FRACTIONAL MAXIMAL OPERATORS ON WEIGHTED VARIABLE LEBESGUE SPACES OVER THE SPACES OF HOMOGENEOUS TYPE

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Abstract. Let (X, d, μ) is a space of homogeneous type, we establish a new class of fractional-type variable weights $A_{p(\cdot),q(\cdot)}(X)$. Then, we get the new weighted strong-type and weak-type characterizations for fractional maximal operators M_η on weighted variable Lebesgue spaces over (X, d, μ). This study generalizes the results by Cruz-Uribe–Fiorenza–Neugebauer [12] (2012), Bernardis–Dalmasso–Pradolini [4] (2014), Cruz-Uribe–Shukla [14] (2018), and Cruz-Uribe–Cummings [9] (2022).

Contents

1. Introduction

In this paper, we focus on the boundedness of fractional maximal operators M_η on weighted Lebesgue spaces with variable exponents over the spaces of homogeneous type $L^{p(\cdot)}(X, \omega)$. This work is based on the theory of boundedness on weighted Lebesgue Spaces $L^{p(\cdot)}(\omega)$ and some recent work by Cruz-Uribe et al. (see Theorems **A-G** below). The theory of maximal operators was first studied by Muckenhoupt et al. [27, 28], and a series of far-reaching results were obtained. Since then, the weighted theory of maximal operators can be regarded as the generalization of the work of Muckenhoupt et al.

Date: April 25, 2024.

2020 *Mathematics Subject Classification.* 42B25, 42B35.

Key words and phrases. Weights; Variable exponents; Fractional maximal operators; spaces of homogeneous type.

$(\mathbb{R}^n, |\cdot|, dx)$ is a special case of the spaces of the homogeneous type, of which we give some definitions and properties as follows.

Definition 1.1. *For a positive function $d : X \times X \to [0,\infty)$, X is a set, the quasi-metric space (X, d) satisfies the following conditions:*

(1) When $x = y$, $d(x, y) = 0$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) For all $x, y, z \in X$, there is a constant $A_0 \geq 1$ such that $d(x, y) \leq A_0(d(x, z) + d(z, y))$.

Definition 1.2. *Let μ be a measure of a space X. For a quasi-metric ball $B(x, r)$ and any $r > 0$, if μ satisfies doubling condition, then there exists a doubling constant $C_\mu \geq 1$, such that*

$$0 < \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty,$$

Definition 1.3. *For a non-empty set X with a qusi-metric d, a triple (X, d, μ) is said to be a space of homogeneous type if μ is a regular measure which satisfies doubling condition on the σ-algebra, generated by open sets and quasi-metric balls.*

Considering a measurable function $p : E \to [1,\infty)$ on a subset $E \subseteq X$, we define $p_-(E) = \text{ess}\inf_{x\in E} p(x)$ and $p_+(E) = \text{ess}\sup_{x\in E} p(x)$, with p_- and p_+ specifically denoting these quantities over the entire space X. Furthermore, we introduce some sets of measurable functions based on these definitions.

$$\mathscr{P}(E) = \{p(\cdot) : \mathrm{E} \to [1,\infty) \text{ is measurable: } 1 < p_-(E) \leq p_+(E) < \infty\};$$
$$\mathscr{P}_1(E) = \{p(\cdot) : \mathrm{E} \to [1,\infty) \text{ is measurable: } 1 \leq p_-(E) \leq p_+(E) < \infty\};$$
$$\mathscr{P}_0(E) = \{p(\cdot) : \mathrm{E} \to [0,\infty) \text{ is measurable: } 0 < p_-(E) \leq p_+(E) < \infty\}.$$

Obviously, $\mathscr{P}(E) \subseteq \mathscr{P}_1(E) \subseteq \mathscr{P}_0(E)$. When $E = X$, we write $\mathscr{P}(X)$ by $\mathscr{P}$ for convenience.

Definition 1.4. *Let $1 \leq p_- \leq p_+ \leq \infty$, the variable exponent Lebesgue spaces with Luxemburg norm is defined as*

$$L^{p(\cdot)}(X) = \{f : \|f\|_{L^{p(\cdot)}(X)} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(\frac{f}{\lambda}) \leq 1\} < \infty\},$$

where $\rho_{p(\cdot)}(f) = \int_X |f(x)|^{p(x)}dx + \|f\|_{L^\infty(X_\infty)}$. We always abbreviate $\|\cdot\|_{L^{p(\cdot)}(X)}$ to $\|\cdot\|_{p(\cdot)}$. For every ball $B \subseteq X$, if $\rho_{p(\cdot)}(f\chi_B) < \infty$, then f is said to be locally $p(\cdot)$-integrable.

In fact, the above spaces are Banach spaces (precisely, ball Banach function spaces), to which readers can refer [11].

Let ω be a weight function on X. The variable exponent weighted Lebesgue spaces are defined by

$$L^{p(\cdot)}(X, \omega) = \{f : \|f\|_{L^{p(\cdot)}(X,\omega)} := \|\omega f\|_{L^{p(\cdot)}(X)} < \infty\}.$$

Definition 1.5. *For any $x, y \in X$ and $d(x, y) < \frac{1}{2}$, we say $p(\cdot) \in LH_0$, if*

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + 1/d(x, y))}. \tag{1.1}$$

We say $p(\cdot) \in LH_\infty$ (respect to a point $x_0 \in X$), if there exists $p_\infty \in X$, for any $x \in \mathbb{R}^n$,

$$|p(x) - p_\infty| \lesssim \frac{1}{\log(e + d(x, x_0))}. \tag{1.2}$$

We denote the globally log-Hölder continuous functions by $LH = LH_0 \cap LH_\infty$.

According to the above definition, it seems to relate to the choice of the point x_0. However, through [1], we can know that such a choice is immaterial.

Lemma 1.6. *For any $y_0 \in X$, if $p(\cdot) \in LH_\infty$ with respect to $x_0 \in X$, then $p(\cdot) \in LH_\infty$ with respect to y_0.*

If x_0 is not chosen definitely, we always suppose that X has an arbitrary given point x_0.

The fractional maximal operator M_η on the spaces of homogeneous type is defined as

$$M_\eta f(x) = \sup_{B \subseteq X} |B|^{\eta - 1} \int_B |f(y)| d\mu \cdot \chi_B(x).$$

When $X = \mathbb{R}^n$, we take $\eta = \frac{\alpha}{n}$ and write M_η by M_α. We now give the definition of fractional-type weights $A_{p(\cdot),q(\cdot)}(X)$ and present some foundational results.

Definition 1.7. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0, 1)$. We say a weight $\omega \in A_{p(\cdot),q(\cdot)}(X)$, if*

$$[\omega]_{A_{p(\cdot),q(\cdot)}(X)} := \sup_{B \subseteq X} \mu(B)^{\eta - 1} \left\|\omega \chi_B\right\|_{q(\cdot)} \left\|\omega^{-1} \chi_B\right\|_{p'(\cdot)} < \infty.$$

Remark 1.8. *The above discussion introduce a broader category of weights, implying that the $A_{p(\cdot),q(\cdot)}$ set can be deduced as many particular instances under specific conditions.*

(1) If $\eta = 0$, then $A_{p(\cdot),q(\cdot)}(X) = A_{p(\cdot)}(X)$ introduced in [9].
(2) If $p(\cdot) \equiv p$, then $A_{p(\cdot),q(\cdot)}(X) = A_{p,q}(X)$.
(3) If $p(\cdot) \equiv p$ and $\eta = 0$, then $A_{p(\cdot),q(\cdot)}(X) = A_p(X)$.
(4) If $X = \mathbb{R}^n$, then $A_{p(\cdot),q(\cdot)}(X) = A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ introduced in [4].
(5) If $X = \mathbb{R}^n$ and $\eta = 0$, then $A_{p(\cdot),q(\cdot)}(X) = A_{p(\cdot)}(\mathbb{R}^n)$ introduced in [12].
(6) If $X = \mathbb{R}^n$ and $p(\cdot) \equiv p$, then $A_{p(\cdot),q(\cdot)}(X) = A_{p,q}(\mathbb{R}^n)$ introduced in [28].
(7) If $X = \mathbb{R}^n$, $p(\cdot) \equiv p$, and $\eta = 0$, then $A_{p(\cdot),q(\cdot)}(X) = A_p(\mathbb{R}^n)$ introduced in [27].

In this paper, we always abbreviate $A_{p(\cdot),q(\cdot)}(X)$ to $A_{p(\cdot),q(\cdot)}$, $A_{p(\cdot)}(X)$ to $A_{p(\cdot)}$, and $A_p(X)$ to A_p. It is easy to observe that

$$\left[\omega^{-1}\right]_{A_{q'(\cdot),p'(\cdot)}} = [\omega]_{A_{p(\cdot),q(\cdot)}}.$$

By Hölder's inequality, we have

$$[\omega]_{A_{q(\cdot)}} \leq [\omega]_{A_{p(\cdot),q(\cdot)}}, \tag{1.3}$$

$$\left[\omega^{-1}\right]_{A_{p'(\cdot)}} \leq [\omega]_{A_{p(\cdot),q(\cdot)}}. \tag{1.4}$$

We introduce some background and motivation regarding the main results of this paper.

$$\begin{tikzcd}
 & A_p(X) \arrow[r] \arrow[ddr] & A_{p(\cdot)}(X) \arrow[dr] & \\
A_p(\mathbb{R}^n) \arrow[r] \arrow[ur] \arrow[dr] & A_{p(\cdot)}(\mathbb{R}^n) \arrow[r] \arrow[ur] & A_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \arrow[r] & A_{p(\cdot),q(\cdot)}(X) \\
 & A_{p,q}(\mathbb{R}^n) \arrow[r] \arrow[ur] & A_{p,q}(X) \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. The relationships between weights

Since 1972 and 1974, Muckenhoupt et al. [27,28] studied the characterization of $A_p(\mathbb{R}^n)$ and $A_{p,q}(\mathbb{R}^n)$ by maximal operators M and fractional maximal operators M_α respectively. Many people began to pay attention to the relationship between the characterization of weights and maximal operators.

In 2012, Cruz-Uribe, Fiorenza, and Neugebauer [12] firstly studied the characterization of $A_{p(\cdot)}(\mathbb{R}^n)$ by maximal operators M.

Theorem A([12]). *Let $p(\cdot) \in \mathscr{P} \cap LH$ and ω is a weight. Then M is bounded on $L^{p(\cdot)}(\omega)$ if and only if $\omega \in A_{p(\cdot)}(\mathbb{R}^n)$.*

Theorem B([12]). *Let $p(\cdot) \in \mathscr{P}_1 \cap LH$ and ω is a weight. Then M is bounded from $L^{p(\cdot)}(\omega)$ to $WL^{p(\cdot)}(\omega)$ if and only if $\omega \in A_{p(\cdot)}(\mathbb{R}^n)$.*

In 2014, Bernardis, Dalmasso, and Pradolini [4] proved the characterizations for $A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by fractional maximal operators M_α as follows.

Theorem C([14]). *Let $p(\cdot), q(\cdot) \in \mathscr{P} \cap LH$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n} \in [0,1)$, and ω is a weight. Then M_α is bounded from $L^{p(\cdot)}(\omega)$ to $L^{q(\cdot)}(\omega)$ if and only if $\omega \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.*

In 2018, Cruz-Uribe and Shukla [14] obtained the following results, which solve the problem of boundedness of fractional maximal operators on variable Lebesgue spaces over the spaces of homogeneous type.

Theorem D([14]). *Let $p(\cdot), q(\cdot) \in \mathscr{P} \cap LH$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0,1)$ Then M_η is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Additionally, if $\mu(X) < +\infty$, the requirement $p(\cdot) \in LH$ can be substituted with $p(\cdot) \in LH_0$.

