γ_5 subtleties in distinct regularizations: the Bumblebee model example

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Abstract

We examine the subtleties of regularization schemes in four-dimensional space ($4S$), related in particular to the introduction of the γ_5 matrix. To illustrate we use a "Bumblebee" model featuring dynamically induced Lorentz symmetry violation. The analysis centers on how different regularization methods affect the solutions to the gap equation in this model. We highlight the resolution of ambiguities associated with the γ_5 matrix in ultraviolet divergent integrals by employing an enhanced Implicit Regularization (IREG) method. This method extends IREG to a quasi-four-dimensional space, $Q4S = 4S \oplus X$, drawing parallels with the consistent approach of Dimensional Reduction (DRED). Comparative analysis is conducted against results from the 't Hooft-Veltman regularization scheme, conventional IREG in strict $4S$, and sharp momentum cutoff techniques. Our results illustrate a scheme to compute γ_5 interactions in physical dimension of divergent amplitudes, confirming the approach in [1].

1 Introduction

The Standard Model of Particle Physics (SM) is a successful quantum theoretical framework that describes three among four of fundamental particle interactions, namely electromagnetism, the weak force, and the strong nuclear force — excluding only gravity. Lorentz symmetry is at the core of the SM framework. A profound connection between Lorentz symmetry and charge, parity, and time reversal (CPT) symmetry invariance is unveiled by the CPT theorem: any Lorentz-invariant local quantum field theory with Hermitian Hamiltonian must have CPT symmetry. This is important because CPT symmetry imposes stringent constraints on the permissible interactions and processes in nature. However, whilst an interacting theory that violates CPT necessarily violates Lorentz invariance, it is possible to have Lorentz violation without CPT violation [2]. On the other hand, observable manifestations of an underlying unified quantum gravity theory may present signals of Planck scale physics associated with Lorentz symmetry breaking [3]. Indeed, various quantum-gravity approaches result in a scenario where Lorentz symmetry is broken.

Despite the absence of a complete quantum theory of gravity, investigating Lorentz symmetry violation (LV) remains pertinent. LV carries observable implications, e.g. in cosmology and the early universe. Some theoretical frameworks propose scenarios where Lorentz symmetry is dynamically broken, potentially influencing cosmic microwave background radiation and the large-scale structure of the universe [4–6]. In this sense, SM Extensions (SME) serve as effective field theories models for investigating specific instances of LV providing valuable insights for probing the fundamental nature of spacetime. In practice, this approach introduces correction terms in the Lagrangian that explicitly break Lorentz symmetry, thereby enabling the generation of measurable deviations from the Standard Model predictions. Usually such LV terms are introduced in the model through spontaneous (SSB) or dynamical (DSB) symmetry breaking. [7–20]. Spontaneous Symmetry Breaking (SSB) plays a crucial role, for instance, in the unified framework of weak and electromagnetic forces. Conversely, Dynamical Symmetry Breaking (DSB) was suggested by Bjorken in 1963 as a method for the "dynamical generation of quantum electrodynamics" (QED). This approach sought to reproduce the observable effects of standard QED without presupposing the local $U(1)$ gauge invariance. DSB is an important alternative to the Higgs mechanism and plays central role in QCD chiral symmetry breaking.

In this contribution we study a dynamically generated Lorentz symmetry breaking model inspired by a four-fermion field model effective potential generated as a quantum correction. The minimum of the potential is determined by a tadpole finite contribution which is superficially divergent and contains γ_5 Dirac matrices. It is well known that the evaluation of divergent integrals involving γ_5 matrices potentially leads to regularization dependence. Such ambiguities in defining the γ_5 algebra across different regularizations can lead to discrepancies in the evaluation of these integrals, impacting the physical predictions. On the other hand, regularizations that operate in the physical dimension should, in principle, be exempt from such problems. This expectation is however too naive, as discussed in detail in [1], where further examples can be found in [21–25].

This work is organized as follows: after introducing the model to be used as our test case, we explore in Section 3 the ambiguities associated with using the γ_5 matrix algebra in four-dimensional space ($4S$) with divergent integrals in IREG, and how IREG ensures result uniqueness. Section 4 details the solutions to the gap equation for the Bumblebee model using IREG in both strict $4S$ and quasi-$4S$ ($Q4S$), comparing these to results from the 't Hooft-Veltman dimensional scheme [26], and two sharp cutoff schemes in $4S$. We present our conclusions in Section 5, which are followed by two appendices: one reviews the basic rules of IREG, and the other presents results for the finite integrals specific to the IREG method used in this analysis.

2 The model

In the late eighties, Kostelecký and Samuel [27] studied a model based on the Einstein-Maxwell action with a potential for the vector field that induces LV when a vector (or tensor) field acquires a non-zero vacuum expectation value. For instance a potential term $1/(2\alpha)(B^\mu B_\mu + b^2)^2$ for a background field B_μ such that $\langle 0|B_\mu|0\rangle = b_\mu \neq 0$ can act as indicator of global Lorentz violation. The potential reaches its minimum when the condition $B^\mu B_\mu = -b^2$ is met, which can be achieved, for instance, by considering a time-like 4-vector $B_\mu = (b, 0, 0, 0)$. Obviously, this choice of B_μ establishes a preferred direction in spacetime.

