On the topology of $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$

Tommaso Rossi*
Dipartimento di Matematica, Università di Roma Tor Vergata

Abstract

This paper contains some results about the topology of $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$, where $\mathcal{M}_{0,n+1}$ is the moduli space of genus zero Riemann surfaces with marked points and $\overline{\mathcal{M}}_{0,n+1}$ is its Deligne-Mumford compactification. We show that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for $n \geq 4$, and they are simply connected for any $n \in \mathbb{N}$. We also present some homology computations: for example we show that the homology of $\mathcal{M}_{0,n+1}/\Sigma_n$ is all torsion and that $\mathcal{M}_{0,p+1}/\Sigma_p$ has no p torsion, where p is a prime. Lastly we compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ for small values of n, proving that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for $n \leq 5$ while $\mathcal{M}_{0,7}/\Sigma_6$ is not.

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1 Introduction

Let $\mathcal{M}_{0,n+1}$ be the moduli space of genus zero Riemann surfaces with $n+1$ marked points and $\overline{\mathcal{M}}_{0,n+1}$ be its Deligne-Mumford compactification. In this paper we study the topology of the quotients $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$. The author's interest in these spaces comes from operad theory: the collections of graded vector spaces (here s denotes the degree shift)

$$Grav \coloneqq \{sH_*(\mathcal{M}_{0,n+1};\mathbb{Q})\}_{n\geq 2} \qquad Hycom \coloneqq \{H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})\}_{n\geq 2}$$

turns out to have a natural operad structure, coming from the geometry of $\mathcal{M}_{0,n+1}$ and $\overline{\mathcal{M}}_{0,n+1}$. These operads are called respectively the Gravity and the Hypercommutative operad and were studied intesively in the nineties by Getzler [9], [10], Barannikov-Kontsevich [2] and Kontsevich-Manin [14]. Algebras over these operads can be thought as a generalization respectively of Lie algebras and commutative algebras (for a precise definition see Paragraph 4.2.1 and Paragraph 4.2.2). The analogy do not stops here: as the Lie operad and the commutative operad are Koszul dual to each other, in the same way *Grav* and *Hycomm* are Koszul dual operads (see [11] and [9]). It turns out (Proposition 4.13 and Proposition 4.16) that

$$\bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q}) \quad \text{and} \quad \bigoplus_{n=2}^{\infty} H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

are respectively the free Gravity algebra and the free Hypercommutative algebra on a generator of degree zero. The main results are listed below:

- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are not topological manifolds for any $n\geq 4$ (Theorem 2.9).
- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are simply connected (Theorems 4.4 and 4.5).
- $\mathcal{M}_{0,n+1}/\Sigma_n$ has the same rational homology of the point (Theorem 4.12).
- $\mathcal{M}_{0,p+1}/\Sigma_p$ and $\mathcal{M}_{0,p+2}/\Sigma_{p+1}$ have no p-torsion (Theorem 5.16), where p is a prime number.
- $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for $n\leq 5$ (see Paragraph 5.5).

Here is the outline of the paper:

Section 2 presents an embedding of $\mathcal{M}_{0,n+1}/\Sigma_n$ into the weighted projective space $\mathbb{P}(n, n-1, \ldots, 2)$ as an open dense subset. This is crucial to prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for $n \geq 4$ (Theorem 2.9). We also describe $\mathcal{M}_{0,n+1}/\Sigma_n$ as the complement of an explicit algebraic subvariety of $\mathbb{P}(n, n-1, \ldots, 2)$.

Section 3 contains a cambinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$ based on cacti. This is useful to do computations when n is small.

Section 4 deals with the computation of the fundamental group and the rational homology of $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$.

Section 5 contains some results about $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p)$, where p is a prime number. Firstly we give an upper bound for the order of a class in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$ (Theorem 5.1). Then we construct a long exact sequence (Proposition 5.5) that express $H^*(X/S^1; \mathbb{F}_p)$ in terms of $H^*_{S^1}(X; \mathbb{F}_p)$, $H^*_{S^1}(X^{\mathbb{Z}/p}; \mathbb{F}_p)$ and $H^*(X^{\mathbb{Z}/p}/S^1; \mathbb{F}_p)$, where X is any suitably nice S^1-space. This sequence is an easy consequence of classical facts about transformation groups, but I was not able to find a reference for it. So I decided to include a detailed construction. Since $\mathcal{M}_{0,n+1}/\Sigma_n$ turns out to be homotopy equivalent to $C_n(\mathbb{C})/S^1$, where $C_n(\mathbb{C})$ is the unordered configuration space of point in the plane, we will use the long exact sequence of Proposition 5.5 to get some information about $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p) \cong H_*(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$. In particular we compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p)$ when $n \neq 0, 1 \bmod p$, and we prove that there is no p torsion in $\mathcal{M}_{0,p+1}/\Sigma_p$ and $\mathcal{M}_{0,p+2}/\Sigma_{p+1}$. We also compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$ for small values of n, and this proves that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for any $n \leq 5$, while it has non trivial homology when $n = 6$.

Section 6 contains the computation of $H^{S^1}_*(C_n(\mathbb{C}^*); \mathbb{F}_p)$ when $n \neq 0 \bmod p$. This computation turns out to be useful to prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for any $n \leq 5$, even if it is an interesting topic on its own.

Notations and conventions: if X is a topological space we will indicate by $F_n(X)$ (resp. $C_n(X)$) the ordered (resp. unordered) configuration space.

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2 Topology of $\mathcal{M}_{0,n+1}/\Sigma_n$

2.1 Orbifold structure

In this paragraph we show that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for $n \geq 4$. For completeness we also discuss the case $n = 3$.

Proposition 2.1. *$\overline{\mathcal{M}}_{0,4}/\Sigma_3$ is homeomorphic to S^2. $\mathcal{M}_{0,4}/\Sigma_3$ is homeomorphic to $S^2 - \{0\}$.*

Proof. $\overline{\mathcal{M}}_{0,4}$ is a compact Riemann surface, which is well known to be homeomorphic to S^2. Since Σ_3 acts faithfully and by biholomorphisms we get that the quotient $\overline{\mathcal{M}}_{0,4}/\Sigma_3$ is again a Riemann surface. By the Riemann-Hurwitz formula we can compute the genus of $\overline{\mathcal{M}}_{0,4}/\Sigma_3$, which turns out to be zero giving the first part of the statement. For $\mathcal{M}_{0,4}/\Sigma_3$ just observe that $\mathcal{M}_{0,4}$ is $\overline{\mathcal{M}}_{0,4}$ minus three points (corresponding to the three stable curves). These points are all identified in $\overline{\mathcal{M}}_{0,4}/\Sigma_3$, so $\mathcal{M}_{0,4}/\Sigma_3$ is $\overline{\mathcal{M}}_{0,4}/\Sigma_3$ minus a point, as claimed. □

We now focus on $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ when $n \geq 4$.

Remark 2.2. Observe that $\mathcal{M}_{0,n+1}$ is homeomorphic to $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$: a point in $\mathcal{M}_{0,n+1}$ is just a configuration of points $(p_0, p_1, \ldots, p_n)$ in the Riemann sphere up to biholomorphisms. Un to rotations we can suppose that p_0 is the point at the infinity $\infty \in \mathbb{C} \cup \{\infty\}$. Deleting this point and using the stereographic projection we obtain a configuration of n points in the complex plane $\mathbb{C}$ up to translation, dilatations and rotations, as claimed. From now on we will identify freely $\mathcal{M}_{0,n+1}$ with $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$.

Proposition 2.3. *There is a homeomorphism*

$$\begin{aligned} \phi : \frac{\mathcal{M}_{0,n+1}}{\Sigma_n} &\to \frac{C_n(\mathbb{C})}{\mathbb{C} \rtimes \mathbb{C}^*} \\ [[z_1, \ldots, z_n]] &\mapsto [\{z_1, \ldots, z_n\}] \end{aligned}$$

where $[[z_1 \ldots, z_n]]$ denotes the class associated to $[(z_1, \ldots, z_n)] \in \mathcal{M}_{0,n+1} \cong F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$.

Proof. Σ_n acts on $F_n(\mathbb{C})$ by permuting the coordinates, and this action commutes with that of $\mathbb{C} \rtimes \mathbb{C}^*$ by translations, rotations and dilations. Therefore the quotients $(F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*)/\Sigma_n = \mathcal{M}_{0,n+1}/\Sigma_n$ and $(F_n(\mathbb{C})/\Sigma_n)/\mathbb{C} \rtimes \mathbb{C}^* = C_n(\mathbb{C})/\Sigma_n$ are homeomorphic.

□

Remark 2.4. The unordered configuration space is naturally a subspace of the symmetric power $SP^n(\mathbb{C})$ which is homeomorphic to $\mathbb{C}^n$. The homeomorphism maps an unordered n-uple $\{z_1, \ldots, z_n\}$ to the coefficients $(a_0, \ldots, a_{n-1})$ of the unique monic polinomial $a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n = (z - z_1) \cdots (z - z_n)$ which have $\{z_1, \ldots, z_n\}$ as roots.

Definition. If $\mathbf{z} := (z_1, \dots, z_n)$ is a n-uple of complex numbers, its **barycenter** is

$$B(\mathbf{z}) := \frac{z_1 + \cdots + z_n}{n}$$

Clearly, the barycenter does not depend on the order of the points $z_1, \dots, z_n$.

Before we get into the topology of $\mathcal{M}_{0,n+1}/\Sigma_n$ we recall the definitions of weighted projective space and lens complex, just to fix the notation.

Definition. Let $(b_0, \dots, b_n)$ be a $(n+1)$-tuple of positive integers. The **weighted projective space** (of weights $(b_0, \dots, b_n)$) is defined as

$$\mathbb{P}(b_0, \dots, b_n) := \mathbb{C}^{n+1} - \{0\}/\sim$$

where $(z_0, \dots, z_n) \sim (t^{b_0} z_0, \dots, t^{b_n} z_n)$ for any $t \in \mathbb{C}^*$.

Definition. Let $(b_0, \dots, b_n)$ be a $(n+1)$-tuple of positive integers and $\zeta := e^{2\pi i/b_n}$. The **lens complex** is defined as

$$L(b_n; b_0, \dots, b_{n-1}) := S^{2n-1}/\sim$$

where $(z_0, \dots, z_{n-1}) \sim (\zeta^{b_0} z_0, \dots, \zeta^{b_{n-1}} z_{n-1})$.

Remark 2.5. The topology of $\mathbb{P}(b_0, \dots, b_n)$ and of $L(b_n; b_0, \dots, b_{n-1})$ is well known. For example, the cohomology rings of these spaces were computed by Kawasaki in [13].

Proposition 2.6. *There is an homeomorphism*

$$\psi : \frac{SP^n(\mathbb{C}) - \Delta}{\mathbb{C} \rtimes \mathbb{C}^*} \to \mathbb{P}(n, n-1, \dots, 2)$$
$$[\{z_1, \dots, z_n\}] \mapsto [a_0 : \cdots : a_{n-2}]$$

where $(a_0, \dots, a_{n-2})$ *are the coefficients of the monic polynomial* $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$ *and* Δ *is the diagonal of* $SP^n(\mathbb{C})$.

Remark 2.7. The coefficient a_{n-1} of the monic polynomial $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$ is 0, indeed

$$a_{n-1} = -z_1 + B(\mathbf{z}) + \cdots - z_n + B(\mathbf{z}) = -(z_1 + \cdots + z_n) + nB(\mathbf{z}) = 0$$

Proof. Consider the map

$$f : SP^n(\mathbb{C}) \to SP^n(\mathbb{C})$$
$$\{z_1, \dots, z_n\} \mapsto \{z_1 - B(\mathbf{z}), \dots, z_n - B(\mathbf{z})\}$$

whose image consists of the unordered n-uples whose barycenter is the origin. By Remark 2.4 there is an homeomorphism $g : SP^n(\mathbb{C}) \to \mathbb{C}^n$ which sends a set of n complex numbers $\{z_1, \dots, z_n\}$ to the coefficients $(a_0, \dots, a_{n-1})$ of the monic polynomial $(z - z_1) \cdots (z - z_n)$. The composite $g \circ f$ maps $\{z_1, \dots, z_n\}$ to the coefficients of the monic polynomial $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$. By Remark 2.7 the image of $g \circ f$ is $\{(a_0, \ldots, a_{n-2}, 0) \mid a_i \in \mathbb{C}\} \cong \mathbb{C}^{n-1}$ therefore we get a map

$$SP^n(\mathbb{C}) - \Delta \xrightarrow{g \circ f} \mathbb{C}^{n-1} - \{0\} \to \mathbb{P}(n, n-1, \ldots, 2)$$

where the last map is the natural projection $\mathbb{C}^{n-1} - \{0\} \to \mathbb{P}(n, n-1, \ldots, 2)$. It is easy to show that this map is constant on the $\mathbb{C} \rtimes \mathbb{C}^*$ orbits, so it induces a bijective map

$$\psi : \frac{SP^n(\mathbb{C}) - \Delta}{\mathbb{C} \rtimes \mathbb{C}^*} \to \mathbb{P}(n, n-1, \ldots, 2)$$

which is an homeomorphism because the source is compact and the target is Hausdorff. ☐

Corollary 2.8. *The composition $\psi \circ \phi : \mathcal{M}_{0,n+1} \to \mathbb{P}(n, n-1, \ldots, 2)$ is an embedding.*

Proof. Just combine Proposition 2.3 and Proposition 2.6. ☐

Theorem 2.9. *$\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not a topological manifolds for any $n \geq 4$.*

Proof. Since $\mathcal{M}_{0,n+1}/\Sigma_n$ is an open subset of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$, it suffices to show that there exist a point in $\mathcal{M}_{0,n+1}/\Sigma_n$ which does not have an Euclidean neighbourhood. Let us consider the point $p \in \mathcal{M}_{0,n+1}/\Sigma_n$ consisting of the roots of the polynomial $z^n + 1$. By the embedding of Corollary 2.8, this corresponds to the point $[1 : 0 : \cdots : 0]$ of $\mathbb{P}(n, n-1, \ldots, 2)$. Since $\mathcal{M}_{0,n+1}/\Sigma_n$ is an open subset of $\mathbb{P}(n, n-1, \ldots, 2)$ our claim is equivalent to show that p does not have an Euclidean neighbourhood contained in $\mathbb{P}(n, n-1, \ldots, 2)$. Consider the open set

$$U_0 \coloneqq \{[1 : a_1 : \cdots : a_{n-2}] \in \mathbb{P}(n, n-1 \ldots, 2)\}$$

If $[1 : a_1 : \cdots : a_{n-2}] = [1 : b_1 : \cdots : b_{n-2}]$ then exists $\lambda \in \mathbb{C}^*$ such that

$$\begin{cases} \lambda^n = 1 \\ \lambda^{n-k} a_k = b_k \text{ for } k = 1, \ldots, n-2 \end{cases}$$

Therefore $U_0 \cong \mathbb{C}^{n-2}/\sim$ where $(a_1, \ldots, a_{n-2}) \sim (b_1, \ldots, b_{n-2})$ if and only if there is a n-th root of unity λ such that $\lambda^{n-k} a_k = b_k$ for all $k = 1, \ldots, n-2$. Therefore U_0 is homeomorphic to a cone on the lens complex $L(n; n-1, \ldots, 2)$. The point $p = [1, 0, \ldots, 0]$ is precisely the vertex of this cone, therefore (by excission) we have

$$\begin{aligned} H_k(\mathbb{P}(n, n-1, \ldots, 2), \mathbb{P}(n, n-1, \ldots, 2) - \{p\}) &= H_k(U_0, U_0 - \{p\}) \\ &= H_{k-1}(U_0 - \{p\}) \\ &= H_{k-1}(L(n; n-1, \ldots, 2)) \end{aligned}$$

If p has an euclidean neighbourhood U, the by excission

$$\begin{aligned} H_k(\mathbb{P}(n, n-1, \ldots, 2), \mathbb{P}(n, n-1, \ldots, 2) - \{p\}) &= H_k(U, U - \{p\}) \\ &= \begin{cases} \mathbb{Z} \text{ if } k = 0, 2(n-2) - 1 \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

But this is a contradiction, because the lens complex $L(n; n-1, \ldots, 2)$ is not a homology sphere if $n \geq 4$. ☐

2.2 An algebro-geometric description

Probably an algebraic-geometry minded reader already noticed that the embedding

$$\psi \circ \phi : \mathcal{M}_{0,n+1}/\Sigma_n \hookrightarrow \mathbb{P}(n, n-1, \ldots, 2) \tag{1}$$

we described in Paragraph 2.1 exhibit $\mathcal{M}_{0,n+1}/\Sigma_n$ as the complement of an algebraic subvariety of $\mathbb{P}(n, n-1, \ldots, 2)$. In this short paragraph we simply make this description explicit. We first recall the definition and the main properties of the discriminant of a polynomial.