Theorem E([14]). *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1 \cap LH$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0,1)$ Then M_η is bounded from $L^{p(\cdot)}(X)$ to $WL^{q(\cdot)}(X)$.*

Moreover, if $\mu(X) < +\infty$, the condition $p(\cdot) \in LH$ may be substituted with $p(\cdot) \in LH_0$.

In 2022, Cruz-Uribe and Cummings [9] demonstrated the following characterizations for $A_{p(\cdot)}(X)$ by maximal operators M.

Theorem F([9]). *Let $p(\cdot) \in \mathscr{P} \cap LH$ and ω is a weight. Then M is bounded on $L^{p(\cdot)}(X, \omega)$ if and only if $\omega \in A_{p(\cdot)}(X)$.*

Theorem G([9]). *Let $p(\cdot) \in \mathscr{P}_1 \cap LH$ and ω is a weight. Then M is bounded from $L^{p(\cdot)}(X, \omega)$ to $WL^{p(\cdot)}(X, \omega)$ if and only if $\omega \in A_{p(\cdot)}(X)$.*

Inspired by the above, it is natural to consider whether $A_{p(\cdot),q(\cdot)}(X)$ can be characterized by fractional maximal operators M_η? The answer to this question is yes. To be precise, we can draw the following conclusions.

Theorem 1.9. *Let $p(\cdot), q(\cdot) \in \mathscr{P} \cap LH$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0, 1)$, and ω is a weight. Then M_η is bounded from $L^{p(\cdot)}(X, \omega)$ to $L^{q(\cdot)}(X, \omega)$ if and only if $\omega \in A_{p(\cdot),q(\cdot)}(X)$.*

Theorem 1.10. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1 \cap LH$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0, 1)$, and ω is a weight. Then M_η is bounded from $L^{p(\cdot)}(X, \omega)$ to $WL^{q(\cdot)}(X, \omega)$ if and only if $\omega \in A_{p(\cdot),q(\cdot)}(X)$.*

Remark 1.11. *It is obvious that Theorems 1.9 and 1.10 generalizes Theorems* **A-G**.

We still need introduce some notations which will be used in this paper.

For some positive constant C independent of appropriate parameters, $A \lesssim B$ means that $A \le CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. What's more $A \lesssim_{\alpha,\beta} B$ means that $A \le C_{\alpha,\beta}B$, where $C_{\alpha,\beta}$ is dependent on α, β. Given an open set $E \subseteq \mathbb{R}^n$ and a measurable function $p(\cdot) : \mathrm{E} \to [1, \infty)$, $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. A weight is defined as a locally integrable function $\omega : X \to [0, \infty]$ satisfying $0 < \omega(x) < \infty$ for almost every $x \in X$. For a given weight ω, its associated measure is established as $d\omega(x) = \omega(x)d\mu(x)$.For a subset $E \subseteq X$, the weighted average integral of a function f is represented by

$$\fint_E f(x)d\omega = \frac{1}{\omega(E)} \int_E f(x)\omega(x)d\mu.$$

Through out this paper, in Section 2, we give some lemmas for variable Lebesgue spaces, weights, and Dyadic Analysis respectively, which play a important roles for the proof of our main theorems. In Section 3, we prove Theorems 1.9 and 1.10.

2. Preliminaries

2.1. Variable Lebesgue spaces.

This subsection includes some foundational lemmas of variable Lebesgue spaces over the spaces of the homogeneous type. The first lemma is called "Lower Mass Bound".

Lemma 2.1 ([9], Lemma 2.1). *For all $0 < r < R$ and any $y \in B(x, R)$, there exists a positive constant $C = C_X$, such that*

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \ge C \left(\frac{r}{R}\right)^{\log_2 C_\mu}$$

Lemma 2.2 ([5], Lemma 1.9). *$\mu(X)<\infty$ if and only if X is bounded, which means there exist a ball $B\subseteq X$, such that $X=B$.*

Lemma 2.3 ([9], Lemma 2.11). *Let $p(\cdot)\in LH$, then $\sup_{B\subseteq X}\mu(B)^{p_-(B)-p_+(B)}\lesssim 1$.*

The following lemmas are initial parts supporting our main results. Actually, some lemmas' proofs are identical to their Euclidean case, and readers can refer to [9, 11, 18] for more details. Hence, we will omit some of them for the brevity of this paper.

Lemma 2.4 ([11], Proposition 2.21). *Let $p(\cdot)\in\mathscr{P}_1$, then*

$$\int_X\left(\frac{|f(x)|}{\|f\|_{p(\cdot)}}\right)^{p(x)}d\mu=1.$$

Lemma 2.5 ([11], Corollary 2.23). *Let $\Omega\subseteq X$ and $p(\cdot)\in\mathscr{P}_1(\Omega)$.*

If $\|f\|_{L^{p(\cdot)}(\Omega)}\leq 1$, then

$$\|f\|_{p(\cdot)}^{p_+(\Omega)}\leq\int_\Omega|f(x)|^{p(x)}d\mu\leq\|f\|_{p(\cdot)}^{p_-(\Omega)}.$$

If $\|f\|_{L^{p(\cdot)}(\Omega)}\geq 1$, then

$$\|f\|_{p(\cdot)}^{p_-(\Omega)}\leq\int_\Omega|f(x)|^{p(x)}d\mu\leq\|f\|_{p(\cdot)}^{p_+(\Omega)}.$$

Moverover, we have $\|f\|_{p(\cdot)}\leq C_1$ if and only if $\int_\Omega|f(x)|^{p(x)}d\mu\leq C_2$. When either $C_1=1$ or $C_2=1$, the other constant is also to be 1.

Lemma 2.6 ([9], Lemma 2.6). *Let $p(\cdot)\in\mathscr{P}_1$, then the bounded functions with bounded support are dense in $L^{p(\cdot)}(X)$. Furthermore, any nonnegative function f in $L^{p(\cdot)}(X)$ can be approximated as the limit of an increasing sequence.*

Lemma 2.7 ([11], Theorem 2.59). *Let $p(\cdot)\in\mathscr{P}_1$. For a sequence of non-negative measureable functions, denoted as $\{f_k\}_{k=1}^\infty$ and increasing pointwise almost everywhere to a function $f\in L^{p(\cdot)}$, we can deduce that $\|f_k\|_{p(\cdot)}\to\|f\|_{p(\cdot)}$.*

Lemma 2.8 ([9], Lemma 2.10). *For any point $y\in G$, G is a subset of X, and two exponents $p_1(\cdot)$ and $p_2(\cdot)$, if there exists a constant $C_0>0$, such that*

$$|p_1(y)-p_2(y)|\leq\frac{C_0}{\log\left(e+d\left(x_0,y\right)\right)}$$

Then there exists a constant $C=C_{t,C_0}$ such that

$$\int_G|f(y)|^{p_1(y)}u(y)d\mu\leq C\int_G|f(y)|^{p_2(y)}u(y)d\mu+\int_G\frac{1}{\left(e+d\left(x_0,y\right)\right)^{ts_-(G)}}u(y)d\mu \tag{2.1}$$

for all functions f with $|f(y)|\leq 1$ and every $t\geq 1$.

2.2. Properties of weights.

This subsection is aimed to exploring the properties of the $A_{p(\cdot),q(\cdot)}$ condition within spaces of homogeneous type. The following lemma reflects the properties of A_∞, defined by $\bigcup_{p\geq 1} A_p$, whose proof are similar to that of [21, Theorem 7.3.3].

Lemma 2.9. *If ω is a weight function, then the following conditions are equivalent:*

(1) $\omega \in A_\infty$.
(2) There exist constants $\epsilon > 0$ and $C_2 > 1$ such that

$$\frac{\mu(E)}{\mu(B)} \leq C_2 \left(\frac{\omega(E)}{\omega(B)}\right)^\epsilon,$$

for any ball B and its measurable subset E.
(3) The measure $d\nu(x) = \omega(x)d\mu(x)$ satisfies doubling condition and there exist constants $\delta > 0$ and $C_1 > 1$ such that

$$\frac{\omega(E)}{\omega(B)} \leq C_1 \left(\frac{\mu(E)}{\mu(B)}\right)^\delta,$$

for any ball B and its measurable subset E.

The following Hölder's inequality is very useful.

Lemma 2.10 ([11], Theorem 2.26)**.** *Let $p(\cdot) \in \mathscr{P}_1$, then*

$$\int_X |f(x)g(x)|d\mu \leq 4\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

To apply the properties introduced in the above, this study employs the $A_{p(\cdot),q(\cdot)}$ condition for the construction of a weight W, see Lemma 2.13, within the A_∞ class. And the following lemmas are necessary for this purpose.

Lemma 2.11. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0, 1)$. For any ball B and its measurable subset E, if $\omega \in A_{p(\cdot),q(\cdot)}$, then*

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \leq 16[\omega]_{A_{p(\cdot),q(\cdot)}} \frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}}$$

Proof. By Hölder's inequality and the $A_{p(\cdot),q(\cdot)}$ condition (Definition 1.7),

$$\begin{aligned}\mu(E) &= \int_X \omega(x)\chi_E\omega(x)^{-1}\chi_B d\mu \\ &\leq 4\left\|\omega\chi_E\right\|_{q(\cdot)}\left\|\omega^{-1}\chi_B\right\|_{q'(\cdot)} \\ &\leq 16\mu(E)^\eta\left\|\omega\chi_E\right\|_{q(\cdot)}\left\|\omega^{-1}\chi_B\right\|_{p'(\cdot)}\end{aligned}$$

Thus,

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \leq 16[\omega]_{A_{p(\cdot),q(\cdot)}} \frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}}$$

$\square$

The next lemma plays a important role in our proof, which is dedicated to the proof of (3.14).

Lemma 2.12 ([9], Lemma 3.3)**.** *Let $p(\cdot) \in \mathscr{P}_1 \cap LH$ and $\omega \in A_{p(\cdot)}$. Then*

$$\sup_{B \subseteq X} \|\omega\chi_B\|_{p(\cdot)}^{p_-(B)-p_+(B)} \lesssim 1.$$

Lemma 2.13. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1 \cap LH$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0,1)$, and $\omega \in A_{p(\cdot),q(\cdot)}$. Then $W(\cdot) := \omega(\cdot)^{q(\cdot)} \in A_\infty$.*

Proof. Fix a ball B and a measurable set $E \subseteq B$. According to Lemma 2.9, in order to proof this lemma, it is sufficient to prove

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \lesssim \left(\frac{W(E)}{W(B)}\right)^{1/q_+}. \tag{2.2}$$

We will prove this in three cases.

