The Stress-Flex Conjecture

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Abstract

Recently, it has been proven that a tensegrity framework that arises from coning the one-skeleton of a convex polytope is rigid. Since such frameworks are not always infinitesimally rigid, this leaves open the question as to whether they are at least prestress stable. We prove here that this holds subject to an intriguing new conjecture about coned polytope frameworks, that we call the *stress-flex conjecture*. Multiple numerical experiments suggest that this conjecture is true, and most surprisingly, seems to hold even beyond convexity and also for higher genus polytopes.

1 Introduction

It was recently proven by Winter [9] that if one takes the one-skeleton of a convex polytope in $\mathbb{R}^d$ and cones it over an interior point, the resulting framework is rigid. It is rigid as a bar-framework, where the edge lengths must stay constant, and it is even rigid as a tensegrity framework, where the "polytope edges" are allowed to get shorter and the "cone edges" are allowed to get longer.

As an example, consider the graph G with 8 vertices and 12 edges obtained from the one-skeleton of a cube in $\mathbb{R}^3$. Cone this graph over a 9th point to obtain the graph G^*. If we forget about the cube's geometry and just choose a generic placement $\hat{\mathbf{p}}$ of the 9 vertices in $\mathbb{R}^3$, the pair $(G^*, \hat{\mathbf{p}})$, considered as a bar framework, will be flexible; it will have a non-trivial infinitesimal flex and no (non-zero) equilibrium stresses. Let us now consider the (non-generic) frameworks $(G^*, \hat{\mathbf{p}})$, where we use the original geometry of the cube for the first 8 vertices and place the the cone point anywhere in $\mathbb{R}^3$. Such a bar framework will have a one dimensional equilibrium stress space and a two dimensional space of infinitesimal flexes complementary to the trivial flexes. The actual rigidity of such a bar-framework appears to be a subtle phenomenon and depends on exactly where the cone point is placed. For some placements of the cone point *outside* of the cube, we have done numerical experiments that suggest that the bar framework will be flexible. For some other such placements outside of the cube, we find that the framework is (certifiably) rigid. In this context, the result of [9] tells us that whenever the cone point is *inside* the cube, the framework must be rigid, both as a bar framework and even as a (less constrained) tensegrity framework.

In the rigidity literature [2] there is a condition called prestress stability, which is stronger than rigidity (though weaker than infinitesimal rigidity). Prestress stability is the main property used to analyze frameworks that are not infinitesimally rigid but that might be rigid. The property can be efficiently tested by solving an appropriate semi-definite program.

After establishing rigidity for interior coned polytopes, in [9, Question 5.2.] the paper asks whether they are also always prestress stable. Establishing prestress stability would place the rigidity of coned polytope frameworks more clearly within the understood rigidity landscape. There are also practical implications. When a framework is prestress stable, one can also use eigen-analysis to obtain certain bounds on the robustness of this rigidity to slight violations of the length constraints of the tensegrity and slight alterations to the original configuration (see e.g., [4, Corollary 3.3]).

In this note we explore this question, and show that it will be answered in the affirmative if one is able to prove a conjecture we pose below, that we call the *stress-flex conjecture*. This perhaps odd-looking conjecture is of intrinsic interest and appears to hold under numerous numerical experiments in three dimensions, even beyond the scope needed for prestress stability. So we are fairly hopeful that the stress-flex conjecture, and thereby prestress stability, can be verified.

In terms of techniques, [9] relies on a result of Izmestiev [5] that generalizes a three-dimensional result of Lovász [7]. This result associates a special n-by-n matrix M to a convex polytope P (with n vertices) in $\mathbb{R}^d$ that contains the origin in its interior. Among other things, M has a single negative eigenvalue. In this note, we add an extra row and column to M to obtain $(n+1)$-by-$(n+1)$ matrix Ω with a single negative eigenvalue. By construction, this matrix will be a stress matrix for the framework of the coned polytope. This stress matrix is then used in our exploration of prestress stability.

Acknowledgements

2   Preliminaries

Here we quickly review the basic rigidity definitions that we will need. The key reference is [2].

