SYMMETRY RESULTS FOR A NONLOCAL EIGENVALUE PROBLEM

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Abstract. In this paper, we study the optimal constant in the nonlocal Poincaré-Wirtinger inequality in $(a,b)\subset\mathbb{R}$:

$$\lambda_\alpha(p,q,r)\left(\int_a^b |u|^q dx\right)^{\frac{p}{q}} \le \int_a^b |u'|^p dx + \alpha\left|\int_a^b |u|^{r-2}u\,dx\right|^{\frac{p}{r-1}},$$

where $\alpha\in\mathbb{R}$, $p,q,r>1$ such that $\frac{4}{5}p\le q\le p$ and $\frac{q}{2}+1\le r\le q+\frac{q}{p}$. This problem can be casted as a nonlocal minimum problem, whose Euler-Lagrange associated equation contains an integral term of the unknown function over the whole interval of definition. Furthermore, the problem can be also seen as an eigenvalue problem.

We show that there exists a critical value $\alpha_C=\alpha_C(p,q,r)$ such that the minimizers are even with constant sign when $\alpha\le\alpha_C$ and are odd when $\alpha\ge\alpha_C$.
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1. Introduction

Let $a,b,\lambda\in\mathbb{R}$, in this paper we study a nonlinear generalization of the celebrated one-dimensional inequality

$$\lambda\int_a^b |u|^2 \mathrm{d}x \le \int_a^b |u'|^2 \mathrm{d}x \quad \forall u\in C^1(a,b); \tag{1}$$

that is the Poincaré inequality when

$$u(a)=u(b)=0 \tag{2}$$

and that is the Wirtinger inequality when

$$\int_a^b u\ \mathrm{d}x = 0. \tag{3}$$

The best constant λ in both Poincaré (1)-(2) and Wirtinger inequality (1)-(3) is obtained for

$$\lambda=\left(\frac{\pi}{b-a}\right)^2.$$

When both Dirichlet (2) and Neumann (3) boundary condition holds, we speak of twisted boundary conditions [BB, FH]. The best constant in the Twisted inequality (1)-(2)-(3) is obtained for

$$\lambda_T=\left(\frac{2\pi}{b-a}\right)^2.$$

Given $p, q, r > 1$, the generalized Poincaré inequalty (see e.g. [GGR] and reference therein) states that there exists a constant $\lambda_P(p,q)$ such that

$$\lambda_P(p,q)\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}} \le \int_a^b |u'|^p \mathrm{d}x \quad \forall u \in W^{1,p}(a,b) \text{ s.t. } u(a)=u(b)=0. \tag{4}$$

When $p = q = 2$, we come back to the classical Poincaré inequality (1)-(2). Moreover, the optimal constant in (4) is also the minimum for the variational problem

$$\lambda_P(p,q) = \min_{\substack{W_0^{1,p}(a,b) \\ u \not\equiv 0}} \frac{\int_a^b |u'|^p \mathrm{d}x}{\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}}}$$

and the minimizing functions are even functions with constant sign. It is easily seen that $\lambda_P(p,q)$ is an homogeneous Dirichlet Laplacian eigenvalue (see e.g. [LE, Th. 3.3]).

On the other hand, the generalized Wirtinger inequality states that there exists a constant $\lambda_W(p,q,r)$ such that

$$\lambda_W(p,q,r)\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}} \le \int_a^b |u'|^p \mathrm{d}x \quad \forall u \in W^{1,p}(a,b) \text{ s.t. } \int_a^b |u|^{r-2}u \ \mathrm{d}x = 0. \tag{5}$$

When $p = q = r = 2$, we come back to the classical Wirtinger inequality (1)-(3). Moreover, the optimal constant in (5) is also the minimum for the variational problem

$$\lambda_W(p,q,r) = \min_{\substack{W^{1,p}(a,b) \\ \int_a^b |u|^{r-2}u \ \mathrm{d}x = 0 \\ u \not\equiv 0}} \frac{\int_a^b |u'|^p \mathrm{d}x}{\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}}}$$

and the minimizing functions are odd functions. For the exact value of $\lambda_W(p,q,r)$ see [GN]. It is easily seen that $\lambda_W(p,q,r)$ is a Neumann Laplacian eigenvalue (see e.g. [LE, Th. 3.4]).

Then, when both the generalized Dirichlet and Neumann boundary condition hold, the generalized Twisted inequality states that there exist a constant $\lambda_T(p,q,r)$ such that

$$\lambda_T(p,q,r)\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}} \le \int_a^b |u'|^p \mathrm{d}x \quad \forall u \in W^{1,p}(a,b) \quad \text{s.t. } u(a)=u(b)=0 \text{ and } \int_a^b |u|^{r-2}u \mathrm{d}x = 0. \tag{6}$$

When $p = q = r = 2$, we come back to the classical Twisted inequality (1)-(2)-(3).

Moreover, the optimal constant in (6) is also the minimum for the variational problem

$$\lambda_T(p,q,r) = \min_{\substack{W_0^{1,p}(a,b) \\ \int_a^b |u|^{r-2}u \ \mathrm{d}x = 0 \\ u \not\equiv 0}} \frac{\int_a^b |u'|^p \mathrm{d}x}{\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}}} \tag{7}$$

and the minimizing functions are odd functions.

Now, let us consider $\alpha \in \mathbb{R}$. The main aim of this paper is to unify and extend the study of Poincaré, Wirtinger and Twisted inequalities by introducing a penalization term. Specifically, we consider the following inequality:

(8)
$$\lambda_\alpha(p,q,r)\left(\int_a^b |u|^q \mathrm{d}x\right)^{\frac{p}{q}} \leq \int_a^b |u'|^p \mathrm{d}x + \alpha \left|\int_a^b |u|^{r-2}u \mathrm{d}x\right|^{\frac{p}{r-1}} \quad \forall u \in W^{1,p}(a,b)$$
$$\text{s.t. } u(a) = u(b) = 0.$$

It is easily seen that, when $\alpha = 0$, the nonlocal inequality (8) is the Poincaré inequality (4); meanwhile when $\alpha \to +\infty$, tends to the Twisted inequality (6).

The optimal constant in (8) corresponds to the value realizing the minimum in the following eigenvalue problem

$$\lambda_\alpha(p,q,r) = \inf\left\{\mathcal{Q}_\alpha[u],\ u \in W_0^{1,p}(a,b),\ u \not\equiv 0\right\}, \tag{9}$$

where

$$\mathcal{Q}_\alpha[u] := \frac{\int_a^b |u'|^p dx + \alpha \left|\int_a^b |u|^{r-2}u\, dx\right|^{\frac{p}{r-1}}}{\left(\int_a^b |u|^q dx\right)^{\frac{p}{q}}}. \tag{10}$$

This kind of problems leads in general to non standard associated Euler-Lagrange equations, that are known in literature as non-local, because they depends on the value that the unknown function assumes on the whole domain throughout the integral over (a,b). Specifically

$$\begin{cases} -(|y'|^{p-2}y')' + \alpha|\gamma|^{\frac{p}{r-1}-2}\gamma|y|^{r-2} = \lambda_\alpha(p,q,r)\,||y||_q^{p-q}|y|^{q-2}y \text{ in }]a,b[\\ y(a) = y(b) = 0, \end{cases}$$

where $\gamma = \int_a^b |y|^{r-2}y\, dx$, except for some trivial cases detailed in Section 2.

