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Constructions of Spherical 3-Designs

Béla Bajnok

Department of Mathematics and Computer Science, Gettysburg College, Gettysburg, PA 17325-1486, USA. e-mail:bbajnok@cc.gettysburg.edu

Abstract. Spherical t-designs are Chebyshev-type averaging sets on the d-sphere $S^d \subset R^{d+1}$ which are exact for polynomials of degree at most t. This concept was introduced in 1977 by Delsarte, Goethals, and Seidel, who also found the minimum possible size of such designs, in particular, that the number of points in a 3-design on S^d must be at least $n \geq 2d+2$. In this paper we give explicit constructions for spherical 3-designs on S^d consisting of n points for $d = 1$ and $n \geq 4$; $d = 2$ and $n = 6, 8, \geq 10$; $d = 3$ and $n = 8, \geq 10$; $d = 4$ and $n = 10, 12, \geq 14$; $d \geq 5$ and $n \geq 5(d+1)/2$ odd or $n \geq 2d+2$ even. We also provide some evidence that 3-designs of other sizes do not exist. We will introduce and apply a concept from additive number theory generalizing the classical Sidon-sequences. Namely, we study sets of integers S for which the congruence $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_t x_t \equiv 0 \bmod n$, where $\varepsilon_i = 0, \pm 1$ and $x_i \in S$ $(i = 1, 2, \ldots, t)$, only holds in the trivial cases. We call such sets Sidon-type sets of strength t, and denote their maximum cardinality by $s(n,t)$. We find a lower bound for $s(n,3)$, and show how Sidon-type sets of strength 3 can be used to construct spherical 3-designs. We also conjecture that our lower bound gives the true value of $s(n,3)$ (this has been verified for $n \leq 125$).

1. Introduction

We are interested in finding finite "well balanced" point sets on the surface of the unit d-sphere $S^d \subset R^{d+1}$. While it may be clear that vertices of regular polygons form such sets on the circle S^1, there is no natural way to generalize this for $d \geq 2$. Of the numerous possible criteria for measuring how "well balanced" our point set is (see e.g. [10]), one of the most useful and interesting one is that of the spherical design, as introduced in a monumental paper by Delsarte, Goethals, and Seidel in 1977 [11].

A *spherical t-design* on S^d is a finite set of points $X \subset S^d$ for which the Chebyshev-type quadrature formula

$$\frac{1}{\sigma_d(S^d)} \int_{S^d} f(x)d\sigma_d(x) \approx \frac{1}{|X|} \sum_{x \in X} f(x)$$

is exact for all polynomials $f(x) = f(x_0, x_1, \ldots, x_d)$ of degree at most t (σ_d denotes the surface measure on S^d). In other words, X is a spherical t-design of

S^d, if for every polynomial $f(x)$ of degree t or less, the average value of $f(x)$ over the whole sphere is equal to the arithmetic average of its values on the finite set X. General references on spherical designs include [11], [6], [5], and [22].

The existence of spherical designs for every t, d, and large enough $n = |X|$ was first proved by Seymour and Zaslavsky in 1984 [25], and general constructions were first given by the author in 1990 [3].

In [11], Delsarte, Goethals, and Seidel also proved that a spherical t-design on S^d must have cardinality

$$n \geq N_d(t) = \binom{\lfloor t/2 \rfloor + d}{d} + \binom{\lfloor (t-1)/2 \rfloor + d}{d}.$$

A spherical t-design on S^d with cardinality $N_d(t)$ is called *tight*. In 1980 Bannai and Damerell [7], [8] proved that tight spherical designs for $d \geq 2$ exist only for $t = 1, 2, 3, 4, 5, 7$ or 11. All tight t-designs are known, except for $t = 4, 5$, and 7. In particular, there is a unique tight spherical 11-design ($d = 23$ and $n = 196, 560$).

Let $M_d(t)$ denote the minimum size of a spherical t-design on S^d, and let $M'_d(t)$ denote the smallest integer such that for every $n \geq M'_d(t)$, t-designs on S^d exist on n nodes. We have $N_d(t) \leq M_d(t) \leq M'_d(t)$. Values of $M_d(t)$ and $M'_d(t)$ are generally unknown when $d \geq 2$ and $t \geq 3$. For an upper bound on $M_d(t)$ and $M'_d(t)$ see [5].