(i) When $\|\omega\chi_B\|_{q(\cdot)} \le 1$. By Lemma 2.11,

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \lesssim \frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}} = \frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}^{q_-(B)/q_+(B)} \|\omega\chi_B\|_{p(\cdot)}^{1-q_-(B)/q_+(B)}}.$$

From Lemma 2.5, we have that $\|\omega\chi_E\|_{q(\cdot)} \le W(E)^{1/q_+(B)}$ and $\|\omega\chi_B\|_{q(\cdot)}^{q_-(B)} \ge W(B)$. It follows from Lemma 2.12 and (1.3) that

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \lesssim \left(\frac{W(E)}{W(B)}\right)^{1/q_+(B)} \|\omega\chi_B\|_{q(\cdot)}^{q_-(B)/q_+(B)-1} \lesssim \left(\frac{W(E)}{W(B)}\right)^{1/q_+}.$$

(ii) When $\|\omega\chi_E\|_{q(\cdot)} \le 1 \le \|\omega\chi_B\|_{q(\cdot)}$, by Lemmas 2.11 and 2.5 again, we have

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta} \lesssim \frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}} \lesssim \frac{W(E)^{1/q_+}}{W(B)^{1/q_+(B)}} \le \left(\frac{W(E)}{W(B)}\right)^{\frac{1}{q_+}}.$$

(iii) When $1 < \|\omega\chi_E\|_{q(\cdot)} \le \|\omega\chi_B\|_{q(\cdot)}$, define $\lambda = \|\omega\chi_B\|_{q(\cdot)} \ge \|\omega\chi_E\|_{q(\cdot)}$ and substitute the measure $d\mu$ with $W(x)d\mu$. Through Lemma 2.8, there is a constant C_t satisfies

$$\int_B \frac{W(x)}{\lambda^{q_\infty}} d\mu \le C_t \int_B \frac{W(x)}{\lambda^{q(x)}} d\mu + \int_B \frac{W(x)}{(e + d(x_0, x))^{tq_\infty}} d\mu. \tag{2.3}$$

By Lemma 2.4, we can know that the first term on the right side is less than 1. Therefore we now need to prove the second term also satisfies this bound, when we take large enough t, independent of B. For a finite $W(X)$,

$$\int_X \frac{W(x)}{(e + d(x_0, x))^{tq_\infty}} d\mu \le Ce^{-tq_\infty} W(X).$$

If $W(X)=\infty$, we define $B_k=B(x_0,2^k)$ and it follows from Lemmas 2.5 and 2.7 that $\lim_{k\to\infty}\|\omega\chi_{B_k}\|_{p(\cdot)}=\infty$. Lemma 2.5 provides

$$\begin{aligned}\int_X \frac{W(x)}{(e+d(x_0,x))^{tq_\infty}}d\mu &\lesssim_t e^{-tq_\infty}W(B_0)+\sum_{k=1}^{\infty}\int_{B_k\setminus B_{k-1}}\frac{W(x)}{(e+d(x_0,x))^{tq_\infty}}d\mu\\ &\le e^{-tq_\infty}W(B_0)+\sum_{k=1}^{\infty}2^{-ktq_\infty}W(B_k)\\ &\le e^{-tq_\infty}W(B_0)+\sum_{k=1}^{\infty}2^{-ktq_\infty}\max\{\|\omega\chi_{B_k}\|_{q(\cdot)}^{q_+},\|\omega\chi_{B_k}\|_{q(\cdot)}^{q_-}\}\\ &\lesssim e^{-tq_\infty}W(B_0)+\sum_{k=1}^{\infty}2^{-ktq_\infty}\|\omega\chi_{B_k}\|_{q(\cdot)}^{q_+},\end{aligned}$$

where the last inequality is derived from this fact that since $\lim_{k\to\infty}\|\omega\chi_{B_k}\|_{p(\cdot)}=\infty$, then there exists $N>0$, for any $k>N$, we have $\|\omega\chi_{B_k}\|_{p(\cdot)}>1$. By Lemma 2.11,

$$\|\omega\chi_{B_k}\|_{q(\cdot)}\le C\left(\frac{\mu(B_k)}{\mu(B_0)}\right)^{1-\eta}\|\omega\chi_{B_0}\|_{q(\cdot)}\le C2^{k(1-\eta)\log_2 C_\mu}.$$

Hence, we have

$$\int_X \frac{W(x)}{(e+d(x_0,x))^{tq_\infty}}d\mu\lesssim e^{-tq_\infty}W(B_0)+\sum_{k=1}^{\infty}2^{kq_+(1-\eta)\log_2 C_\mu-ktq_\infty}. \tag{2.4}$$

When $t>\frac{q_+(1-\eta)\log_2 C_\mu}{q_\infty}$, the sum is convergent. The right-hand side of (2.3) becomes bounded, which means that

$$W(B)^{1/q_\infty}\lesssim\|\omega\chi_B\|_{q(\cdot)}. \tag{2.5}$$

Replacing B by E and $q(\cdot)$ by q_∞, we get

$$1\le\int_E\frac{W(x)}{\lambda^{q(x)}}d\mu\le C_t\int_E\frac{W(x)}{\lambda^{q_\infty}}d\mu+\int_E\frac{W(x)}{(e+d(x_0,x))^{tq_\infty}}d\mu.$$

It follows from the above that

$$\lambda^{q_\infty}=\|\omega\chi_E\|_{q(\cdot)}^{q_\infty}\lesssim W(E). \tag{2.6}$$

Then by Lemma 2.11,

$$\left(\frac{\mu(E)}{\mu(B)}\right)^{1-\eta}\lesssim\frac{\|\omega\chi_E\|_{q(\cdot)}}{\|\omega\chi_B\|_{q(\cdot)}}\lesssim\left(\frac{W(E)}{W(B)}\right)^{1/q_\infty}\le\left(\frac{W(E)}{W(B)}\right)^{1/q_+}.$$

□

It follows instantly from the proof of Lemma 2.13 that

Lemma 2.14. *Let $p(\cdot),q(\cdot)\in\mathscr{P}_1\cap LH$ and $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\eta\in[0,1)$. If $\omega\in A_{p(\cdot),q(\cdot)}$ satisfying $\|\omega\chi_B\|_{q(\cdot)}\ge 1$ for some ball B, then $\|\omega\chi_B\|_{q(\cdot)}\approx W(B)^{1/q_\infty}$.*

Lemma 2.15. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_1 \cap LH$ and $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0,1)$, then $1 \in A_{p(\cdot),q(\cdot)}$.*

Proof. If $\mu(B) \leq 1$, it follows from Lemma 2.5 that $\|\chi_B\|_{q(\cdot)}^{q_+(B)} \leq \mu(B)$ and $\|\chi_B\|_{p'(\cdot)}^{(p')_+(B)} \leq \mu(B)$. By Lemma 2.3,

$$\mu(B)^{\eta-1}\|\chi_B\|_{q(\cdot)}\|\chi_B\|_{p'(\cdot)} \leq \mu(B)^{\frac{1}{q_+(B)}+\eta-\frac{1}{p_+(B)}} = \mu(B)^{\frac{p_-(B)-p_+(B)}{p_+(B)p_-(B)}} \leq C.$$

If $\mu(B) > 1$, it follows from the proof of Lemma 2.14 that

$$\mu(B)^{\eta-1}\|\chi_B\|_{q(\cdot)}\|\chi_B\|_{p'(\cdot)} \leq C.$$

□

Remark 2.16. *In the proof of the main theorems, we will always combine the above lemmas with (1.3) and (1.4) to apply it.*

2.3. **Dyadic Analysis.**

This classical dyadic cubes defined as

$$Q = 2^k([0,1)^n + m), \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n.$$

These constructs play an essential role in constructing our main theorem. The following discussion adopts the framework of dyadic cubes as formulated by Hytönen and Kairema [20], as explicated in [3].

Lemma 2.17 ([3], Theorem 2.1)**.** *There exist a family $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$, composed of subsets of X, such that:*

(1) For cubes $Q_1, Q_2 \in \mathcal{D}$, either $Q_1 \cap Q_2 = \varnothing$, $Q_1 \subseteq Q_2$, or $Q_2 \subseteq Q_1$.
(2) The cubes $Q \in \mathcal{D}_k$ are pairwise disjoint. And for any $k \in \mathbb{Z}$, $X = \bigcup_{Q\in\mathcal{D}_k} Q$. We call $\mathcal{D}_k$ as the kth generation.
(3) For any $Q_1 \in \mathcal{D}_k$, there always exists at least one child of Q_1 in $\mathcal{D}_{k+1}$, such that $Q_2 \subseteq Q_1$, and there always exists exactly one parent of Q_1 in $\mathcal{D}_{k-1}$, such that $Q_1 \subseteq Q_3$.
(4) If Q_2 is a child of Q_1, then for a constant $0 < \epsilon < 1$, depended on the set X, $\mu(Q_2) \geq \epsilon\mu(Q_1)$.
(5) For every $k \in \mathbb{Z}$ and $Q \in \mathcal{D}_k$, there exists constants C_d and $d_0 > 1$, such that

$$B\left(x_c(Q), d_0^k\right) \subseteq Q \subseteq B\left(x_c(Q), C_d d_0^k\right),$$

where x_c denotes the centre of cube $Q \in \mathcal{D}$.

We call the family $\mathcal{D}$ as dyadic grid and the cubes $Q \in \mathcal{D}$ as dyadic cubes.

Frequently, the sets of cubes and balls are interchangeable, as demonstrated by the equivalent formulation of the $A_{p(\cdot),q(\cdot)}$ condition.

Lemma 2.18. *Let $p(\cdot), q(\cdot) \in \mathscr{P}_0 \cap LH$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \eta \in [0,1)$, and $\mathcal{D}$ is a dyadic grid. If $\omega \in A_{p(\cdot),q(\cdot)}$, then $\omega \in A^{\mathcal{D}}_{p(\cdot),q(\cdot)}$, where*

$$[\omega]_{A^{\mathcal{D}}_{p(\cdot),q(\cdot)}} := \sup_{Q\in\mathcal{D}} \mu(Q)^{\eta-1}\|\omega\chi_Q\|_{q(\cdot)}\left\|\omega^{-1}\chi_Q\right\|_{p'(\cdot)} < \infty.$$

Proof. Using Theorem 2.17 with fixing $Q \in \mathcal{D}_k$ and Lemma 2.1,

$$\|\omega\chi_Q\|_{q(\cdot)} \|\omega^{-1}\chi_Q\|_{p'(\cdot)} \leq \left\|\omega\chi_{B\left(x_c(Q),C_d d_0^k\right)}\right\|_{q(\cdot)} \left\|\omega^{-1}\chi_{B\left(x_c(Q),C_d r d_0^k\right)}\right\|_{p'(\cdot)}$$
$$\lesssim \mu\left(B\left(x_c(Q),Cd_0^k\right)\right)^{1-\eta} \lesssim \mu\left(B\left(x_c(Q),d_0^k\right)\right)^{1-\eta} \lesssim \mu(Q)^{1-\eta}.$$

□

In the proof of Lemma 2.18, we initially expand cubes to encompass balls, subsequently applying the lower mass bound (see Lemma 2.1) to switch back to cube dimensions. Additionally, this approach allows for the maximal operator to be efficiently reformulated in dyadic terms.

Definition 2.19. *Let $\eta \in [0,1)$, σ is a weight, and $\mathcal{D}$ is a dyadic grid. Define the weighted dyadic fractional maximal operator $M_{\eta,\sigma}^{\mathcal{D}}$ by*

$$M_{\eta,\sigma}^{\mathcal{D}}f(x) = \sup_{x\in Q\in\mathcal{D}} \sigma(Q)^{\eta-1}\int_Q |f(y)|\sigma d\mu.$$

When $\eta = 0, M_{0,\sigma}^{\mathcal{D}} = M_\sigma^{\mathcal{D}}$, which is a weighted dyadic maximal operator. When $\sigma = 1, M_{\eta,\sigma}^{\mathcal{D}} = M_\eta^{\mathcal{D}}$, which is a dyadic fractional maximal operator.

The following lemma can guarantee that we always transform a proof involving M_η into that for $M_\eta^{\mathcal{D}_i}$.

Lemma 2.20 ([22], Lemma 7.8)**.** *Let $\eta \in [0,1)$, there exists a finite family $\{\mathcal{D}_i\}_{i=1}^N$ of dyadic grids such that*

$$M_\eta f(x) \approx \sum_{i=1}^N M_\eta^{\mathcal{D}_i} f(x),$$

where the implicit constants depend only X, μ, and η.

Then the following lemma first appears in [9], which is a key tool used in after proof.

Lemma 2.21 ([9], Lemma 4.4)**.** *Let $\mathcal{D}$ is a dyadic grid, σ is a weight, and $1 < p < \infty$. Then the dyadic maximal operator $M_\sigma^{\mathcal{D}}$ is bounded on $L^p(X,\sigma)$, which is also bounded from $L^1(X,\sigma)$ to $WL^1(X,\sigma)$.*

We now present the fractional-type Calderón-Zygmund decomposition on the spaces of homogeneous type as follows.