Problems of this type date back to at least the 1837 papers by Duhamel [Du] and Liouville [Lio] on thermo-elasticity. Moreover, these nonlocal problems have been treated in the study of the reaction-diffusion equations describing chemical processes (see [F2, S]) or Brownian motion with random jumps (see [Pin]). They have been the object of much study over the last 25 years [F1, F2, FV], particularly by considering the minimization of the nonlocal one-parameter problem, both in n-dimensional ([BFNT]) and in one dimensional ([DPP1, DPP2]) case.

In higher dimensions, problem (9) has been treated ([BFNT]) only in the case when $p = q = r = 2$. The authors have obtained a saturation phenomenon when a volume constraint holds; specifically, they show that the optimal shape is the ball (up to a critical value of the parameter α) or the union of two equal balls (for supercritical values). Analogous result holds when a Finsler metric replace the Euclidean one [Pis]. A subsequent research area, not investigated in this paper, consists in the study of the existence of threshold values below or above which the symmetry of optimal domains is broken, as e.g. in [BDNT, BCGM, N2].

In one dimension, the study of the nonlocal problem (9) rely on the study of the generalized Wirtinger inequality started by the pionering work of Dacorogna Gangbo and Subia [DGS] for $q \leq 2p$ and $r = 2$. Then in [E, BKN, BK, N1, CD, GN] variuous range has been analyzed and finally in [GGR] the issue of symmetry/non symmetry has been completely settled. Specifically, the symmetry of the minimizers holds when $q \leq (2r-1)p$ and no odd function can be a minimizers when $q > (2r-1)p$.

Our aim is the study of the symmetry properties of the minimizers of (9) and, as a consequence, to give some informations on $\lambda_\alpha(p,q,r)$. In order to study the full range of the exponents $p,q,r>1$, we recall that, up to our knowledge, the nonlocal problem (9) has been treated only for $p=q=2$ and $2 \leq r \leq 3$ in [DPP1] and for $p=q\geq 2$ and $\frac{p}{2}+1 \leq r \leq \frac{p}{2}$ in [DPP2]. In these ranges, the minimizers of (9) are symmetric (even or odd) and a saturation phenomenon occurs. Particularly, for subcritical values of the parameter α, the minimizers are even functions with constant sign, meanwhile, for supercritical values, the minimizers are odd sign-changing functions. It is worth investigating in which ranges a symmetry breaking is expected to hold.

Throughout this paper, for the sake of simplicity, we will study the problem in the interval $(-1,1)$ instead of (a,b). The general case can be easily recovered since the nonlocal eigenvalue admits the following rescaling

$$\lambda_\alpha(p,q,r;(a,b)) = \left[\left(\frac{2}{b-a}\right)^{\frac{1}{p'}+\frac{1}{q}}\right]^p \lambda_{\tilde\alpha}\left(p,q,r;(-1,1)\right),$$

with $\tilde\alpha = \left(\frac{b-a}{2}\right)^{\left(\frac{1}{r-1}+\frac{1}{p'}\right)p}\alpha$.

In the present paper, we extend the range of treatable exponents and prove the following saturation phenomenon.

Theorem 1.1. *Let $p,q,r>1$ be such that $\frac{4}{5}p \leq q \leq p$ and $\frac{q}{2}+1 \leq r \leq q+\frac{q}{p}$. Then there exists a positive number $\alpha_C = \alpha_C(p,q,r)$ such that:*

(i) *if $\alpha < \alpha_C$, then $\lambda_\alpha(p,q,r) < \lambda_T(p,q,r)$;*
(ii) *If $\alpha \geq \alpha_C$, then $\lambda_\alpha(p,q,r) = \lambda_T(p,q,r)$.*

In addition, we prove the following symmetry results for the solutions of problem (9). We refer to Section 2 for the definition of the generalized trigonometric function $\sin_{p,q}(\cdot)$.

Theorem 1.2. *Let $p,q,r>1$ be such that $\frac{4}{5}p \leq q \leq p$ and $\frac{q}{2}+1 \leq r \leq q+\frac{q}{p}$.*

(i) *If $\alpha < \alpha_C$, then any minimizer y of $\lambda_\alpha(p,q,r)$ is an even function with constant sign in $(-1,1)$.*
(ii) *If $\alpha > \alpha_C$, the function $y(x) = \sin_{p,q}(\lambda_T(p,q,r)x)$, $x \in (-1,1)$, is the unique minimizer, up to a multiplicative constant, of $\lambda_\alpha(p,q,r)$. Hence it is an odd function, $\int_{-1}^1 |y|^{r-2}y\,dx = 0$, and $\overline{x}=0$ is the only point in $(-1,1)$ such that $y(\overline{x})=0$.*
(iii) *If $\alpha = \alpha_C$, then $\lambda_{\alpha_C}(p,q,r)$ admits both a positive minimizer and the minimizer $y(x) = \sin_{p,q}(\pi_{p,q}x)$, up to a multiplicative constant. Moreover, if $r > \frac{q}{2}+1$ any minimizer has constant sign or it is odd.*

Furthermore, if $r=p+1$, then $\alpha_C(p,q,p+1) = \frac{2^p-1}{2^p}\frac{q}{p'}\left(\frac{2p'}{p'+q}\right)^{1-\frac{p}{q}}\pi_{p,q}^p$.

The outline of the paper follows. In Section 2, we provide some recalls on the nonlocal eigenvalue problem we are dealing with; in Section 3, we study the properties of an auxiliary function useful to give some representations of the eigenvalue and the eigenfunctions of problem (9); in Section 4, we give the proof of the main Theorems.

2. The eigenvalue problem

In this Section, we firstly recall some results on the generalized trigonometric functions and then some properties of the eigenvalue problem (9).

2.1. **The $p-q$-circular functions.**

We briefly summarize some properties the p-trigonometric functions for any fixed $1<p<+\infty$ (refer e.g. [LE, Lin, Pe]). These functions generalize the familiar trigonometric functions and coincide with them when $p=2$.

Let us consider the function $F_p:[0,1]\to\mathbb{R}$ defined as

$$F_p(x)=\int_0^x \frac{dt}{(1-t^p)^{\frac{1}{p}}}.$$

Denote by $z(s)$ the inverse function of F which is defined on the interval $\left[0,\frac{\pi_p}{2}\right]$, where

$$\pi_p=2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}.$$

Therefore, the p-sine function $\sin_p$ is defined as the following periodic extension of $z(t)$:

$$\sin_p(t)=\begin{cases} z(t) & \text{if } t\in\left[0,\dfrac{\pi_p}{2}\right],\\ z(\pi_p-t) & \text{if } t\in\left[\dfrac{\pi_p}{2},\pi_p\right],\\ -\sin_p(-t) & \text{if } t\in[-\pi_p,0]. \end{cases}$$

It is extended periodically to all $\mathbb{R}$, with period $2\pi_p$. Furthermore, the p-cosine function is defined by

$$\cos_p(t)=\frac{d}{dt}\sin_p(t)$$

and is a $2\pi_p$-periodic and odd function.

To further extend the definitions of trigonometric functions, let us consider $p,q>1$ and set

$$\pi_{p,q}:=2\int_0^1 \frac{1}{(1-t^q)^{\frac{1}{p}}}dt=\frac{2}{q}B\left(\frac{1}{p'},\frac{1}{q}\right)=\frac{2}{q}\frac{\Gamma\left(\frac{1}{p'}\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{p'}+\frac{1}{q}\right)},$$

where B and Γ are the beta and the gamma function, respectively.