Definition. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients. The **discriminant** of $p(z)$ is defined as

$$Disc(p(z)) := \frac{(-1)^{n(n-1)/2}}{a_n} \cdot Res(p(z), p'(z))$$

where $Res(p(z), p'(z))$ is the resultant of $p(z)$ and its derivative $p'(z)$, i.e. it is the determinant of the Sylvester matrix of $p(z)$ and $p'(z)$. In particular, $Disc(p(z))$ is a homogeneus polynomial in $a_0, \ldots, a_n$ of degree $2n-2$.

Two important properties of the discriminant are the following:

1. $Disc(p(z)) = 0$ if an only if $p(z)$ has a multiple root.
2. If $a_0^{i_0} a_1^{i_1} \ldots a_n^{i_n}$ is a monomial of $Disc(p(z))$, then $i_0, \ldots, i_n$ satisfy the equation

$$ni_0 + (n-1)i_1 + \cdots + i_{n-1} = n(n-1) \tag{2}$$

   In particular, if $a_n = 1$ we get that $Disc(p(z)) \in \mathbb{C}[a_0, \ldots, a_{n-1}]$ is a weighted homogeneus polynomial (with a_i variable of weight $n-i$) and it makes sense to talk about the zero locus of $Disc(p(z))$ inside $\mathbb{P}(n, n-1, \ldots, 1)$.

Proposition 2.10. *Let $p(z) = a_0 + a_1 z + \cdots + a_{n-2} z^{n-2} + z^n$ be a generic polynomial. The embedding*

$$\mathcal{M}_{0,n+1}/\Sigma_n \hookrightarrow \mathbb{P}(n, n-1, \ldots, 2)$$

maps the space $\mathcal{M}_{0,n+1}/\Sigma_n$ to the complement of $\{Disc(p(z)) = 0\} \subseteq \mathbb{P}(n, n-1, \ldots, 2)$.

Proof. If we identify $\mathcal{M}_{0,n+1}/\Sigma_n$ with $C_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$, then the embedding maps a configuration $[\{z_1, \ldots, z_n\}]$ to the coefficients $[a_0 : \cdots : a_{n-2}]$ of the polynomial $p(z) = (z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$. The complement of the image of this embedding consists exactly of the points $[a_0 : \cdots : a_{n-2}] \in \mathbb{P}(n, n-1, \ldots, 2)$ such that the polynomial $a_0 + a_1 z + \ldots a_{n-2} z^{n-2} + z^n$ has a multiple root. But this is equivalent to say that the discriminant of such a polynomial is zero. Equation 2 tell us that $Disc(p(z))$ is a weighted polynomial with variables a_i of weight $n-i$, thus its zero locus is well defined. $\square$

3 A combinatorial model using cacti

The main goal of this section is to provide a combinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$. The key observation is that $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$, the quotient of the unordered configuration space by the circle action. Starting from the homotopy equivalence between the ordered configuration space $F_n(\mathbb{C})$ and the space of cacti (see the work by McClure-Smith [16]) we will build a CW-complex which is homotopy equivalent to $C_n(\mathbb{C})/S^1$.

3.1 Cacti

In this paragraph we recall the definition and properties of the space of cacti. A combinatorial construction of this space was introduced by McClure and Smith [15], while geometric constructions are due to Voronov [22] and Kaufmann [12]. Salvatore compared the two approaches in [20].

Definition (The space of cacti)**.** Let $\mathcal{C}_n$ be the set of partitions x of S^1 into n closed 1-manifolds $I_1(x), \ldots, I_n(x)$ such that:

- They have equal measure.
- They have pairwise disjoint interiors.
- There is no cyclically ordered sequence of points (z_1, z_2, z_3, z_4) in S^1 such that $z_1, z_3 \in I_j(x)$, $z_2, z_4 \in I_k(x)$ and $j \neq k$.

We can equip this set with a topology by defining a metric on it: for any $x, y \in \mathcal{C}_n$ we set

$$d(x, y) = \sum_{j=1}^{n} \mu(I_j(x) - \mathring{I}_j(y))$$

where μ denotes the measure. We will call $\mathcal{C}_n$ (with this topology) the **space of based cacti with n-lobes**. See Figure 1 for an example.

Remark 3.1. $\mathcal{C}_n$ is called the space of cacti for the following reason: given $x \in \mathcal{C}_n$, let us define a relation $\sim$ on S^1: two points $z_1, z_2 \in S^1$ are equivalent if there is an index $j \in \{0, \ldots, n\}$ such that z_1 and z_2 are the boundary points of the same connected component of $S^1 - \mathring{I}_j$. The quotient space $c(x) := S^1/\sim$ by this relation is a pointed space (the basepoint is just the image of $1 \in S^1$) called the **cactus** associated to x: topologically it is a configuration of n-circles in the plane, called lobes, whose dual graph is a tree. The dual graph is a graph with two kind of vertices: a white vertex for any lobe and a black vertex for any intersection point between two lobes. An edges connects a white vertex w to a black vertex b if b represent the intersection point of the lobe corresponding to w with some other lobe. See Figure 2 for some examples. Note that the lobes are the image of the 1-manifolds $I_1(x), \ldots, I_n(x)$ under the quotient map $S^1 \to S^1/\sim$. In what follows we will freely identify a partition $x \in \mathcal{C}_n$ and its associated cactus $c(x)$.

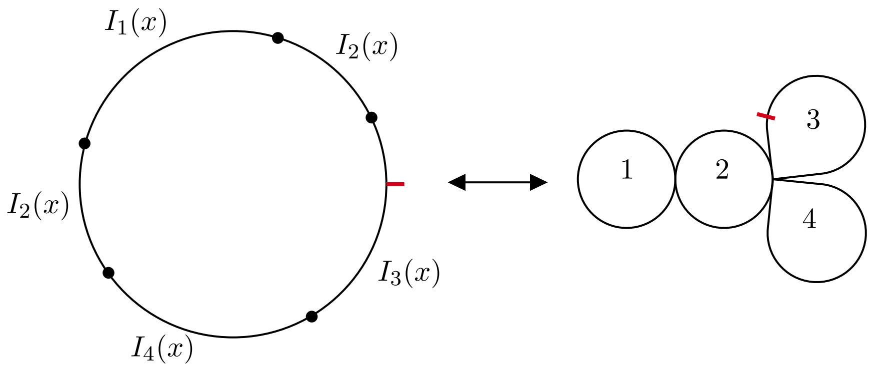


Figure 1: On the left there is an element $x \in \mathcal{C}_4$, on the right its associated cactus $c(x)$. The basepoint of the circle S^1 is depicted in red and corresponds to a basepoint on the cactus $c(x)$ (which we depict as a red spine).

Cell decomposition: to any point $x \in \mathcal{C}_n$ we can associate a sequence $(X_1, \dots, X_l)$ of numbers in $\{1, \dots, n\}$ by the following procedure: start from the point $1 \in S^1$ and move along the circle clockwise. The sequence $(X_1, \dots, X_l)$ is obtained by writing one after the other the indices of all the 1-manifolds you encounter and has the following properties:

- All values between 1 and n appear.
- $X_i \neq X_{i+1}$ for every $i = 1, \dots, l-1$.
- There is no subsequence of the form (i, j, i, j) with $i \neq j$.

This sequence is just a combinatorial way to encode the shape of the cactus $c(x)$. For any such a sequence $(X_1, \dots, X_l)$, the subspace of $\mathcal{C}_n$ consisting of partitions whose associated sequence is $(X_1, \dots, X_l)$ turns out to be a product of simplices. More precisely, if m_i is the cardinality of $\{j \in \{1, \dots, l\} \mid X_j = i\}$, $i = 1, \dots, n$, then

$$(X_1, \dots, X_l) \cong \prod_{i=1}^{n} \Delta^{m_i - 1}$$

For an example look at the cactus of Figure 1: it belongs to the cell $(3, 4, 2, 1, 2, 3) \cong \Delta^0 \times \Delta^1 \times \Delta^1 \times \Delta^0$. Intuitively all the cacti $c(x)$ associated to partitions $x \in (X_1, \dots, X_l)$ have the same shape. So we will represent pictorially a cell by drawing a cactus, meaning that the cell contains all the partitions $x \in \mathcal{C}_n$ whose associated cactus $c(x)$ has that shape. From this point of view, the parameters of a cell $(X_1, \dots, X_l)$ can be thought as the lengths of the arcs between two consecutive intersection points of lobes. The boundary of a cell is obtained by collapsing some of these arcs. See Figure 3 for some examples. This gives a regular CW-decomposition of $\mathcal{C}_n$.

Remark 3.2. There are two relevant groups acting on $\mathcal{C}_n$: S^1 acts by rotating the basepoint of a cactus, Σ_n acts by relabelling the lobes.

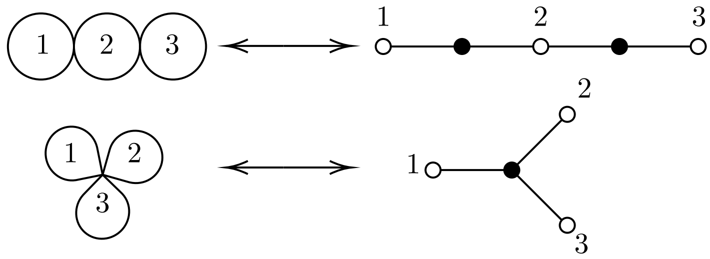


Figure 2: On the left there are some cacti (without basepoint), on the right the corresponding dual graphs.

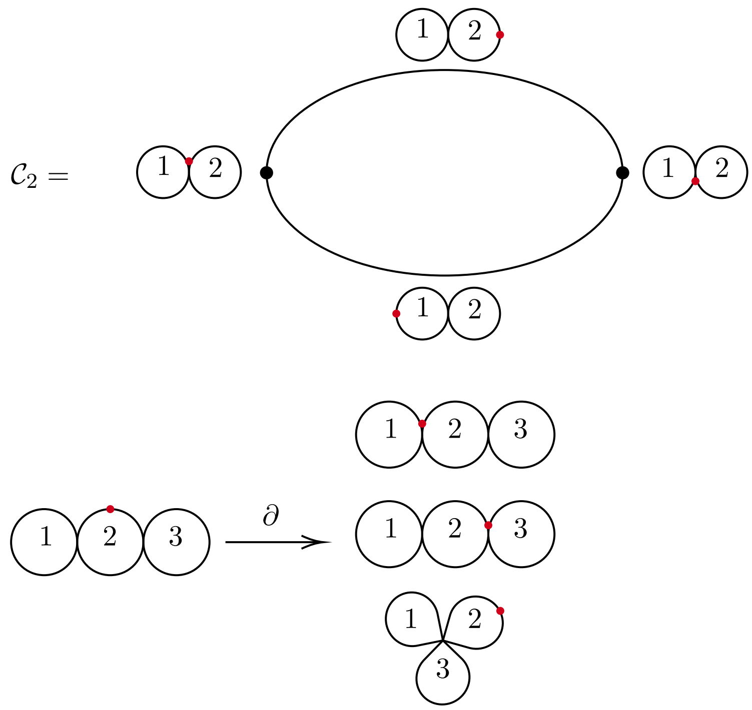


Figure 3: On top where is a full description of $\mathcal{C}_2 \cong S^1$: there are two zero cells $(2,1)$ (on the left) and $(1,2)$ (on the right). The 1-cells are $(2,1,2)$ (on the top) and $(1,2,1)$ (on the bottom). Below we see the cell $(2,3,2,1,2) \cong \Delta^0 \times \Delta^2 \times \Delta^0$ of $\mathcal{C}_3$ and the codimension one cells in its boundary.

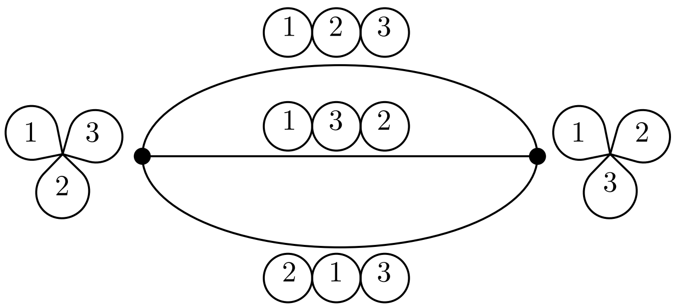


Figure 4: This picture shows the CW-complex $\mathcal{C}_3/S^1 \simeq \mathcal{M}_{0,4}$. There are two zero cells and three edges.