Lemma 2.22. *Let $\eta \in [0,1)$, $\mathcal{D}$ is a dyadic grid on X, and $\sigma \in A_\infty$. Set $\mu(X) = \infty$. If $f \in L^1_{loc}(\sigma)$ satisfying $\lim\limits_{j\to\infty} \sigma(Q_j)^{\eta-1}\int_{Q_j}|f|\,\sigma d\mu = 0$ for any nested sequence $\{Q_j \in \mathcal{D}_j\}_{j=0}^\infty$, where Q_{j+1} is a child of Q_j, then for any $\lambda > 0$, there exists a (possibly empty) collection of mutually disjoint dyadic cubes $\{Q_j\}$, called Calderón-Zygmund cubes for f at the height λ, and a constant $C_{CZ} > 1$, which is independent of λ and dependent of $\mathcal{D}, X, \sigma$, such that*

$$X_{\eta,\lambda}^{\mathcal{D}} := \left\{x \in X : M_{\eta,\sigma}^{\mathcal{D}}f(x) > \lambda\right\} = \bigcup_j Q_j.$$

Moreover, for each j,

$$\lambda < \sigma(Q_j)^{\eta-1} \int_{Q_j} |f| \, \sigma d\mu \leq C_{CZ}\lambda. \tag{2.7}$$

Now, suppose that $\left\{Q_j^k\right\}$ is the Calderón-Zygmund cubes at height a^k for each $k \in \mathbb{Z}$ and $a > C_{CZ}$. These sets, $E_j^k := Q_j^k \setminus X_{\eta,a^{k+1}}^{\mathcal{D}}$, are mutually disjoint for all indices j and k, such that

$$\left(1 - \left(\frac{C_{cz}}{a}\right)^{\frac{1}{1-\eta}}\right) \sigma(Q_j^k) \leq \sigma(E_j^k) \leq \sigma(Q_j^k). \tag{2.8}$$

If set $\mu(X) < \infty$, then Calderón-Zygmund cubes can be established for every function $f \in L_{loc}^1(\sigma)$ at any height $\lambda > \lambda_0 := \int_X |f| \, \sigma d\mu$, meanwhile, (2.7) also holds. Under these conditions, the sets E_j^k are pairwise disjoint with (2.8) holds, for $k > \log_a \lambda_0$.

Proof. The first case is that $\mu(X) = \infty$. We only need to consider $X_{\eta,\lambda}^D \neq \emptyset$. Otherwise, we can take $\{Q_j\}$ to be the empty sets.

As the property of the dyadic cube in Theorem 2.17, for every $x \in X_{\eta,\lambda}^D$, there exists a dyadic cube Q_k^x of each generation $k > 0$, such that $x \in Q_k^x$ and $M_{\eta,\sigma}^{\mathcal{D}} f(x) > \lambda$. So there exist k, such that

$$\sigma(Q_k^x)^{\eta-1} \int_{Q_k^x} |f(y)| d\sigma > \lambda. \tag{2.9}$$

Since $\lim_{k\to\infty} \sigma(Q_k^x)^{\eta-1} \int_{Q_k^x} |f(y)| d\sigma = 0$, then there are only finite k such that (2.9) holds. Select k to be the smallest integer such that (2.9) holds, in this case, we denote the cube with generation k by Q_x. What's more, the set $\left\{Q_x : x \in X_{\eta,\lambda}^{\mathcal{D}}\right\}$ can be enumerated as $\{Q_j\}$ due to there are countable dyadic cubes. If $Q_i \cap Q_j \neq \emptyset$, without loss of generality, we define $Q_i \subseteq Q_j$. Moreover, by the maximality, $Q_i = Q_j$. Thus, the set $\left\{Q_x : x \in X_{\eta,\lambda}^{\mathcal{D}}\right\} := \{Q_j\}$ is countably non-overlapping maximal dyadic cubes. Hence, $X_\lambda^{\mathcal{D}} \subseteq \bigcup_j Q_j$.

On the other hand, if $z \in Q_x$, for some $x \in X_{\eta,\lambda}^{\mathcal{D}}$, then

$$\lambda < \sigma(Q_x)^{\eta-1} \int_{Q_x} |f(y)| d\sigma \leq M_{\eta,\sigma}^{\mathcal{D}} f(z).$$

Thus, $X_\lambda^{\mathcal{D}} = \bigcup_j Q_j$.

Next, we will prove (2.7). The left inequality of (2.7) holds since the choice of Q_j. For the second inequality, by the maximality of each Q_j, we can deduce that its parent $\tilde{Q}_j$ satisfies

$$\sigma(\tilde{Q}_j)^{\eta-1} \int_{Q_j} |f(y)| d\sigma \leq \lambda$$

It follows from Lemma 2.17 and 2.1 that

$$\sigma(Q_j)^{\eta-1} \int_{Q_j} |f(y)| d\sigma \leq \frac{\sigma\left(\tilde{Q}_j\right)}{\sigma\left(Q_j\right)} \lambda \leq \frac{\sigma\left(B\left(x_c\left(\tilde{Q}_j\right), C d_0^{k+1}\right)\right)}{\sigma\left(B\left(x_c\left(Q_j\right), d_0^k\right)\right)} \lambda \leq C d_0^{\log_2 C_\mu} \lambda.$$

Consequencely, (2.7) holds.

Setting $a > C_{CZ}$, we define the Calderón-Zygmund cubes $\{Q_j^k\}$ at heights a^k for $k \in \mathbb{Z}$. We abbreviate $X_{\eta,a^k}^{\mathcal{D}}$ to X_k. Given Q_i^{k+1} and for any $x \in Q_i^{k+1}$, we have $Q_i^{k+1} \in \{Q_k^x\}$ (defined as above). It follows that there must be an index j for which $Q_i^{k+1} \subseteq Q_j^k$.

Next, we want to show that the E_j^k are pairwise disjoint for all j, k. Setting $k_1 \leq k_2$, it suffices to prove that $E_{j_1}^{k_1} \cap E_{j_2}^{k_2} = \emptyset$ for $E_{j_1}^{k_1} \neq E_{j_2}^{k_2}$. If $k_1 = k_2$ and $j_1 \neq j_2$, then $Q_{j_1}^{k_1} \cap Q_{j_2}^{k_2} = \emptyset$ can deduce the desired results. If $k_1 < k_2$, then $E_{j_1}^{k_1} \subseteq (X_{k_1+1})^c \subseteq (X_{k_2})^c$ and $E_{j_2}^{k_2} \subseteq X_{k_2}$ can deduce the desired results.

Finally, we will prove that $\sigma(Q_j^k) \approx \sigma(E_j^k)$. It follows obviously from that

$$\begin{aligned}\sigma(Q_j^k \cap X_{k+1})^{1-\eta} &= \left(\sum_{i:Q_i^{k+1} \subseteq Q_j^k} \sigma(Q_i^{k+1}) \right)^{1-\eta} \leq \sum_{i:Q_i^{k+1} \subseteq Q_j^k} \left(\sigma(Q_i^{k+1})\right)^{1-\eta} \\ &\leq \frac{1}{a^{k+1}} \sum_{i:Q_i^{k+1} \subseteq Q_j^k} \int_{Q_i^{k+1}} |f| \, d\sigma \leq \frac{1}{a^{k+1}} \int_{Q_j^k} |f| \, d\sigma \leq \frac{C_{cz}}{a} \sigma(Q_j^k)^{1-\eta}.\end{aligned}$$

Note that $\sigma(Q_j^k) = \sigma(Q_j^k \cap X_{k+1}) + \sigma(E_j^k)$, then we have

$$\left(1 - \left(\frac{C_{cz}}{a}\right)^{\frac{1}{1-\eta}}\right) \sigma(Q_j^k) \leq \sigma(E_j^k) \leq \sigma(Q_j^k).$$

□

3. The proof of Theorem 1.9 and 1.10

3.1. **Necessity.** In this subsection, we want to prove the Necessity of Theorem 1.9. But actually, we prove the stronger claim, which is the necessity of Theorem 1.10. In this proof, somewhere, we will use the sufficiency of Theorem 1.10, whose proof can be referred to the next subsection 3.2. Now, we supppose that M_η is bounded from $L^{p(\cdot)}(X, \omega)$ to $WL^{q(\cdot)}(X, \omega)$, which means that

$$\sup_{t>0} \left\| t\omega\chi_{\{x \in X: M_\eta f(x) > t\}} \right\|_{q(\cdot)} \lesssim \|\omega f\|_{p(\cdot)}. \tag{3.1}$$

For following, it suffices to prove that $\omega \in A_{p(\cdot),q(\cdot)}$.

Firstly, we claim that for every $B \subseteq X$,

$$\|\omega\chi_B\|_{q(\cdot)} < \infty. \tag{3.2}$$

If $\|\omega\chi_B\|_{q(\cdot)} = \infty$. For any $x \in B$, there exist $E \subseteq B$, such that $x \in E$. For any $t < \mu(B)^{\eta-1}\mu(E)$, then $M_\eta\chi_E(x) \geq \mu(B)^{\eta-1}\mu(E)\chi_B(x) > t$. Moreover, it follows from (3.1) that

$$\infty = t\|\omega\chi_B\|_{q(\cdot)} \leq \left\|t\omega\chi_{\{x \in X: M_\eta\chi_E(x) > t\}}\right\|_{q(\cdot)} \lesssim \|\omega\chi_E\|_{p(\cdot)} \lesssim \mu(E)^\eta \|\omega\chi_E\|_{q(\cdot)} \leq \infty.$$

By Lemma 2.5, we have

$$\mu(E)^{-1}\int_E \omega(x)^{q(x)}\,d\mu=\infty.$$

When $E\to\{x\}$, by the Lebesgue Differentiation Theorem, see [2, Theorem 1.4], we can find that $\omega(x)^{q(x)}=\infty$ for almost every x. This result clearly contridicts with the definition of a weight and therefore (3.2) is valid.

Secondly, we will show that $\omega\in A_{p(\cdot),q(\cdot)}$.

Case 1: $\|\omega^{-1}\chi_B\|_{p'(\cdot)}<\infty$**.**

In this case, since the homogeneity, we assume that $\|\omega^{-1}\chi_B\|_{p'(\cdot)}=1$. It suffices to prove that

$$\sup_{B\subseteq X}\mu(B)^{\eta-1}\|\omega\chi_B\|_{q(\cdot)}\lesssim 1 \tag{3.3}$$

We define the following sets

$$B_0\equiv\{x\in B\,:\,p'(x)<\infty\},\qquad B_\infty\equiv\{x\in B\,:\,p'(x)=\infty\}.$$

By the definition of the norm, for any $\lambda\in(\frac{1}{2},1)$,

$$1\le\rho_{p'(\cdot)}\left(\frac{\omega^{-1}\chi_B}{\lambda}\right)=\int_{B_0}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu+\lambda^{-1}\big\|\omega^{-1}\chi_{B_\infty}\big\|_\infty.$$

At least one of the two terms on the right-hand side is not less than $\frac{1}{2}$. Furthermore, one of the following two situations must be true: either $\|\omega^{-1}\chi_{B_\infty}\|_\infty\ge\frac{1}{2}$, or given $\lambda_0\in(\frac{1}{2},1)$, then $\int_{B_0}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu\ge\frac{1}{2}$ for any $\lambda\in[\lambda_0,1)$.