This definition coincides with π_p when $p=q$. Therefore the function $\sin_{p,q}$ is defined on the interval $[0,\frac{\pi_{p,q}}{2}]$ as the inverse of $F_{p,q}:[0,1]\to\mathbb{R}$ given by

$$F_{p,q}(x)=\int_0^x \frac{1}{(1-t^q)^{\frac{1}{p}}}dx$$

and extended to the real line by the usual process involving the symmetry and the $2\pi_{p,q}$ periodicity.

Finally, we recall from [LE, Thm. 3.3], that any eigenvalue and eigenfunction of the 1-dimensional Dirichlet p,q-Laplacian eigenvalue problem:

$$\begin{cases} -(|y'|^{p-2}y')'=\lambda|y|^{q-2}y & \text{in }]-1,1[\\ y(-1)=y(1)=0. \end{cases} \tag{11}$$

are of the form

$$\lambda_n = c_1 \frac{q}{p'} \left(\frac{n\pi_{p,q}}{2}\right) \quad \text{and} \quad y_n(x) = c_2 \sin_{p,q}\left(\frac{n\pi_{p,q}}{2}(x+1)\right) \quad \forall n \in \mathbb{N}$$

respectively, for $c_1, c_2 \in \mathbb{R}$. Clearly, when $p = q$, we fall in the case of the p-Laplacian problem.

2.2. **The eigenvalue problem.** We firstly show some properties of the solution of the eigenvalue problem (9).

Proposition 2.1. *Let $\alpha \in \mathbb{R}$, $p, q, r > 1$ be such that $q \le p$ and $\frac{q}{2} + 1 \le r \le q + \frac{q}{p}$. Then, problem (9) admits a solution in $W_0^{1,p}(-1,1)$ and any minimizer y of (9) is a solution of the following Dirichlet homogeneous problem*

(12)
$$\begin{cases} -(|y'|^{p-2}y')' + \alpha|\gamma|^{\frac{p}{r-1}-2}\gamma|y|^{r-2} = \lambda_\alpha(p,q,r)\,||y||_q^{p-q}|y|^{q-2}y \quad \text{in }]-1,1[\\ \\ y(-1) = y(1) = 0, \end{cases}$$

where

$$\gamma = \begin{cases} 0 \text{ if both } r = p+1 \text{ and } \displaystyle\int_{-1}^{1} |y|^{r-2}y\ dx = 0, \\ \displaystyle\int_{-1}^{1} |y|^{r-2}y\ dx \text{ otherwise.} \end{cases}$$

Moreover, $y, y'|y'|^{p-2} \in C^1[-1,1]$.

Proof. Standard methods of Calculus of Variations prove the existence of a minimizer. Let us observe that, since $p \ge q \ge r - \frac{q}{p} \ge r - 1$, we have that $p \ge r - 1$. If $p > r - 1$, the functional $\mathcal{Q}_\alpha[\cdot]$ in (10) is differentiable and hence the associated Euler-Lagrange equation leads to (12); meanwhile, when $p = r - 1$, the problem (9) coincides with problem (7); hence $\gamma = 0$ and we get the conclusion.

Finally, the fact that $y, y'|y'|^{p-2} \in C^1[-1,1]$ is easily seen from (12). □

At this stage, we analyze the monotonicity and asymptotic properties of the eigenvalue (9) with respect to the parameter α.

Proposition 2.2. *Let $\alpha \in \mathbb{R}$, $p, q, r > 1$ be such that $q \le p$ and $\frac{q}{2} + 1 \le r \le q + \frac{q}{p}$. Then the function $\alpha \in \mathbb{R} \mapsto \lambda_\alpha(p,q,r)$ is Lipschitz continuous, non-decreasing with respect to $\alpha \in \mathbb{R}$ and*

$$\lim_{\alpha \to -\infty} \lambda_\alpha(p,q,r) = -\infty, \qquad \lim_{\alpha \to +\infty} \lambda_\alpha(p,q,r) = \lambda_T(p,q,r).$$

Proof. Let us fix $\varepsilon > 0$, then by using the Hölder inequality, we have

$$\mathcal{Q}_{\alpha+\varepsilon}[u] \le \mathcal{Q}_\alpha[u] + \varepsilon \frac{\left(\int_{-1}^{1} |u|^{r-1}\ dx\right)^{\frac{p}{r-1}}}{\int_{-1}^{1} |u|^p\ dx} \le \mathcal{Q}_\alpha[u] + 2^{\frac{p(q-r+1)}{q(r-1)}}\varepsilon.$$

Therefore, we gain the following chain of inequalities

$$\mathcal{Q}_\alpha[u] \le \mathcal{Q}_{\alpha+\varepsilon}[u] \le \mathcal{Q}_\alpha[u] + 2^{\frac{p(q-r+1)}{q(r-1)}}\varepsilon \quad \forall\, \varepsilon > 0.$$

By taking the minimum for any $u \in W_0^{1,p}(-1,1)$, we have

$$\lambda_\alpha(p,q,r) \le \lambda_{\alpha+\varepsilon}(p,q,r) \le \lambda_\alpha(p,q,r) + 2^{\frac{p(q-r+1)}{q(r-1)}}\varepsilon \quad \forall \varepsilon > 0,$$

that implies the desired Lipschitz continuity and monotonicity.

Now, let us consider a positive admissible function $\varphi \in W_0^{1,p}(-1,1)$. Then, we have that $\mathcal{Q}_\alpha[\varphi] \to -\infty \quad$ as $\alpha \to -\infty$ and, since $\lambda_\alpha(p,q,r) \le \mathcal{Q}_\alpha[\varphi]$, we have that

$$\lim_{\alpha\to-\infty} \lambda_\alpha(p,q,r) = -\infty.$$

Finally, let us consider a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \to +\infty$. Since $\lambda_\alpha(p,q,r)$ is decreasing with respect to α, we have that $\lambda_\alpha(p,q,r) \le \lambda_T(p,q,r)$ for any $\alpha \in \mathbb{R}$. Let us denote $u_n = u_{\alpha_n}$ the normalized ($||u_n||_q = 1$) minimizer in $W_0^{1,p}$ of (9) when the value of the parameter is α_n; we have that

$$\lambda_{\alpha_k}(p,q,r) = \int_{-1}^1 |u_n'|^p \ dx + \alpha_n \left(\int_{-1}^1 |u_n|^{r-2} u_n \ dx \right)^{\frac{p}{r-1}} \le \lambda_T(p,q,r).$$