The important thing about cacti is that they are a very small cellular model for the configuration space $F_n(\mathbb{C})$:

Theorem 3.3 (McClure-Smith, [16])**.** *The space of cacti $\mathcal{C}_n$ is $(S^1 \times \Sigma_n)$-equivariantly homotopy equivalent to $F_n(\mathbb{C})$.*

Corollary 3.4. *The moduli space $\mathcal{M}_{0,n+1}$ is homotopy equivalent to $\mathcal{C}_n/S^1$.*

Proof. Recall that $\mathcal{M}_{0,n+1}$ is homeomorphic to $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$. By Theorem 3.3 $\mathcal{C}_n/S^1$ is homotopy equivalent to $F_n(\mathbb{C})/S^1$. Therefore we get that

$$\mathcal{M}_{0,n+1} \cong \frac{F_n(\mathbb{C})}{\mathbb{C} \rtimes \mathbb{C}^*} \simeq \frac{F_n(\mathbb{C})}{S^1} \simeq \frac{\mathcal{C}_n}{S^1}$$

where the first homotopy equivalence holds because the group $\mathbb{C} \rtimes \mathbb{C}^*$ is homotopy equivalent to S^1. □

Remark 3.5. The quotient $\mathcal{C}_n/S^1$ is still a regular CW-complex, and its cells are described by the same combinatorics of $\mathcal{C}_n$ except that we do not have a basepoint on our cacti (see Figure 4 for an example). We call $\mathcal{C}_n/S^1$ the **space of unbased cacti**.

3.2 A combinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$

In this paragraph we describe a small combinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$. We will use this model in Paragraph 5.5 to compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ for small values of n.

Proposition 3.6. *$\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $\mathcal{C}_n/(S^1 \times \Sigma_n)$.*

Proof. By Proposition 3.3 $F_n(\mathbb{C})$ is $(S^1 \times \Sigma_n)$-equivariantly homotopy equivalent to $\mathcal{C}_n$, so we have a homotopy equivalence between the quotients

$$F_n(\mathbb{C})/(S^1 \times \Sigma_n) \simeq \mathcal{C}_n/(S^1 \times \Sigma_n)$$

The left hand space is homotopy equivalent to $\mathcal{M}_{0,n+1}/\Sigma_n$, and this proves the statement. □

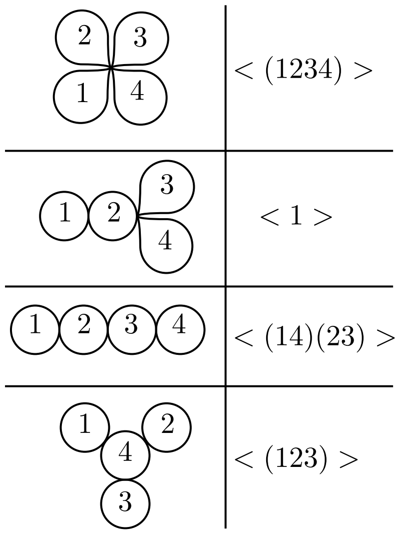


Figure 5: On the left there are some cells of $\mathcal{C}_4/S^1$. On the right is represented the stabilizer of the cell respect to the Σ_4 action by relabelling the lobes. In the first row we see a 0-dimensional cell, whose stabilizer is the cyclic group of order four generated by $(1234) \in \Sigma_4$. In the second row there is a 1-cell, which has trivial stabilizer. The last two cells are two dimensional with stabilizer respectively a cyclic group of order two and three.

Remark 3.7. As we know from Theorem 3.3 the space of cacti $\mathcal{C}_n$ is $(S^1 \times \Sigma_n)$-equivariantly homotopy equivalent to the ordered configuration space $F_n(\mathbb{C})$. Therefore we have a homotopy equivalence between $\mathcal{C}_n/(S^1 \times \Sigma_n)$ and $C_n(\mathbb{C})/S^1$. Combining this with Proposition 3.6 we conclude that $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$. We will freely switch between $\mathcal{M}_{0,n+1}/\Sigma_n$ and $C_n(\mathbb{C})/S^1$ depending on the situation.

Now let us describe in details the space $\mathcal{C}_n/(S^1 \times \Sigma_n)$: recall that the space of cacti $\mathcal{C}_n$ is a regular CW-complex whose cells are described by cacti (with numbered lobes and a basepoint). The boundary of a cell is described by collapsing arcs. The $(S^1 \times \Sigma_n)$-action is encoded as follows:

- S^1 acts by rotating the basepoint.
- Σ_n acts by relabelling the lobes.

To better understand the quotient $\mathcal{C}_n/(S^1 \times \Sigma_n)$ it is useful to first consider the space of unbased cactus $\mathcal{C}_n/S^1 \simeq \mathcal{M}_{0,n+1}$ and then quotient by the Σ_n-action: $\mathcal{C}_n/S^1$ is a CW-complex where a cell is given by an unbased cactus, and the boundary of each cell is obtained by collapsing arcs as in $\mathcal{C}_k$. Now we quotient by the Σ_n-action: this corresponds to relabelling the lobes. This time we have some cells with non-trivial stabilizer and we need to take this into account (some examples of these cells are depicted in Figure 5). To sum up, $\mathcal{C}_n/(S^1 \times \Sigma_n)$ is built up by two types of cells, those with trivial stabilizer ("not symmetric cells") and those with non-trivial stabilizer ("symmetric cells"):

1. **Non symmetric cells:** take a cactus c (without basepoint and labelling of the lobes) and fixes an arbitrary labelling of the lobes. Being a non symmetric cell means that the only permutation of Σ_n which fix c as a labelled cactus is the identity. The cell $\sigma(c)$ associated to c is given by:

$$\sigma(c) := \prod_{i=1}^{n} \Delta^{n_i - 1}$$

where n_i is the number of intersection points of the i-th lobe with the other lobes. As for the space of cacti $\mathcal{C}_n$ the parameters of Δ^{n_i-1} represent the length of the arcs between two intersection points. The boundary of $\sigma(c)$ is given by sending to zero some parameter, i.e. collapsing some arc.

2. **Symmetric cells:** let c be a cactus (without basepoint and labelling of the lobes) and fix an arbitrary labelling of the lobes. Being a symmetric cell means that the isotropy group of c as a labelled cactus is a non trivial subroup $G_c \leq \Sigma_n$. This cactus gives us a cell

$$\sigma(c) := \frac{\prod_{i=1}^{n} \Delta^{n_i - 1}}{G_c}$$

In other words, $\sigma(c)$ is the quotient of the cell associated to c as a labelled cactus (with an arbitrary fixed labelling) by the action of its isotropy group. The boundary of $\sigma(c)$ is given by obtained by sending to zero some parameter, i.e. collapsing some arc.

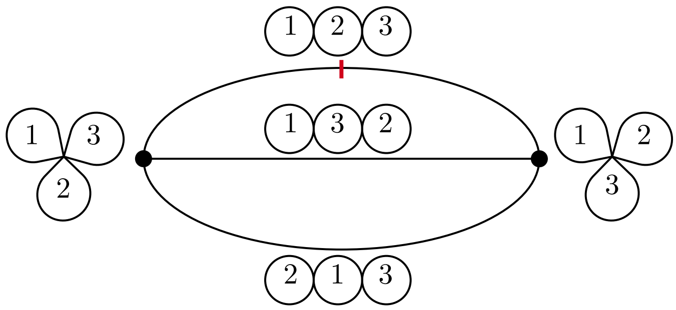
We conclude this paragraph by discussing some explicit examples.

Example (n=3)**.** There are only two cactus (unlabelled, without basepoint) with three lobes, let us call them c_0 and c_1 (see Figure 6). c_0 is fixed by a rotation of $2\pi/3$, so its stabilizer is $\mathbb{Z}/3$. The corresponding cell is a point. c_1 is fixed by a rotation of π, so its stabilizer is $\mathbb{Z}/2$. The corresponding cell $\sigma(c_1)$ is obtained from $\Delta^1 = \{(t_0, t_1) \in [0,1]^2 \mid t_0 + t_1 = 1\}$ quotienting by the relation $(t_0, t_1) \sim (t_1, t_0)$. Therefore $\mathcal{C}_3/(S^1 \times \Sigma_3)$ is contractible.

Example (n=4)**.** There are four cactus (unlabelled, without basepoint) with four lobes, let us call them c_0, c_1, c_2 and c_3 (see Figure 5 where such cacti are depicted with an arbitrary labelling of the lobes). c_0 is fixed by a rotation of $2\pi/4$, so its stabilizer is $\mathbb{Z}/4$. The corresponding cell is a point. c_1 is not fixed by any rotation, so its corresponding cell $\sigma(c_1)$ is a copy of Δ^1. c_2 is fixed by a rotation of π, so its stabilizer is $\mathbb{Z}/2$. The corresponding cell $\sigma(c_2)$ is obtained from $\Delta^1 \times \Delta^1$ by imposing the relation

$$(t_0, t_1) \times (s_0, s_1) \sim (s_1, s_0) \times (t_1, t_0)$$

Finally, c_3 is fixed by a rotation of $2\pi/3$, so its stabilizer is $\mathbb{Z}/3$. The corresponding cell $\sigma(c_3)$ is obtained from Δ^2 by quotienting the $\mathbb{Z}/3$-action. To be explicit, the generator of $\mathbb{Z}/3$ acts on Δ^2 by permuting cyclically the coordinates (t_0, t_1, t_2).

$\mathcal{C}_3/S^1 =$ 

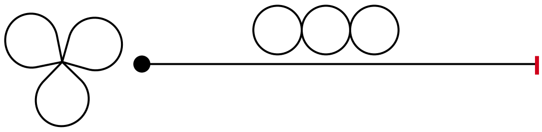
$\mathcal{C}_3/\left(S^1 \times \Sigma_3\right) =$ 

Figure 6: On top of this picture we see the CW-complex $\mathcal{C}_3/S^1$; the red segment indicate the middle of the 1-cell. $\mathcal{C}_3/(S^1 \times \Sigma_3)$ is depicted below and it is obtained from $\mathcal{C}_3/S^1$ by quotienting the Σ_3-action: the two 0-cells of $\mathcal{C}_3/S^1$ are identified, and the same happens for the three 1-cells. Since the one cells have as stabilizer $\mathbb{Z}/2$, we have an additional identification: we need to glue together the two halves of any 1-cell.

4 Fundamental group and rational homology

4.1 Fundamental group

In this paragraph we prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are simply connected. First we recall some facts on the fundamental group of orbit spaces.

4.1.1 On the fundamental group of orbit spaces

The following results can be found in the paper of B. Noohi [17], which extend a previous work by M.A. Armstrong [1]. The goal of this paragraph is to explain the relation between $\pi_1(X)$ and $\pi_1(X/G)$, when G is a topological group acting on a nice topological space X. By nice topological space we mean that X is connected, locally path-connected, semilocally simply-connected. We also assume that the orbit space X/G is semilocally simply-connected. From now on we will always assume that we are in this situation.

Remark 4.1. The statement of the next Theorem contain the notion of *action with slice property*, which is discussed in [18] and [17]. Instead of giving the precise definition we give some important examples where this property is satisfied:

1. If G is a finite group then any action of G on X has the slice property.

2. G Lie group (not necessarily compact), X locally compact. If the map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is proper, then the action has the slice property.

3. If G is compact Lie group acting on a completely regular space, then the action of G on X has the slice property.

Theorem 4.2 ([17], p. 23)**.** *Let G be a topological group acting on a topological space X which is connected, locally path-connected and semilocally simply-connected. We also assume that X/G is semilocally simply connected. Fix a base point $x_0 \in X$ and let $[x_0]$ be its image in X/G. Suppose that the action has the slice property. If all stabilizer groups G_x are locally path-connected then we have an exact sequence*

$$\pi_1(X, x_0) \longrightarrow \pi_1(X/G, [x_0]) \longrightarrow \pi_0(G)/I \longrightarrow 1$$

where $I \subset \pi_0(G)$ is the subgroup generated by the path components of G containing an element g which fix some point of X.

Remark 4.3. The first map in the exact sequence above is the one induced by the projection $p : X \to X/G$. The second map is defined as follows: given a loop α in X/G we can lift it to a path $\tilde{\alpha}$ in X beginning at the basepont x_0 and ending at $\tilde{\alpha}(1)$. Since α is a loop in X/G we have that $\tilde{\alpha}(1) = g_\alpha x_0$ for some $g_\alpha \in G$. Let us denote by $[g_\alpha]$ the element in $\pi_0(G)/I$ which corresponds to the path component of G containing g_α. It is easy to verify that the assignment

$$\begin{aligned} \pi_1(X/G, [x_0]) &\to \pi_0(G)/I \\ [\alpha] &\mapsto [g_\alpha] \end{aligned}$$

is a well defined function.

4.1.2 Computations

We now apply the results of the previous paragraph to compute the fundamental group of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$.

Theorem 4.4. *The quotient $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ is simply connected.*

Proof. Since $\overline{\mathcal{M}}_{0,n+1}$ is simply connected, Theorem 4.2 gives us an isomorphism between $\pi_1(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n)$ and Σ_n/I. Let $(i, j) \in \Sigma_n$ be a transposition. It has at least one fixed point since it fixes a configuration whose clustering is of the form $((i, j), 1 \dots, \hat{i}, \dots, \hat{j}, \dots, n)$. Therefore the subgroup I contains all the transpositions, so it must be Σ_n itself. This concludes the proof. $\square$

Theorem 4.5. *The quotient $\mathcal{M}_{0,n+1}/\Sigma_n$ is simply connected.*

Proof. The space $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$, therefore it suffice to prove statement for this last space. Let us denote by $p : C_n(\mathbb{C}) \to C_n(\mathbb{C})/S^1$ the quotient map. The proof follows several steps:

1. Since S^1 is connected we get that $\pi_0(G)/I = 1$ in the exact sequence of Theorem 4.2. This implies that $p_* : B_n \to \pi_1(C_n(\mathbb{C})/S^1)$ is surjective. Therefore we have a short exact sequence

$$1 \longrightarrow Ker(p_*) \hookrightarrow B_n \xrightarrow{p_*} \pi_1(C_n(\mathbb{C})/S^1) \longrightarrow 1$$

2. $\pi_1(C_n(\mathbb{C})/S^1)$ is abelian: the previous step tell us that $\pi_1(C_n(\mathbb{C})/S^1)$ is a quotient of the braid group, in particular it is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ and in addition to the relations of B_n there are some extra relations coming from $Ker(p_*)$. Let us make explicit some of these relations: the braid $\Delta := \sigma_1(\sigma_2\sigma_1)\ldots(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1)$ belongs to $Ker(p_*)$ since it is the given by a rotation of π. Therefore we have the relation

$$\sigma_i = \sigma_i\Delta = \Delta\sigma_{n-i} = \sigma_{n-i}$$

Now suppose n is odd. The previous relation enable us to prove that σ_i and σ_{i+1} commute in $\pi_1(C_n(\mathbb{C})/S^1)$:

$$\sigma_i\sigma_{i+1} = \sigma_{n-i}\sigma_{i+1} = \sigma_{i+1}\sigma_{n-i} = \sigma_{i+1}\sigma_i$$

where the middle equality holds because n is odd and therefore $n - i \neq i$ for each $i = 1, \ldots, n-1$. To get the statement in the case $n = 2k$ we can do the same procedure to show that σ_i and σ_{i+1} commute for each $i \neq k$. So it remains to prove the equality $\sigma_k\sigma_{k+1} = \sigma_{k+1}\sigma_k$: combining $\sigma_{k+1}\sigma_{k+2}\sigma_{k+1} = \sigma_{k+2}\sigma_{k+1}\sigma_{k+2}$ and $\sigma_{k+2}\sigma_{k+1} = \sigma_{k+1}\sigma_{k+2}$ we get that $\sigma_{k+2} = \sigma_{k+1}$. Therefore $\sigma_k\sigma_{k+1} = \sigma_k\sigma_{k+2} = \sigma_{k+2}\sigma_k = \sigma_{k+1}\sigma_k$.