For definiteness, consider the Bumblebee-type model given by the of Lagrangian in which we have a Lagrange-multiplier potential proportional to a positive parameter λ:

$$\mathcal{L}_B = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial_\mu\gamma^\mu - m - e\not{B}\gamma_5)\psi - \frac{\lambda}{4}(B_\mu B^\mu - \alpha^2)^2. \tag{1}$$

Here $F_{\mu\nu}F^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Performing a shift $B_\mu \to b_\mu + A_\mu$, with $\langle A_\mu \rangle = 0$, yields:

$$\mathcal{L}_B = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\not{\partial} - m - e(\not{A} + \not{b})\gamma_5]\psi - \frac{\lambda}{4}\left(A_\mu A^\mu + 2A_\mu b^\mu\right)^2, \tag{2}$$

in which the term in b_μ explicitly violates Lorentz symmetry.

A different venue to arrive at this model comes by following the work of Coleman and Weinberg [28] that established that spontaneous symmetry breaking can be generated at quantum level. It has been demonstrated in the literature that Bumblebee-type potentials, analogous to those found in the Bumblebee model of gauge symmetry breaking, can arise dynamically within certain theoretical frameworks. [29]. In particular, considering as a starting point an interaction term consisting of a fermion bilinear of massless fields transforming as an axial-vector under Lorentz transformations [29]:

$$\begin{aligned}\mathcal{L} &= \bar{\psi} i\partial\!\!\!/ \psi - \frac{e^2}{2g^2}(\bar{\psi}\gamma^\mu\gamma^5\psi)^2 + \frac{g^2}{2}\left[B_\mu - \frac{e}{g^2}\bar{\psi}\gamma^\mu\gamma^5\psi\right]^2 \\ &= \frac{g^2}{2}B_\mu B^\mu + \bar{\psi}(i\partial\!\!\!/ - eB\!\!\!/\gamma^5)\psi, \end{aligned} \tag{3}$$

in which the term $-\frac{e^2}{2g^2}(\bar{\psi}\gamma^\mu\gamma^5\psi)^2$ has been cancelled out. Following [26], after some manipulations we arrive at the effective potential for the Bumblebee field:

$$V_{\text{eff}} = -\frac{g^2}{2}B_\mu B^\mu + i\text{tr}\int \frac{d^4k}{(2\pi)^4}\ln(k\!\!\!/ - eB\!\!\!/\gamma^5). \tag{4}$$

The nontrivial minimum of this potential is obtained by imposing:

$$\left.\frac{dV_{\text{eff}}}{dB_\mu}\right|_{eB_\mu = b_\mu} = -\frac{g^2}{e}b^\mu - i\Pi^\mu = 0, \tag{5}$$

where

$$\Pi^\mu = \text{tr}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k\!\!\!/ - b\!\!\!/\gamma^5}(-ie)\gamma^\mu\gamma_5 \tag{6}$$

is the one-loop tadpole amplitude. It is evaluated in Ref. [26] to

$$\Pi^\mu = \frac{ie^2b^2}{3\pi^2}b^\mu, \tag{7}$$

leading to a gap equation:

$$\left.\frac{dV_{\text{eff}}}{dB_\mu}\right|_{eB_\mu = b_\mu} = \left(-\frac{1}{G} + \frac{b^2}{3\pi^2}\right)eb_\mu = 0. \tag{8}$$

It has the non-trivial solution

$$b^2 = \frac{3\pi^2}{G}, \tag{9}$$

with $G > 0$ ($G < 0$) for timelike (spacelike) b_μ. On the other hand, the effective potential reads

$$V_{\text{eff}} = -\frac{e^2b^2}{6\pi^2}B^2 + \frac{e^4}{12\pi^2}B^4 + c, \tag{10}$$

where c is an integration constant. In this way, the Bumblebee potential in equation 1 is reproduced by choosing $c = b^4/(12\pi^2)$ and $\lambda = e^4/(3\pi^2)$.

This illustrates that within the four-fermion model, a bumblebee potential featuring nontrivial minima can arise due to quantum corrections, which in turn leads to dynamical Lorentz symmetry breaking within this context. To ensure the framework's consistency, it is necessary for Π^μ to be finite, as this condition is essential for obtaining a physical solution that corresponds to the potential's minimum.

In this work, we explore how the coefficients in eq. (10) depend on a coherent treatment of the γ_5 algebra by various regularization techniques. Our primary attention is on regularizations that are applied in the physical dimension, as these can notably simplify higher-order calculations if a consistent approach to chiral theories is adopted.

3 γ_5 matrix algebra in implicit regularization

Handling the γ_5 Dirac matrix in conventional dimensional regularization (CDR) is problematic, as expanding its specific algebra to arbitrary dimensions D breaks chiral symmetry. To resolve this, authors have devised new schemes for precise calculations beyond leading order [30–45], redefining algebraic rules for extended Lorentz tensors and gamma matrices, enforcing constraints order by order, and adding counterterms to restore symmetries according to quantum action principles [46].

On the other hand, refining theoretical predictions in particle physics requires effective computational methods for calculating Feynman amplitudes beyond the next-to-leading order (NLO). For some processes, high-precision measurements already call for theoretical calculations up to at least four loop order [47] and serve to probe extensions of the Standard Model. In this sense exploring non-dimensional regularization schemes offers a promising approach to simplify computations following some already successful mixed schemes such as dimensional reduction (DRED) [48, 49] and four-dimensional helicity (FDH) [50, 51]. Among such schemes, notable examples include four-dimensional formulation (FDF) [52], four dimensional regularization (FDR) [53], the four dimensional unsubtracted (FDU) method [54, 55] and implicit regularization (IREG) [56–59], see [60, 61] for reviews.