This implies that, up to a subsequence, u_n strongly converges in $L^p(-1,1)$ and weakly in $W_0^{1,p}(-1,1)$ to a function $u \in W_0^{1,p}(-1,1)$ such that $\|u\|_{L^p} = 1$. On one hand, we have that

$$\left(\int_{-1}^1 |u_n|^{r-2} u_n \ dx \right)^{\frac{p}{r-1}} \le \frac{\lambda_T(p,q,r)}{\alpha_n} \to 0 \quad \text{as } n \to +\infty$$

which means that $\int_{-1}^1 |u|^{r-2} u \ dx = 0$. On the other hand, since u is an admissible function for (7), by using the lower semicontinuity, we have that

$$\begin{aligned} \lambda_T(p,q,r) \le \int_{-1}^1 |u'|^p \ dx &\le \liminf_{n\to+\infty} \left[\int_{-1}^1 |u_n'|^p \ dx + \alpha_n \left(\int_{-1}^1 |u_n|^{r-2} u_n \ dx \right)^{\frac{p}{r-1}} \right] \\ &= \lim_{n\to+\infty} \lambda_{\alpha_n}(p,q,r) \le \lambda_T(p,q,r) \end{aligned}$$

and hence the conclusion follows. $\square$

3. The auxiliary function H

In this Section, we study the behavior of an auxiliary function on which is based the proof of the main results (Theorems 1.1 and 1.2). We consider the following integral function:

$$H(m,p,q,r) : (m,p,q,r) \in [0,1] \times]1,+\infty[\times \left[\frac{4}{5}p, p\right] \times \left[\frac{q}{2}+1, q+\frac{q}{p}\right] \mapsto \mathbb{R},$$

defined as

$$\begin{aligned} (13)\quad H(n,p,q,r) &:= \int_{-m}^1 \frac{dy}{[1 - R(m,q,r)(1-|y|^{r-2}y) - |y|^q]^{\frac{1}{p}}} \\ &= \int_0^1 \frac{dy}{[1 - R(m,q,r)(1-y^{r-1}) - y^q]^{\frac{1}{p}}} + \int_0^1 \frac{m dy}{[1 - R(m,q,r)(1+m^{r-1}y^{r-1}) - m^q y^q]^{\frac{1}{p}}} \end{aligned}$$

where $R(m,q,r) = \frac{1-m^q}{1+m^{r-1}}$.

It will be also very useful in the sequel to consider h, the integrand function of H, that is defined as

$$\begin{aligned} &h(m,p,q,r,y) := \\ &\frac{1}{[1 - R(m,q,r)(1-y^{r-1}) - y^q]^{\frac{1}{p}}} + \frac{m}{[1 - R(n,q,r)(1+m^{r-1}y^{r-1}) - m^q y^q]^{\frac{1}{p}}}, \end{aligned}$$

for any $y \in [0,1[$, except when $m = y = 0$.

We will prove the monotonicity of the auxiliary function with respect to r (Lemma 3.1) and then with respect to m (Lemma 3.2), to finally provide some useful estimates for the function H (Proposition 3.3).

Regarding the monotonicity with respect to r, we prefer to study the function h.

Lemma 3.1. *Let $p, q, r > 1$ be such that $p \geq q$ and $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$. For any $y \in [0, 1[$ and*

- *for any fixed $m \in [0, 1[$, the function $h(m, p, q, r, y)$ is strictly increasing with respect to r.*
- *for $m = 1$, the function $h(1, p, q, r, y)$ is constant with respect to r.*

Proof. We divide the proof into three steps: in the first step we compute the expression of the derivative of h with respect to r for any $m \in]0, 1[$ and $y \in]0, 1[$; in the second step we study the sign of the aforementioned derivative; in the third step we analyze the cases excluded by the previous steps. From now on, for the sake of simplicity, we set $R = R(m, q, r)$.

Step 1 (The derivative of h). Let us start by considering the case when $m \in]0, 1[$ and $y \in]0, 1[$. Differentiating h with respect to r, we have

$$\partial_r h(m, p, q, r, y) = -\frac{1}{p} \frac{(1 - y^{r-1})\partial_r R + R\, y^{r-1} \log y}{[1 - R(1 - y^{r-1}) - y^q]^{\frac{p+1}{p}}} + \\ - \frac{m}{p} \frac{[-(1 + m^{r-1}y^{r-1})\partial_r R - R\, m^{r-1}y^{r-1}(\log m + \log y)]}{[1 - R(1 + m^{r-1}y^{r-1}) - m^q y^q]^{\frac{p+1}{p}}}.$$

Therefore, in order to compute the derivative of h with respect to r, we need to differentiate R with respect to r. We have

$$\partial_r R = -\frac{1 - m^q}{(1 + m^{r-1})^2} m^{r-1} \log m$$

and hence

(14)

$$\partial_r h(m, p, q, r, y) = -\frac{1}{p} \frac{1 - n^q}{(1 + m^{r-1})^2} \left\{ \frac{(1 - y^{r-1})m^{r-1} \log n + y^{r-1}(1 + m^{r-1}) \log y}{[1 - R(1 - y^{r-1}) - y^q]^{\frac{p+1}{p}}} + \right. \\ \left. + m^r \frac{(1 + m^{r-1}y^{r-1}) \log m - (1 + m^{r-1})y^{r-1}(\log m + \log y)}{[1 - R(1 + m^{r-1}y^{r-1}) - m^q y^q]^{\frac{p+1}{p}}} \right\}.$$

Step 2 (The monotonicity of h). It is easily seen that the numerator of the first ratio, in the curly brackets of (14), is negative. If the numerator of the second ratio is also negative, we get the desired monotonicity. Otherwise, if this second numerator is positive, let us observe that

$$m^q(1 - R(1 - y^{r-1}) - y^q) \leq [1 - R(1 + m^{r-1}y^{r-1}) - m^q y^q],$$

that implies:

$$\partial_r h(m,p,q,r,y) \geq -\frac{1}{p}\frac{1-m^q}{(1+m^{r-1})^2}\left\{\frac{(1-y^{r-1})m^{r-1}\log m + y^{r-1}(1+m^{r-1})\log y}{m^{\frac{q(p-1)}{p}}\left[1-R(1+m^{r-1}y^{r-1})-m^q y^q\right]^{\frac{p+1}{p}}}+\right.$$
$$\left. + m^r\frac{(1+m^{r-1}y^{r-1})\log m-(1+m^{r-1})y^{r-1}(\log m+\log y)}{\left[1-R(1+m^{r-1}y^{r-1})-m^q y^q\right]^{\frac{p+1}{p}}}\right\}. \tag{15}$$

Hence, by setting

$$g(m,p,q,r,y) := \Big[-(1-y^{r-1})m^{r-1}\log m - y^{r-1}(1+m^{r-1})\log y\Big]+$$
$$+\Big[(y^{r-1}-1)\log m + (1+m^{r-1})y^{r-1}\log y\Big]m^{r-\frac{q(p+1)}{p}},$$

we have that (15) can be written as

$$\partial_r h(m,p,q,r,y) \geq \frac{1}{p}\frac{1-m^q}{(1+m^{r-1})^2 m^{\frac{q(p+1)}{p}}}g(m,p,q,r,y). \tag{16}$$

To prove the positivity of $\partial_r h$, we will show that

$$g(m,p,q,r,y) > 0, \tag{17}$$

by proving that g is decreasing for any $y \in]0,1[$. By differentiating g with respect to y, we obtain

$$\begin{aligned}\partial_y g(m,p,q,r,y) &= \Big[(r-1)y^{r-2}m^{r-1}\log m-(r-1)y^{r-2}(1+m^{r-1})\log y - y^{r-2}(1+m^r)\Big]\\ &\quad+\Big[(r-1)y^{r-2}\log m + (1+m^{r-1})((r-1)y^{r-2}\log y + y^{r-2})\Big]m^{r-\frac{q(p+1)}{p}}\\ &= y^{r-2}\Big[(r-1)(m^{r-1}+m^{r-\frac{q(p+1)}{p}})\log m + (r-1)(1+m^{r-1})\left(m^{r-\frac{q(p+1)}{p}}-1\right)\log y\\ &\quad+(1+m^{r-1})\left(m^{r-\frac{q(p+1)}{p}}-1\right)\Big].\end{aligned}$$