Definition 2.1. *A* configuration $\mathbf{p}$ *of n points in $\mathbb{R}^d$ is an ordered set of n points, $\mathbf{p}_i \in \mathbb{R}^d$.*

Definition 2.2. *A* tensegrity framework $(G, \mathbf{p})$ *in $\mathbb{R}^d$ is a configuration $\mathbf{p}$ of n points $\mathbb{R}^d$, and a labeled graph G on n vertices. The edges of G are labeled as "cables", "struts" and "bars". We denote by $(\bar{G}, \mathbf{p})$ the associated* bar framework*, where all of the labels in G are changed to be "bars".*

Definition 2.3. *A tensegrity framework $(G, \mathbf{p})$ is (locally)* rigid *if for every configuration $\mathbf{q}$ of n points in $\mathbb{R}^d$ that is sufficiently close to $\mathbf{p}$[1] and such that $\mathbf{q}$ is not congruent to $\mathbf{p}$, we have the property that the Euclidean lengths of the cables are not decreased, the lengths of the struts are not increased and the lengths of the bars are not changed.*

Definition 2.4. *Let $(\bar{G}, \mathbf{p})$ be a bar framework in $\mathbb{R}^d$. We say that a configuration $\mathbf{p}'$ of n vectors in $\mathbb{R}^d$ is an* infinitesimal flex *for the framework if for all edges ij, we have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$.*

An infinitesimal flex is trivial *if there is a d-by-d skew symmetric matrix S and a vector $\mathbf{t}$ such that for all i, $\mathbf{p}'_i = S\mathbf{p}_i + \mathbf{t}$. (Such a flex arises as the time derivative of a continuous isometry of $\mathbb{R}^d$.)*

If the only infinitesimal flexes for $(\bar{G}, \hat{\mathbf{p}})$ are trivial ones, then the bar framework is said to be *infinitesimally rigid* which implies that it must be rigid. The converse is not necessarily true. (There is also an infinitesimal notion in the context of tensegrity frameworks, but we will not go into this.)

Next we develop the idea of prestress stability. The rough idea is to establish rigidity by defining a certain energy function over $\mathbf{p}$ that only depends on the edge lengths, and then show that $\mathbf{p}$ is a critical point such that the energy has a positive definite Hessian. When all of the dust settles, we are left with the following definitions and theorem.

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[1] Formally, there exists an ϵ so that for every $\mathbf{q}$ within this distance to $\mathbf{p}$, the property holds.

Definition 2.5. *Let $(G, \mathbf{p})$ be a tensegrity with n vertices. An (equilibrium)* stress matrix *Ω of $(G, \mathbf{p})$ is an n-by-n symmetric matrix with the following properties:*

(i) $\Omega_{ij} = 0$ when $i \neq j$ and ij is not an edge of G.

(ii) $\Omega \mathbf{p} = 0$, where we think of $\mathbf{p}$ as an n-by-d matrix.

(iii) $\Omega \mathbf{1} = 0$, where $\mathbf{1}$ is a vector of all ones.

The stress is called strictly proper *if $\Omega_{ij} < 0$ when ij is a cable and $\Omega_{ij} > 0$ when ij is a strut.*

Definition 2.6 (See [2, Proposition 3.4.2].)**.** *A tensegrity framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is* prestress stable *if it has a strictly proper equilibrium stress matrix Ω, such that for every non-trivial infinitesimal flex $\mathbf{p}'$ of its bar-framework $(\bar{G}, \mathbf{p})$, we have*

$$\mathrm{tr}(\mathbf{p}'^t \Omega \mathbf{p}') > 0 \tag{1}$$

where we think of $\mathbf{p}'$ as an n-by-d matrix.

Finally, we recall the following.

Theorem 2.7 ([2, Proposition 3.3.2])**.** *If a tensegrity framework is prestress stable, then it is rigid.*

The converse does not always hold.

3   Izmestiev Stress

Let $P \subset \mathbb{R}^d$ be a convex polytope with n vertices, with a full affine span and with the origin in its interior. Let $\mathbf{p}$ be its vertex configuration and let G be the graph of its one-skeleton. Izmestiev [5] constructs an n-by-n symmetric matrix M, that we call the *Izmestiev matrix*, with the following properties:

(i) $M_{ij} = 0$ when $i \neq j$ and ij is not an edge of G.

(ii) $M_{ij} < 0$ when ij is an edge of G.

(iii) $M\mathbf{p} = 0$, where we think of $\mathbf{p}$ as an n-by-d matrix.

(iv) M has rank $n - d$.

(v) M has exactly one negative eigenvalue.