In conventional dimensional methods (CDR), a primary challenge comes from the relation between the emergence of anomalous terms and the breakdown of cyclicity — a characteristic typically preserved in finite dimensions [62]. While in CDR $\{\gamma_5, \gamma_\mu\} = 0$ is typically preserved in $D \neq 4$, in the Breitenlohner, Maison, ´t Hooft and Veltman (BMHV) extension [31] $\{\gamma_5, \gamma_\mu\} \neq 0$ and, as a byproduct, gauge and BRST invariance should be consistently restored by symmetry-restoring counterterms at all orders [63–65]. One might anticipate that for four-dimensional regularization schemes, the γ_5 matrix issue would not arise. Nonetheless, there exist subtleties related to finite ambiguities when integrating over internal momenta. One example is symmetric integration in the internal loop momenta [66]. Such operation alters the Lorentz structure and consequently the γ_5 Clifford algebra by introducing spurious terms [1, 22, 23, 25]. In a nutshell, the contraction of Lorentz indices does not commute with renormalization in these non-dimensional schemes. To circumvent this problem, Bruque and collaborators [1] proposed a scheme in which Dirac matrices, with the exception of γ_5, are defined in a quasi-dimensional space $Q4S = QdS \oplus Q(2\epsilon)S$ [46]. However, in contrast to DRED (where the momenta still need to be treated in QdS) the amplitude momenta are also defined in $Q4S$. One defines $Q4S = 4S \oplus X$, X being an auxiliary space, which needs not be explicitly defined [1]. In the specific case of IREG, the inconsistencies boil down to the contraction of internal momenta in Feynman amplitudes. Let us illustrate this point with a toy integral[1] involving the internal momentum k and symmetric integration: $k^2 \to k_\alpha k_\beta g^{\alpha\beta}$ or $k_\alpha k_\beta \to \frac{k^2}{4} g_{\alpha\beta}$. On one hand,

$$\int_k \frac{k^2}{k^2(k-p)^2} = \int_k \frac{1}{(k-p)^2} = \lim_{\mu^2 \to 0} \int_k \frac{1}{(k-p)^2 - \mu^2} = \lim_{\mu^2 \to 0} \int_k \frac{1}{k^2 - \mu^2} = 0 \tag{11}$$

whereas, performing symmetric integration,

$$\begin{aligned} g^{\alpha\beta} \int_k \frac{k_\alpha k_\beta}{k^2(k-p)^2} &= g^{\alpha\beta} \left\{ \left(\frac{p_\alpha p_\beta}{3} - \frac{g_{\alpha\beta} p^2}{12} \right) \left[I_{\log}(\lambda^2) - b \ln\left(-\frac{p^2}{\lambda^2} \right) + \frac{13b}{6} \right] - \frac{g_{\alpha\beta} b p^2}{24} \right\} \\ &= -\frac{b}{6} p^2, \end{aligned} \tag{12}$$

where $b = i/(4\pi)^2$, and $\int_k \equiv \int d^4k/(2\pi)^4$, clearly showing an ambiguity in the evaluation, which is dependent on regularization since symmetric integration is not applicable in general [66]. In the γ_5 matrix algebra, such ambiguity emerges in non-dimensional regularization schemes upon

[1] A concise summary of IREG's fundamental rules can be found in the appendix.

consistently requiring $\{\gamma_\mu, \gamma_5\} = 0$:

$$\int_k \frac{\not{k}\gamma_5\not{k}}{k^2(k-p)^2} = \begin{cases} 0, & \text{using eq. (11) or} \\ \frac{b}{6}p^2\gamma_5, & \text{using eq.(12) .} \end{cases} \tag{13}$$

In order to avoid these ambiguities, one defines

$$\gamma_5 = -\frac{i}{4!}\epsilon_{abcd}\bar{\gamma}^a\bar{\gamma}^b\bar{\gamma}^c\bar{\gamma}^d \tag{14}$$

where we use an overbar to denote an object pertaining to $4S$. Since the Dirac matrices are defined in $Q4S$, they are required to satisfy:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}; \quad \{\bar{\gamma}_\mu, \bar{\gamma}_\nu\} = \{\gamma_\mu, \bar{\gamma}_\nu\} = 2\bar{g}_{\mu\nu}\mathbb{1}; \quad \gamma_\mu\gamma^\mu = \gamma_\mu\bar{\gamma}^\mu = 4\,\mathbb{1} \tag{15}$$

$$\{\bar{\gamma}_\mu, \hat{\gamma}_\nu\} = 0; \quad \{\gamma_\mu, \hat{\gamma}_\nu\} = \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\hat{g}_{\mu\nu}\mathbb{1}; \quad \gamma_\mu\hat{\gamma}^\mu = \bar{\gamma}_\mu\hat{\gamma}^\mu = \hat{\gamma}_\mu\hat{\gamma}^\mu = 0; \tag{16}$$

$$\{\bar{\gamma}_\mu, \gamma_5\} = 0; \quad \{\gamma_\mu, \gamma_5\} = 2\gamma_5\hat{\gamma}_\mu; \quad [\hat{\gamma}_\mu, \gamma_5] = 0 \tag{17}$$

where hatted objects belong to X space. In view of the above properties, equation (13) is evaluated as [25]

$$\begin{aligned} \int_k \frac{\not{k}\gamma_5\not{k}}{k^2(k-p)^2} &= 2\gamma_5\int_k \frac{\hat{\not{k}}\not{k}}{k^2(k-p)^2} - \int_k \frac{\gamma_5 k^2}{k^2(k-p)^2} \\ &= \gamma_5\int_k \frac{k^2}{k^2(k-p)^2} - 2\gamma_5\int_k \frac{\bar{k}^2}{k^2(k-p)^2} \\ &= -2\bar{g}_{ab}\gamma_5\int_k \frac{\bar{k}^a\bar{k}^b}{k^2(k-p)^2} = \frac{bp^2}{3}\gamma_5. \end{aligned} \tag{18}$$

Notice that the above result differs from the two naive options of equation 13. Since this is just a toy integral, it is not possible to connect any of the approaches to the (possible) breaking of Ward identities. However, only the last approach, extending the theory to the $Q4S$ space, is completely free of ambiguities (particularly at multiloop calculations), since it does not require reading points to be defined [39, 40, 67], for instance [2].