This derivative is negative if and only if

$$\log y < -\frac{\left(m^{r-1}+m^{r-\frac{q(p+1)}{p}}\right)\log m}{(1+m^{r-1})\left(m^{r-\frac{q(p+1)}{p}}-1\right)}-\frac{1}{r-1}. \tag{18}$$

Since the left-hand term is negative, then if the the right-hand side of (18) is nonnegative, then the inequality (18) holds. To this aim, we wiil equivalently show that

$$f(m,p,q,r) := -\left(m^{r-1}+m^{r-\frac{q(p+1)}{p}}\right)\log m - \frac{1}{r-1}\left(1+m^{r-1}\right)\left(m^{r-\frac{q(p+1)}{p}}-1\right) > 0. \tag{19}$$

Therefore, we have

$$\begin{aligned} f(m,p,q,r) &= \left(m^{r-1} + m^{r-\frac{q(p+1)}{p}}\right)\log\frac{1}{m} + \frac{1}{r-1}\left(1 + m^{r-1} - n^{r-\frac{q(p+1)}{p}} - m^{2r-1-\frac{q(p+1)}{p}}\right) \\ &= m^{r-1}\left(\log\frac{1}{n} + \frac{1}{r-1}\right) + m^{r-\frac{q(p+1)}{p}}\left(\log\frac{1}{m} - \frac{1}{r-1}\right) + \frac{1}{r-1}\left(1 - m^{2r-1-\frac{q(p+1)}{p}}\right) \\ &\geq m^{r-1}\left(\log\frac{1}{m} + \frac{1}{r-1}\right) + m^{r-\frac{q(p+1)}{p}}\left(\log\frac{1}{m} - \frac{1}{r-1}\right) \\ &= m^{r-\frac{q(p+1)}{p}}\left(m^{\frac{q(p+1)}{p}-1}\left(\log\frac{1}{m} + \frac{1}{r-1}\right) + \log\frac{1}{m} - \frac{1}{r-1}\right). \end{aligned}$$

Hence, then the positivity of f as in (19) follows by

$$e(m,p,q,r) := m^{\frac{q(p+1)}{p}-1}\left(\log\frac{1}{m} + \frac{1}{r-1}\right) + \log\frac{1}{m} - \frac{1}{r-1} > 0. \tag{20}$$

Since $e(1,p,q,r) = 0$, to prove (20), we show that e is decreasing with respect to m; indeed, we have

$$\partial_m e(n,p,q,r) = m^{\frac{q(p+1)}{p}-2}\left(\log\left(\frac{1}{m^{\frac{q(p+1)}{p}-1}}\right) + \frac{\frac{q(p+1)}{p}-1}{r-1} - 1 - \frac{1}{m^{\frac{q(p+1)}{p}-1}}\right)$$

that is negative since $\log z < z-1$ when $z > 1$ and $m^{\frac{q(p+1)}{p}-1} < 1$, $p \geq q$ and $r \geq \frac{q}{2}+1$.

Hence (20), (19), (18) and (17) are satisfied and recalling the behavior h from (16), this implies that

$$\partial_r h(m,p,q,r,y) \geq \frac{1}{p}\frac{1-m^q}{(1+m^{r-1})^2} g(m,p,q,r,y) > \frac{1}{p}\frac{1-m^q}{(1+m^{r-1})^2} g(m,p,q,r,1) = 0.$$

when $m \in]0,1[$ and $y \in]0,1[$.

Step 3 (The trivial cases) We observe that if $m = 0$, then $R = 1$ and

$$h(0,p,q,r,y) = \frac{1}{(y^{r-1} - y^q)^{\frac{1}{p}}},$$

that is strictly increasing with respect to r.

Meanwhile, if $m = 1$, then $R = 0$ and

$$h(1,p,q,r,y) = \frac{2}{(1-y^q)^{\frac{1}{p}}},$$

that is constant with respect to r.

Finally, when $y = 0$, we have

$$h(m,p,q,r,0) = \frac{1+m}{1-R},$$

that is strictly increasing with respect to r. □

At this stage, to prove the monotonicity of H with respect to m, we argue using a change of variables similarly as in [GGR]. Before providing the result, let us explicitly note that, in the previous Lemma, we have only supposed that $q \leq p$ and $\frac{q}{2}+1 \leq r \leq q + \frac{q}{p}$. These two conditions implies that $q \geq \frac{2p}{p+2}$ but, for the following result, we need to suppose a bit more: $q \geq \frac{4}{5}p$, that is also the assumption we use to prove the main Theorems.

Lemma 3.2. *Let $p, q > 1$ be such that $\frac{4}{5}p \le q \le p$, then $\partial_m H(m, p, q, \frac{q}{2}+1) \le 0$ for any $m \in]0,1[$.*

Proof. Let us consider the following nonnegative functions

$$A(m,y) := n^{\frac{q}{2}} + (1-m^{\frac{q}{2}})y^{\frac{q}{2}} - y^q, \quad \forall\ (m,y) \in [0,1]^2,$$
$$B(m,y) := n^{\frac{q}{2}} - (1-m^{\frac{q}{2}})m^{\frac{q}{2}}y^{\frac{q}{2}} - m^q y^q, \quad \forall\ (m,y) \in [0,1]^2.$$

Moreover, let us observe that

$$R\left(m, q, \frac{q}{2}+1\right) = 1 - m^{\frac{q}{2}}, \quad \forall\ m \in [0,1].$$

Hence $K(m) = \int_0^1 \left(A(m,y)^{-\frac{1}{p}} + mB(m,y)^{-\frac{1}{p}}\right) dy$ and

$$K'(m) = -\frac{1}{p}\int_0^1 \left(A(m,y)^{-\frac{1}{p}-1}\partial_m A(m,y) + B(m,y)^{-\frac{1}{p}-1}\left(-pB(m,y) + m\partial_m B(m,y)\right)\right) dy.$$

For sake of simplicity, we set

$$K(m) := H\left(m, p, q, \frac{q}{2}+1\right).$$

Differentiating $A(m,y)$ and $B(m,y)$ with respect to m, we obtain

$$\begin{aligned}\partial_m A(m,y) &= \frac{q}{2}m^{\frac{q}{2}-1}(1-y^{\frac{q}{2}})\\ -pB(m,y) + mB_m(m,y) &= \left(\frac{q}{2}-p\right)m^{\frac{q}{2}}(1-y^{\frac{q}{2}}) + (q-p)\, m^q(y^{\frac{q}{2}} - y^q).\end{aligned}$$

Hence we have

$$K'(m) = m^{\frac{q}{2}-1}\int_0^1 -\frac{q}{2p}\frac{1-y^{\frac{q}{2}}}{A(m,y)^{\frac{1}{p}+1}} + \left(1-\frac{q}{2p}\right)\frac{m(1-y^{\frac{q}{2}})}{B(m,y)^{\frac{1}{p}+1}} + \left(1-\frac{q}{p}\right)\frac{m^{\frac{q}{2}+1}(y^{\frac{q}{2}}-y^q)}{B(m,y)^{\frac{1}{p}+1}}\ dy.$$