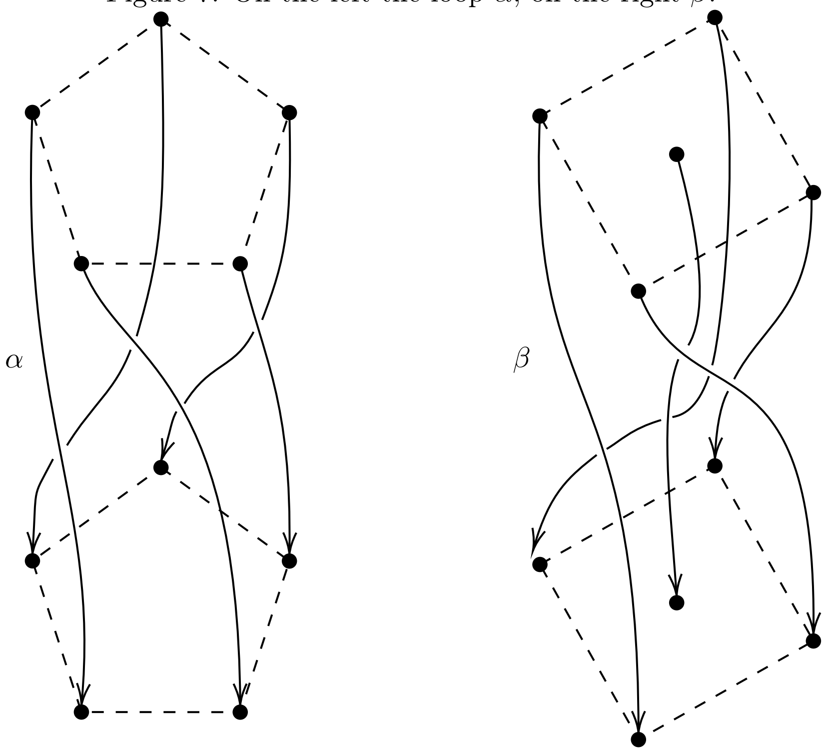
3. $\pi_1(C_n(\mathbb{C})/S^1)$ is generated by σ_1: by the previous point we know that $\pi_1(C_n(\mathbb{C})/S^1)$ is abelian. Combining this fact with the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ we get $\sigma_i = \sigma_{i+1}$ for all $i = 1, \ldots, n-2$. Thus $\pi_1(C_n(\mathbb{C})/S^1)$ is generated by σ_1.

4. $\pi_1(C_n(\mathbb{C})/S^1)$ is the trivial group: by the previous discussion we know that $\pi_1(C_n(\mathbb{C})/S^1)$ is an abelian group, so it is isomorphic to $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$ by Hurewicz. Moreover $\pi_1(C_n(\mathbb{C})/S^1)$ it is generated by σ_1. So it suffice to show that $\sigma_1 = 0$. We proceed as follows: consider the n-th roots of unity $\{\zeta_1, \ldots, \zeta_n\} \in C_n(\mathbb{C})$. Now take the loop

$$\begin{aligned} \alpha : [0,1] &\to C_n(\mathbb{C}) \\ t &\mapsto \{e^{2t\pi i/n}\zeta_1, \ldots, e^{2t\pi i/n}\zeta_n\} \end{aligned}$$

In plain words α rotates the n-agon $\{\zeta_1, \ldots, \zeta_n\}$ counter-clockwise between 0 and $2\pi/n$ degrees. It is easy to see that this loop represent the class $(n -$

Figure 7: On the left the loop α, on the right β.



$1)\sigma_1 \in H_1(C_n(\mathbb{C});\mathbb{Z})$ (see figure 7 for a picture). Therefore $p_*(\alpha)$ is $(n-1)$ times the generator of $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. But $p_*(\alpha)$ is a constant loop, so it is the zero class in homology. Therefore we get the equation $(n-1)\sigma_1 = 0$ in $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. Similarly, let $\{\zeta_1, \dots, \zeta_{n-1}\}$ be the set of $(n-1)$-th roots of unity. Consider the loop

$$\beta : [0,1] \to C_n(\mathbb{C})$$
$$t \mapsto \{e^{2t\pi i/(n-1)}\zeta_1, \dots, e^{2t\pi i/(n-1)}\zeta_{n-1}, 0\}$$

In plain words α rotates the configuration $\{\zeta_1, \dots, \zeta_{n-1}, 0\}$ counter-clockwise between 0 and $2\pi/(n-1)$ degrees. It is easy to see that this loop represent the class $n\sigma_1 \in H_1(C_n(\mathbb{C});\mathbb{Z})$ (see Figure 7). Therefore $p_*(\alpha)$ is n times the generator of $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. But $p_*(\alpha)$ is a constant loop, so it is the zero class in homology. Therefore we get the equation $n\sigma_1 = 0$ in $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. Finally we can conclude: we know that $\pi_1(C_n(\mathbb{C})/S^1) \cong H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$ is a cyclic group with generator σ_1. However $n\sigma_1 = 0 = (n-1)\sigma_1$, therefore $\sigma_1 = 0$.

□

4.2 Rational homology

This paragraph contains some results on the rational homology of $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$. To do such computations we will use some operad theory, which will be reviewed in the next paragraphs.

4.2.1 The Gravity operad

Let X be a S^1-space. The action $\theta : S^1 \times X \to X$ induces an operator $\Delta : H_*(X;\mathbb{Z}) \to H_{*+1}(X;\mathbb{Z})$ by the composition

$$H_*(X) \longrightarrow H_*(S^1) \otimes H_*(X) \xrightarrow{\times} H_*(S^1 \times X) \xrightarrow{\theta_*} H_*(X)$$

where the first map take a class $x \in H_*(X)$ and send it to $[S^1] \otimes x$. We will call Δ the **BV-operator** (see [8]). In what follows all the homology groups are taken with integer coefficients, unless otherwise stated. To easy the notation we sometimes write $H_*(X)$ instead of $H_*(X;\mathbb{Z})$.

Definition (Getzler, [8])**.** Let $\mathcal{D}_2$ be the little two disk operad. S^1 acts on $\mathcal{D}_2(n)$ by rotations, so we get a BV-operator $\Delta : H_*(\mathcal{D}_2(n)) \to H_{*+1}(\mathcal{D}_2(n))$. This map is compatible with the operadic structure and induces a morphism of operads $\Delta : H_*(\mathcal{D}_2) \to H_{*+1}(\mathcal{D}_2)$. The kernel of this map is a sub-operad of $H_*(\mathcal{D}_2)$, called the **Gravity operad**. We will denote it by *Grav*.

Remark 4.6. S^1 acts freely on $\mathcal{D}_2(n)$ so we can identify the kernel of $\Delta : H_*(\mathcal{D}_2(n)) \to H_{*+1}(\mathcal{D}_2(n))$ with $H_*^{S^1}(\mathcal{D}_2(n)) \cong sH_*(\mathcal{M}_{0,n+1})$, where this last isomorphism holds because $\mathcal{M}_{0,n+1}$ and $\mathcal{D}_2(n)/S^1$ are homotopy equivalent (Corollary 3.4). To sum up we have the following identification:

$$Grav(n) = sH_*(\mathcal{M}_{0,n+1})$$

Remark 4.7. The action of Σ_{n+1} on $\mathcal{M}_{0,n+1}$ by relabelling the points induces an action in homology, making *Grav* a cyclic operad.

Unlike many familiar operads, the Gravity operad is not generated by a finite number of operations. However, it has a nice presentation with infinitely many generators:

Theorem 4.8 (Getzler, [10])**.** *As an operad Grav is generated by (graded) symmetric operations of degree one*

$$\{a_1, \ldots, a_n\} \in Grav(n) \quad \text{for } n \geq 2$$

Geometrically, $\{a_1, \ldots, a_n\}$ corresponds to the generator of $H_0(\mathcal{M}_{0,n+1}, \mathbb{Z})$. These operations (called brackets) satisfy the so called generalized Jacobi relations: for any $k \geq 2$ and $l \in \mathbb{N}$

$$\sum_{1 \leq i < j \leq k} (-1)^{\epsilon(i,j)} \{\{a_i, a_j\}, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k, b_1, \ldots, b_l\} = \{\{a_1, \ldots, a_k\}, b_1, \ldots, b_l\} \tag{3}$$

where the right hand term is interpreted as zero if $l = 0$ and $\epsilon(i,j) = (|a_1| + \cdots + |a_{i-1}|)|a_i| + (|a_1| + \cdots + |a_{j-1}|)|a_j| + |a_i||a_j|$.

Definition. A **Gravity algebra** (in the category of chain complexes over $\mathbb{Z}$) is an algebra over the Gravity operad. To be explicit, it is a chain complex (A, d_A) together with graded symmetric chain maps $\{-,\ldots,-\} : A^{\otimes k} \to A$ of degree one such that for $k \geq 3$, $l \geq 0$ and $a_1,\ldots,a_k,b_1,\ldots,b_l \in A$ Equation 3 is satisfied.

Definition. Let $(A, d_A) \in Ch(\mathbb{Z})$ be a chain complex. The **free Gravity algebra** on (A, d_A) is the following chain complex:

$$Grav(A) \coloneqq \bigoplus_{n=2}^{\infty} Grav(n) \otimes_{\Sigma_n} A^{\otimes n}$$

Remark 4.9. (Free) Gravity algebras can be defined in the same way in the category of chain complexes over some commutative ring R. In this paper we will focus in the case $R = \mathbb{Q}$.

4.2.2 The Hypercommutative operad

Let $\overline{\mathcal{M}}_{0,n+1}$ be the Deligne-Mumford compactification of the moduli space of genus zero Riemann surfaces with marked points. For any $m, n \in \mathbb{N}$ and $i = 1,\ldots,n$ we can define *gluing maps*

$$\begin{gathered}\circ_i : \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}\\ (C_1, p_0,\ldots,p_n) \times (C_2, q_0,\ldots,q_m) \mapsto (C, p_0,\ldots,p_{i-1}, q_1,\ldots,q_m, p_{i+1},\ldots,p_n)\end{gathered}$$

where C is the curve obtained from $C_1 \sqcup C_2$ identifying p_i and q_0 and introducing a nodal singularity.

Definition. Let us denote by $\overline{\mathcal{M}}(n) \coloneqq \overline{\mathcal{M}}_{0,n+1}$. Σ_n acts on $\overline{\mathcal{M}}(n)$ by permuting the marked points $p_1,\ldots,p_n$ of a stable curve $(C, p_0,\ldots,p_n) \in \mathcal{M}_{0,n+1}$. The gluing maps defined above define an operad structure on the collection $\overline{\mathcal{M}} \coloneqq \{\overline{\mathcal{M}}(n)\}_{n\geq 2}$. We call $\overline{\mathcal{M}}$ the **Deligne-Mumford operad**.

Since homology is a lax monoidal functor, the operadic structure of $\overline{\mathcal{M}}$ induce an operadic structure on $H_*(\overline{\mathcal{M}};\mathbb{Z})$, giving an operad in graded abelian groups:

Definition. The **Hypercommutative operad** $Hycom$ is a (cyclic) operad in graded abelian groups whose arity n operations are given by

$$Hycom(n) \coloneqq H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Z})$$

As an operad, it is generated by (graded) symmetric operations of degree $2(n-2)$

$$(a_1,\ldots,a_n) \in H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Z}) \quad n \geq 2$$

which satisfy the following *generalized associativity relations*:

$$\sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} \pm((a,b,x_{S_1}),c,x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} \pm(a,(b,c,x_{S_1}),x_{S_2}) \tag{4}$$

where if $S = \{s_1,\ldots,s_k\}$ is a subset of $\{1,\ldots,n\}$, x_S is an abbreviation for $x_{s_1},\ldots,x_{s_k}$. The sign in front of each summand is determined by the Koszul sign rule. We report below some explicit examples of these relations:

- $n = 0$: we get $((a,b),c) = (a,(b,c))$, i.e. the binary operation is associative (and commutative).

- $n = 1$: $((a,b),c,d) + (-1)^{|c||d|}((a,b,d),c) = (a,(b,c),d) + (a,(b,c,d))$.

Further details can be found in [9] and [14].

Remark 4.10. Geometrically, the n-ary operation $(a_1,\dots,a_n)$ corresponds to the fundamental class $[\overline{\mathcal{M}}_{0,n+1}]$.

Definition. An **hypercommutative algebra** (in the category of chain complexes over $\mathbb{Z}$) is just an algebra over the Hypercommutative operad. Explicitly, it is a chain complex (A, d_A) equipped by (graded) symmetric products $(-,\cdots,-) : A^{\otimes n} \to A$ of degree $2(n-2)$, such that Equation 4 is satisfied for any choice of variables $a, b, c, x_1, \dots, x_n \in A$.

Definition. Let $(A, d_A) \in Ch(\mathbb{Z})$ be a chain complex. The **free Gravity algebra** on (A, d_A) is the following chain complex:

$$Hycom(A) := \bigoplus_{n=2}^{\infty} Hycom(n) \otimes_{\Sigma_n} A^{\otimes n}$$

Remark 4.11. (Free) Hypercommutative algebras can be defined in the same way in the category of chain complexes over some commutative ring R. In this paper we will focus in the case $R = \mathbb{Q}$.