4 The bumblebee model from distinct regularizations

In [26], Π^μ in equation 6 was evaluated by employing the exact propagator and using dimensional regularization with 't Hooft-Veltmann prescription [30] following an analytical extension from 4 to a D-dimensional spacetime. Moreover Dirac matrices are required to obey $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g_{\mu\nu}g^{\mu\nu} = D$. Dirac matrices γ^μ and the metric tensor $g^{\mu\nu}$ are split as

$$\begin{aligned} \gamma^\mu &= \bar{\gamma}^\mu + \hat{\gamma}^\mu, \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} + \hat{g}^{\mu\nu}, \end{aligned} \tag{19}$$

that is, into 4-dimensional parts (expressed with a bar) and $(D-4)$-dimensional parts (hatted), so that now the Dirac matrices satisfy the relations

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}, \quad \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu}, \quad \{\bar{\gamma}^\mu, \hat{\gamma}^\nu\} = 0, \tag{20}$$

and the metric tensors obey

$$\bar{g}_{\mu\nu}\bar{g}^{\mu\nu} = 4, \quad \hat{g}_{\mu\nu}\hat{g}^{\mu\nu} = D-4, \quad \bar{g}_{\mu\nu}\hat{g}^{\mu\nu} = 0. \tag{21}$$

[2]The main drawback is that chiral symmetry is inevitably broken, as happens in the BMHV scheme. It can, nevertheless, be restored by including finite counterterms (see [1, 25] for examples in IREG).

In this way one has

$$[\hat{\gamma}^\mu, \gamma^5] = 0 \tag{22}$$

and the anticommutation relation

$$\{\bar{\gamma}^\mu, \gamma^5\} = 0. \tag{23}$$

is maintained.

We proceed to study how eq.(6) can be evaluated in four-dimensional regularization schemes, such as IREG. Before discussing the gap equation, though, we will take a step back and discuss how the Feynmann rules are extracted from eq.(2).

This task may be non trivial in the presence of γ_5, which is only defined in the physical dimension. In particular, if the regularization to be adopted is defined in a different dimension, the Lagrangian (and thereof the Feynman rules) must also be extended to this dimension. In particular, once γ_μ is defined in the extended dimension, the property $\{\gamma_\mu, \gamma_5\} = 0$ may not hold. For the bumblebee model in particular, this implies that fermionic propagator (D_B) will not be written in terms of the left/right propagators (P_L/P_R) only. Consider, for instance, that $\{\gamma^\mu, \gamma^5\} = 2\hat{\gamma}^\mu$, where $\hat{\gamma}^\mu$ is defined in the extra space X defined in the previous section. At first sight, we have

$$D_B = \frac{1}{\not{k} - \not{b}\gamma^5}, \quad D_B^{-1} = \left((\not{k} - \not{b})P_R + (\not{k} + \not{b})P_L\right) \ . \tag{24}$$

However, it is straightforward to show that

$$D_B D_B^{-1} = 1 - \left(\frac{1}{\not{k} - \not{b}} + \frac{1}{\not{k} + \not{b}}\right)(\hat{\not{k}} - \hat{\not{b}}\gamma^5) \ . \tag{25}$$

Thus, D_B^{-1} cannot be the inverse propagator, unless $\{\gamma^\mu, \gamma^5\} = 0$ holds.

In the following, consider that the regularization to be employed is defined in the physical dimension, implying that $\{\gamma^\mu, \gamma^5\} = 0$ holds and the gap equation can be written as

$$\begin{aligned}
\Pi^\mu = Tr\left[\int_k \frac{1}{\not{k} - \not{b}\gamma^5}\gamma^\mu\gamma^5\right] &= Tr\left[\int_k \left(\frac{1}{\not{k} - \not{b}}P_L + \frac{1}{\not{k} + \not{b}}P_R\right)\gamma^\mu\gamma^5\right] \\
&= 2\int_k \frac{k^\mu - b^\mu}{(k-b)^2} - 2\int_k \frac{k^\mu + b^\mu}{(k+b)^2} \\
&= 2\int \frac{k^\mu[(k+b)^2 - (k-b)^2] - b^\mu[(k+b)^2 + (k-b)^2]}{(k-b)^2(k+b)^2}
\end{aligned} \tag{26}$$

After employing Feynman parametrization, we obtain

$$\Pi^\mu = Tr\left[\int_k \frac{1}{\not{k} - \not{b}\gamma^5}\gamma^\mu\gamma^5\right] = -2\int_0^1 dx \int_k \frac{2b^\mu(k^2 - \Delta) - 4k^\mu(b\cdot k)}{(k^2 - \Delta)^2} \tag{27}$$

where $\Delta = 4b^2x(x-1) + \mu^2$, and μ^2 is a fictitious mass introduced in the propagators of eq.26. This result should be compared with the one obtained in the framework of Dimensional Regularization [26]

$$Tr\left[\int_k \frac{1}{\not{k} - \not{b}\gamma^5}\gamma^\mu\gamma^5\right] = -2\mu^{4-D}\int_0^1 dx \int \frac{dk^d}{(2\pi)^d}\frac{2b^\mu(k^2 - M^2 - 2\hat{k}^2) - 4k^\mu(b\cdot k)}{(k^2 - M^2)^2} \tag{28}$$

where $M^2 = 4b^2x(x-1)$. The main difference is the appearance of a term containing $\hat{k}^2$.