To prove the nonpositivity of the integral, we have to show that
(21)
$$\frac{q}{2p}\int_0^1 \frac{1-y^{\frac{q}{2}}}{A(n,y)^{\frac{p+1}{p}}}\ dy \ge \int_0^1 \left(1-\frac{q}{2p}\right)\frac{n(1-y^{\frac{q}{2}})}{B(n,y)^{\frac{1}{p}+1}} + \left(1-\frac{q}{p}\right)\frac{n^{\frac{q}{2}+1}(y^{\frac{q}{2}}-y^q)}{B(n,y)^{\frac{1}{p}+1}}\ dy.$$

Following the ideas of [GGR], for all $n \in (0,1)$ and $z \in (0,1)$, we set

$$\delta(z) := [1 - (1-m^{\frac{q}{2}})z^{\frac{q}{2}}]^{\frac{2}{q}}$$

and

$$\ell(z) := \frac{mz}{\delta(z)}.$$

It holds that $\ell(0) = 0$, $\ell(1) = 1$ and

$$\ell'(z) := \frac{m}{\delta(z)^{\frac{q}{2}+1}}.$$

The function ℓ is strictly increasing and, keeping the change of variables $y = \ell(z)$ into account, the inequality (21) follows if we prove that

$$\frac{q}{2p}\int_0^1 \frac{1-m^{\frac{q}{2}}z^{\frac{q}{2}}\delta(z)^{-\frac{q}{2}}}{\left(m^{\frac{q}{2}}+(1-m^{\frac{q}{2}})m^{\frac{q}{2}}z^{\frac{q}{2}}\delta(z)^{-\frac{q}{2}}-m^q y^q\delta(z)^{-q}\right)^{\frac{p+1}{p}}}\cdot\frac{m}{\delta(z)^{\frac{q}{2}+1}}\,dz$$
$$\geq \int_0^1 \frac{\left(1-\frac{q}{2p}\right)m(1-y^{\frac{q}{2}})+\left(1-\frac{q}{p}\right)m^{\frac{q}{2}+1}(y^{\frac{q}{2}}-y^q)}{\left(m^{\frac{q}{2}}-(1-n^{\frac{q}{2}})m^{\frac{q}{2}}z^{\frac{q}{2}}-m^q z^q\right)^{\frac{p+1}{p}}}\,dz,$$

Since it is easily checked that $1-z^{\frac{q}{2}} = \delta(z)^{\frac{q}{2}} - m^{\frac{q}{2}}z^{\frac{q}{2}}$ and

$$m^{\frac{q}{2}} - (1-m^{\frac{q}{2}})n^{\frac{q}{2}}y^{\frac{q}{2}} - m^q y^q = \delta(y)^q\left(m^{\frac{q}{2}}+(1-m^{\frac{q}{2}})\frac{m^{\frac{q}{2}}y^{\frac{q}{2}}}{\delta(y)^{\frac{q}{2}}}-\frac{m^q y^q}{\delta(y)^q}\right),$$

the conclusion follows. □

The two previous Lemmata yield to the following estimates for the function H.

Proposition 3.3. *Let $p,q,r>1$ be such that $\frac{4}{5}p \leq q \leq p$.*

(i) *If $\frac{q}{2}+1 \leq r \leq q+\frac{q}{p}$, then*
$$H\left(m,p,q,r\right) \geq \pi_{p,q}$$
for any $m \in [0,1]$.

(ii) *If $\frac{q}{2}+1 < r \leq q+\frac{q}{p}$, then*
$$H\left(n,p,q,r\right) = \pi_{p,q}$$
if and only if $m=1$.

Proof. Case (i). If $m=1$, by the definition (13) of H, we have that

$$H(1,p,q,r) = 2\int_0^1 \frac{dy}{(1-y^q)^{\frac{1}{p}}} = \pi_{p,q}. \tag{22}$$

If $m=0$, it is easily seen that

$$H(0,p,q,r) = \int_0^1 \frac{dy}{(y^{r-1}-y^q)^{\frac{1}{p}}} \geq \int_0^1 \frac{dy}{(1-y^q)^{\frac{1}{p}}} = \pi_{p,q}. \tag{23}$$

When $0<m<1$, by Lemmata 3.1 and 3.2, we have

$$H(m,p,q,r) \geq H\left(m,p,q,\frac{q}{2}+1\right) \geq H(1,p,q,r) = \pi_{p,q}. \tag{24}$$

Case (ii). The sufficient condition follows by (22), meanwhile we the necessary condition follows by observing that, if $m=0$, the inequality in (23) is strict and, if $m \in (0,1)$, the first inequality in (24) is strict. □

4. Proof of the main Theorems

A key role in the proof of the main result is played by the sign-changing minimizers. When this kind of solution occurs, both the eigenvalue and the eigenfunctions admit a representation throughout the function H introduced in the previous section.

Proposition 4.1. *Let $p, q, r > 1$ be such that $\frac{4}{5}p \le q \le p$ and $\frac{q}{2} + 1 \le r \le q + \frac{q}{p}$ and suppose that there exists $\alpha > 0$ such that $\lambda_\alpha(p, q, r)$ admits a minimizer y that changes sign in $[-1, 1]$. Then the following properties hold.*

(i) *The minimizer y has exactly one maximum point η_M in $[-1, 1]$, has exactly one minimum point η_m in $[-1, 1]$ and, up to a multiplicative constant, satisfies*

$$y(\eta_M) = 1 = \max_{[-1,1]} y(x), \quad y(\eta_m) = -m = \min_{[-1,1]} y(x), \quad \text{with } m \in]0, 1]. \tag{25}$$

(ii) *If $y_+ \ge 0$ and $y_- \le 0$ are, respectively, the positive and negative part of y, then y_+ and y_- are, respectively, symmetric about $x = \eta_M$ and $x = \eta_m$.*

(iii) *There exists a unique zero of y in $]-1, 1[$.*

(iv) *The following representations hold*

$$\lambda_\alpha(p, q, r) = \frac{q}{p'} ||y||_q^{q-p} H^p (m, p, q, r),$$
$$||y||_q = \left[\frac{r - 1 + p'}{q + p'} \gamma + (1 - R(m, q, r)) \frac{2p'}{p' + q} \right]^{\frac{1}{q}}.$$

(v) $\lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$.

Proof. For the sake of simplicity, throughout the proof, we will write $\lambda = \lambda_\alpha(p, q, r)$. We can multiply the sign-changing minimizer y of λ times a suitable (positive or negative) constant such that (25) is verified.