The case $d = 1$ is completely settled; it is easy to see that vertices of a regular n-gon with $n \geq t + 1$ give a spherical t-design on the circle, hence $N_1(t) = M_1(t)+ M'_1(t) = t + 1$. (Hong [19] proved in 1982 that these are the unique t-designs on S^1 when $t + 1 \leq n \leq 2t + 1$.)

Much work has been done for $d = 2$. It is well known that $N_2(t) = M_2(t)$ if and only if $t = 1$ (2 antipodal points), $t = 2$ (4 vertices of a regular tetrahedron), $t = 3$ (the regular octahedron), or $t = 5$ (the icosahedron). For $t = 4$ we have $N_2(4) = 9$, and there are designs of sizes $n = 12, 14$, and $n \geq 16$ [17]. Hardin and Sloane [17] also exhibit numerical evidence that a 4-design on S^2 does not exist for $n = 10, 11, 13$, and 15; hence the conjectures $M_2(4) = 12$ and $M'_2(4) = 16$. Recent papers of Reznick [23] and Hardin and Sloane [18] give constructions for $t = 5$ (in which case $N_2(5) = M_2(5) = 12$) for $n = 12, 16, 18, 20$, and $n \geq 22$, and conjecture that this list is complete, hence that $M'_2(5) = 22$. In [18] Hardin and Sloane also provide numeric evidence for what they believe is a complete set of possible sizes for $t = 6, 7, 8, 9, 10, 11$, and 12. Their work indicates that for these values of t, $M'_2(t) - M_2(t)$ varies greatly between 2 ($t = 12$) and 12 ($t = 7$).

Keeping t constant and letting the dimension vary, we first note that $N_d(1) = M_d(1) = M'_d(1) = 2$ for every $d \geq 1$. Mimura [21] settled the case $t = 2$ in 1990: He proved that $M_d(2) = N_d(2) = d + 2$, and that $M'_d(2) = d + 2$ when d is odd and $M'_d(2) = d + 4$ when d is even. Much less has been known when $t \geq 3$. For $t = 3$ the author conjectured that 3-designs on S^2 do not exist on $n = 7$ or 9 points ($N_2(3) = 6$), and this was recently supported by a powerful computer search done by Hardin and Sloane [18]. In [17] Hardin and Sloane also present numerical evidence for values of $M_d(4)$ and $M'_d(4)$ for $d \leq 7$. If their conjectures are valid, then $M_d(4) = M'_d(4)$ for $d = 3, 4, 6$, and 7, but $M'_d(4) - M'_d(4) = 12$ for $d = 5$.

The goal of this paper is to provide constructions for 3-designs on S^d for all values of n for which such designs exist of size n. Our results are summarized in the table below.

d	$NN_d(3) = M_d(3)$	n
1	4	≥ 4
2	6	6, 8, ≥ 10
3	8	8, ≥ 10
4	10	10, 12, ≥ 14
5	12	12, ≥ 14
6	14	14, 16, ≥ 18
7	16	16, 18, ≥ 20
8	18	18, 20, ≥ 22
9	20	20, 22, ≥ 24
≥ 5	$2d + 2$	$\geq 2d + 2$ & even, $\geq 5(d + 1)/2$ & odd

We believe that our list above is complete. In particular, we conjecture that $M'_d(3) = \lfloor 5d/2 + 3 \rfloor_2$, where $d \neq 2$ or 4 and $\lfloor x \rfloor_2$ is the largest even integer not greater than x.

We will employ methods similar to those used in [1], [21], and [23]. We will also introduce and apply a concept from additive number theory generalizing the famous but not yet completely understood Sidon-sequences. A *Sidon-sequence*, as first studied by Sidon in 1993 [24], is a sequence of distinct integers $\{x_1, x_2, \ldots\}$ with the property that the sums $x_i + x_j$ are all distinct or, equivalently, that the equation $x_i + x_j - x_k - x_l = 0$ is satisfied only in the trivial case of $\{i, j\} = \{k, l\}$.

It follows from a 1941 paper of Erdös and Turán [14] (and was independently proved by Lindström in 1969 [20]) that in the interval $[1, n]$, a Sidon-sequence can have at most $n^{1/2} + n^{1/4} + 1$ elements. In 1944 Erdös [12] and Chowla [9] independently proved that a Sidon-sequence in $[1, n]$ with at least $n^1 - n^{5/16}$ elements can indeed be found. It is a \$1,000 Erdös problem to prove or disprove that the correct maximal cardinality differs from $\sqrt{n}$ by a constant. These and other results on Sidon-sets and related questions can be found in Erdös's and Freud's excellent survey [13], as well as in [15] and [16].