Suppose that the first situation holds. Set $s>\|\omega^{-1}\chi_{B_\infty}\|_\infty=\underset{x\in B_\infty}{essinf}\,\omega(x)$, there exists a subset $E\subseteq B_\infty$ with $\mu(E)>0$, such that $\mu(E)^{-1}\omega(E)\le s$. Note that $p(\cdot)$ is equal to 1 on B_∞, then $\|\omega\chi_E\|_{p(\cdot)}=\omega(E)$. Then, for all $t<\mu(B)^{\eta-1}\mu(E)$, we have $M_\eta\chi_E(x)\ge\mu(B)^{\eta-1}\mu(E)\chi_B(x)>t\chi_B(x)$. Thus, it follows from (3.1) that

$$t\|\omega\chi_B\|_{q(\cdot)}\le\big\|t\omega\chi_{\{x\in X:M_\eta\chi_E(x)>t\}}\big\|_{q(\cdot)}\lesssim\|\omega\chi_E\|_{p(\cdot)}=\omega(E).$$

Letting $t\to\mu(B)^{\eta-1}\mu(E)$, we get that $\mu(B)^{\eta-1}\mu(E)\parallel\omega\chi_B\|_{q(\cdot)}\lesssim\omega(E)$. Then,

$$\mu(B)^{\eta-1}\parallel\omega\chi_B\|_{q(\cdot)}\lesssim\mu(E)^{-1}\omega(E)\le s$$

Letting $s\to\parallel\omega^{-1}\chi_{B_\infty}\parallel_\infty^{-1}$, we have

$$\mu(B)^{\eta-1}\|\omega\chi_B\|_{q(\cdot)}\lesssim\big\|\omega^{-1}\chi_{B_\infty}\big\|_\infty^{-1}\le 2,$$

then (3.3) is valid.

When the second situation holds, we define $B_R=\{x\in B_0\,:\,p'(x)<R\}$, for any $R>1$. By Lemma 2.7, there exists R that close to ∞ sufficiently, such that

$\int_{B_R}\left(\frac{\omega(x)^{-1}}{\lambda_0}\right)^{p'(x)}d\mu > \frac{1}{3}$. It follows from $\|\omega^{-1}\chi_B\|_{p'(\cdot)}=1$ and Lemma 2.5 that

$$\int_{B_R}\left(\frac{\omega(x)^{-1}}{\lambda_0}\right)^{p'(x)}d\mu \le \int_{B_R}\left(\frac{2}{\lambda_0}\right)^{p'(x)}\left(\frac{\omega(x)^{-1}}{2}\right)^{p'(x)}d\mu \le \left(\frac{2}{\lambda_0}\right)^{R}<\infty.$$

We need to use the following auxiliary function

$$G(\lambda)=\int_{B_R}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu,$$

where $\frac{1}{3}<G(\lambda_0)<\infty$. The Lebesgue dominated convergence theorem can deduce that G is continuous on $[\lambda_0,1]$.

For any $\lambda\in[\lambda_0,1)$, if $G(1)\ge\frac{1}{3}$, by Lemma 2.5,

$$\frac{1}{3\lambda}\le\frac{1}{\lambda}\int_{B_R}\omega(x)^{-p'(x)}\,d\mu\le G(\lambda)\le\lambda^{-R}<\infty.$$

Let λ sufficiently close to 1, then $\lambda^{-R}\le 2$ and

$$\frac{1}{3}\le\int_{F_R}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu\le 2. \tag{3.4}$$

If $G(1)<\frac{1}{3}$, by continuity of G, there exists $\lambda\in(\lambda_0,1)$ such that $G(\lambda)=\frac{1}{3}$. Then (3.4) holds for this λ as well.

Fixed λ and let

$$f(x)=\omega(x)^{-p'(x)}\lambda^{1-p'(x)}\chi_{B_R}.$$

Then

$$\rho_{p(\cdot)}(\omega f)=\int_{B_R}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu\le 2.$$

By Lemma 2.5, $\|\omega f\|_{p(\cdot)}\le 2^{\frac{1}{(p')_-}}$. For any $x\in B$,

$$M_\eta f(x)\ge\mu(B)^{\eta-1}\int_B f\,d\mu=\lambda\mu(B)^{\eta-1}\int_{B_R}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu\ge\frac{\lambda}{3}\mu(B)^{\eta-1}.$$

For any $t<\frac{\lambda}{3}\mu(B)^{\eta-1}$, it follows from (3.1) that

$$t\|\omega\chi_B\|_{q(\cdot)}\le\left\|t\omega\chi_{\{x\in X:M_\eta f(x)>t\}}\right\|_{q(\cdot)}\lesssim\|\omega f\|_{p(\cdot)}\le 2^{\frac{1}{(p')_-}}.$$

Letting $t\to\frac{\lambda}{3}\mu(B)^{\eta-1}$, (3.3) is valid.

Case 2: $\|\omega^{-1}\chi_B\|_{p'(\cdot)}=\infty$**.**

In this case, we will use the perturbation method to prove.

Given $\epsilon>0$, denote the weight $\omega_\epsilon(x)=\omega(x)+\epsilon$. Then $\omega_\epsilon^{-1}\le\epsilon^{-1}<\infty$ and so $\|\omega_\epsilon^{-1}\chi_B\|_{p'(\cdot)}<\infty$. It follows immediately from the sufficiency of Theorem 1.9, Lemma 2.15, and (3.1) that

$$\begin{aligned}t\left\|\omega_\epsilon\chi_{\{x\in X:M_\eta f(x)>t\}}\right\|_{q(\cdot)}&\le t\left\|\omega\chi_{\{x\in X:M_\eta f(x)>t\}}\right\|_{q(\cdot)}+\epsilon t\left\|\chi_{\{x\in X:M_\eta f(x)>t\}}\right\|_{q(\cdot)}\\&\lesssim\|\omega f\|_{p(\cdot)}+\epsilon\|f\|_{p(\cdot)}\le 2\|\omega_\epsilon f\|_{p(\cdot)}\end{aligned}$$

This shows that ω_ϵ satisfies (3.1). When $\|\omega^{-1}\chi_B\|_{p'(\cdot)} < \infty$, it follows from (3.3) that

$$\sup_{B\subseteq X} \mu(B)^{\eta-1}\|\omega_\epsilon\chi_B\|_{q(\cdot)}\left\|\omega_\epsilon^{-1}\chi_B\right\|_{p'(\cdot)} \leq K,$$

where K is actually independent of ϵ. Thus,

$$\mu(B)^{\eta-1}\|\omega\chi_B\|_{q(\cdot)}\left\|\omega_\epsilon^{-1}\chi_B\right\|_{p'(\cdot)} \leq \mu(B)^{\eta-1}\|\omega_\epsilon\chi_B\|_{q(\cdot)}\left\|\omega_\epsilon^{-1}\chi_B\right\|_{p'(\cdot)} \leq K.$$

Letting $\epsilon \to 0$, by Lemma 2.7 and ω_ϵ^{-1} increases to ω^{-1}, we have that $[\omega]_{A_{p(\cdot),q(\cdot)}} \leq K$.

This finishes the necessity of Theorems 1.9 and 1.10.

3.2. **Sufficiency.**

The purpose of this section is to prove the sufficiency of Theorem 1.9, which implicits the sufficiency of Theorem 1.10. We will discuss the case for $\mu(X) < \infty$ at the end of this subsection. The initial focus will be on cases where $\mu(X) = \infty$.

Case 1: $\mu(X) = \infty$. We first simplify some details with three steps.

Step 1. Lemma 2.20 implies that to establish the boundedness of M_η, it is sufficient to demonstrate the boundedness of $M_\eta^{\mathcal{D}}$. By the homogeneity and Lemma 2.6, it suffices to consider that f is a nonnegative function with $\|\omega f\|_{p(\cdot)} = 1$.

Step 2. We introduce the weights $W(\cdot) = \omega(\cdot)^{q(\cdot)}$ and $\sigma(\cdot) = \omega(\cdot)^{-p'(\cdot)}$. According to Lemma 2.13 and Lemma 2.9, $W(\cdot)$ and $\sigma(\cdot)$ are both in A_∞ and satisfy the doubling property.

Step 3. Due to Lemma 2.22, it suffices to show that for any nested sequence $\{Q_k \in \mathcal{D}_k\}_{k=1}^\infty$ with $Q_k \subseteq Q_{k-1}$, we have

$$\lim_{k\to\infty} \mu(Q_k)^{\eta-1}\int_{Q_k} f d\mu = 0. \tag{3.5}$$

Indeed, since W is doubling, if we fix a sequence with $k = 1$, then

$$W(Q_1) \leq W(B(x_c(Q_1), C_d d_0)) \leq C_W^{\log_2 C_d} W(B(x_c(Q_1), d_0)).$$

By Lemma 2.17, for any k, with the similar argument, we have

$$\frac{1}{W(Q_k)} \lesssim \frac{1}{W\left(B\left(x_c(Q_k), C_d d_0^k\right)\right)}$$

Using lemma 2.9 combining above two estimates, we get

$$\frac{W(Q_1)}{W(Q_k)} \lesssim \frac{W(B(x_c(Q_1), d_0))}{W\left(B\left(x_c(Q_k), C_d d_0^k\right)\right)} \lesssim \left(\frac{\mu(B(x_c(Q_1), d_0))}{\mu\left(B\left(x_c(Q_k), C_d d_0^k\right)\right)}\right)^\delta.$$

If we rearrange and apply Lemma 2.1 (the lower mass bound),

$$\mu\left(B\left(x_c(Q_1), C d_0^k\right)\right)^\delta \lesssim \mu\left(B\left(x_c(Q_k), C_d d_0^k\right)\right)^\delta \lesssim W(Q_k).$$

By continuity of μ and the fact $X = \lim\limits_{k\to\infty} B\left(x_c(Q_1), C d_0^k\right)$, we have $\lim\limits_{k\to\infty} W(Q_k) = \infty$.

By the condition of $A_{p(\cdot),q(\cdot)}$ and Lemma 2.10, and Lemma 2.5,

$$\mu(Q_k)^{\eta-1}\int_{Q_k} f d\mu \lesssim [\omega]_{A_{p(\cdot),q(\cdot)}} \|\omega f\|_{p(\cdot)} \|\omega\chi_{Q_k}\|_{q(\cdot)}^{-1} \lesssim \|\omega\chi_{Q_k}\|_{q(\cdot)}^{-1}.$$

Since Lemma 2.5 implies $\lim\limits_{k\to\infty} W(Q_k) = \lim\limits_{k\to\infty} \|\omega\chi_{Q_k}\|_{q(\cdot)} = \infty$, (3.5) is valid.

Next, we decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\{f\sigma^{-1}>1\}}$ and $f_2 = f\chi_{\{f\sigma^{-1}\leq 1\}}$. Lemma 2.5 can deduce that

$$\int_X |f_i(x)|^{p(x)}\omega(x)^{p(x)} d\mu \leq \|f_i\omega\|_{p(\cdot)} \leq \|f\omega\|_{p(\cdot)} = 1, \quad i = 1,2. \tag{3.6}$$

By Lemma 2.5 again and the sublinearity of $M_\eta^{\mathcal{D}}$, it suffices to show that

$$\int_X \left(M_\eta^{\mathcal{D}} f_i(x)\right)^{q(x)} \omega(x)^{q(x)} d\mu \lesssim 1, \quad i = 1,2, \tag{3.7}$$

where the implicit constant is independent on f.