4.2.3 Computations

We start with the computation of the rational homology of $\mathcal{M}_{0,n+1}$:

Theorem 4.12. $H_k(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q}) = \begin{cases} \mathbb{Q} \ \textit{if } k = 0 \\ 0 \ \textit{otherwise} \end{cases}$

To proof of this Theorem is elementary but a bit long, so we will subdivide it in several steps. The key observation is that up to a shift in degrees the vector space

$$\bigoplus_{n=2}^{\infty} H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

carries a very nice algebraic structure:

Proposition 4.13. *Let us denote by $Grav(x)$ the free gravity algebra on a generator x of degree zero (over $\mathbb{Q}$). Then*

$$Grav(x) = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

Proof. Let $\mathbb{Q}x$ be the graded vector space of dimension 1 spanned by x. By definition

$$Grav(x) \coloneqq \bigoplus_{n=2}^{\infty} Grav(n) \otimes_{\Sigma_n} (\mathbb{Q}x)^{\otimes n}$$

where $Grav(n) = sH_*(\mathcal{M}_{0,n+1}, \mathbb{Q})$. Explicitly, if $p \in H_*(\mathcal{M}_{0,n+1}, \mathbb{Q})$ and $\sigma \in \Sigma_n$ we have the identification $(\sigma \cdot p) \otimes x \otimes \cdots \otimes x = p \otimes \sigma \cdot (x \otimes \cdots \otimes x)$. Since x is a variable of degree zero Σ_n acts trivially on $(\mathbb{Q}x)^{\otimes n}$, so we get the identification

$$(\sigma p) \otimes x \otimes \cdots \otimes x \sim p \otimes x \otimes \cdots \otimes x$$

Therefore $Grav(x) = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}; \mathbb{Q})_{\Sigma_n} = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Q})$. □

Lemma 4.14. *In $Grav(x)$, we have $\{\{x, x\}, x, \ldots, x\} = 0$.*

Proof. Let us consider generalized Jacobi relation (Equation 3) with $l = 0$ and $n \geq 3$:

$$\sum_{1 \leq i < j \leq n} \pm\{\{a_i, a_j\}, a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_n\} = 0 \tag{5}$$

The relevant case for us is when $x_k = x$ for all $k = 1, \ldots, n$ and x is a variable of degree zero. In this case all the signs are positive and we get

$$\binom{n}{2}\{\{x, x\}, x, \ldots, x\} = 0$$

and the statement follows.

□

Corollary 4.15. *The only non trivial elements in $Grav(x)$ are of the form $\{x, \ldots, x\}$.*

Proof. Let us consider the generalized Jacobi relation associated to any $k \in \mathbb{N}$ and $l > 0$:

$$\sum_{\leq i < j \leq k} \pm\{\{a_i, a_j\}, a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_k, b_1, \ldots, b_l\} = \{\{a_1, \ldots, a_k\}, b_1, \ldots, b_l\}$$

If all the variables a_i, b_i are equal to a variable x of degree zero, the left hand side of the equation written above is 0 because of the previous Lemma. It follows that for all $k \geq 3$, $l > 0$ we have

$$\{\underbrace{\{x, \ldots, x\}}_{k}, x, \ldots, x\} = 0$$

Therefore all the iterated brackets are 0 and the only non trivial elements of $Grav(x)$ are of the form $\{x, \ldots, x\}$. □

Now we are ready to prove Theorem 4.12:

Proof. (of Theorem 4.12) By the previous Corollary, in $Grav(x)$ the only non trivial elements are of the arity k brackets $\{x, \dots, x\}$. Since x has degree 0 and the arity k bracket $\{-, \dots, -\}$ has degree one, the element $\{x, \dots, x\}$ has degree one. By Proposition 4.13 we conclude that the arity k bracket $\{x, \dots, x\}$ is the suspension of the generator of $H_0(\mathcal{M}_{0,n+1}/\Sigma_n, \mathbb{Q})$. Since these are the only non zero elements of $Grav(x)$, the statement of Theorem 4.12 follows. $\square$

A similar approach can be used to compute $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n; \mathbb{Q})$, as the following Proposition suggests:

Proposition 4.16. *Let us denote by $Hycom(x)$ the free hypercommutative algebra on one generator x of degree zero (over $\mathbb{Q}$). Then we have*

$$Hycom(x) = \bigoplus_{n=2}^{\infty} H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n; \mathbb{Q})$$

Proof. Let $\mathbb{Q}x$ be the graded vector space of dimension 1 spanned by x. By definition

$$Hycom(x) := \bigoplus_{n=2}^{\infty} Hycom(n) \otimes_{\Sigma_n} (\mathbb{Q}x)^{\otimes n}$$

where $Hycom(n) = H_*(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Q})$. Explicitly, if $p \in H_*(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Q})$ and $\sigma \in \Sigma_n$ we have the identification $(\sigma \cdot p) \otimes x \otimes \dots \otimes x = p \otimes \sigma \cdot (x \otimes \dots \otimes x)$. The right hand side is equal to $p \otimes x \otimes \dots \otimes x$ since x is a degree zero variable. It follows that

$$(\sigma p) \otimes x \otimes \dots \otimes x \sim p \otimes x \otimes \dots \otimes x$$

Finally, $Hycom(x) = \bigoplus_{n=2}^{\infty} H_*(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Q})_{\Sigma_n} = \bigoplus_{n=2}^{\infty} H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n; \mathbb{Q})$. $\square$

Remark 4.17. This Proposition is the *Koszul dual* (in the sense of Ginzburg-Kapranov [11], see also [9]) of Proposition 4.13. While $\mathcal{M}_{0,n+1}/\Sigma_n$ has the rational homology of a point, the case of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ is highly non trivial. I started thinking about this problem in the Fall 2022 and some computations I did at that time are reported below. However, in March 2023 the problem was solved in the paper [5]: $\overline{\mathcal{M}}_{0,n+1}$ can be constructed by iterated blow-ups from $\mathbb{C}P^{n-2}$. Since the centers of these blow-ups are Σ_n-equivariant, the blow-up formula for cohomology allows the authors to compute $H^*(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Q})$ as Σ_n-representation. By taking the Σ_n-invariant part they get a formula for the Poincarè polynomial of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$, even if the combinatorics involved looks difficult. So, if you want to know the ranks of the homology of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ you have to count some combinatorial objects, but there is not any closed formula/recursion that may help you in this task. It would be interesting to see if our operadic approach can bring to a better formula for these ranks. I thank Professor Vladimir Dotsenko for pointing out to me the paper [5].

Remark 4.18. In principle Proposition 4.16 gives us a complete computation of $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n; \mathbb{Q})$: indeed the Hypercommutative operad has a nice presentation in terms of generators and relations (see Paragraph 4.2.2); however there are infinitely many of them, making explicit computations a bit involved.

We end this paragraph by stating some results about the ranks of $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$.

Proposition 4.19. *The vector spaces* $H_i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ *and* $H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ *have the same dimension.*

Proof. By Poincarè duality we have an isomorphism

$$\begin{aligned} \varphi : H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q}) &\to H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q}) \\ \psi &\mapsto [\overline{\mathcal{M}}_{0,n+1}] \cap \psi \end{aligned}$$

If $g \in \Sigma_n$, then

$$\begin{aligned} \varphi(g\cdot\psi) &= [\overline{\mathcal{M}}_{0,n+1}] \cap g^*\psi \\ &= (g_*)^{-1} g_*([\overline{\mathcal{M}}_{0,n+1}] \cap g^*\psi) \\ &= (g_*)^{-1}(g_*([\overline{\mathcal{M}}_{0,n+1}]) \cap \psi) \\ &= (g_*)^{-1}([\overline{\mathcal{M}}_{0,n+1}] \cap \psi) = g^{-1}\cdot\varphi(\psi) \end{aligned}$$

where the fourth equality holds because Σ_n acts on $\overline{\mathcal{M}}_{0,n+1}$ by complex automorphism, so each $g \in \Sigma_n$ preserves the orientation and, as a consequence, the fundamental class. Therefore the Poincarè duality isomorphism induce an isomorphism between the coinvariants

$$H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})_{\Sigma_n} \xrightarrow{\cong} H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})_{\Sigma_n}$$

The right hand term is isomorphic to $H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ Since we are working rationally. The left hand term is isomorphic (by the norm map) to the invariants $H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})^{\Sigma_n} \cong H^i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$. Now the statement follows by the Universal Coefficient Theorem which allow us to identify $H^i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ with $H_i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$. ☐

A simple application of Proposition 4.19 and Proposition 4.16 is the following computation:

Corollary 4.20. $H_2(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ *has dimension* $n-2$.

Proof. By Proposition 4.19 $H_2(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ is isomorphic to $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$, so it suffices to compute the dimension of this last vector space. By the presentation of the Hypercommutative operad, $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ is generated by classes of the form

$$(\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k\text{ times}}) \qquad k=2,\dots,n-1$$

Now we claim the these classes are actually a basis for $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$: indeed in this case the generalized associativity relations are of the form

$$\sum_{k=2}^{n-1}\binom{n-3}{k-2}(\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k\text{ times}}) = \sum_{k=2}^{n-1}\binom{n-3}{k-2}(x,\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k-1\text{ times}})$$

But x and the symmetric products $(x, \ldots, x)$ are all variables of even degree, so we can switch them without producing signs. Therefore the right hand term of the equality above is really the same as the left hand term, so the generalized associativity is actually the trivial relation $0 = 0$. The claim is now proved since there are no other relations. $\square$

More generally, one can combine Proposition 4.16 and Proposition 4.19 to compute $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n; \mathbb{Q})$ for small values of n. We summarize the results in the following table:

n	Poincarè polynomial
2	1
3	$1 + t^2$
4	$1 + 2t^2 + t^4$
5	$1 + 3t^2 + 3t^4 + t^6$
6	$1 + 4t^2 + 7t^4 + 4t^6 + t^8$

5 On the torsion of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$

5.1 An upper bound on the order of the elements

Theorem 4.12 tell us that rationally $\mathcal{M}_{0,n+1}/\Sigma_n$ has the same homology of a point. In particular each class in the integral homology is a torsion class. This paragraph contains some partial answers to the question "what kind of torsion appears in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$?".

Theorem 5.1. *Let x be a class in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$. Then the order of x divides $n!$.*

Proof. The quotient map $p : \mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n+1}/\Sigma_n$ is a $n!$ fold ramified covering in the sense of [21]. Therefore we have a transfer map $\tau : H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z}) \to H_*(\mathcal{M}_{0,n+1}; Z)$ such that the diagram

$$\begin{tikzcd}
H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z}) \arrow[rr, "\cdot n!"] \arrow[dr, "\tau"'] & & H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z}) \\
& H_*(\mathcal{M}_{0,n+1}; \mathbb{Z}) \arrow[ur, "p_*"'] &
\end{tikzcd}$$

commutes. Given $x \in H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$ we know that it is a torsion class; however $H_*(\mathcal{M}_{0,n+1}; \mathbb{Z})$ is torsion free, so τ is the zero map and we get the statement. $\square$

In order to get more information on the torsion of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$ we will follow this approach:

- First of all we focus on the calculation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p)$, with p be any prime number. The advantage is that now we have coefficients in a field and we can use linear algebra.

- Secondly we note that $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$, which is a bit easier to handle (Remark 3.7).
- To compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p) \cong H_*(C_n(\mathbb{C})/S^1;\mathbb{F}_p)$ the main idea is that we can compare it to $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$, which is known by the computation of [19]. I thank my advisor Paolo Salvatore to suggesting this approach to me.

We will get a complete calculation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ when $n \neq 0, 1 \bmod p$ and $n = p, p+1$. The other cases seems more complicated. Moreover we will prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for any $n \leq 5$. Before going into the details of these computations we discuss a bit the relation between homotopy quotients and strict quotients.

5.2 Homotopy quotients vs strict quotients

Let X be an Hausdorff space equipped with an S^1-action. In particular, being Hausdorff ensures that the isotropy groups G_x are closed subgroups, i.e. $G_x = S^1$ or it is a finite group. In this paragraph we discuss the relation between the homotopy quotient X_{S^1} and the strict quotient X/S^1. Most of the results presented are easy consequences of the theory of transformation groups, which is developed in the books [4] and [7]. I believe that what follows is well known to the experts in the field, but I was not able to find it in the classical references cited above.

Proposition 5.2 ([4], p. 371)**.** *Let X be a S^1-space, $A \subseteq X$ be a closed invariant subspace and F be an abelian group of coefficients. Suppose that for any $x \in X - (A \cup X^{S^1})$ we have $H^i(BG_x;F) = 0$ for all $i > 0$. Then for any $i \in \mathbb{N}$ the map $f : X_{S^1} \to X/S^1$ induces an isomorphism*

$$f^* : H^i(X/S^1, A/S^1 \cup X^{S^1}; F) \to H^i_{S^1}(X, A \cup X^{S^1}; F)$$

If we fix $\mathbb{F}_p$ as coefficients and $A = X^{\mathbb{Z}/p}$ we get:

Corollary 5.3. *Let X be an S^1-space. Then for any $i \in \mathbb{N}$ the map $f : X_{S^1} \to X/S^1$ induces an isomorphism*

$$f^* : H^i(X/S^1, X^{\mathbb{Z}/p}/S^1; \mathbb{F}_p) \to H^i_{S^1}(X, X^{\mathbb{Z}/p}; \mathbb{F}_p)$$

Remark 5.4. If there are not $\mathbb{Z}/p$-fixed points, then we have

$$H^i(X/S^1;\mathbb{F}_p) \cong H^i_{S^1}(X;\mathbb{F}_p)$$

Intuitively we can justify this statement like this: if there are no $\mathbb{Z}/p$-fixed points the S^1-action on X behave as it would be free, so there is no difference (in (co)homology mod p) between the homotopy quotient and the strict quotient.

We can use this Corollary to get a *Mayer-Vietoris* sequence as follows: the map $f : X_{S^1} \to X/S^1$ sends $(X^{\mathbb{Z}/p})_{S^1}$ to $X^{\mathbb{Z}/p}/S^1$ therefore we get a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(\frac{X}{S^1}, \frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d] & H^i(\frac{X}{S^1}) \arrow[r] \arrow[d] & H^i(\frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d] & H^{i+1}(\frac{X}{S^1}, \frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^i_{S^1}(X, X^{\mathbb{Z}/p}) \arrow[r] & H^i_{S^1}(X) \arrow[r] & H^i_{S^1}(X^{\mathbb{Z}/p}) \arrow[r] & H^i_{S^1}(X, X^{\mathbb{Z}/p}) \arrow[r] & \cdots
\end{tikzcd}$$

where the red vertical arrows are isomorphisms by the previous Corollary. Therefore we get:

Proposition 5.5 (Mayer-Vietoris)**.** *Fix p a prime and use $\mathbb{F}_p$ as field of coefficients for cohomology. Then we have a long exact sequence*

$$\cdots \longrightarrow H^i(\tfrac{X}{S^1}) \longrightarrow H^i(\tfrac{X^{\mathbb{Z}/p}}{S^1}) \oplus H^i_{S^1}(X) \longrightarrow H^i_{S^1}(X^{\mathbb{Z}/p}) \longrightarrow H^{i+1}(\tfrac{X}{S^1}) \longrightarrow \cdots$$

Remark 5.6. Proposition 5.5 continues to hold if we replace $X^{\mathbb{Z}/p}$ with X^{S^1} and use rational coefficients for cohomology.