In this paper we are interested in the following generalization. Let S be a set of integers, and suppose that the congruence $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_t x_t \equiv 0 \mod n$, where $\varepsilon_i = 0, \pm 1$ and $x_i \in S$ for $i = 1, 2, \ldots, t$, only holds in the trivial case, that is when $\varepsilon_i = 0$ for all $i = 1, 2, \ldots, t$ or when the same x_i appears with both a coefficient of 1 and of -1. We here call such sets *Sidon-type sets of strength* t, and denote their maximum cardinality (they clearly must be finite) by $s(n, t)$. It is obvious that $s(n, 1) = n - 1$, and it is also easy to see that $s(n, 2) = \lfloor (n - 1)/2 \rfloor$. Here we find the following lower bound for $s(n, 3)$: (i)$s(n, 3) \geq \lfloor n/4 \rfloor$ is n is even; (ii) $s(n, 3) \geq \lfloor (n + 1)/6 \rfloor$ if n is odd and has no divisors congruent to 5 mod 6; and (iii) $s(n, 3) \geq \frac{(p+1)n}{6p}$ if n is odd and p is its smallest divisor which is congruent to 5 mod 6. We show how Sidon-type sets of strength 3 can be used to construct spherical 3-designs. We also conjecture that our lower bound gives the true value of $s(n, 3)$ (this has been verified for $n \leq 125$), which in part supports our conjecture for $M_d'(3)$ above. Note also that a Sidon-type set of strength 4 forms a Sidon-sequence in $[1, n]$, hence $s(n, 4) \le n^{1/2} + n^{1/4} + 1$.

2. Harmonic Polynomials

To construct spherical designs, we will use the following equivalent definition, cf. [11]:

A finite subset X of S^d is a spherical t-design, if and only if

$$\sum_{x \in X} f(x) = 0$$

for all homogeneous harmonic polynomials $f(x_0, x_1, \ldots, x_d)$ with $1 \le \deg f \le t$.

A polynomial $f(x_0, x_1, \ldots, x_d)$ is called *harmonic* if it satisfies Laplace's equation $\Delta f = 0$. The set of homogeneous harmonic polynomials of degree s forms a vector space $Harm_{d+1}(s)$, with

$$\dim Harm_{d+1}(s) = \binom{s+d}{d} - \binom{s+d-2}{d}.$$

In particular, for $s \le 3$, we see that Φ_s forms a basis for $Harm_{d+1}(s)$ where

$$\Phi_1 = \{x_i | 0 \le i \le d\},$$

$$\Phi_2 = \{x_i x_j | 0 \le i < j \le d\} \cup \{x_i^2 - x_{i+1}^2 | 0 \le i \le d-1\}, \text{ and}$$

$$\Phi_3 = \{x_i x_j x_k | 0 \le i < j < k \le d\} \cup \{x_i^3 - 3x_i x_j^2 | 0 \le i \ne j \le d\}.$$

We associate matrices with spherical designs in the following way. For a set $X = \{u_k = (u_{ok}, u_{1k}, \ldots, u_{dk}) \in R^{d+1} | 1 \le k \le n\}$ we consider the $(d+1) \times n$ matrix U with column vectors $u_1, u_2, \ldots, u_n$.

For a polynomial $f(x_0, x_1, \ldots, x_d)$ we define

$$f(U) = \sum_{k=1}^{n} f(u_k).$$

With these notations, *X is a spherical t-design, if and only if*

$$(*) \qquad \begin{aligned} &\sum_{i=0}^{d} u_{ik}^2 = 1 \quad \textit{for } 1 \le k \le n, \quad \textit{and} \\ &f(U) = 0 \quad \textit{for every polynomial } f \in \bigcup_{s=1}^{t} \Phi_s. \end{aligned}$$

3. Antipodal Designs

It is well known and most obvious that vertices of the generalized regular octahedra form (tight) 3-designs on S^d:

Construction 3.1. *The matrix* $(I\,{-I})$ *provides a spherical* 3*-design on* S^d *of size* $2d+2$. *Here* I *is the* $d+1$ *by* $d+1$ *identity matrix.*

More generally, antipodal point sets on S^d (sets where $x \in S^d$ implies $-x \in S^d$) can be used to construct spherical 3-designs. Equations (*) show that if t is even and A is the matrix of a t-design on S^d, then (the set of column vectors of) the matrix $(A\,{-A})$ provides a $(t+1)$-design on S^d. Since 2-designs on S^d exist for sizes $n \geq d+2$ when d is odd and for $n = d+2, n \geq d+4$ when d is even [21], we immediately have

Proposition 3.2. *Let* n *be an even integer such that* $n \geq 2d+4$, *except for* $n = 2d+6$ *when* d *is even. Then a spherical* 3*-design on* S^d *of size* n *exists.*

Primarily with the cases of even d in mind, we provide the following

Construction 3.3. *Suppose that* A *is the matrix of a* 2*-design on* S^{d-1} *of size* n_1, J *is the* 1 *by* n_1 *matrix of all* 1*'s*, $\alpha = \sqrt{d/(d+1)}$, *and* $\delta = \sqrt{1/(d+1)}$. *Then* $M = \left(\begin{smallmatrix} \alpha A & -\alpha A \\ \delta J & -\delta J \end{smallmatrix}\right)$ *is a* 3*-design of size* $2n_1$ *on* S^d.

Proof. For $A = (u_{ik})_{0 \leq i \leq d-1, 1 \leq k \leq n_1}$ we have

$$\sum_{i=0}^{d-1} u_{ik}^2 = 1 \quad \text{for } 1 \leq k \leq n_1, \quad \text{and}$$
$$\sum_{k=0}^{n_1} u_{ik}^2 - u_{i+1,k}^2 = 0 \quad \text{for } 0 \leq i \leq d-2.$$

Therefore, M satisfies (*) for $t = 3$ if and only if the equations

$$\alpha^2 + \delta^2 = 1 \text{ and } \alpha^2 \frac{2n_1}{d} - \delta^2 \cdot 2n_1 = 0$$

hold. These two equations are equivalent to

$$\alpha^2 = \frac{d}{d+1} \text{ and } \delta^2 = \frac{1}{d+1}. \qquad \square$$

As a corollary, we get

Proposition 3.4. *Let* n *be an even integer such that* $n \geq 2d+2$, *except for* $n \geq 2d+4$ *when* d *is odd. Then a spherical* 3*-design on* S^d *of size* n *exists.*

4. Sidon-Type Sets

For other constructions of spherical 3-designs, we will use what we call Sidon-type sets of strength 3.

Let R be a ring with identity, S a subset of R, and t a positive integer. We say that S is a *Sidon-type set of strength* t in R if no non-trivial-trivial sum of the form

$$\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_t x_t,$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t = 0, \pm 1$ and $x_1, x_2, \ldots, x_t$ are (not necessarily distinct) elements of S, equals 0. We call such a sum non-trivial if no x_i appears in it with both a coefficient of 1 and -1, and if at least one ε_i is non-zero $(i = 1, 2, \ldots, t)$.

Here we are only interested in Sidon-type sets in Z_n, and we think of these sets as integer subsets of the interval $[1, n]$. The cardinality of a largest Sidon-type set of strength t in Z_n will be denoted by $s(n, t)$. It is obvious that $s(n, 1) = n - 1$ (take all integers from 1 to $n - 1$), and it is easy to see that $s(n, 2) = \lfloor (n-1)/2 \rfloor$ (S cannot contain both x and $n - x$, but it can consist of all integers up to $\lfloor (n-1)/2 \rfloor$). For $t = 3$ we give a constructive proof for the following.

Theorem 4.1.

(i) $s(n, 3) \geq \lfloor n/4 \rfloor$ *if n is even*;
(ii) $s(n, 3) \geq \lfloor (n+1)/6 \rfloor$ *if n is odd and has no divisors congruent to* 5 *mod* 6; *and*
(iii) $s(n, 3) \geq \frac{(p+1)n}{6p}$ *if n is odd and p is its smallest divisor which is congruent to* 5 *mod* 6.

Proof. We can always take all the odd integers up to (but not including) $n/3$, proving (ii). When n is even, we can take all the odd integers up to (but not including) $n/2$, which proves (i).