We proceed to compute eq.27 using IREG. The main idea is to extract the UV divergent part, in a way that the integrals only depend on internal momenta. This can be achieved employing the identity as many times as necessary

$$\frac{1}{k^2 - M^2 - \mu^2} = \frac{1}{k^2 - \mu^2}\left(1 + \frac{M^2}{k^2 - M^2 - \mu^2}.\right) \tag{29}$$

For instance,

$$I_2 = \int_k \frac{1}{k^2-\Delta} = \underbrace{\int_k \frac{1}{k^2-\mu^2}}_{\text{Quad Div}} + \underbrace{\int_k \frac{M^2}{(k^2-\mu^2)^2}}_{\text{Log Div}} + \underbrace{\int_k \frac{M^4}{(k^2-\mu^2)^2(k^2-M^2-\mu^2)}}_{\text{Finite}} \tag{30}$$

$$I_2^{\mu\nu} = \int_k \frac{k_\mu k_\nu}{(k^2-\Delta)^2} = \underbrace{\int_k \frac{k_\mu k_\nu}{(k^2-\mu^2)^2}}_{\text{Quad Div}} + \underbrace{\int_k \frac{2k_\mu k_\nu M^2}{(k^2-\mu^2)^3}}_{\text{Log Div}} + \\ + \underbrace{\int_k \frac{2k_\mu k_\nu M^4}{(k^2-\mu^2)^3(k^2-M^2-\mu^2)}}_{\text{Finite}} + \underbrace{\int_k \frac{k_\mu k_\nu M^4}{(k^2-\mu^2)^2(k^2-M^2-\mu^2)^2}}_{\text{Finite}} \tag{31}$$

In terms of the integrals above, the gap equation can be written as

$$\Pi^\mu = Tr\left[\int_k \frac{1}{\not k - \not b\gamma^5}\gamma^\mu\gamma^5\right] = -2\int_0^1 dx\,(2b^\mu I_2 - 4b_\alpha I_2^{\mu\alpha})\ . \tag{32}$$

By employing the identities

$$\int_k \frac{k_\mu k_\nu}{(k^2-\mu^2)^2} = \frac{g_{\mu\nu}}{2}\int_k \frac{1}{(k^2-\mu^2)}\,, \qquad \int_k \frac{k_\mu k_\nu}{(k^2-\mu^2)^3} = \frac{g_{\mu\nu}}{4}\int_k \frac{1}{(k^2-\mu^2)^2}\,, \tag{33}$$

it is immediate to notice that all divergences will cancel in the gap equation. Regarding the finite pieces, they will also cancel, as shown in appendix A. Therefore, if the regularization is defined in the physical dimension (in particular if $\{\gamma^\mu,\gamma^5\} = 0$ does hold), we obtain a null result for the gap equation, in disagreement with [26].

There are, however, subtleties when defining a regularization in the physical dimension. In particular, if the regularization complies with shift invariance, it can be shown [1] that $f(\{\gamma^\mu,\gamma^5\})I_{\mu\nu} \neq 0$. Here $f()$ stems for an expression containing Dirac matrices (for instance a Dirac Trace), and $I_{\mu\nu}$ is a divergent integral. The crucial point is that both indexes of the integral $I_{\mu\nu}$ are contracted when multiplying $f()$. From a more formal point of view, the fact that $\{\gamma^\mu,\gamma^5\} \neq 0$ does not hold under regularization can be incorporated by extending the dimension of the underlying Lagrangian, while γ_5 is still only defined in the physical dimension. In the framework of Dimensional Regularization, this stands for the BHMV approach [63–65]. For IREG, a similar approach can be envisaged [1].

Once we define the theory on Q4S, we can treat the propagator of the Bumblebee theory (eq. 2), which is given by

$$D_B = \frac{1}{\not k - \not b\gamma^5}. \tag{34}$$

Notice that the $\not k$, $\not b$ were extended to Q4S, while γ_5 stays in 4S. Using the properties regarding Dirac matrices in Q4S (see eqs.(15)-(17)), it can be shown that the propagator may also be expressed as

$$D_B = \frac{1}{\not k - \not b\gamma^5} = \frac{k^2+\overline{b}^2+(2k\cdot\overline{b}+[\hat{\not k},\overline{\not b}])\gamma^5}{(k-\overline{b})^2(k+\overline{b})^2-4\hat{k}^2\overline{b}^2}(\not k+\overline{\not b}\gamma^5)\ . \tag{35}$$

Thus, the gap equation is now obtained as

$$\Pi^\mu = Tr\left[\int_k \frac{1}{\not k - \not b\gamma^5}\gamma^\mu\gamma^5\right] = Tr\left[\int_k \frac{k^2+\overline{b}^2+(2k\cdot\overline{b}+[\hat{\not k},\overline{\not b}])\gamma^5}{(k-\overline{b})^2(k+\overline{b})^2-4\hat{k}^2\overline{b}^2}(\not k+\overline{\not b}\gamma^5)\gamma^\mu\gamma^5\right]\ . \tag{36}$$