We are going to use Proposition 5.5 for $X = C_n(\mathbb{C})$ in order to do some computations of $H_*(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$. Before going on let us observe that the space of fixed points $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is well understood:

Lemma 5.7. *Let $n = pq$ or $n = pq + 1$. Then the fixed points $C_n(\mathbb{C})^{\mathbb{Z}/p}$ are homeomorphic to $C_q(\mathbb{C}^*)$.*

Proof. Let us prove the statement when $n = pq$, the other case in similar. Let us denote by $\zeta := e^{i2\pi/p}$ the generator of $\mathbb{Z}/p$. Consider the quotient space

$$H := \{z \in \mathbb{C}^* \mid arg(z) \in [0, 2\pi/p]\}/\sim$$

where $\sim$ identifies a point $z \in \{z \in \mathbb{C}^* \mid arg(z) = 0\}$ with ζz. So H is homeomorphic to $\mathbb{C}^*$. Now observe that any configuration in $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is of the form $\{z_1, \zeta z_1, \ldots \zeta^{p-1} z_1, \ldots, z_q, \zeta z_q, \ldots \zeta^{p-1} z_q\}$, where $z_1, \ldots, z_q$ are distinct points in $\{z \in \mathbb{C}^* \mid arg(z) \in [0, 2\pi/p)\}$. The association $\{z_1, \zeta z_1, \ldots \zeta^{p-1} z_1, \ldots, z_q, \zeta z_q, \ldots \zeta^{p-1} z_q\} \mapsto \{z_1, \ldots, z_q\}$ defines a continuous map

$$f : C_n(\mathbb{C})^{\mathbb{Z}/p} \to C_q(H)$$

Conversely, if we have a configuration $\{z_1, \ldots, z_q\} \in C_q(H)$, we can produce a configuration of $C_n(\mathbb{C})^{\mathbb{Z}/p}$ by taking the $\mathbb{Z}/p$-orbits of every point. More precisely, the association $\{z_1, \ldots, z_q\} \mapsto \{z_1, \zeta z_1, \ldots \zeta^{p-1} z_1, \ldots, z_q, \zeta z_q, \ldots \zeta^{p-1} z_q\}$ defines a continuous function $C_q(H) \to C_n(\mathbb{C})^{\mathbb{Z}/p}$, which is the inverse of f. See Figure 8 for a pictorial description of f. ☐

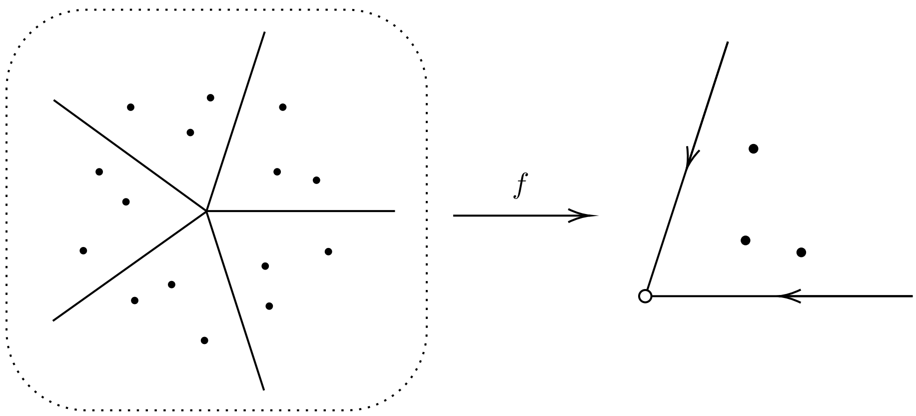


Figure 8: This picture shows how the homeomorphism $f : C_{15}(\mathbb{C})^{\mathbb{Z}/5} \to C_3(\mathbb{C}^*)$ works.

5.3 Recollection on $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$

In this paragraph we recall the computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ for any $n \in \mathbb{N}$ and any prime p (see [19] for details). Combining this with the Mayer-Vietoris sequence of Proposition 5.5 we will be able to compute $H_*(C_n(\mathbb{C})/S^1;\mathbb{F}_p) \cong H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ is some cases.
The computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ was done using the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$$

The homology of the fiber is known, thanks to the work of F. Cohen (see [6]):

Theorem 5.8 (Cohen, [6]). *Consider the disjoint union $C(\mathbb{C}) := \coprod_{n\in\mathbb{N}} C_n(\mathbb{C})$ and let p be any prime. Then $H_*(C(\mathbb{C});\mathbb{F}_p)$ has the following form:*

$p = 2$: *$H_*(C(\mathbb{C});\mathbb{F}_2)$ is the free graded commutative algebra on classes $\{Q^i(a)\}_{i\in\mathbb{N}}$, where $Q^i(a)$ is a class of degree $2^i - 1$ in $H_*(C_{2^i}(\mathbb{C});\mathbb{F}_2)$.*

$p \neq 2$: *$H_*(C(\mathbb{C});\mathbb{F}_p)$ is the free graded commutative algebra on classes*

$$\{a, [a,a], Q^i[a,a], \beta Q^i[a,a]\}_{i\geq 1}$$

where a is the generator of $H_0(C_1(\mathbb{C});\mathbb{F}_p)$, $[a,a]$ is the generator of $H_1(C_2(\mathbb{C});\mathbb{F}_p)$ and $Q^i[a,a]$ (resp. $\beta Q^i[a,a]$) is a class of degree $2p^i - 1$ (resp. $2p^i - 2$) in $H_*(C_{2p^i}(\mathbb{C});\mathbb{F}_p)$.

The differential of the second page of the homological spectral sequence can be determined by the following Proposition:

Proposition 5.9 ([19]). *Let X be a topological space of finite type equipped with an S^1 action. Fix $\mathbb{F}$ a field of coefficients for (co)homology. Then the differential d^2 of* *the second page of the homological spectral sequence associated to* $X \hookrightarrow X_{S^1} \to BS^1$ *is given by*

$$d^2(x \otimes y_{2i}) = \begin{cases} 0 \ \textit{if } i = 0 \\ \Delta(x) \otimes y_{2i-2} \ \textit{otherwise} \end{cases}$$

where y_{2i} *is the generator of* $H_{2i}(BS^1; \mathbb{F})$ *and* $\Delta : H_*(X; \mathbb{F}) \to H_{*+1}(X; \mathbb{F})$ *is the* BV*-operator.*

Consider a class $a^k[a,a]^l x \in H_*(C_n(\mathbb{C}); \mathbb{F}_p)$, where $k \in \mathbb{N}$, $l = 0, 1$ and x is a monomial which contains only the letters $\{Q^i[a,a], \beta Q^i[a,a]\}_{i \geq 1}$. It turns out that the operator Δ acts as follows:

$$\Delta(a^k x) = k(k-1)a^{k-2}[a,a]x \qquad \Delta(a^k[a,a]x) = 0 \tag{6}$$

Therefore the second page of the homological spectral sequence associated to

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$$

is quite explicit and we are ready to state the main Theorems of [19]:

Theorem 5.10. *Let* p *be a prime,* $n \in \mathbb{N}$ *such that* $n = 0, 1$ *mod* p*. Then the (co)homological spectral sequence (with* $\mathbb{F}_p$*-coefficients) associated to the fibration*

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$$

degenerates at the second page. In particular

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_p) \otimes H_*(BS^1; \mathbb{F}_p)$$

Theorem 5.11. *Let* p *be an odd prime,* $n \in \mathbb{N}$ *such that* $n \neq 0, 1$ *mod* p*. The homological spectral sequence (with* $\mathbb{F}_p$*-coefficients) associated to the fibration*

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$$

degenerates at the third page. In particular

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_p) \cong coker(\Delta)$$

where $\Delta : H_*(C_n(\mathbb{C}); \mathbb{F}_p) \to H_{*+1}(C_n(\mathbb{C}); \mathbb{F}_p)$ *is the* BV*-operator.*

Remark 5.12. We can explicitly describe $coker(\Delta)$: a basis of this vector space is given by (the image of) classes in $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ which do not contain the bracket $[a,a]$ (Equation 6).

5.4 Computations in some special cases

From now on we fix $\mathbb{F}_p$ as field of coefficients for homology, where p is any prime number. We are interested in the quotient $\mathcal{M}_{0,n+1}/\Sigma_n$, which is homotopy equivalent to $C_n(\mathbb{C})/S^1$. Corollary 5.3 allow us to do a very nice computation:

Theorem 5.13. *If $n \neq 0, 1$ mod p the map $f : C_n(\mathbb{C})_{S^1} \to C_n(\mathbb{C})/S^1$ induces an isomorphism*

$$f^* : H^i(C_n(\mathbb{C})/S^1; \mathbb{F}_p) \to H^i_{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$$

Proof. Just observe that if $n \neq 0, 1$ mod p there are no $\mathbb{Z}/p$-fixed points. □

Now suppose that p divides n (or $n-1$). We would like to apply the Mayer-Vietoris sequence of Proposition 5.5

$$\cdots \longrightarrow H^i(\tfrac{C_n(\mathbb{C})}{S^1}) \longrightarrow H^i(\tfrac{C_n(\mathbb{C})^{\mathbb{Z}/p}}{S^1}) \oplus H^i_{S^1}(C_n(\mathbb{C})) \longrightarrow H^i_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow \cdots \tag{7}$$

to compute of $H^i(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$, but in this generality the problem is hard. Indeed we do not know $H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1; \mathbb{F}_p)$, which could be as difficult to compute as $H^*(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$. The only advantage is that $C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1$ is a much smaller space with respect to $C_n(\mathbb{C})/S^1$. Moreover, even if we would be able to do such computation it remains to understand the maps that fit into the Mayer-Vietoris sequence, which can be an even harder problem.

Remark 5.14. In the Mayer-Vietoris sequence above the terms $H^i_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ can be computed easily: consider the map of fibrations

$$\begin{tikzcd}
C_n(\mathbb{C})^{\mathbb{Z}/p} \arrow[r] \arrow[d, hook] & C_n(\mathbb{C})^{\mathbb{Z}/p} \arrow[d, hook] \\
\left(C_n(\mathbb{C})^{\mathbb{Z}/p}\right)_{\mathbb{Z}/p} \arrow[r] \arrow[d] & \left(C_n(\mathbb{C})^{\mathbb{Z}/p}\right)_{S^1} \arrow[d] \\
B\mathbb{Z}/p \arrow[r] & BS^1
\end{tikzcd}$$

and observe that the mod p Serre spectral sequence of the left fibration degenerates at the second page. The map of cohomological spectral sequences induced by the map of fibrations above is surjective at the E_2 page so the Serre spectral sequence of the right fibration degenerates at the second page as well. Therefore $H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p) = H^*(BS^1; \mathbb{F}_p) \otimes H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ as $\mathbb{F}_p$-vector space. Finally we note that $H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ is known: $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is homeomorphic to the space of unordered configurations of points in $\mathbb{C}^*$ (Lemma 5.7) and the homology of this space is known thanks to the work of F. Cohen [6] (see also Proposition 6.3 for a computation of these homology groups).

Remark 5.15. In some cases we can compute all the terms in the Mayer-Vietoris sequence 7. For example take $n = pq$ (or $n = pq + 1$) with $(p, q) = 1$. The terms $H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1; \mathbb{F}_p)$ can be computed as follows: first recall that $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is homeomorphic to $C_q(\mathbb{C}^*)$ (Lemma 5.7), therefore $C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1 \cong C_q(\mathbb{C}^*)/S^1$. Since p does not divide q the subspace $C_q(\mathbb{C}^*)^{\mathbb{Z}/p}$ is empty and by Remark 5.4 we deduce that

$$H^*(C_q(\mathbb{C}^*)/S^1; \mathbb{F}_p) \cong H^*_{S^1}(C_q(\mathbb{C}^*); \mathbb{F}_p)$$

The advantage is that $H^*_{S^1}(C_q(\mathbb{C}^*);\mathbb{F}_p)$ can be computed by the Serre spectral sequence associated to the fibration

$$C_q(\mathbb{C}^*) \hookrightarrow C_q(\mathbb{C}^*)_{S^1} \to BS^1$$

We will postpone this computation to Section 6 (Theorem 6.4).

We end this paragraph with an acyclicity result:

Theorem 5.16. *Let p be a prime. Then*

$$H^*(C_p(\mathbb{C})/S^1;\mathbb{F}_p) \cong H^*(C_{p+1}(\mathbb{C})/S^1;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p \text{ if } * = 0 \\ 0 \text{ otherwise} \end{cases}$$

In order to prove this Theorem we need a preliminary Lemma:

Lemma 5.17. *Let $n \in \mathbb{N}$ and p be a prime. Suppose $n = 0, 1 \bmod p$. If $i : C_n(\mathbb{C})^{\mathbb{Z}/p} \hookrightarrow C_n(\mathbb{C})$ is the inclusion then*

$$i^* : H^k_{S^1}(C_n(\mathbb{C});\mathbb{F}_p) \to H^k_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$$

is a monomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^k_{S^1}(C_n(\mathbb{C})) & \xrightarrow{\quad i^* \quad} & H^k_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}) \\ \downarrow & & \downarrow \\ \left(S^{-1}H^*_{S^1}(C_n(\mathbb{C}))\right)^k & \xrightarrow{i^*} & \left(S^{-1}H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p})\right)^k \end{array}$$

where S is the multiplicatively closed subset $\{1, c, c^2, c^3, \dots\} \subseteq H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[c]$, with c be a variable of degree two. By the Localization Theorem (see [7, Theorem 4.2 p.198]) the bottom horizontal arrow is an isomorphism. The cohomological version of Theorem 5.10 tells us that the mod p spectral sequence associated to the fibration $C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$ degenerates at the E_2 page, so $H^*_{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ is a free $H^*(BS^1)$-module. Therefore the left vertical arrow is a monomorphism, and this proves the statement. $\square$

Proof. (of Theorem 5.16) We do the case of $C_p(\mathbb{C})$, the other one is completely analogous. First of all note that $C_p(\mathbb{C})^{\mathbb{Z}/p} \cong C_1(\mathbb{C}^*) = \mathbb{C}^*$. Therefore

$$C_p(\mathbb{C})^{\mathbb{Z}/p}/S^1 \cong \mathbb{C}^*/S^1$$

and we conclude that $C_p(\mathbb{C})^{\mathbb{Z}/p}/S^1$ is contractible. If $i \geq 1$ the Mayer-Vietoris sequence 7 becomes

$$\cdots \longrightarrow H^i(\tfrac{C_p(\mathbb{C})}{S^1}) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow \cdots$$