Our goal is to use M to construct an equilibrium stress Ω for the framework $(G^*, \hat{\mathbf{p}})$ obtained by coning the one-skeleton of P over the origin. Let $\alpha^t := -\mathbf{1}^t M$ and $b := -\sum_i \alpha_i = \mathbf{1}^t M \mathbf{1}$. The α_i are also known as the (unnormalized) Wachspress coordinates of the origin with respect to P [9], and so each is greater than 0, and b is negative. We define the $(n+1)$-by-$(n+1)$ matrix in block form as follows:

$$\Omega := \begin{pmatrix} M & \alpha \\ \alpha^t & b \end{pmatrix}$$

The added row/column are linearly dependent on M, and thus the rank of Ω is also $n - d$, and its nullity is $d + 1$. Since M has one negative eigenvalue, and Ω just has an extra 0 eigenvalue, from the eigenvalue interlacing theorem, Ω must also have exactly one negative eigenvalue.

Let $\mathbf{p}_{n+1}$ be placed at the origin and let $\hat{\mathbf{p}} := [\mathbf{p}, \mathbf{p}_{n+1}]$ be the configuration of $n+1$ points in $\mathbb{R}^d$. Then, since $\mathbf{p}_{n+1} = 0$ we have $\Omega\hat{\mathbf{p}} = 0$. By our definition of α and b, we have $\Omega\mathbf{1} = 0$. Let us label the edges of G^* from the polytope as cables and the coned edges as struts. We see that Ω is a strictly proper equilibrium stress for $(G^*, \hat{\mathbf{p}})$. We call this an *Izmestiev stress*.

Finally we note that the space of stresses for a framework does not change under translation in $\mathbb{R}^d$. Thus we can start with P as any convex polytope with n vertices and a full affine span in $\mathbb{R}^d$ and with $\mathbf{p}_{n+1}$ any point in the interior of P. We can create its associated coned tensegrity, and it must have an Izmestiev stress.

4   One Negative Eigenvalue

Prestress stability, and in particular Equation (1), is based on positivity of a quadratic energy. Our stress matrix Ω has one negative eigenvalue that we need to work around. Our main tool for doing this is the following lemma.

Lemma 4.1. *Let Ω be an $(n+1)$-by-$(n+1)$ real symmetric matrix. Assume the following:*

(i) Ω has exactly one negative eigenvalue.

(ii) $\Omega_{n+1,n+1} < 0$.

If $\hat{\mathbf{x}} \in \mathbb{R}^{n+1}$ has the property that the $(n+1)$st entry of $\Omega\hat{\mathbf{x}}$ equals 0, then $\hat{\mathbf{x}}^t\Omega\hat{\mathbf{x}} \geq 0$.

Proof. Let $\mathbf{e}$ be the $(n+1)$st indicator vector. By assumption $\mathbf{e}^t\Omega\mathbf{e} < 0$, so the vector, $\mathbf{e}$ is in the negative cone of Ω. Meanwhile, since the last entry of $\Omega\hat{\mathbf{x}}$ equals 0, we have $\mathbf{e}^t\Omega\hat{\mathbf{x}} = 0$, i.e., $\hat{\mathbf{x}}$ is Ω-orthogonal to $\mathbf{e}$. Together, since Ω has only a single negative eigenvalue, this implies $\hat{\mathbf{x}}^t\Omega\hat{\mathbf{x}} \geq 0$.   □

Similar ideas to Lemma 4.1 are used in [6].

5   The Stress-Flex Conjecture

The Izmestiev stress Ω satisfies the assumptions of Lemma 4.1. If we can show that each of the d spatial coordinates of $\hat{\mathbf{p}}'$ have the properties of the vector $\hat{\mathbf{x}}$ in that lemma, then we can apply this to establish prestress stability. The following conjectures just that:

Conjecture 5.1 (The *stress-flex conjecture*)**.** *Let P be a convex polytope with n vertices with a full dimensional span in $\mathbb{R}^d$. Let $(G^*, \hat{\mathbf{p}})$ be the tensegrity framework on $n+1$ vertices defined by putting cables on the one-skeleton of P and struts from the vertices of P to a selected cone point in the interior of P. Let Ω be its Izemstiev stress. Let $\hat{\mathbf{p}}'$ be an infinitesimal flex of its bar framework. Then the last row of $\Omega\hat{\mathbf{p}}'$ equals zero.*

When the last row of $\Omega\hat{\mathbf{p}}'$ equals zero, we say that the *stress-flex condition* is satisfied. Note that this condition is always satisfied for trivial infinitesimal flexes, since in this case $\hat{\mathbf{p}}'$ is an affine image of $\hat{\mathbf{p}}$ and so $\Omega\hat{\mathbf{p}}' = 0$.