Estimate for f_1: Let $k \in \mathbb{Z}$ and $a > C_{CZ}$, define

$$X_k = \left\{x \in X : M_\eta^{\mathcal{D}} f_1(x) > a^{k+1}\right\}.$$

Since $f \in L^1_{loc}$ and $\lim\limits_{k\to\infty} \mu(Q_k)^{\eta-1}\int_{Q_k} f d\mu = 0$, $M_\eta^{\mathcal{D}} f_1$ is finite almost everywhere, then

$$\left\{x \in X : M_\eta^{\mathcal{D}} f_1(x) > 0\right\} = \bigcup_{k\in\mathbb{Z}} X_k \backslash X_{k+1}.$$

Let $\{Q_j^k\}$ be the CZ cubes of f_1 at height a^k. Then by Lemma 2.22, for every k,

$$X_k = \bigcup_j Q_j^k. \tag{3.8}$$

Set $E_j^k = Q_j^k \backslash X_{k+1}$, we find that

$$X_k \backslash X_{k+1} = \bigcup_j E_j^k.$$

It is obviously to get that

$$\begin{aligned}
&\int_X M_\eta^{\mathcal{D}} f_1(x)^{q(x)}\omega(x)^{q(x)} d\mu \\
=&\sum_k \int_{X_k\backslash X_{k+1}} M_\eta^{\mathcal{D}} f_1(x)^{q(x)}\omega(x)^{q(x)} d\mu \\
\lesssim&\sum_k \int_{X_k\backslash X_{k+1}} a^{kq(x)}\omega(x)^{q(x)} d\mu \\
\lesssim&\sum_{k,j} \int_{E_j^k} \left(\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y) d\mu\right)^{q(x)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)} d\mu.
\end{aligned} \tag{3.9}$$

Through the definition of f_1, σ and (3.6),

$$\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)d\mu \le \int_{Q_j^k} \left(f_1(y)\sigma(y)^{-1}\right)^{p(y)} \sigma(y)d\mu \le \int_{Q_j^k} (f_1(y)\omega(y))^{p(y)}d\mu \le 1.$$

Then,

$$\begin{aligned}
&\sum_{k,j}\int_{E_j^k}\left(\int_{Q_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q(x)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu\\
\le&\sum_{k,j}\left(\int_{Q_j^k}\left(f_1(y)\sigma(y)^{-1}\right)^{p(y)/p_-\left(Q_j^k\right)}\sigma(y)d\mu\right)^{q_-\left(Q_j^k\right)}\int_{E_j^k}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu.
\end{aligned}$$

Next, it follows from Hölder's inequality that the above

$$\lesssim\sum_{k,j}\left(\fint_{Q_j^k}\left(f_1(y)\sigma(y)^{-1}\right)^{p(y)/p_-}\sigma(y)d\mu\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}p_-}\int_{E_j^k}\sigma\left(Q_j^k\right)^{q_-\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu. \tag{3.10}$$

We claim that

$$\int_{E_j^k}\sigma\left(Q_j^k\right)^{q_-\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu\lesssim\sigma\left(Q_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}}. \tag{3.11}$$

Since $\mu\left(Q_j^k\right)\approx\mu\left(E_j^k\right)$ and $\sigma\in A_\infty$, by Lemma 2.13 applied to $\omega^{-1}\in A_{q'(\cdot),p'(\cdot)}$, Lemma 2.22 and Lemma 2.9, we obtain $\sigma\left(Q_j^k\right)\approx\sigma\left(E_j^k\right)$. Thus, (3.11) can deduce that (3.10) is bounded by

$$\begin{aligned}
&\sum_{k,j}\left(\fint_{Q_j^k}\left(f_1(y)\sigma(y)^{-1}\right)^{p(y)/p_-}\sigma(y)d\mu\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}p_-}\sigma\left(E_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}p_-}\\
\lesssim&\sum_{k,j}\left(\int_{E_j^k}M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)\,d\mu\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}}\\
\lesssim&\sum_{\theta=1,\frac{q_+}{p_-}}\sum_{k,j}\left(\int_{E_j^k}M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)d\mu\right)^{\theta}\\
\le&\sum_{\theta=1,\frac{q_+}{p_-}}\left(\sum_{k,j}\int_{E_j^k}M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)d\mu\right)^{\theta}\\
\le&\sum_{\theta=1,\frac{q_+}{p_-}}\left(\int_X M_\sigma^{\mathcal{D}}((f_1\sigma^{-1})^{p(\cdot)/p_-})(x)^{p_-}\sigma(x)d\mu\right)^{\theta}.
\end{aligned}$$

By Lemma 2.21 and (3.7), the above is bounded by $\sum_{\theta=1,\frac{q_+}{p_-}} \left(\int_X (\omega(x)f_1(x))^{p(x)}d\mu\right)^\theta \lesssim 1$.

Next, we will verify (3.11) to finish the estimate for f_1. In fact, the left-hand side of (3.11) can be rewrite by

$$\left(\frac{\sigma\left(Q_j^k\right)}{\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}}\right)^{q_-\left(Q_j^k\right)} \int_{Q_j^k} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q_-\left(Q_j^k\right)-q(x)} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)}d\mu. \tag{3.12}$$

To prove (3.11), it suffices to prove that

$$\int_{Q_j^k} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)}d\mu \lesssim 1, \tag{3.13}$$

$$\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q_-\left(Q_j^k\right)-q(x)} \lesssim 1, \tag{3.14}$$

$$\left(\frac{\sigma\left(Q_j^k\right)}{\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}}\right)^{q_-\left(Q_j^k\right)} \lesssim \sigma\left(Q_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}}. \tag{3.15}$$

Firstly, (3.13) follows instantly from the condition of $A_{p(\cdot),q(\cdot)}$ and Lemma 2.5. Secondly, we will prove (3.14) as follows.

Assume that $\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)} < 1$, otherwise, there is nothing to prove. Then,

$$p(x) - p_-(Q_j^k) \approx \frac{1}{p_-(Q_j^k)} - \frac{1}{p(x)} = \frac{1}{q_-(Q_j^k)} - \frac{1}{q(x)} \approx q(x) - q_-(Q_j^k), \tag{3.16}$$

which only depend on $p(\cdot)$ and η. Moreover, we have

$$\begin{aligned} q(x) - q_-\left(Q_j^k\right) &= \frac{q'(x)}{q'(x)-1} - \frac{(q')_+\left(Q_j^k\right)}{(q')_+\left(Q_j^k\right)-1} \\ &= \frac{(q')_+\left(Q_j^k\right) - q'(x)}{[q'(x)-1]\left[(q')_+\left(Q_j^k\right)-1\right]} \\ &\lesssim (q')_+\left(Q_j^k\right) - (q')_-\left(Q_j^k\right) \\ &\approx (p')_+\left(Q_j^k\right) - (p')_-\left(Q_j^k\right), \end{aligned}$$

where the last step holds since we used (3.16) and the implicit constants only depend on $p(\cdot)$ and η. Thus, (3.14) follows immediately from Lemma 2.12 (applied to cubes) and (1.4).

Last, we prove (3.15) as follows. If $\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)} > 1$, then by Lemma 2.5,

$$\left(\frac{\sigma\left(Q_j^k\right)}{\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}}\right)^{q_-\left(Q_j^k\right)} \leq \left(\sigma\left(Q_j^k\right)^{1-1/(p')+\left(Q_j^k\right)}\right)^{q_-\left(Q_j^k\right)} = \sigma\left(Q_j^k\right)^{\frac{q_-\left(Q_j^k\right)}{p_-\left(Q_j^k\right)}}.$$

If $\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)} \leq 1$, then applying Lemma 2.5 and Lemma 2.12,

$$\begin{aligned}
\left(\frac{\sigma\left(Q_j^k\right)}{\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}}\right)^{q_-\left(Q_j^k\right)} &\leq \left(\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{(p')_-\left(Q_j^k\right)-1}\right)^{q_-\left(Q_j^k\right)} \\
&\leq \left(\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{(p')_-\left(Q_j^k\right)-1+(p')_+\left(Q_j^k\right)-(p')_+\left(Q_j^k\right)}\right)^{q_-\left(Q_j^k\right)} \\
&\lesssim \left(\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{(p')_+\left(Q_j^k\right)-1}\right)^{q_-\left(Q_j^k\right)} \\
&\lesssim \left(\sigma\left(Q_j^k\right)^{\frac{(p')_+\left(Q_j^k\right)-1}{(p')_+\left(Q_j^k\right)}}\right)^{q_-\left(Q_j^k\right)} \\
&\lesssim \sigma\left(Q_j^k\right)^{\frac{q_-\left(Q_j^k\right)}{p_-\left(Q_j^k\right)}}.
\end{aligned}$$

Eventually, (3.11) is valid and then we finish the proof of (3.6) for f_1.

Estimate for f_2: Initially, we notice that $1, \sigma$, and W are in A_∞. Considering $\{Q_j^k\}$ as the Calderón-Zygmund dyadic cubes for f_2 relative to μ, and selecting a nested tower of cubes $\{Q_{k,0}\}$, it is observed that the measures $\mu(Q_{k,0}), \sigma(Q_{k,0})$, and $W(Q_{k,0})$ all tend towards infinity. We will often use the doubling property for A_∞ in following.

Finding a cube $Q_{k_0,0} =: Q_0 \in \mathcal{D}_{k_0}$ s.t. $\mu\left(Q_0\right), W\left(Q_0\right)$ and $\sigma\left(Q_0\right) \geq 1$ and fixing a LH_∞ base point $x_0 = x_c\left(Q_0\right)$, by Lemma 1.6. Define $N_0 = 2A_0C_d$ and the sets

$$\begin{aligned}
\mathscr{F} &= \left\{(k,j) \in \mathbb{Z}\times\mathbb{Z} : Q_j^k \subseteq Q_0\right\}; \\
\mathscr{G} &= \left\{(k,j) \in \mathbb{Z}\times\mathbb{Z} : Q_j^k \nsubseteq Q_0 \text{ and } d\left(x_0, x_c\left(Q_j^k\right)\right) < N_0 d_0^k\right\}; \\
\mathscr{H} &= \left\{(k,j) \in \mathbb{Z}\times\mathbb{Z} : Q_j^k \nsubseteq Q_0 \text{ and } d\left(x_0, x_c\left(Q_j^k\right)\right) \geq N_0 d_0^k\right\}.
\end{aligned}$$

By the same argument of getting (3.9), and replacing f_1 with f_2, we have

$$\int_X M^{\mathcal{D}} f_2(x)^{q(x)}\omega(x)^{q(x)}d\mu \lesssim \sum_{k,j}\int_{E_j^k}\left(\fint_{Q_j^k} f_2(y)\sigma(y)\sigma(y)^{-1}d\mu\right)^{q(x)}\left(\mu\left(Q_j^k\right)^\eta\omega(x)\right)^{q(x)}d\mu.$$

We decompose $\sum_{k,j}$ into $\sum_{(k,j)\in\mathscr{F}} = I_1$, $\sum_{(k,j)\in\mathscr{G}} = I_2$ and $\sum_{(k,j)\in\mathscr{H}} = I_3$.

Estimate for I_1: Noting that $f_2\sigma^{-1} \le 1$ allows us to remove f_2 from consideration. Subsequently, by applying (3.10), we obtain

$$\begin{aligned}
I_1 &\le \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \left(\fint_{Q_j^k} \sigma(y) d\mu \right)^{q(x)} \left(\mu\left(Q_j^k\right)^{\eta} \omega(x)\right)^{q(x)} d\mu \\
&\le \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \sigma\left(Q_j^k\right)^{q(x)-q_-\left(Q_j^k\right)} \sigma\left(Q_j^k\right)^{q_-\left(Q_j^k\right)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)} d\mu \\
&\le \sum_{(k,j)\in\mathscr{F}} \left(1+\sigma\left(Q_j^k\right)\right)^{q_+\left(Q_j^k\right)-q_-\left(Q_j^k\right)} \int_{E_j^k} \sigma\left(Q_j^k\right)^{q_-\left(Q_j^k\right)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)} d\mu \\
&\lesssim (1+\sigma(Q_0))^{q_+-q_-} \sum_{(k,j)\in\mathscr{F}} \sigma\left(Q_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}} \\
&\lesssim (1+\sigma(Q_0))^{q_+-q_-} \sum_{\theta=1,\frac{q_+}{p_-}} \sum_{(k,j)\in\mathscr{F}} \left(\sigma\left(Q_j^k\right)^{\theta}\right) \\
&\lesssim (1+\sigma(Q_0))^{q_+-q_-} \sum_{\theta=1,\frac{q_+}{p_-}} \left(\sum_{(k,j)\in\mathscr{F}} \sigma\left(E_j^k\right) \right)^{\theta} \\
&\le (1+\sigma(Q_0))^{q_+-q_-} \sum_{\theta=1,\frac{q_+}{p_-}} \sigma(Q_0)^{\theta},
\end{aligned}$$

where the implicit constants are independent on Q_j^k and f.