Now suppose that n is odd and that there is a non-negative integer q such that $p = 6q + 5$ divides n. We define

$$S = \{ap + 2b + 1 | \quad a = 0, 1, \ldots, n/p - 1, \quad b = 0, 1, \ldots, q\}.$$

We see that S has cardinality $(p + 1)n/(6p)$. To verify that S is a Sidon-type set of strength 3, suppose that n divides

$$\begin{aligned} x &= \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 \\ &= (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 x_3)p + 2(\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3) + \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \end{aligned}$$

This implies that

$$y = 2(\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 x_3) + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

is divisible by p, but since $|y| \leq 6q + 3$ and $p = 6q + 5$, this can only happen if $y = 0$. Since 0 is an even number, either all $\varepsilon' s$ are equal to 0 (a trivial sum), or exactly one ε, say ε_1, is 0. In the latter case, since $b_2, b_3 \geq 0$, we must have $\varepsilon_2 = -\varepsilon_3$, which implies that $b_2 = b_3$. In this case we also get

$$|X| = |(\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3)p| = |a_2 - a_3|p \leq \left(\frac{n}{p} - 1\right)p < n,$$

so n can only divide x if $x = 0$. But then $a_2 = a_3$, hence $x_2 = x_3$, and we again have the trivial sum. $\square$

We performed a computer search for values of $s(n,3)$ for $n \le 125$, and found that in all cases the true value agreed with the lower bound found in Theorem 4.1. Therefore we state

Conjecture 4.2. *Theorem* 4.1 *gives the exact value of* $s(n,3)$. *In particular,*

(i) $s(n,3) \le n/4$, *with equality if and only if n is divisible by* 4; *and*
(ii) *if n is odd, then* $s(n,3) \le n/5$, *with equality if and only if n is divisible by* 5.

5. Regular 3-Designs

It is well known and is easy to check that the vertices of a regular n-gon where $n \ge t+1$ form a t-design on S^1. In this section we investigate a generalization of this for dimensions $d \ge 1$, where d is odd.

For positive integers m and n we define the vectors $s(m)$ and $c(m)$ in R^n to be

$$s(m) = \left(\sin\left(\frac{2\pi}{n}\,m\right), \sin\left(\frac{2\pi}{n}\,2m\right), \ldots, \sin\left(\frac{2\pi}{n}\,nm\right)\right) \text{ and}$$
$$c(m) = \left(\cos\left(\frac{2\pi}{n}\,m\right), \cos\left(\frac{2\pi}{n}\,2m\right), \ldots, \cos\left(\frac{2\pi}{n}\,nm\right)\right).$$

Now let $e > 0$ and $m_1, m_2, \ldots, m_e$ be integers, and set $S = \{m_1, m_2, \ldots, m_e\}$. We define the matrix $A(S)$ to be the $2(e) \times n$ matrix with row vectors $s(m_1)$, $c(m_1)$, $s(m_2)$, $c(m_2), \ldots, s(m_e)$, $c(m_e)$.

Lemma 5.1. *Let e be a positive integer,* $s = 1$, 2, *or* 3, *and suppose that* $S = \{m_1, m_2, \ldots, m_e\}$ *is a Sidon-type set of strength s. We define the matrix* $A = A(S)$ *as above. If* $f\colon R^{2e} \to R$ *is a polynomial such that* $f \in \Phi_s$, *then* $f(A) = 0$.

Proof. The statement is well known for $s = 1$: setting $z_j = \cos(\frac{2\pi}{n} m_j) + i\sin(\frac{2\pi}{n} m_j)$, we see that, since $z_j \ne 1$ when m_j is not divisible by n, we have $\sum_{k=1}^{n} z_j^k = 0$ for every $j = 1, 2, \ldots, e$.

Our claims for the cases of $s = 2$ when f is square-free and of $s = 3$ (and f any homogeneous cubic polynomial) follow from the $s = 1$ case after repeated use of the trigonometric identities

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)],$$
$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)], \text{ and}$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)].$$

Finally, when $f = x_i^2 - x_{i+1}^2$, $i = 0, 1, \ldots, d-1$, we get

$$f(A) = \frac{1}{2}\sum_{k=1}^{n} \cos 0 - \frac{1}{2}\sum_{k=1}^{n} \cos 0 = 0. \qquad \square$$

We have the following corollary.