By Lemma 5.17 the map $H^k_{S^1}(C_p(\mathbb{C});\mathbb{F}_p) \to H^k_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ is a monomorphism, therefore the Mayer-Vietoris sequence splits (for $i \geq 1$) as:

$$0 \longrightarrow H^i_{S^1}(C_p(\mathbb{C})) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow H^{i+1}(\tfrac{C_p(\mathbb{C})}{S^1}) \longrightarrow 0$$

To conclude the proof we show that $H^i_{S^1}(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ are vector spaces of the same dimension: by Theorem 5.10 we have $H^*_{S^1}(C_p(\mathbb{C});\mathbb{F}_p) \cong H^*(C_p(\mathbb{C});\mathbb{F}_p)\otimes H^*(BS^1;\mathbb{F}_p)$ as $H^*(BS^1;\mathbb{F}_p)$-module. Similarly $H^*_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p) \cong H^*(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)\otimes H^*(BS^1;\mathbb{F}_p)$ by Remark 5.14. Finally we observe that $H_*(C_p(\mathbb{C});\mathbb{F}_p)$ is generated by a class of degree zero and one of degree one (i.e. a^p and $[a,a]a^{p-2}$) therefore $H^i(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p) \cong H^i(S^1;\mathbb{F}_p)$ have the same dimension for each $i \in \mathbb{N}$. As a consequence we get that $H^i_{S^1}(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ have the same dimension, as claimed. $\square$

5.5 Examples

In what follows we use the results of the previous paragraph to do some explicit computations of $H_*(C_n(\mathbb{C})/S^1;\mathbb{F}_p)$. In particular, we will see that the quotients $C_n(\mathbb{C})/S^1$ are contractible until $n = 6$, which is the first non contractible space.

n=1: $C_1(\mathbb{C})/S^1$ is homeomorphic to a half line, so it is contractible.

n=2: by the combinatorial model explained in Section 3 $C_2(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex with only one cell of dimension zero, so it is contractible.

n=3: by the combinatorial model explained in Section 3 $C_3(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex of dimension one (see also Figure 6), so there are no homology classes of degrees strictly greater than 1. But $C_3(\mathbb{C})/S^1$ is simply connected, so $H_1(C_3(\mathbb{C})/S^1;\mathbb{Z}) = 0$ and this implies that $C_3(\mathbb{C})/S^1$ is contractible.

n=4: We prove that $C_4(\mathbb{C})/S^1$ is contractible by showing that $H_i(C_4(\mathbb{C})/S^1;\mathbb{Z}) = 0$ for any $i \geq 1$. This is enough to prove the claim since $C_4(\mathbb{C})/S^1$ is homotopic to a CW complex of dimension two, and it is simply connected. By the cellular model we know that there are no homology classes of degrees strictly greater than 2. By Theorem 4.12 we know that the Euler characteristic must be one, so

$$1 = b_0 - b_1 + b_2$$

where b_i is i-th Betti number. But $C_4(\mathbb{C})/S^1$ is simply connected, so $b_1 = 0$ and combining this fact with the equation above we get that $b_2 = 0$. Therefore $C_4(\mathbb{C})/S^1$ is contractible.

n=5: We will prove that $C_5(\mathbb{C})/S^1$ is contractible by showing that for any prime p and any $i \geq 1$ $H_i(C_5(\mathbb{C})/S^1; \mathbb{F}_p) = 0$. The cellular model tells us that $C_5(\mathbb{C})/S^1$ has no homology classes of degrees strictly greater than three. By Theorem 5.1 the order of any class in $H_*(C_5(\mathbb{C}); \mathbb{Z})$ divides 5!. So the only interesting cases are when we take $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_5$ as coefficients for homology.

- $\mathbb{F}_2$-coefficients: by Proposition 5.5 we have a Mayer-Vietoris sequence

$$\cdots \longrightarrow H^i(\tfrac{C_5(\mathbb{C})}{S^1}) \longrightarrow H^i(\tfrac{C_5(\mathbb{C})^{\mathbb{Z}/2}}{S^1}) \oplus H^i_{S^1}(C_5(\mathbb{C})) \longrightarrow H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}) \longrightarrow \cdots$$

Now observe that $C_5(\mathbb{C})^{\mathbb{Z}/2}$ is homeomorphic $C_2(\mathbb{C}^*)$, whose homology with $\mathbb{F}_2$-coefficients is generated by the classes listed below:

Homology class	Degree
a^2b	0
$[b,a]a$	1
$b \cdot Qa$	1
$[[b,a],a]$	2

For the notation and more details about $H_*(C_n(\mathbb{C}^*); \mathbb{F}_p)$ see Proposition 6.3. Moreover, $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to a circle (see Figure 9 and its caption for an explanation). The second pages of the Serre spectral sequences that compute $H^*_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2)$ and $H^*_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)$ are displayed below:

$$\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2)}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)}
\end{array}$$

As we know, both the spectral sequences degenerate at the second page (Theorem 5.10 and Remark 5.14) and summing over the diagonals we get the ranks of $H^i_{S^1}(C_5(\mathbb{C}))$ and $H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2})$. By Lemma 5.17 the inclusion $i : C_5(\mathbb{C})^{\mathbb{Z}/2} \hookrightarrow C_5(\mathbb{C})$ induces a monomorphism $i^* : H^i_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2) \to H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)$ in each degree, so we can conclude that it is an isomorphism for any $i \geq 2$ by looking at the ranks. If we put all these information in the Mayer-Vietoris sequence we conclude immediately that

$$H^i(C_5(\mathbb{C})/S^1; \mathbb{F}_2) = 0 \quad \text{for all } i \geq 3$$

Now $C_5(\mathbb{C})/S^1$ is simply connected, so the first (co)homology group is zero. By the same argument as before using the Euler characteristic we can conclude that $H^2(C_5(\mathbb{C})/S^1; \mathbb{F}_2) = 0$ as well. So there is no 2-torsion in $H_*(C_5(\mathbb{C})/S^1; \mathbb{Z})$.

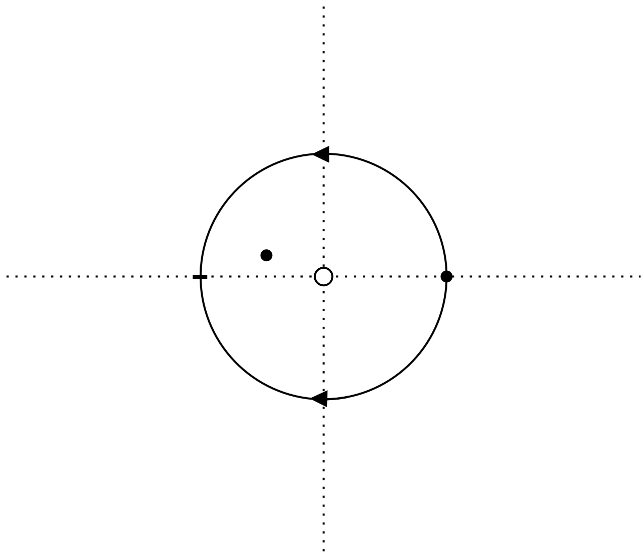


Figure 9: This picture explains the homotopy equivalence between $C_2(\mathbb{C}^*)/S^1$ and S^1: first of all note that $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to $C_2(\mathbb{C}^*)/\mathbb{C}^*$. Now observe that any configuration $\{z_1, z_2\} \in C_2(\mathbb{C}^*)$ is equivalent (up to rotations and dilations) to a configuration of the form $\{1, z\}$, where $z \in D^2 - \{0, 1\}$. If $|z_1| \neq |z_2|$ there is a unique representative $\{1, z\}$ of the class $[z_1, z_2] \in C_2(\mathbb{C}^*)/\mathbb{C}^*$. In the case $|z_1| = |z_2|$ there are two representatives of $[z_1, z_2]$: $\{1, z_1 z_2^{-1}\}$ and $\{1, z_2 z_1^{-1}\}$. Therefore $C_2(\mathbb{C}^*)/\mathbb{C}^*$ is homeomorphic to the space obtained from $D^2 - \{0, 1\}$ by gluing the boundary of the disk according to the relation $z \sim z^{-1}$. This space is homeomorphic to S^2 minus two points, so we can conclude that $C_2(\mathbb{C}^*)/\mathbb{C}^*$ is homotopy equivalent to a circle.

- $\mathbb{F}_3$-coefficients: by Theorem 5.13 $H_*(C_5(\mathbb{C})/S^1;\mathbb{F}_3) \cong H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$. Theorem 5.11 tells us that $H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$ is isomorphic to the subspace of $H_*(C_5(\mathbb{C});\mathbb{F}_3)$ spanned by the classes which do not contain the bracket $[a,a]$. In this case $H_*(C_5(\mathbb{C});\mathbb{F}_3)$ has only two classes: a^5 of degree zero and $a^3[a,a]$ of degree one, so we can conclude that $H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$ is trivial. Hence there is no 3-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

- $\mathbb{F}_5$-coefficients: By Theorem 5.16 $H^i(C_5(\mathbb{C})/S^1;\mathbb{F}_5) = 0$ for any $i \geq 1$. Thus there is no 5-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

n=6: as we said at the beginning of this paragraph, $C_6(\mathbb{C})/S^1$ is the first non contractible space. In particular its homology with $\mathbb{F}_3$-coefficients will be not trivial. By Theorem 5.1 the order of any class in $H_*(C_6(\mathbb{C})/S^1;\mathbb{Z})$ divides 6!. So the only interesting coefficients for homology are $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_5$. From the cellular model described in Section 3 we know that $C_6(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex of dimension 4, therefore $H^i(C_6(\mathbb{C})/S^1;\mathbb{Z}) = 0$ for any $i \geq 5$. Moreover $C_6(\mathbb{C})/S^1$ is simply connected and its Euler characteristic is equal to one, so we get $0 = b_2 - b_3 + b_4$.

- $\mathbb{F}_5$-coefficients: by Theorem 5.16 $H^i(C_6(\mathbb{C})/S^1;\mathbb{F}_5) = 0$ for any $i \geq 1$, so there is no 5-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

- $\mathbb{F}_3$-coefficients: consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^i(\tfrac{C_6(\mathbb{C})}{S^1}) \longrightarrow H^i(\tfrac{C_6(\mathbb{C})^{\mathbb{Z}/3}}{S^1}) \oplus H^i_{S^1}(C_6(\mathbb{C})) \longrightarrow H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3}) \longrightarrow \cdots$$

  Now observe that $C_6(\mathbb{C})^{\mathbb{Z}/3}$ is homeomorphic $C_2(\mathbb{C}^*)$, whose homology with $\mathbb{F}_3$-coefficients is generated by the classes listed below:

Homology class	Degree
ba^2	0
$b[a,a]$	1
$[b,a]a$	1
$[[b,a],a]$	2

  As we already know $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to a circle, so it has no classes of degrees greater that two. The second pages of the Serre spectral sequences that compute $H^*_{S^1}(C_6(\mathbb{C});\mathbb{F}_3)$ and $H^*_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)$ are displayed below:

$$
\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_6(\mathbb{C});\mathbb{F}_3)}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)}
\end{array}
$$

As we know, both the spectral sequences degenerate at the second page and summing over the diagonals we get the ranks of $H^i_{S^1}(C_6(\mathbb{C}))$ and $H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3})$. By Lemma 5.17 the inclusion $i : C_6(\mathbb{C})^{\mathbb{Z}/3} \hookrightarrow C_6(\mathbb{C})$ induces a monomorphism $i^* : H^i_{S^1}(C_6(\mathbb{C});\mathbb{F}_3) \to H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)$ in each degree, so we can conclude that it is an isomorphism for any $i \geq 4$ by looking at the ranks. If we put all these information in the Mayer-Vietoris sequence we conclude immediately that

$$H^i(C_6(\mathbb{C})/S^1;\mathbb{F}_3) = \begin{cases} 0 & \text{for all } i \geq 5 \text{ and } i = 1,2 \\ \mathbb{F}_3 & \text{for } i = 0,3,4 \end{cases}$$

Thus $C_6(\mathbb{C})/S^1$ is not contractible, since there are 3-torsion classes in $H_*(C_6(\mathbb{C})/S^1;\mathbb{Z})$.

6 Extra: computation of $H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ when $p \nmid n$

In this section we compute $H^*_{S^1}(C_n(\mathbb{C}^*);\mathbb{F}_p)$ in the case $n \neq 0 \bmod p$. We will do this by analyzing the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}^*) \hookrightarrow C_n(\mathbb{C}^*)_{S^1} \to BS^1$$

We start by reviewing the homology of the fiber.

6.1 Preliminaries on $H_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$

Definition (Bödigheimer, [3]). Let M be a manifold and $(X,*)$ be a based CW-complex, not necessarily connected. The space of configurations in M with labels in X is defined as

$$C(M;X) := \bigsqcup_{n\in\mathbb{N}} F_n(M) \times_{\Sigma_n} X^n/\sim$$

where $(p_1,\ldots,p_n;x_1,\ldots,x_n) \sim (p_1,\ldots,\hat{p}_i,\ldots,p_n;x_1,\ldots,\hat{x}_i,\ldots,x_n)$ if $x_i = *$.

When $M = \mathbb{R}^n$ the homology of $C(\mathbb{R}^n;X)$ is known, thanks to the work of Cohen [6]. The idea is the following: $C(\mathbb{R}^n;X)$ is homotopy equivalent to the free $\mathcal{D}_n$-algebra on X, where $\mathcal{D}_n$ is the little n-disk operad. Therefore $H_*(C(\mathbb{R}^n; X); \mathbb{F}_p)$ can be described as a functor of $H_*(X; \mathbb{F}_p)$. For the purpose of this work it is enough to recall the results in the case $n = 2$.

Definition. Let p be a prime number. Fix a basis $\mathcal{B}$ of $H_*(X; \mathbb{F}_p)/[*]$, where $[*] \in H_0(X; \mathbb{F}_p)$ is the class of the basepoint. We define a **basic bracket of weight k** inductively as follows:

- A basic bracket of weight 1 is just an element $a \in \mathcal{B}$. Its degree is by definition the homological degree of a. Observe that any class of $H_*(X; \mathbb{F}_p)$ can be seen as a class in $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$.
- By induction assume that the basic brackets of weight j have been defined and equipped with a total ordering compatible with weight for $j < k$. Then a basic bracket of weight k is a homology class $[a, b] \in H_*(C(\mathbb{C}; X); \mathbb{F}_p)$, where $[-, -]$ is the Browder bracket and a, b are basic brackets such that:

  1. $weight(a) + weight(b) = k$.
  2. $a < b$ and if $b = [c, d]$ then $c \leq a$.

  The degree of $[a, b]$ is by definition $deg(a) + deg(b) + 1$.

In the case $p \neq 2$ we also include as basic brackets classes of the form $[a, a]$, where a is a basic bracket of even degree.