By multiplying equation in (12) for y' and integrating in $]-1, 1[$, we get

$$\frac{1}{p'}|y'|^p + \frac{\lambda ||y||_q^{p-q}}{q}|y|^q = \frac{\alpha|\gamma|^{\frac{p}{r-1}-2}\gamma}{r-1}|y|^{r-2}y + c \tag{26}$$

for a suitable constant c, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Therefore, since $y'(\eta_M) = 0$ and $y(\eta_M) = 1$, $y'(\eta_m) = 0$ and $y(\eta_m) = -m$, we have

$$c = \frac{\lambda ||y||_q^{p-q}}{q} - \frac{\alpha|\gamma|^{\frac{p}{r-1}-2}\gamma}{r-1} = \frac{\lambda ||y||_q^{p-q}}{q} m^q + \frac{\alpha|\gamma|^{\frac{p}{r-1}-2}\gamma}{r-1} m^{r-1}.$$

Hence, we obtain

$$\begin{cases} \frac{\alpha|\gamma|^{\frac{p}{r-1}-2}\gamma}{r-1} = \frac{\lambda ||y||_q^{p-q}}{q} R(m, q, r) \\ c = \frac{\lambda ||y||_q^{p-q}}{q} (1 - R(m, q, r)). \end{cases} \tag{27}$$

So, equation (26) can be written as

$$\frac{1}{p'}|y'|^p + \frac{\lambda ||y||_q^{p-q}}{q}|y|^q = \frac{\lambda ||y||_q^{p-q}}{q} R(m, q, r) |y|^{r-2} y + \frac{\lambda ||y||_q^{p-q}}{q} (1 - R(m, q, r)). \tag{28}$$

and as

$$|y'|^p = \frac{p'\lambda||y||_q^{p-q}}{q}\left(1 - R\left(m,q,r\right)\left(1 - |y|^{r-2}y\right) - |y|^q\right). \tag{29}$$

It is easy to see that the number of zeros of y has to be finite, hence let

$$-1 = \zeta_1 < \ldots < \zeta_j < \zeta_{j+1} < \ldots < \zeta_n = 1$$

be the zeroes of y and (see also [CD]) that

$$y'(x) = 0 \iff y(x) = -m \text{ or } y(x) = 1. \tag{30}$$

If we set

$$\mu(y) := 1 - R\left(m,q,r\right)\left(1 - |y|^{r-2}y\right) - y^q, \quad y \in \left[-m, 1\right],$$

then (29) gives

$$|y'|^p = \frac{p'\lambda||y||_q^{p-q}}{q}\mu\left(y\right). \tag{31}$$

Let us observe that $\mu(-m) = \mu(1) = 0$. Being $q \geq r - 1$ by assumption, it is easily seen that for any $\overline{y}$ such that $\mu'(\overline{y}) = 0$ then $\mu(\overline{y}) > 0$. Hence, μ does not vanish in $]-m, 1[$ and, therefore, by (31), $y'(x) \neq 0$ if $y(x) \neq 1$ and $y(x) \neq -m$, that proves (30).

This implies that y has no other local minima or maxima in $]-1,1[$, that in any interval $]\zeta_j, \zeta_{j+1}[$ where $y > 0$ there is a unique maximum point and that in any interval $]\zeta_j, \zeta_{j+1}[$ where $y < 0$ there is a unique minimum point.

Then the properties *(i)*, *(ii)* and *(iii)* follows by adapting the argument of [DGS, Lemma 2.6], see also [DPP1] for the case $p = 2$. We remark that they can be also proved by using a symmetrization argument, by rearranging the functions y^+ and y^- and using the Pólya-Szegő inequality and the properties of rearrangements (see also, for example, [BFNT] and [DP]). Specifically, one can prove that

- in any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y and in which $y = y^+ > 0$, has the same length; in any of such intervals, y^+ is symmetric about $x = \frac{\zeta_j + \zeta_{j+1}}{2}$;
- in any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y and in which $y = y^- < 0$ has the same length; in any of such intervals, y^- is symmetric about $x = \frac{\zeta_j + \zeta_{j+1}}{2}$;
- there is a unique zero of y in $]-1,1[$.

In order to show *(iv)*, it is not restrictive to suppose the order relation $\eta_M < \eta_m$ between the unique maximum and the unique minimum point of y. It is easily seen ([DGS, Lem. 2.6]) that $\eta_M - \eta_m = 1$, with $y' < 0$ in $]\eta_M, \eta_m[$. Then, from (29), we have

$$\frac{-y'}{\left[1 - R\left(m,q,r\right)\left(1 - |y|^{r-2}y\right) - y^q\right]^{\frac{1}{p}}} = \left(\frac{p'\lambda||y||_q^{p-q}}{q}\right)^{\frac{1}{p}} \quad \text{in }]\eta_M, \eta_m[.$$

Then, integrating between η_M and $\eta_{\bar{m}}$, we have

(32)

$$\lambda = \frac{q}{p'}||y||_q^{q-p}\left[\int_{-m}^{1} \frac{dz}{\left[1 - R\left(m,q,r\right)\left(1 - |y|^{r-2}y\right) - y^q\right]^{\frac{1}{p}}}\right]^p = \frac{q}{p'}||y||_q^{q-p}H^p\left(m,p,q,r\right),$$

that is the first part of *(iv)*. The second part follows by integrating (28) over $(-1,1)$ and recalling that $||y'||_p^p + \alpha|\gamma|^{\frac{p}{r-1}} = \lambda||y||_q^p$.

Finally, since by Proposition 2.2 we know that $\lim_{\alpha\to+\infty}\lambda_\alpha(p,q,r)=\lambda_T(p,q,r)$ and since the relation (32) does not depends by α, we have

$$\frac{q}{p'}||y||_q^{q-p}H^p\left(m,p,q,r\right)=\lim_{\alpha\to+\infty}\lambda_\alpha(p,q,r)=\lambda_T(p,q,r),$$

that gives *(v)*. □

At this stage, we are in position to state that each sign-changing minimizer of problem (9) is a symmetric and zero average function.

Proposition 4.2. *Let $p,q,r>1$ be such that $\frac{4}{5}p\le q\le p$ and suppose that there exists $\alpha>0$ such that $\lambda_\alpha(p,q,r)$ admits a minimizer y that changes sign in $[-1,1]$ and satisfies the conditions in* (25)*.*

- (i) *If $\frac{q}{2}+1<r\le q+\frac{q}{p}$, then*
  - (a) $\int_{-1}^{1}|y|^{r-2}y\,dx=0$;
  - (b) $y(x)=C\sin_{p,q}(\lambda_T(p,q,r)x)$*, with* $C\in\mathbb{R}\setminus\{0\}$*;*
  - (c) *the only point* $\bar{x}\in]-1,1[$ *where y vanishes is* $\overline{x}=0$*.*
- (ii) *If $r=\frac{q}{2}+1$ and $\int_{-1}^{1}|y|^{r-2}y\,dx=0$, then $y(x)=C\sin_{p,q}(\lambda_T(p,q,r)x)$, with $C\in\mathbb{R}\setminus\{0\}$, and the only point in $\bar{x}\in]-1,1[$ where y vanishes is $\overline{x}=0$.*

Proof. In the case that $\frac{q}{2}+1<r\le q+1$, we know from [CD, Thm. 1.1] the exact value of the best constant in the Twisted inequality (and let us note that there is no dependence by the parameter r). Therefore, by Proposition 4.1*(iv)* and *(v)*, we have

$$\begin{aligned}\lambda_T(p,q,r)&=\left[\left(\frac{1}{p'}\right)^{\frac{1}{q}}\left(\frac{1}{q}\right)^{\frac{1}{p'}}\left(\frac{2}{p'+q}\right)^{\frac{1}{p}-\frac{1}{q}}q\right]^p\pi_{p,q}^p=\frac{q}{p'}\left(\frac{2p'}{p'+q}\right)^{1-\frac{p}{q}}\pi_{p,q}^p\\&\le\frac{q}{p'}\left[\frac{r-1+p'}{q+p'}\gamma+(1-R(m,q,r))\frac{2p'}{p'+q}\right]^{1-\frac{p}{q}}H^p\left(m,p,q,r\right)\\&=\lambda_\alpha(p,q,r)=\lambda_T(p,q,r).\end{aligned}\tag{33}$$

Hence, since by Proposition 3.3*(ii)* we know that $H\left(m,p,q,r\right)=\pi_{p,q}$ if and only if $m=1$, the strict decrease of R with respect to m and the first identity of (27) gives that

$$\int_{-1}^{1}|y|^{r-2}ydx=0,\tag{34}$$

that is *(a)*. To prove *(b),(c)*, let us explicitly observe that, when (34) holds, then y solves problem (11) with $\lambda=\lambda_T(p,q,r)||y||_q^{p-q}$. Hence $y(x)=C\sin_{p,q}(\pi_{p,q}x)$, with $C\in\mathbb{R}\setminus\{0\}$.