Estimate for I_2: Set $B_j^k = B(x_c(Q_j^k), A_0(C_d+1)N_0 d_0^k)$. For $(k,j) \in \mathscr{G}$, as $Q_j^k \not\subseteq Q_0$, if $x_c(Q_j^k) \in Q_0$, then by Lemma 2.17, $Q_0 \subseteq Q_j^k \subseteq B_j^k$. If $x_c(Q_j^k) \notin Q_0$, noting that $Q_0 \supseteq B(x_0, d_0^{k_0})$, we have

$$d_0^{k_0} \le d(x_0, x_c(Q_j^k)) \le N_0 d_0^k.$$

By Lemma 2.17 again, since $x_0 \in B(x_c(Q_j^k), N_0 d_0^k)$ and $Q_0 \subseteq B(x_0, C_d d_0^{k_0})$, then for every $x \in Q_0$,

$$d(x, x_c(Q_j^k)) \le A_0(d(x,x_0) + d(x_0, x_c(Q_j^k))) \le A_0(C_d d_0^{k_0} + N_0 d_0^k) \le A_0(C_d+1)N_0 d_0^k.$$

Hence, for any $(k,j) \in \mathscr{G}$, $Q_0 \subseteq B_j^k$. Furthermore, $W(B_j^k), \sigma(B_j^k) \ge 1$. Note also that by doubling property and Lemma 2.17, $\mu(Q_j^k) \approx \mu(B_j^k)$.

Lemma 2.5 can deduce that $\|\omega^{-1}\chi_{Q_0}\|_{p'(\cdot)} \ge 1$, since $\sigma(Q_0) \ge 1$. By (2.2), (1.4), and Lemma 2.14, it follows that

$$\mu\left(Q_j^k\right)^{-1} \approx \mu\left(B_j^k\right)^{-1} \lesssim \mu(Q_0)^{-1} \left(\frac{\sigma(Q_0)}{\sigma\left(B_j^k\right)} \right)^{\frac{1}{(1-\eta)p'_\infty}} \approx \left\|\omega^{-1}\chi_{B_j^k}\right\|_{p'(\cdot)}^{\frac{1}{(\eta-1)}} \lesssim \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{\frac{1}{(\eta-1)}}.$$

Together with the above and Lemma 2.10, we have

$$\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k} f_2(y)d\mu \lesssim \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{-1}\|f_2\omega\|_{p(\cdot)}\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)} \lesssim 1.$$

It follows immediately from Lemma 2.8 that

$$\begin{aligned} I_2 &\lesssim \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\left(C^{-1}\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k} f_2(y)d\mu\right)^{q(x)}\omega(x)^{q(x)}d\mu \\ &\le C_t\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\left(\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k} f_2(y)d\mu\right)^{q_\infty}\omega(x)^{q(x)}d\mu + \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\frac{W(x)}{\left(e+d\left(x_0,x\right)\right)^{tq_-}}d\mu \end{aligned} \tag{3.17}$$

Similar to getting (2.4), we can choose t sufficiently large to obtain

$$\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\frac{W(x)}{\left(e+d\left(x_0,x\right)\right)^{tq_-}}d\mu \le \int_X\frac{W(x)}{\left(e+d\left(x_0,x\right)\right)^{tq_-}}d\mu \le 1 \tag{3.18}$$

To finish the estimation of I_2, it suffices to estimate the first term of (3.17). Therefore, we have

$$\begin{aligned} &\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\left(\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k} f_2(y)d\mu\right)^{q_\infty}\omega(x)^{q(x)}d\mu \\ &= \sum_{(k,j)\in\mathscr{G}}\left(\frac{1}{\sigma\left(Q_j^k\right)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty}\left(\frac{\sigma\left(Q_j^k\right)}{\mu\left(Q_j^k\right)^{1-\eta}}\right)^{q_\infty}W\left(E_j^k\right). \end{aligned}$$

Next, we claim that

$$\left(\frac{\sigma(Q_j^k)}{\mu(Q_j^k)^{1-\eta}}\right)^{q_\infty} \lesssim \frac{\sigma(Q_j^k)}{W(Q_j^k)}. \tag{3.19}$$

Indeed, applying (2.5) to $(\sigma,\, p'(\cdot))$ and $(W,\, q(\cdot))$ for cubes, and by $A_{p(\cdot),q(\cdot)}$ condition, it follows that

$$\sigma(Q_j^k)^{q_\infty-1} \lesssim \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q_\infty} \lesssim \left(\frac{\mu(Q_j^k)^{1-\eta}}{\left\|\omega\chi_{Q_j^k}\right\|_{q(\cdot)}}\right)^{q_\infty} \lesssim \frac{\mu(Q_j^k)^{(1-\eta)q_\infty}}{W(Q_j^k)}$$

Thus, (3.19) follows obviously from the rearrangement.

In the following, we proceed to estimate the first term of (2.4).

$$
\begin{aligned}
&\sum_{(k,j)\in\mathscr{G}}\left(\frac{1}{\sigma\left(Q_j^k\right)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty}\left(\frac{\sigma\left(Q_j^k\right)}{\mu\left(Q_j^k\right)^{1-\eta}}\right)^{q_\infty} W\left(E_j^k\right)\\
\lesssim&\sum_{(k,j)\in\mathscr{G}}\left(\frac{1}{\sigma\left(Q_j^k\right)}\int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty}\sigma\left(Q_j^k\right)\\
\lesssim&\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k} M_\sigma\left(f_2\sigma^{-1}\right)(x)^{q_\infty}\sigma(x)d\mu.\\
\leq&\int_X M_\sigma\left(f_2\sigma^{-1}\right)(x)^{q_\infty}\sigma(x)d\mu && (3.20)\\
\lesssim&\int_X\left(f_2(x)\sigma(x)^{-1}\right)^{q_\infty}\sigma(x)d\mu && (3.21)\\
\leq&\int_X\left(f_2(x)\sigma(x)^{-1}\right)^{p_\infty}\sigma(x)d\mu. && (3.22)\\
\leq& C_t\left(\int_X\left(f_2(x)\sigma(x)^{-1}\right)^{p(x)}\sigma(x)d\mu+\int_X\frac{\sigma(x)}{\left(e+d\left(x_0,x\right)\right)^{tp_-}}d\mu\right) && (3.23)\\
\leq& C_t\left(\int_X f_2(x)^{p(x)}\omega(x)^{p(x)}d\mu+\int_X\frac{\sigma(x)}{\left(e+d\left(x_0,x\right)\right)^{tp_-}}d\mu\right). && (3.24)
\end{aligned}
$$

where (3.21) comes from Lemma 2.21, (3.22) holds due to the fact that $f_2\sigma^{-1}\leq 1$, and (3.23) is valid due to Lemma 2.8. Then, the second term of (3.24) is similar to (3.18) and we just replace W with σ. Thus, I_2 is bounded by a constant due to (3.6).

Estimate for I_3: Firstly, we claim that

$$\sup_{x\in Q_j^k} d\left(x_0,x\right)\approx\inf_{x\in Q_j^k} d\left(x_0,x\right) \tag{3.25}$$

where the implicit constant is independent on Q_j^k.for some constant $R\geq 1$ which is independent of k and j. In our analysis, the validity of inequality (3.25) will be established through substitution of Q_j^k with the ball $A_j^k=N_0^{-1}B_j^k$, which encompasses Q_j^k. For this purpose, we fix a pair (k,j) within $\mathscr{H}$ and choose an arbitrary x from A_j^k. We get that

$$d(x,x_0)\leq A_0[d(x,x_c(Q_j^k))+d(x_0,x_c(Q_j^k))]\leq A_0[C_d d_0^k+d(x_0,x_c(Q_j^k))]\leq\left(A_0+\frac{1}{2}\right)d(x_0,x_c(Q_j^k)).$$

In the other hand,

$$d(x_0,x_c(Q_j^k))\leq A_0[d(x_0,x)+d(x,x_c(Q_j^k))]=\frac{1}{2}N_0d_0^k+A_0d(x_0,x)\leq\frac{1}{2}d(x_0,x_c(Q_j^k))+A_0d(x_0,x).$$

Then, we obtain that

$$d\left(x_0,x_c\left(Q_j^k\right)\right)\leq 2A_0d\left(x_0,x\right).$$

Consequently, (3.25) holds.

To proceed with the estimation of I_3, it becomes necessary to partition $\mathscr{H}$ into two distinct subsets,

$$\mathscr{H}_1=\left\{(k,j)\in\mathscr{H}:\sigma\left(Q_j^k\right)\le 1\right\},\quad \mathscr{H}_2=\left\{(k,j)\in\mathscr{H}:\sigma\left(Q_j^k\right)>1\right\}.$$

Initially, we aggregate over $\mathscr{H}_1$. Consider x_+ within $\overline{A_j^k}$, chosen such that $q_+(A_j^k)=q(x_+)$, a selection made possible by the continuity of $q(\cdot)$ in LH_0. Subsequently, in accordance with the LH_∞ criterion and inequality (3.25), it holds for almost every x in Q_j^k that,

$$\begin{aligned}0\le q_+\left(Q_j^k\right)-q(x)&\le\left|q\left(x_+\right)-q_\infty\right|+\left|q(x)-q_\infty\right|\\&\le\frac{C_\infty}{\log\left(e+d\left(x_0,x_+\right)\right)}+\frac{C_\infty}{\log\left(e+d\left(x_0,x\right)\right)}\\&\approx\frac{1}{\log\left(e+d\left(x_0,x\right)\right)}\end{aligned}$$

By Lemma 2.8 and (3.18), we derive

$$\begin{aligned}&\sum_{(k,j)\in\mathscr{H}_1}\int_{E_j^k}\left(\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k}f_2(y)d\mu\right)^{q(x)}\omega(x)^{q(x)}d\mu\\&\lesssim\left(\sum_{(k,j)\in\mathscr{H}_1}\int_{E_j^k}\left(\mu\left(Q_j^k\right)^{\eta-1}\int_{Q_j^k}f_2(y)d\mu\right)^{q_+\left(Q_j^k\right)}\omega(x)^{q(x)}d\mu\right)+1\end{aligned}\tag{3.26}$$

It follows from Lemma 2.3 that

$$\mu\left(Q_j^k\right)^{q(x)-q_+\left(Q_j^k\right)}\lesssim\left(\mu\left(Q_j^k\right)^{q_+\left(Q_j^k\right)}+\mu\left(Q_j^k\right)^{q_-\left(Q_j^k\right)}\right)\mu\left(Q_j^k\right)^{-q_+\left(Q_j^k\right)}\lesssim 1.$$

The first term of (3.26) is bounded by

$$\sum_{(k,j)\in\mathscr{H}_1}\int_{E_j^k}\left(\frac{1}{\sigma\left(Q_j^k\right)}\int_{Q_j^k}f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_+\left(Q_j^k\right)}\sigma\left(Q_j^k\right)^{q_+\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu.$$