Theorem 6.1 (Cohen, [6])**.** *Let p be any prime, and $Q : H_q(C(\mathbb{C}; X); \mathbb{F}_p) \to H_{pq+p-1}(C(\mathbb{C}; X); \mathbb{F}_p)$ be the first Dyer-Lashof operation (when p is odd it acts only on classes of odd degree q). Then $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ has the following form:*

$p = 2$**:** *$H_*(C(\mathbb{C}; X); \mathbb{F}_2)$ is the free graded commutative algebra on classes $Q^i(x)$, where Q^i denotes the i-th iteration of Q and x is a basic bracket.*

$p \neq 2$**:** *$H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ is the free graded commutative algebra on classes $Q^i(x)$ and $\beta Q^i(x)$, where β is the Bockstein operator, Q^i denotes the i-th iteration of Q and x is a basic bracket of odd degree.*

Now consider the discrete space $S^0 \vee S^0 = \{*, a, b\}$, where $*$ is the basepoint. The labelled configuration space $C(\mathbb{C}; S^0 \vee S^0)$ is a disjoint union of components $\{C_{n\circ + m\bullet}(\mathbb{C})\}_{n,m \in \mathbb{N}}$ consisting of all configurations of m black particles and n white particles, where by *black particle* (resp. *white particle*) we mean a point in the plane with label a (resp. b). A tipical element of $C(\mathbb{C}; S^0 \vee S^0)$ is depicted in Figure 10.

Remark 6.2. $C_n(\mathbb{C}^*)$ is homotopy equivalent to the configuration space of n black particles and one white particle in the plane. Therefore $H_*(C_n(\mathbb{C}^*); \mathbb{F}_p)$ will be the subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$ spanned by those classes that involve only n black particles and one white particle.

Let us denote by a (resp. b) the class in $H_0(S^0 \vee S^0)$ which represent a black particle (resp. a white particle). Then $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$ can be computed using Theorem 6.1. The following Proposition just identifies $H_*(C_n(\mathbb{C}^*); \mathbb{F}_p)$ as a subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0); \mathbb{F}_p)$ (for details see [19]):

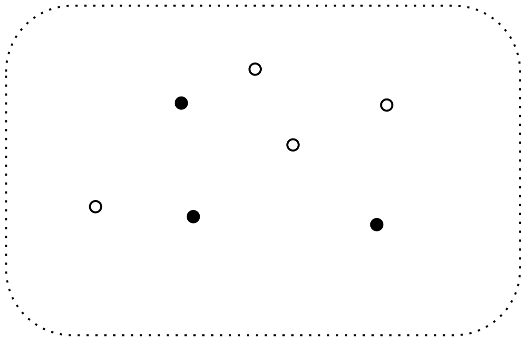


Figure 10: In this picture we see a point of $C_{4\circ+3\bullet}(\mathbb{C})$.

Proposition 6.3. *Let $n \in \mathbb{N}$ and p be a prime. Then*

$$H_*(C_n(\mathbb{C}^*); \mathbb{F}_p) = \bigoplus_{k=0}^{n} b_k \cdot H_*(C_{n-k}(\mathbb{C}); \mathbb{F}_p)$$

where

$$\begin{aligned} b_0 &:= b \\ b_1 &:= [b, a] \\ b_2 &:= [[b, a], a] \\ b_3 &:= [[[b, a], a], a] \\ &\dots \end{aligned}$$

Example. The generators of $H_*(C_3(\mathbb{C}^*); \mathbb{F}_2)$ are listed in the first column of this table:

Homology class x	Degree
ba^3	0
$[b, a]a^2$	1
$baQa$	1
$[b, a]Qa$	2
$[[b, a], a]a$	2
$[[[b, a], a], a]$	3

6.2 Computation

The main result of this paragraph is the following:

Theorem 6.4. *Let p be any prime and $n \in \mathbb{N}$ be a natural number not divisible by p. Then*

$$H_*^{S^1}(C_n(\mathbb{C}^*); \mathbb{F}_p) \cong coker(\Delta)$$

where $\Delta : H_*(C_n(\mathbb{C}^*)) \to H_{*+1}(C_n(\mathbb{C}^*))$ is the BV-operator.

Remark 6.5. This Theorem gives us an explicit description of $H_*^{S^1}(C_n(\mathbb{C}^*);\mathbb{F}_p)$: by Proposition 6.3 we get that any class in $H_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ is a linear combination of monomials of the form $b_k \cdot x$ where $x \in H_*(C_{n-k}(\mathbb{C});\mathbb{F}_p)$. Therefore if we want to compute Δ on $H_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ it suffices to know its value on these generators. Since

$$\Delta(xy) = \Delta(x)y + (-1)^x x\Delta(y) + (-1)^x[x,y]$$

(see [8]) it suffices to know how Δ acts on the brackets $\{b_k\}_{k\in\mathbb{N}}$ and on the classes $x \in H_*(C_{n-k}(\mathbb{C}^*);\mathbb{F}_p)$. b_k is a top dimensional class in $H_*(C_k(\mathbb{C}^*);\mathbb{F}_p)$ therefore

$$\Delta(b_k) = 0 \quad k \in \mathbb{N}$$

On the other hand, $\Delta(x) = 0$ if $n = 0,1 \mod p$ (Theorem 5.10), while if $n \neq 0,1 \mod p$ $\Delta(x)$ is given by the formulas 6. For example, let us suppose $n = 3$ and $p = 2$. The generators of $H_*(C_3(\mathbb{C}^*);\mathbb{F}_2)$ are listed in the first column of this table:

Homology class x	Degree	$\Delta(x)$
ba^3	0	$[b,a]a^2$
$[b,a]a^2$	1	0
$baQa$	1	$[b,a]Qa + [[b,a],a]a$
$[b,a]Qa$	2	$[[[b,a],a],a]$
$[[b,a],a]a$	2	$[[[b,a],a],a]$
$[[[b,a],a],a]$	3	0

In the last column we see the image of each generator through Δ. Therefore

$$H_i^{S^1}(C_3(\mathbb{C}^*);\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 \text{ if } i = 0,1,2 \\ 0 \text{ otherwise} \end{cases}$$

The idea behind the proof of Theorem 6.4 is to analyse the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}^*) \hookrightarrow C_n(\mathbb{C}^*)_{S^1} \to BS^1$$

As usual it is useful to compare the spectral sequence to one that is better understood. To define the comparison spectral sequence we need to introduce some notation:

Definition. Let us denote by $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ the space $F_n(\mathbb{C}^*)/\Sigma_{n-1}$, where Σ_{n-1} is embedded in Σ_n as the subgroup that fix n. As the notation suggests, we can think of a point of $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ as a configuration in $\mathbb{C}^*$ of $n-1$ indistinguishable black particles and one white particle.

S^1 acts by rotations on $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ and we have a n-fold covering

$$p : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_n(\mathbb{C}^*)$$

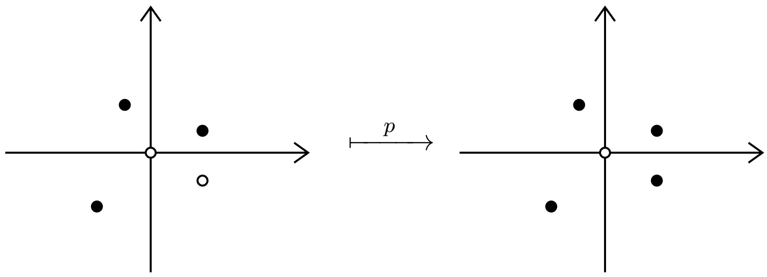


Figure 11: The n-fold covering $p : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_n(\mathbb{C}^*)$

which colors black the white particle (see Figure 11). Since p is S^1-equivariant we get a map of fibrations which will be the key to prove Theorem 6.4:

$$\begin{tikzcd}
C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[r] \arrow[d, hook] & C_n(\mathbb{C}^*) \arrow[d, hook] \\
C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} \arrow[r] \arrow[d] & C_n(\mathbb{C}^*)_{S^1} \arrow[d] \\
BS^1 \arrow[r] & BS^1
\end{tikzcd}$$

Lemma 6.6. *Fix a field of coefficients $\mathbb{F}$ for (co)homology. Then the (co)homological spectral sequence associated to the fibration*

$$C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \hookrightarrow C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} \to BS^1$$

degenerates at page E^3. More precisely, we have

$$E^3_{i,j} = \begin{cases} H_j(C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1) \text{ if } i = 0 \\ 0 \text{ if } i > 0 \end{cases}$$

Proof. We first observe a few elementary facts:

1. S^1 acts freely on $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$, therefore the homotopy quotient $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1}$ and the strict quotient $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$ are homotopy equivalent.

2. The continuous map

$$s : F_n(\mathbb{C}^*) \to F_n(\mathbb{C}^*)$$
$$(z_1, \dots, z_n) \mapsto \left(\frac{z_n}{|z_n|}\right)^{-1} \cdot (z_1, \dots, z_n)$$

is $(\Sigma_{n-1} \times S^1)$-equivariant, so it induces a map

$$\begin{tikzcd}
F_n(\mathbb{C}^*) \arrow[r, "s"] \arrow[d] & F_n(\mathbb{C}^*) \arrow[r] & C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \\
C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1 \arrow[urr, dashed] & &
\end{tikzcd}$$

which is a section to the S^1-principal bundle $\pi : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$. Therefore $C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \cong S^1 \times C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$.

In what follows we will write X instead of $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$ to shorten the notation. From the second point we can rewrite the fibration of the statement as

$$S^1 \times X \overset{i}{\hookrightarrow} X \to BS^1$$

where $i : S^1 \times X \to X$ is the projection on the second factor. Let us work cohomologically in order to exploit the multiplicativity of the spectral sequence: the second page looks as follows:

$$E_2 = \frac{\mathbb{F}[a]}{(a^2)} \otimes H^*(X;\mathbb{F}) \otimes H^*(BS^1;\mathbb{F})$$

where a is a generator of $H^1(S^1)$. The classes $x \in H^*(X) \subseteq E_2^{0,*}$ are infinite cycles because they belong to the image of $i^* : H^*(X) \to H^*(S^1 \times X)$. This observation implies that $E_3 = E_\infty$, because the only multiplicative generator which can have non zero differentials is a, which is a class in $E_2^{0,1}$, and therefore $d_n(a) = 0$ for $n \geq 3$. Now observe that $d_2(a) \neq 0$: if it were zero then $E_2 = E_\infty$ but this is a contraddiction because $H^*(X;\mathbb{F})$ is finite dimensional. Let us call $c := d_2(a)$; with this notation we have $H^*(BS^1;\mathbb{F}) \cong \mathbb{F}[c]$ and the differential of the second page is given by

$$\begin{aligned} d_2(a \cdot x \cdot c^i) &= x \cdot c^{i+1} \\ d_2(x \cdot c^i) &= 0 \end{aligned}$$

for any $x \in H^*(X;\mathbb{F})$ and any $i \in \mathbb{N}$. This implies the (dual) statement. □

Now we are ready to prove Theorem 6.4:

Proof. (of Theorem 6.4) Consider the Serre spectral sequence with $\mathbb{F}_p$-coefficients associated to the fibration

$$C_n(\mathbb{C}^*) \hookrightarrow C_n(\mathbb{C}^*)_{S^1} \to BS^1$$

We claim that $E^3_{p,q} = 0$ for any $p > 0$, $q \in \mathbb{N}$, therefore the spectral sequence degenerates at the third page and

$$H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p) \cong E^3_{0,*} \cong coker(\Delta)$$

where the last isomorphisms follows directly from Proposition 5.9. Consider the map of fibrations

$$\begin{array}{ccc} C_{(n-1)\bullet+\circ}(\mathbb{C}^*) & \xrightarrow{p} & C_n(\mathbb{C}^*) \\ \big\downarrow & & \big\downarrow \\ C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} & \longrightarrow & C_n(\mathbb{C}^*)_{S^1} \\ \big\downarrow & & \big\downarrow \\ BS^1 & \longrightarrow & BS^1 \end{array}$$

Let us denote by (E_*, d_*) (resp. (E'_*, d'_*)) the homological spectral sequence associated to the right (resp. left) fibration . Fix a cycle $x \in E_2^{i,j}$ with $i > 0$, and let $\tau : H_*(C_n(\mathbb{C}^*); \mathbb{F}_p) \to H_*(C_{(n-1)\bullet+\circ}(\mathbb{C}^*); \mathbb{F}_p)$ be the transfer map associated to the n-fold covering p. We are going to show that x does not survive at page E_3, i.e. page E_3 contains elements only in the first column and this proves the claim. The key property we need to prove the Theorem is:

$$(\tau \otimes 1) \circ d_2 = d'_2 \circ (\tau \otimes 1) \tag{8}$$

We refer to Corollary 6.7 for a proof of this equation. If we assume this result, then we have that $d'_2 \circ (\tau \otimes 1)(x) = (\tau \otimes 1) \circ d_2(x) = 0$ because x is a cycle. By the previous Lemma $(\tau \otimes 1)(x)$ does not survive at E'_3, so there exists $y \in E'_2$ such that $d'_2(y) = (\tau \otimes 1)(x)$. Applying $p_* \otimes 1$ we obtain

$$(p_* \otimes 1)(d'_2(y)) = (p_* \otimes 1) \circ (\tau \otimes 1)(x) = n \cdot x$$

But n is not zero in $\mathbb{F}_p$ so we can divide by n and get

$$d_2 \circ (p_* \otimes 1)(y/n) = (p_* \otimes 1) \circ d'_2(y/n) = x$$

which is exactly what we claimed. □

We end this paragraph justifying Equation 8:

Corollary 6.7. *The equality 8 holds.*

Proof. By Proposition 5.9 d_2 and d'_2 are given by the operator Δ, so it suffices to prove the equality

$$\tau\Delta = \Delta\tau$$

Consider the pullback square

$$\begin{tikzcd}
S^1 \times C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[r, "\theta'"] \arrow[d, "1\times p"] & C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[d, "p"] \\
S^1 \times C_n(\mathbb{C}^*) \arrow[r, "\theta"] & C_n(\mathbb{C}^*)
\end{tikzcd}$$

where the horizontal arrows are the S^1-actions. Let us denote by τ (resp. τ') the homological transfer associated to right (resp. left) vertical arrow. Fix a class $y \in H_*(C_n(\mathbb{C}^*))$ and consider its cross product $[S^1] \times y \in H_*(S^1 \times C_n(\mathbb{C}^*))$; by the naturality of the transfer under pullbacks we get

$$\begin{aligned}
\tau\Delta(y) &= \tau\theta_*([S^1] \times y) \\
&= \theta'_*\tau'([S^1] \times y) \\
&= \theta'_*([S^1] \times \tau(y)) = \Delta\tau(y)
\end{aligned}$$

□

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