Construction 5.2. *Suppose that d is an odd positive integer, $e = (d+1)/2$, $s = 1$, 2, or 3, and that $S = \{m_1, m_2, \ldots, m_e\}$ is a Sidon-type set of strength s. Then the n column vectors of the matrix $M(S) = \sqrt{2/(d+1)} \cdot A(S)$ form a spherical s-design on S^d.*

The s-designs constructed with Construction 5.2 will be called *regular s-designs*. We note that Lemma 5.1 and Construction 5.2 are false for strengths $s \geq 4$ (see [1]). As Construction 5.2 provides spherical 3-designs on S^d with $s(n,3) \geq (d+1)/2$, we have

Proposition 5.3. *Suppose that d is an odd positive integer. Regular 3-designs of size n on S^d exist when*

(i) *n is even and $n \geq 2d+2$;*
(ii) *n is odd and $n \geq 3d+2$; and*
(iii) *n is odd and $n \geq \frac{p}{p+1}(3d+3)$, where p is a divisor of n which is congruent to 5 mod 6.*

In particular, there are regular 3-*designs of size n on S^d when n is an odd integer which is divisible by 5 and $n \geq 5(d+1)/2$.*

Conjecture 4.2 implies that Proposition 5.3 characterizes all values of n for which regular 3-designs exist on S^d. In particular, we believe that no regular 3-design exists for odd values of n with $n < 5(d+1)/2$ (this has been verified for $d \leq 49$).

6. Other Spherical 3-Designs

We have just seen constructions for 3-designs on S^d for all odd values of n when $n \geq 5(d+1)/2$, d is odd, and n is divisible by 5. In this section we will construct 3-designs on S^d of size n for every odd value of n with $n \geq \max\{5(d+1)/2, 2d+7\}$.

Construction 6.1. *Let d_1 and d_2 be positive even integers with $d = d_1 + d_2 - 1$, let n_1 and n_2 be positive integers with $n = n_1 + n_2$, and suppose that $d_1 n_1 \geq d_2 n_2$. Suppose further that A is (the matrix of, see section 2) a regular 3-design of size n_1 on S^d, and that C is a regular 3-design of size n_2 on $S^{d_1 - 1}$. Then the n column vectors of the matrix $M = \begin{pmatrix} \alpha A_1 & C \\ \beta A_2 & 0 \end{pmatrix}$, where $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and α and β are defined below, form a 3-design on S^d.*

Proof. M satisfies (*) if and only if the equations

$$\alpha^2 \frac{d_1}{d_1+d_2} + \beta^2 \frac{d_2}{d_1+d_2} = 1 \text{ and}$$

$$\frac{n_2}{d_1} + \alpha^2 \frac{n_1}{d_1+d_2} - \beta^2 \frac{n_1}{d_1+d_2} = 0$$

hold (see proof of Construction 3.3). The two equations are equivalent to

$$\alpha^2 = 1 - \frac{d_2 n_2}{d_1 n_1} \text{ and } \beta^2 = 1 + \frac{n_2}{n_1}.$$

We see that α is real iff $d_1 n_1 \geq d_2 n_2$, as was assumed. □

A corollary is the following.

Proposition 6.2. *Let d and n be odd integers such that $n \geq \max\{5(d+1)/2, 2d+7\}$. Then there are 3-designs of size n on S^d.*

Proof. The construction is given in 6.1 when taking $d_1 = 2$, $d_2 = d-1$, $n_1 = n-5$, and $n_2 = 5$. The necessary inequalities all hold: $d_1 n_1 \geq d_2 n_2$, $n_1 \geq 2d+2$ (n_1 is even, see Proposition 5.3 (i)), and $n_2 \geq 4$ (C is a 3-design on the circle, see Proposition 5.3 (ii)). □

We now turn to the case when d is even.

Construction 6.3. *Let d_1 and d_2 be positive even integers with $d = d_1 + d_2$, let n_1 and n_2 be positive integers with $n = 2n_1 + n_2$, and suppose that $2d_1 n_1 \geq (d_2+1)n_2$. Suppose further that A is a regular 2-design of size n_1 on S^{d-1}, and that C is a regular 3-design of size n_2 on S^{d_1-1}. Then the n column vectors of the matrix $M = \begin{pmatrix} \alpha A_1 & -\alpha A_1 & C \\ \beta A_2 & -\beta A_2 & 0 \\ \delta J & -\delta J & 0 \end{pmatrix}$, where $A = \binom{A_1}{A_2}$, J is the 1 by n_1 matrix of all 1's, and α, β, and* δ are defined below, form a 3-design on S^d.