After a tedious, yet straightforward calculation, we obtain

$$\Pi^\mu(b) = -2\int_0^1 dx\int_k \frac{2\overline{b}^\mu(k^2-\Delta)-4k^\mu\left(\overline{b}\cdot k\right)}{(k^2-\Delta)^2} + 8\int_0^1 dx\int_k \frac{\overline{b}^\mu\hat{k}^2}{(k^2-\Delta)^2} \tag{37}$$

which should be compared against eq.27. The only difference from the first term to the gap equation written in the physical dimension (eq.27) is that the internal momentum is now defined in Q4S. However, all the steps leading to the integrals $I_2^{\mu\nu}$ and I_2 still hold. Therefore, we arrive at the same conclusion as before, the first term is null, and the gap equation is given simply by

$$\Pi^\mu(b) = 8\int_0^1 dx \int_k \frac{\overline{b}^\mu \hat{k}^2}{(k^2-\Delta)^2} \, . \tag{38}$$

In order to evaluate this integral, we notice that $\hat{k}^2 = k^2 - \bar{k}^2$. In the framework of IREG, we have the property below (required to fulfill shift invariance in Q4S) [1]

$$\int_k k^2 f(k) \neq g_{\alpha\beta} \int_k k^\alpha k^\beta f(k) \, . \tag{39}$$

This ultimately implies that Lorentz contraction and regularization do not commute, once the internal momenta are in Q4S. On the other hand, for contracted internal momenta in 4S, we have

$$\int_k \bar{k}^2 f(k) = \bar{g}_{\alpha\beta} \int_k k^\alpha k^\beta f(k) \, . \tag{40}$$

Thus, the gap equation can be expressed as

$$\Pi^\mu(b) = 8\overline{b}^\mu \left[\int_0^1 dx \int_k \frac{k^2}{(k^2-\Delta)^2} - \bar{g}_{\alpha\beta} \int_0^1 dx \int_k \frac{k^\alpha k^\beta}{(k^2-\Delta)^2} \right] \, . \tag{41}$$

Therefore these integrals should again be decomposed in its finite and basic divergent contributions. With $N(k) = k^2 - \bar{g}_{\alpha\beta} k^\alpha k^\beta$ and recalling that $\Delta = M^2 + \mu^2$ one has

$$\begin{aligned} \int_k \frac{N(k)}{(k^2-\Delta)^2} = & \underbrace{\int_k \frac{N(k)}{(k^2-\mu^2)^2}}_{\text{Quad Div}} + \underbrace{\int_k \frac{2M^2 N(k)}{(k^2-\mu^2)^3}}_{\text{Log Div}} + \underbrace{\int_k \frac{2M^4 N(k)}{(k^2-\mu^2)^3 (k^2 - M^2 - \mu^2)}}_{\text{Finite}} \\ & + \underbrace{\int_k \frac{M^4 (N(k)}{(k^2-\mu^2)^2 (k^2 - M^2 - \mu^2)^2}}_{\text{Finite}} \, . \end{aligned} \tag{42}$$

For the finite pieces the contraction displayed in eq. 40 can be applied leading immediately to a cancellation of these terms.

Regarding the divergent pieces one uses $N(k) = (k^2 - \mu^2) + \mu^2 - \bar{g}_{\alpha\beta} k^\alpha k^\beta$ to cancel powers in the denominator and be able to use the relations 33, obtaining

$$\Pi^\mu(b)|_{div} \;=\; 8\overline{b}^\mu \int_0^1 dx \left[\int_k \frac{\mu^2}{(k^2-\mu^2)^2} - \int_k \frac{1}{(k^2-\mu^2)} \right] \tag{43}$$

$$+\; 8\overline{b}^\mu \int_0^1 dx \int_k \frac{2M^2\mu^2}{(k^2-\mu^2)^3} . \tag{44}$$

$$\tag{45}$$

After taking the limit $\mu^2 \to 0$, the first two terms are null and the last term yields the finite contribution to the gap equation,

$$\Pi^\mu(b) = 8\overline{b}^\mu \left[\frac{2i}{(4\pi)^2} \frac{b^2}{3} \right] = i \frac{\overline{b}^\mu}{3\pi^2} \, . \tag{46}$$

4.1 Cut-off regularizations

A widely used regularization procedure consists in applying a sharp $4D$ momentum cutoff Λ on the divergent integrals. Here we will discuss two types of cutoff regularization.

First we consider the standard approach, in which the tensor structures involving the loop momentum k are dealt with using symmetric integration. In the case of two Lorentz indices this corresponds to the replacement

$$\int_k \frac{k_\mu k_\nu}{(k^2-M^2)^n} \to \frac{g_{\mu\nu}}{4}\int_k \frac{k^2}{(k^2-M^2)^n} \tag{47}$$

in finite as well as divergent integrals. Here $\int_k = \int \frac{d^4k}{(2\pi)^4}$ and M^2 stands for any scalar dependence on momenta (other than the loop momentum), masses or Feynman parameters. In general this reduction turns out to induce all sorts of violations of symmetries when the substitution is done in divergent integrals.

In the case of the Bumblebee model studied here the integrals to be evaluated, after Dirac trace is taken using the γ_5 algebra in $4D$ and after Feynman parametrization, are identical to the ones of eq. 28, but now evaluated with a sharp cutoff Λ instead. With

$$\int_k^\Lambda \frac{1}{k^2-M^2} = -\frac{i}{(4\pi)^2}(\Lambda^2 - M^2\ln(\frac{\Lambda^2+M^2}{M^2})), \tag{48}$$

$$\int_k^\Lambda \frac{1}{(k^2-M^2)^2} = \frac{i}{(4\pi)^2}(\ln(\frac{\Lambda^2+M^2}{M^2}) - \frac{\Lambda^2}{\Lambda^2+M^2}), \tag{49}$$

a simple calculation using the symmetric integration (47) leads to the result

$$\Pi_\mu = \frac{ib_\mu}{4}\left(\frac{b^2}{3\pi^2} + \frac{\Lambda^2}{2\pi^2}\right). \tag{50}$$

This result differs from the ‘t Hooft Veltman scheme, eq. 46, by a cutoff dependence, which antagonizes with the effective potential requirement for a finite result. Interestingly the finite term has also a different coefficient.