The case *(ii)* easily follows using the same arguments. □

At the previous results give the tools to prove the main Theorems of this paper.

Proof of Theorem 1.1. When $\alpha\le0$, the minimizers of (9) have constant sign; indeed

$$\mathcal{Q}_\alpha[u]\ge\mathcal{Q}_\alpha[|u|],$$

with equality if and only if $u\ge0$ or $u\le0$.

In order to prove the main result, we will show that there exists $\alpha>0$ for which the problem (9) admits a minimizer y that changes sign. By contradiction, we suppose that for any $k\in\mathbb{N}$, there exists a divergent sequence α_k, and a corresponding sequence of nonnegative eigenfunctions $\{y_k\}_{k\in\mathbb{N}}$ relative to $\lambda_{\alpha_k}(p,q,r)$ such that and $\|y_k\|_p = 1$.

By Proposition 2.2, we have that $\lambda_{\alpha_k}(p,q,r) \le \lambda_T(p,q,r)$ and hence, it holds that

$$\int_{-1}^{1} |y_k'|^p \, dx + \alpha_k \left(\int_{-1}^{1} y_k^{r-1} \, dx \right)^{\frac{p}{r-1}} \le \lambda_T(p,q,r). \tag{35}$$

Therefore, y_k converges (up to a subsequence) to a function $y \in W_0^{1,p}(-1,1)$, strongly in $L^p(-1,1)$ and weakly in $W_0^{1,p}(-1,1)$. Moreover $\|y\|_p = 1$ and y is not identically zero. Therefore $\|y\|_{r-1} > 0$ and, letting $\alpha_k \to +\infty$ in (35) we have a contradiction. Therefore we have proved there exists a positive value of α such that the minimum problem (9) admits an eigenfunction y that satisfies $\int_{-1}^{1} |y|^{r-2} y \, dx = 0$. In such a case, $\lambda_\alpha(p,q,r) = \lambda_T(p,q,r)$ and, up to a multiplicative constant, $y = \sin_{p,q}(\pi_{p,q} x)$.

Since, by Proposition 2.1, $\lambda_\alpha(p,q,r)$ is a nondecreasing Lipschitz function in α, we can define

$$\alpha_C = \min\{\alpha \in \mathbb{R} \colon \lambda_\alpha(p,q,r) = \lambda_T(p,q,r)\} = \sup\{\alpha \in \mathbb{R} \colon \lambda_\alpha(p,q,r) < \lambda_T(p,q,r)\},$$

and it is easily verified that this value of the parameter is positive □

Proof of Theorem 1.2. If $\alpha < \alpha_C$, the minimizers corresponding to $\lambda_\alpha(p,q,r)$ have constant sign, otherwise $\lambda_\alpha(p,q,r) = \lambda_T(p,q,r)$. When $\alpha > \alpha_C$, then any minimizer y corresponding to α is such that $\int_{-1}^{1} |y|^{r-2} y \, dx = 0$. Indeed, if we assume, by contradiction, that there exist $\bar\alpha > \alpha_C$ and $\bar y$ such that $\int_{-1}^{1} |\bar y|^{r-2} \bar y \, dx > 0$, $\|y\|_p = 1$ and $\mathcal{Q}_{\bar\alpha}[\bar y] = \lambda_{\bar\alpha}(p,q,r)$, then

$$\begin{aligned} \mathcal{Q}_{\bar\alpha-\varepsilon}[\bar y] &= \mathcal{Q}_{\bar\alpha}[\bar y] - \varepsilon \left(\int_{-1}^{1} |\bar y|^{r-2} \bar y \, dx \right)^{\frac{p}{r-1}} \\ &= \lambda_{\bar\alpha}(p,q,r) - \varepsilon \left(\int_{-1}^{1} |\bar y|^{r-2} \bar y \, dx \right)^{\frac{p}{r-1}} < \lambda_{\bar\alpha}(p,q,r). \end{aligned}$$

Hence, for ε sufficiently small, $\lambda_T(p,q,r) = \lambda_{\alpha_C}(p,q,r) \le \lambda_{\bar\alpha-\varepsilon}(p,q,r) < \lambda_{\bar\alpha}(p,q,r)$ and this is absurd. Finally, by Proposition 4.2, the proof of of *(i)* and *(ii)* follows. Regarding *(iii)*, it is not difficult to see, by means of approximating sequences, that $\lambda_{\alpha_C}(p,q,r)$ admits both a nonnegative minimizer and a minimizer with vanishing r-average.

To conclude the proof of Theorem 1.2, we have to study the behavior of the solutions when $r = p+1$. When $\alpha = \alpha_C(p,q,p+1)$, the corresponding positive minimizer y is a solution of

$$\begin{cases} (|y'|^{p-2} y')' + \lambda_T(p,q,p+1) ||y||_q^{p-q} y^{q-1} = \alpha_C(p,q,p+1) y^{q-1} & \text{in }]-1,1[\\ y(-1) = y(1) = 0. \end{cases}$$

The positivity of the eigenfunction guarantees that (refer also to (33)):

$$\lambda_T(p,q,p+1) ||y||_q^{p-q} - \alpha_C(p,q,p+1) = \lambda_0(p,q,p+1) ||y||_q^{p-q} = \frac{q}{p'} \left(\frac{2p'}{p'+q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p,$$

hence $\alpha_C(p,q,p+1) = \frac{2^p-1}{2^p} \frac{q}{p'} \left(\frac{2p'}{p'+q} \right)^{1-\frac{p}{q}} \pi_{p,q}^p$. □

Remark 4.3. When the exponents p, q, r satisfy the same assumptions of the main Theorems, we obtain the following lower bound on $\alpha_C(p, q, r)$:

$$\alpha_C(p,q,r) \geq \frac{2^p - 1}{2^{\frac{p}{r-1}+p-1}} \frac{q}{p'} \left(\frac{2p'}{p'+q} \right)^{1-\frac{p}{q}} \pi_{p,q}^p. \tag{36}$$

To get the estimate (36), we use the monotonicity of $\lambda_\alpha(p, q, r)$ with respect to α, and consider the test function $u(x) = \sin_{p,q}(\frac{\pi_{p,q}}{2}(x + 1))$. Hence

$$\begin{aligned}\lambda_T(p,q,r) = \lambda_{\alpha_C}(p,q,r) \leq \mathcal{Q}[u,\alpha_C] &= \frac{q}{p'} \left(\frac{2p'}{p'+q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p + \alpha_C \left(\int_{-1}^{1} u^{r-1} dx \right)^{\frac{p}{r-1}} \\ &\leq \frac{q}{p'} \left(\frac{2p'}{p'+q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p + \alpha_C 2^{\frac{p}{r-1}-1}.\end{aligned}$$

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