Through Lemma 2.8 and $f_2\sigma^{-1}\le 1$, the above

$$\begin{aligned}&\lesssim\sum_{(k,j)\in\mathscr{H}_1}\int_{E_j^k}\left(\frac{1}{\sigma\left(Q_j^k\right)}\int_{Q_j^k}f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty}\sigma\left(Q_j^k\right)^{q_+\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu\\&+\sum_{(k,j)\in\mathscr{H}_1}\int_{E_j^k}\sigma\left(Q_j^k\right)^{q_+\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\frac{\omega(x)^{q(x)}}{\left(e+d\left(x_0,x\right)\right)^{tq_-}}d\mu\\&=:J_1+J_2.\end{aligned}$$

To estimate J_2, we note that $\sigma(E_j^k) \approx \sigma(Q_j^k) \leq 1$. By (3.11) and (3.25), we deduce that

$$\begin{aligned}
J_2 &\leq \sum_{(k,j)\in\mathscr{H}_1} \sup_{x\in E_j^k} (e+d(x_0,x))^{-tq_-} \int_{E_j^k} \sigma\left(Q_j^k\right)^{q_-\left(Q_j^k\right)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)} \omega(x)^{q(x)} d\mu \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \sup_{x\in E_j^k} (e+d(x_0,x))^{-tq_-} \sigma\left(E_j^k\right) \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \frac{\sigma(x)}{(e+d(x_0,x))^{tq_-}} d\mu \\
&\leq \int_X \frac{\sigma(x)}{(e+d(x_0,x))^{tq_-}} d\mu \\
&\lesssim 1.
\end{aligned}$$

where the last inequality is the same as the argument for estimating the second term in (3.24). Similarly, it follows obviously from (3.10) that

$$\begin{aligned}
J_1 &\lesssim \sum_{(k,j)\in\mathscr{H}_1} \left(\sigma\left(Q_j^k\right)^{-1} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty} \sigma\left(Q_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}} \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \left(\sigma\left(Q_j^k\right)^{-1} \int_{Q_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q_\infty} \sigma\left(E_j^k\right) \\
&\lesssim \int_X M_\sigma\left(f_2\sigma^{-1}\right)(x)^{q_\infty}\sigma(x)d\mu.
\end{aligned}$$

where the last estimate similar to (3.20), which is bounded by a constant. We finish the estimate for $\mathscr{H}_1$.

Finally, for the case of $\mathscr{H}_2$, by Lemma 2.10, we have

$$\int_{Q_j^k} f_2(y)d\mu \lesssim \|f_2\omega\|_{p(\cdot)} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)} \leq \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}.$$

Applying Lemma 2.8,

$$\begin{aligned}
&\sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)d\mu\right)^{q(x)} (\mu(Q_j^k)^\eta\omega(x))^{q(x)} d\mu \\
\lesssim &\sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(c\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{-1} \int_{Q_j^k} f_2(y)d\mu\right)^{q(x)} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)} d\mu \\
\lesssim &\sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\left\|\omega\chi_{Q_j^k}\right\|_{p'(\cdot)}^{-1} \int_{Q_j^k} f_2(y)d\mu\right)^{q_\infty} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)} d\mu \\
&+ \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \frac{\left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)}}{(e+d(x_0,x))^{tq_-}} d\mu
\end{aligned}$$

$=: K_1 + K_2.$

To estimate K_2, note that $1 \le \sigma\left(Q_j^k\right) \approx \sigma\left(E_j^k\right)$. By (3.13) and (3.25), it follows from that

$$
\begin{aligned}
K_2 &\lesssim \sum_{(k,j)\in\mathscr{H}_2} \sup_{x\in E_j^k} \left(e + d\left(x_0, x\right)\right)^{-tq_-} \int_{E_j^k} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_2} \sup_{x\in E_j^k} \left(e + d\left(x_0, x\right)\right)^{-tq_-} \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_2} \sup_{x\in E_j^k} \left(e + d\left(x_0, x\right)\right)^{-tq_-} \sigma\left(E_j^k\right) \\
&\lesssim \int_X \frac{\sigma(x)}{\left(e + d\left(x_0, x\right)\right)^{tq_-}} d\mu.
\end{aligned}
\tag{3.27}
$$

Actually, (3.27) has been argued in J_2 and I_2 which is bounded by a constant.

To estimate K_1, it follows from (2.5) to get

$$
\left\|w^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{-q_\infty} \sigma(Q_j^k)^{q_\infty} \lesssim \sigma(Q_j^k)^{q_\infty - q_\infty/q'_\infty} = \sigma(Q_j^k)^{\frac{q_\infty}{p_\infty}}.
$$

Since $1 \le \sigma\left(Q_j^k\right) \approx \sigma\left(E_j^k\right)$, by (3.13), we have

$$
\begin{aligned}
& K_1 \\
&= \sum_{(k,j)\in\mathscr{H}_2} \int_{E_j^k} \left(\sigma\left(Q_j^k\right)^{-1} \int_{Q_j^k} f_2(y)d\mu\right)^{q_\infty} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)-q_\infty} \sigma\left(Q_j^k\right)^{q_\infty} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_2} \left(\frac{1}{\sigma\left(Q_j^k\right)} \int_{Q_j^k} f_2(y)d\mu\right)^{q_\infty} \sigma\left(Q_j^k\right)^{\frac{q_\infty}{p_\infty}} \int_{Q_j^k} \left\|\omega^{-1}\chi_{Q_j^k}\right\|_{p'(\cdot)}^{q(x)} \mu(Q_j^k)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu \\
&\lesssim \left(\sum_{(k,j)\in\mathscr{H}_2} \left(\frac{1}{\sigma\left(Q_j^k\right)} \int_{Q_j^k} f_2(y)d\mu\right)^{p_\infty} \sigma\left(E_j^k\right)\right)^{\frac{q_\infty}{p_\infty}} \\
&\lesssim \left(\int_X M_\sigma\left(f_2\sigma^{-1}\right)(x)^{p_\infty}\sigma(x)d\mu\right)^{\frac{q_\infty}{p_\infty}}
\end{aligned}
$$

Next, we use the same method as for estimating (3.20) to make the above estimate bounded by a constant. Thus the estimates for I_3 are completed, which accomplishes the proof of sufficiency for $\mu(X) = \infty$.

Case 2: $\mu(X) < \infty$.

Last but not least, we turn to the case for $\mu(X) < \infty$. In the finite case, the proof is similar to before and we just need to make some changes for Calderón-Zygmund Decomposition. We also consider f is a nonnegative funcion with $\|\omega f\|_{p(\cdot)} = 1$ and decompose $f = f_1 + f_2$ as before. The construction of Calderón-Zygmund cubes at any height $\lambda > \lambda_0 := \mu\left(Q_j^k\right)^{\eta-1} \int_{Q^k} f_i d\mu$.

By Lemma 2.10, Lemma 2.2, and the condition of $A_{p(\cdot),q(\cdot)}$,

$$\lambda_0 \leq 4\mu(X)^{\eta-1}\|f_i\omega\|_{p(\cdot)}\left\|\omega^{-1}\right\|_{p'(\cdot)} \leq 4[\omega]_{A_{p(\cdot),q(\cdot)}}\|\omega\|_{q(\cdot)}^{-1}.$$

In addition, by Lemma 2.5 and Lemma 2.2 again, we can conclude that $\lambda_0 \lesssim 1$.

From Lemma 2.22, set $a = 2C_{CZ}$ and $\left\{Q_j^k\right\}$ is the Calderón-Zygmund cubes of f_i at height a^k, for all integers $k \geq k_0 = [\log_a \lambda_0]$. Then

$$X = X_{\eta,a^{k_0}}^{\mathcal{D}} \bigcup \left(X_{\eta,a^{k_0}}^{\mathcal{D}}\right)^c = \left(\bigcup_{k=k_0}^{\infty} X_{\eta,a^k}^{\mathcal{D}} \backslash X_{\eta,a^{k+1}}^{\mathcal{D}}\right) \bigcup \left(X_{\eta,a^{k_0}}^{\mathcal{D}}\right)^c,$$

where $X_{\eta,a^{k_0}}^{\mathcal{D}} := \left\{x \in X : M_\eta^{\mathcal{D}} f_i(x) > \lambda_0\right\} \subseteq \left\{Q_j^k\right\}$.

It follows instantly from the getting of (3.9) that

$$\begin{aligned}
&\int_X M_\eta^{\mathcal{D}} f_i(x)^{q(x)}\omega(x)^{q(x)}d\mu \\
=&\int_{\left(X_{\eta,a^{k_0}}^{\mathcal{D}}\right)^c} M_\eta^{\mathcal{D}} f_i(x)^{q(x)}\omega(x)^{q(x)}d\mu + \sum_{k=k_0}^{\infty}\int_{X_{\eta,a^k}^{\mathcal{D}}\backslash X_{\eta,a^{k+1}}^{\mathcal{D}}} M_\eta^{\mathcal{D}} f_i(x)^{q(x)}\omega(x)^{q(x)}d\mu \\
\lesssim&\lambda_0 W(X) + \sum_{k\geq k_0,j}\left(\int_{Q_j^k} f_i(y)\sigma(y)^{-1}\sigma(y)d\mu\right)^{q(x)} \mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu.
\end{aligned}$$

The first term is bounded by a constant, which depends only on X, $\mathcal{D}$, ω, η, and $p(\cdot)$. When $i = 1$, the second term is similar to the infinite case. We consider the following for $i = 2$.

After choosing $Q_0 = X$, then $I_2 = I_3 = 0$. Further, since $f_2\sigma^{-1} \leq 1$, $\sigma(Q_j^k) \approx \sigma(E_j^k)$, and (3.11), the second term is bounded by

$$\begin{aligned}
&\sum_{k\geq k_0,j}\int_{E_j^k}\sigma(X)^{q(x)}\left(\frac{\sigma\left(Q_j^k\right)}{\sigma(X)}\right)^{q(x)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu \\
\leq&\left(\sigma(X)^{q_+} + \sigma(X)^{q_-}\right)\sum_{k\geq k_0,j}\int_{E_j^k}\left(\frac{\sigma\left(Q_j^k\right)}{\sigma(X)}\right)^{q_-\left(Q_j^k\right)}\mu\left(Q_j^k\right)^{(\eta-1)q(x)}\omega(x)^{q(x)}d\mu \\
\lesssim&\left(\sigma(X)^{q_+} + \sigma(X)^{q_-}\right)\left(\frac{1}{\sigma(X)^{q_+}} + \frac{1}{\sigma(X)^{q_-}}\right)\sum_{k\geq k_0,j}\sigma\left(E_j^k\right)^{\frac{q_-(Q_j^k)}{p_-(Q_j^k)}} \\
\leq&\frac{\left(\sigma(X)^{q_+} + \sigma(X)^{q_-}\right)^2}{\sigma(X)^{q_++q_-}}\sum_{\theta=1,\frac{q_+}{p_-}}\sum_{k\geq k_0,j}\left(\sigma\left(E_j^k\right)^\theta\right) \\
\leq&\frac{\left(\sigma(X)^{q_+} + \sigma(X)^{q_-}\right)^2}{\sigma(X)^{q_++q_-}}\sum_{\theta=1,\frac{q_+}{p_-}}\left(\sum_{k\geq k_0,j}\sigma\left(E_j^k\right)\right)^\theta
\end{aligned}$$

$$\leq \frac{\left(\sigma(X)^{q_+}+\sigma(X)^{q_-}\right)^2}{\sigma(X)^{q_++q_-}} \sum_{\theta=1,\frac{q_+}{p_-}} \sigma(X)^{\theta}.$$

We accomplish estimate for $i=2$ and finish the proof of sufficiency for $\mu(X)<\infty$.

Acknowledgements The author would like to thank the referees for careful reading and valuable comments, which lead to the improvement of this paper.

Data Availability Our manuscript has no associated data.

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