Secondly we consider the gauge invariant sharp cutoff procedure of [68]. In this particular calculation the main difference to the naive procedure just outlined is that the coefficient that accompanies the reduction to the metric tensor in divergent integrals results from the requirement that the surface terms (ST) relating the difference of the following two quadratic divergences, or the following two logarithmic divergences, vanish

$$\int_k \frac{\partial}{\partial k_\nu}\frac{k^\mu}{k^2-M^2} = g^{\mu\nu}\int_k \frac{1}{k^2-M^2} - 2\int_k \frac{k^\mu k^\nu}{(k^2-M^2)^2} = 0, \tag{51}$$

$$\int_k \frac{\partial}{\partial k_\nu}\frac{k^\mu}{(k^2-M^2)^2} = g^{\mu\nu}\int_k \frac{1}{(k^2-M^2)^2} - 4\int_k^\Lambda \frac{k^\mu k^\nu}{(k^2-M^2)^3} = 0. \tag{52}$$

The vanishing of ST complies with the requirement of momentum routing invariance (the invariance under shifts in the loop momentum) and is at the core of gauge invariance (in the case of IREG the conditions are embodied in eqs. 33). In order to be able to relate to a simple momentum cutoff, the authors identify a set of rules [68] and get

$$\int_k^\Lambda \frac{k^\mu k^\nu}{(k^2-M^2)^2} = -\frac{g^{\mu\nu}}{2}\frac{i}{(4\pi)^2}\left[\Lambda^2 - M^2\ln\left(\frac{\Lambda^2+M^2}{M^2}\right)\right]. \tag{53}$$

One finally obtains that he improved (gauge invariant) sharp cutoff momentum prescription leads to an identical result as in IREG in strict four-dimensions, namely a null result for Π_μ.

5 Conclusions

The development of regularization methods that operate entirely or partially in the physical dimension aims to automate calculations at and beyond next-to-leading order (NLO). Traditional dimensional regularization schemes often increase complexity when handling objects like the γ_5 matrix, which are well-defined only in the physical dimension. This complexity has motivated the adoption of non-dimensional methods. However, it's recognized that not all operations of γ_5 algebra are applicable to divergent integrals without introducing ambiguities. In the case of Implicit Regularization (IREG), inconsistencies at NLO can generally be resolved by either symmetrizing the trace in divergent integrals [21–23] or applying the "right-most position" technique in open fermionic strings [24, 69], both maintaining strict adherence to the physical dimension. In this study, a new layer of complexity is introduced as the γ_5 matrix also appears in the fermionic propagator. Its handling in the physical dimension involves using $\{\gamma_5, \gamma_\mu\} = 0$ before trace symmetrization. We employ a version of "Bumblebee" model, where Lorentz symmetry is ostensibly violated, to test these approaches. Using IREG in the physical dimension results in the gap equation consistently evaluating to zero, indicating no Lorentz violation. This is analogous to results from a gauge-invariant sharp cutoff scheme in $4S$, unlike conventional sharp cutoff regularization that depends on symmetric integration and yields different, cutoff-dependent results. However, extending loop momenta and Clifford algebra to quasi-four-dimensional space $Q4S = 4S \oplus X$, while keeping γ_5 in $4S$, and systematically applying IREG yields finite results comparable to those from the 't Hooft and Veltman (HV) scheme. The $Q4S$ extension offers unique results similar to the BMHV scheme, proving effective when combined with IREG. Despite technically extending beyond the physical dimension, its application remains user-friendly, allowing to perform correctly all integrations in the physical dimension, by appropriately addressing potential symmetry-violating terms through the X space, maintaining otherwise all the established rules of IREG.

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A Finite integrals

Here we list the finite contributions due to the integrals I_2 and $I_2^{\mu\nu}$, eqs. 30 and 31 respectively. They still have to be integrated over the Feynman parameter x, in the final expression 32.

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Defining $A = \int_0^1 dx I_{2fin}$ and $(B+C)^{\mu\nu} = \int_0^1 dx I_{2fin}^{\mu\nu}$, one has

$$A = \int_0^1 dx \int_k \frac{M^4}{(k^2-\mu^2)^2(k^2-M^2-\mu^2)} = \int_0^1 dx M^4 \int_0^1 d\phi \frac{(1-\phi)}{R(x,\phi)} = \frac{-2b^2}{9}\left[8+3\ln\left(\frac{\mu_0}{4}\right)\right] \tag{54}$$

$$\begin{aligned} B^{\mu\nu} &= \int_0^1 dx \int_k \frac{2k_\mu k_\nu M^4}{(k^2-\mu^2)^3(k^2-M^2-\mu^2)} \\ &= -\frac{g_{\mu\nu}}{2}\int_0^1 dx M^4 \int_0^1 d\phi \frac{(1-\phi)^2}{R(x,\phi)} = -g_{\mu\nu}\frac{b^2}{18}\left[19+6\ln\left(\frac{\mu_0}{4}\right)\right] \end{aligned} \tag{55}$$

$$C^{\mu\nu} = \int_0^1 dx \int_k \frac{k_\mu k_\nu M^4}{(k^2-\mu^2)^2(k^2-M^2-\mu^2)^2} = \frac{-g_{\mu\nu}}{2}\int_0^1 dx M^4 \int_0^1 d\phi \frac{\phi(1-\phi)}{R(x,\phi)} = \frac{b^2}{6} g_{\mu\nu}. \tag{56}$$

