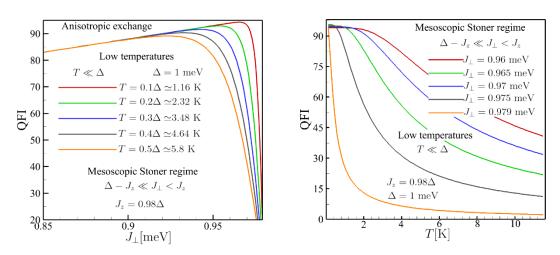


Figure 5: Quantum Fisher information plotted as a function of J_{\perp} for various temperatures (left), and as a function of temperature for $J_z > J_{\perp}$ (right).

Substituting Eq. (26) into Eq. (15), the QFI is obtained as QFI = 4S tanh $\left[\frac{1}{2T}(J_z - J_\perp)(2S - 1)\right]$. At sufficiently low temperatures, the hyperbolic tangent function approaches 1, and the QFI

simplifies to 4S. For a given value of Δ and J_z , the ground state spin increases with increasing J_{\perp} , leading to an increase in QFI (refer to Figure 5-left). In contrast to the high-temperature limit (see Figure 4-right for comparison), here the QFI rises as the system approaches the isotropic point, peaks near this point, and then sharply decreases to zero at $J_z = J_{\perp}$. At this juncture, where the transverse spin susceptibility is zero, the QFI also becomes zero, consistent with its behavior at high temperatures.

In the Ising model case $(J_{\perp} = 0)$, as long as $\Delta > J_z$, the ground state spin is $S = \frac{1}{2}$ (0) for an odd (even) number of electrons. In both scenarios, the QFI evaluates to zero.

Practical Implications and Relevance of the Study

The parameters in the plots have been intentionally scaled to measurable units, with an assumption of an energy level spacing of 1 meV, corresponding to a planar QD or nanoparticle dimension of approximately 100 nm in diameter. This allows the reported plots to reflect meaningful values for experimentalists. For example, the dynamic susceptibility spectra, plotted versus frequency, show that the peaks occur in the range of 1-5 GHz. While this frequency range may pose challenges for measurement in practice, it is worth noting that under applied magnetic fields, this range can be reduced. Moreover, the metallic domain assumption is applicable to manyelectron QDs, nanoparticles, or metallic impurities in a host material. These practical observations provide valuable insights for experimentalists seeking to validate the theoretical predictions in real-world scenarios.

The findings of this study hold practical implications in several key areas of quantum research and technology. Firstly, by elucidating the behavior of QDs under various types of exchange interaction, the study contributes to the development of more efficient means of determining the entanglement in QDs. Understanding the impact of diverse exchange interactions within QDs is paramount for designing QDs with tailored phase stability or coherence properties.

QDs serve as exceptional platforms for exploring quantum effects at the nanoscale, providing unparalleled control over individual quantum states. Entanglement, a fundamental resource for quantum technologies, highlights the system's sensitivity to environmental factors. In this regard, by demonstrating dynamic susceptibility as a proxy to detect and manipulate entanglement in QDs, this study paves the way for the development of high-precision quantum sensors and measurement devices. Such advancements could find applications in fields like magnetometry, navigation, and medical imaging, offering improved sensitivity and accuracy in quantum metrology tools.

Moreover, the study's emphasis on addressing practical challenges posed by stoner instability underscores the necessity of considering these factors in the design and fabrication of quantum devices. By elucidating the influence of various types of exchange interactions on QD behavior, this research informs the development of robust and reliable quantum devices capable of functioning effectively in real-world conditions.

Future Directions: Quantum Metrology and Beyond with AC Susceptibility Measurements

The proposed characterization method offers scalability advantages compared to resource-intensive techniques outlined in references 16 and 19. While traditional methodologies excel in accessing non-local entanglement, a hallmark of topological phases as discussed in references 17 and 18, physical susceptibilities establish closer ties with local operators.

Despite QFI displaying scaling behavior at topological transitions, it falls short of identifying topological phases, as demonstrated in [1] using the Kitaev wire example. This limitation is shared by other entanglement witnesses. 19,20

A key challenge lies in exploring non-local extensions of the proposed protocol, potentially enabling the characterization of topological entanglement²¹ and exotic quantum phases.^{22,23}

The multipartite entanglement detected by QFI serves as a valuable resource for quantum metrology. 17,18 This could expand its application scope to include quantification of entanglement in quantum simulators for many-body problems 24,25,26 and characterization of strongly correlated systems.

For example, one intriguing avenue to explore is whether entanglement influences material properties such as high-temperature superconductivity, where scaling behavior has already been observed in optical conductivity.²⁷ Additionally, an underlying antiferromagnetic critical point²⁸ is expected to exhibit quantum-critical scaling of entanglement, a theory that could be tested in cold atomic-gas experiments.^{29,30}

In both the isotropic and anisotropic QDs we investigated, the null dynamic longitudinal susceptibility indicates that the z-component of the spin operators in the QD remains constant over time. Consequently, the electron-electron interaction does not influence the longitudinal relaxation time (T_1) . Conversely, the transverse components of the spin operators exhibit correlations at different times, resulting in a nonzero dynamic transverse susceptibility and finite entanglement between electrons. This entanglement plays a crucial role in determining the transverse relaxation time (T_2) . Therefore, to design a QD with long relaxation times, efforts should focus on minimizing the entanglement of particles induced by the exchange interaction. Notably, these results were obtained for QDs containing a large number of electrons. In the context of qubit QDs, reducing the number of electrons within the QD becomes essential. It remains an intriguing question whether the transverse relaxation time is still dependent on the exchange interaction of particles in such scenarios.

Conclusion

This study examines the entanglement in QDs, shedding light on various phenomena influenced by confinement, exchange interactions, temperature variations, and proximity to the SI point. Our investigation reveals that magnetization in QDs is significantly influenced by their proximity to the SI point. We observe that charge-spin coupling is strong near the SI point but weakens as it moves away. Magnetization fluctuations near the SI point strengthen charge-spin interactions, highlighting the intricate relationship between magnetization and entanglement. Furthermore, our analysis shows that the canonical partition function and Coulomb blockade regime play crucial roles in governing electron count and charging in QDs. Far from the SI point, charge and spin become decoupled, as evidenced by magnetic susceptibilities showing no charge interaction. Examining the quantum Fisher information (QFI) under various exchange interactions at both low and high temperatures provides additional insights:

- QDs with anisotropic exchange models at high temperatures: As exchange constants approach isotropy, QFI reduces to zero, indicating a decline in entanglement.
- QDs with Ising exchange model at high temperatures: Far below the SI point, weak spin-charge coupling leads to zero entanglement encoded in the dynamical susceptibility. However, approaching the SI point results in increased dynamic transverse fluctuations of magnetization, enhancing multi-partite entanglement in the QD.

- QDs with isotropic or Ising exchange models at low temperatures: QFI vanishes due to the imaginary part of the dynamic susceptibility reducing to zero.
- QDs with anisotropic exchange models at low temperatures: QFI increases with decreasing temperature, reaching a maximum near the isotropic point before decreasing rapidly to zero at the isotropic point. Lowering the temperature leads to an increase in QFI, saturating at low temperatures.

These findings not only deepen our understanding of entanglement in QDs but also have practical implications for quantum research and technology. By elucidating the intricate interplay between exchange interactions, temperature variations, and entanglement, this study lays the groundwork for future investigations aimed at harnessing entanglement for quantum applications.

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