Proof. M satisfies (*) if and only if the equations

$$\alpha^2 \frac{d_1}{d_1+d_2} + \beta^2 \frac{d_2}{d_1+d_2} + \delta^2 = 1,$$

$$\frac{n_2}{d_1} + \alpha^2 \frac{2n_1}{d_1+d_2} - \beta^2 \frac{2n_1}{d_1+d_2} = 0,$$

$$\frac{n_2}{d_1} + \alpha^2 \frac{2n_1}{d_1+d_2} - \delta^2 2n_1 = 0, \text{and}$$

$$\beta^2 \frac{2n_1}{d_1+d_2} - \delta^2 2n_1 = 0$$

hold (see again the proof of Construction 3.3). The four equations are equivalent to the following three:

$$\alpha^2 = \frac{d_1+d_2}{d_1+d_2+1}\left(1 - \frac{(d_2+1)n_2}{2d_1 n_1}\right),$$

$$\beta^2 + \frac{d_1+d_2}{d_1+d_2+1}\left(1 + \frac{n_2}{2n_1}\right), \text{and}$$

$$\delta^2 = \frac{1}{d_1+d_2+1}\left(1 + \frac{n_2}{2n_1}\right).$$

We see that α is real iff $2d_1 n_1 \geq (d_2+1)n_2$, as was assumed. □

This gives us the following corollary.

Proposition 6.4. *Let d be even and n odd, such that $n \geq \max\{5(d+1)/2, 2d+7\}$. Then there are 3-designs of size n on S^d.*

Proof. The construction is given in 6.3 when taking $d_1 = 2$, $d_2 = d-2$, $n_1 = (n-5)/2$, and $n_2 = 5$. The necessary inequalities all hold: $2d_1n_1 \geq (d_2+1)n_2$, $n_1 \geq d+1$ (Construction 5.2 provides regular 2-designs on S^{d-1} when $d-1$ is odd and $s(n_1, 2) = \lfloor (n_1-1)/2 \rfloor \geq d/2$), and finall $n_2 \geq 4$ (see Proposition 5.3(ii)). □

7. Summary of Results

We summarize our results in the following theorem.

Theorem 7.1. *No spherical 3-design exists on S^d of size $n < 2d+2$. Spherical 3-designs on S^d exist when*

(i) *n is even and $n \geq 2(d+1)$;*
(ii) *n is odd and $n \geq 5(d+1)/2$, except for $d = 2$ and $n = 9$, $d = 4$ and $n = 13$.*

Proof. The first statement is a reiteration of $n \geq N_d(3) = 2d+2$ (see [11] for more). The cases when n is even are either from Proposition 3.2 or Proposition 3.4. The cases of $d = 1$ and $d = 3$ of (ii) are stated in Proposition 5.3 (ii), and the case of $d = 5$ and $n = 15$ is a special case of Proposition 5.3 (iii). All other cases in (ii) follow from Propositions 6.2 and 6.4. □

According to Theorem 7.1, the number of different values of n for which the problem is open is $\lfloor (d+2)/4 \rfloor$ when $d \neq 2$ or 4. Only one case is unsettled for $d = 3$ and $d = 5$, and at most two cases are open when $d \leq 9$. We state our

Conjecture 7.2. *Theorem 7.1 gives a complete list of all possible sizes n for which spherical 3-design on S^d exist. In particular, $M'_d(3) = \lfloor 5d/2 + 3 \rfloor_2$, where $d \neq 2$ or 4 and $\lfloor x \rfloor_2$ is the largest even integer not greater than x, $M'_2(3) = 10$, and $M'_4(3) = 14$.*

Conjecture 7.2 is supported by our previously stated belief that no regular 3-designs exist with $n < 5(d+1)/2$ when n is odd (which has been verified for $d \leq 49$); and was also numerically demonstrated for $d = 2$ by Hardin and Sloane [18].

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Note added in proof. An upcoming paper of Boyvalenkov, Danev, and Nikova contains new nonexistence results, such as the nonexistence of a 7 point 3-design on S^2.