where $R(x,\phi) = \mu^2 + M^2\phi$, $M^2 = 4b^2 x(x-1)$ in terms of the Feynman parameters x, ϕ. Here $\mu_0 = \frac{\mu^2}{b^2}$. In these results the limit $\mu^2 \to 0$ has been taken. Notice the occurrence of infrared divergences in integrals A and $B^{\mu\nu}$, they emerge in the process of separating the BDI from the strictly finite UV contributions in the original integrals I_2 and $I_2^{\mu\nu}$, which according to the algorithm 59 increases the powers of loop momentum in the denominator. As expected these cancel in the final result below 57[5].
Inserting these expressions in eq. 32 the final result for the gap equation vanishes in this case

$$\Pi^\mu = -2(2b^\mu A - 4b_\nu (B+C)^{\mu\nu}) = 0 \tag{57}$$

B Overview of Implicit Regularization

In section 3 it is explained how the γ_5 can be consistently treated in connection with IREG. The procedure below outlined assumes implicitly that whenever the γ_5 matrix is present, the operations pertaining to the space $Q4S = 4S \oplus X$ have been performed beforehand.

In this section we present the rules of IREG focusing on one loop order and in the massless limit considered in the present Bumblebee model. A complete n-loop set of rules can be found in [73, 74].

In IREG, the extraction of the UV divergent content of a Feynman amplitude is done by using algebraic identities at the integrand level. This is done in alignment with Bogoliubov's recursion formula [75–77], implying that the way the method defines an UV convergent integral respects locality, Lorentz invariance and unitarity [59]. IREG has been shown to respect abelian gauge invariance to n-loop order [21,78], as well as non-abelian and SUSY symmetries in specific examples up to two-loop order [24, 74, 79–82]. This is achieved in a constrained version of the method, in which surface terms (ST's), which are related to momentum routing of loops in Feynman diagrams, are set to zero. In the realm of applications, processes such as $h \to \gamma\gamma$ [70], $e^-e^+ \to \gamma^* \to q\bar{q}(g)$ [60], and $H \to gg(g)$ [71] were studied at NLO.

In a nutshell, the rules of IREG are summarized as follows: consider a general 1-loop Feynman amplitude where we denote by k the internal (loop) momenta, and p_i the external momenta. To this amplitude, we apply the set of rules:

1. Perform Dirac algebra in the physical dimension;

[5]The parametrization of the infrared divergences adopted is employed successfully in decay and scattering processes in connection with the Kinoshita-Lee-Nauenberg (KLN) theorem in various processes calculated with IREG [60, 61, 69–72]

2. In order to respect numerator/denominator consistency, as described in the reference [1], it is necessary to eliminate terms involving internal momenta squared in the numerator by dividing them out from the denominator. For instance,

$$\int_k \frac{k^2}{k^2(k-p)^2}\bigg|_{\text{IREG}} \neq g^{\alpha\beta}\int_k \frac{k_\alpha k_\beta}{k^2(k-p)^2}\bigg|_{\text{IREG}} \quad \text{where} \quad \int_k \equiv \int d^4k/(2\pi)^4. \tag{58}$$

3. Include a fictitious mass μ^2 in all propagators, where the limit $\mu \to 0$ must be taken at the end of the calculation. In the presence of IR divergences, a logarithm with μ^2 will remain. Assuming that we have an implicit regulator, we apply the following identity in all propagators dependent on the external momenta p_i

$$\frac{1}{(k-p_i)^2-\mu^2} = \sum_{j=0}^{n-1} \frac{(-1)^j(p_i^2-2p_i\cdot k)^j}{(k^2-\mu^2)^{j+1}} + \frac{(-1)^n(p_i^2-2p_i\cdot k)^n}{(k^2-\mu^2)^n\left[(k-p_i)^2-\mu^2\right]}. \tag{59}$$

Here n is chosen such that the UV divergent part only has propagators of the form $(k^2-\mu^2)^{-j}$.

4. Express UV divergencies in terms of Basic Divergent Integrals (BDI's) of the form

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}, \qquad I_{log}^{\nu_1\cdots\nu_{2r}}(\mu^2) \equiv \int_k \frac{k^{\nu_1}\cdots k^{\nu_{2r}}}{(k^2-\mu^2)^{r+2}}. \tag{60}$$

5. Surface terms (weighted differences of loop integrals with the same degree of divergence) should be set to zero on the grounds of momentum routing invariance in the loop of Feynman diagrams. This constrained version automatically preserves gauge invariance. For instance,

$$\int_k \frac{\partial}{\partial k_\mu}\frac{k^\nu}{(k^2-\mu^2)^2} = 4\left[\frac{g_{\mu\nu}}{4}I_{log}(\mu^2) - I_{log}^{\mu\nu}(\mu^2)\right] = 0. \tag{61}$$

Similar identities follow for BDI's with a larger number of free Lorentz indexes, as well as for quadratic divergent integrals, see eqs.(33)-(36) of Ref. [72] for more examples.

6. A renormalization group scale can be introduced by disentangling the UV/IR behavior of BDI's under the limit $\mu \to 0$. This is achieved by employing the identity

$$I_{log}(\mu^2) = I_{log}(\lambda^2) + \frac{i}{(4\pi)^2}\ln\frac{\lambda^2}{\mu^2}, \tag{62}$$

It is possible to absorb the BDI's in the renormalisation constants (without explicit evaluation) [83], and renormalisation functions can be readily computed using

$$\lambda^2\frac{\partial I_{log}(\lambda^2)}{\partial\lambda^2} = -\frac{i}{(4\pi)^2}. \tag{63}$$

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