We have done many numerical experiments in 3D whose results support this conjecture. These results included (simple) convex polytopes defined by the intersection of random planes as well a number of polytopes from the wonderful library at [8].

In fact, this appears to be an even more robust property than stated by the conjecture. First of all, the stress-flex condition experimentally holds even when we place the cone point outside of the polytope. We have also looked at polytopes where the equilibrium stress space of $(G^*, \hat{\mathbf{p}})$

has dimension larger than one. For example, when P is the cuboctahedron, the coned framework has a 4 dimensional stress space and it has a non-trival infinitesimal flex. When P is the rhombic dodecahedron, the coned framework has a 2 dimensional stress space and it has a 3 dimensional space of flexes complementary to the trivial flexes. In these cases, the stress-flex condition holds for any of the stresses, not just the Izmestiev stress.

We have also looked at non-convex polyhedra and non-embedded polyhedra where there are stresses and non-trivial infinitesimal flexes, and the stress-flex condition continues to hold. We have looked at higher genus polyhedra, such as Szilassi Polyhedron. We have even looked at non-orientable polyhedra such as the cubohemioctahedron. In all of these cases, the stress-flex condition still holds. Moreover, the stress-flex condition holds for the hypercube in $\mathbb{R}^4$, coned anywhere.

In light of these experiments, we propose the following *strong stress-flex conjecture*.

Conjecture 5.2. *Let P be any "polytope" with a full dimensional affine span in $\mathbb{R}^d$. Let $\mathbf{p}_{n+1}$ be placed anywhere in $\mathbb{R}^d$. Let Ω be any stress for the bar framework defined using one-skeleton of P coned over $\mathbf{p}_{n+1}$. Let $\hat{\mathbf{p}}'$ be any infinitesimal flex of the bar framework. Then the the stress-flex condition holds.*

In this conjecture, we have chosen to not exactly pin-down what we mean by "polytope", but it seems that we need nothing more than a closed surface with flat faces.

We note that if one starts with a convex polytope coned from inside, and then slides the vertices along their lines to the cone vertex, even though the face-flatness is not maintained, the dimension of the stress space and infinitesimal flex space is maintained. It also can be shown that the property of prestress stability is maintained under such sliding. But we observe that for generic sliding the stress-flex condition is not maintained. So the face-flatness appears to be crucial for the stress-flex condition.

5.1 Another Point of View

It can also be shown that the strong stress-flex conjecture is equivalent to the following: Start with any polytope P with full affine span in $\mathbb{R}^d$. Let $(G, \mathbf{p})$ be the bar framework of its one-skeleton. Let $(G, \mathbf{q})$ be the orthogonal projection to $\mathbb{R}^{d-1}$ by forgetting the last coordinate. Let Ψ be any equilibrium stress matrix for $(G, \mathbf{q})$. Let $\mathbf{q}'$ be any infinitesimal flex of $(G, \mathbf{q})$ in $\mathbb{R}^{d-1}$. Then the following two conditions must hold:

$$(\mathbf{p}^d)^t \Psi \mathbf{q}' = 0$$

where $\mathbf{p}^d$ is the dth spatial coordinate of $\mathbf{p}$ and we think of $\mathbf{q}'$ as an n-by-$(d-1)$ matrix and

$$(\mathbf{p}^d)^t \Psi (\mathbf{q}' \cdot \mathbf{q}) = 0$$

where $(\mathbf{q}' \cdot \mathbf{q})_i := \mathbf{q}'_i \cdot \mathbf{q}_i$.

6 Prestress Stability of a Convex Polytope, Coned From Inside

Finally we tie things together, and prove that under the (weak) stress-flex conjecture, we must have prestress stability. There is not much to do here, other than use the properties of the Izmestiev stress and Lemma 4.1. This will tell us that any infinitesimal flex must be an affine flex. All that is left is to string together some previous results that let us conclude that this affine flex must be a trivial flex.

Theorem 6.1. *Let P be a convex polytope with n vertices with a full dimensional span in $\mathbb{R}^d$. Let $(G^*, \hat{\mathbf{p}})$ be the tensegrity framework on $n+1$ vertices defined by putting cables on the one-skeleton of P and struts from the vertices of P to a selected cone point in the interior of P.*

Assume the (weak) stress-flex conjecture (Conjecture 5.1) is true, then the resulting framework is prestress stable.

Proof. From Section 3, $(G^*, \hat{\mathbf{p}})$ must have an Izmestiev stress. This stress has exactly one negative eigenvalue and its lower-right entry is negative. Under the stress-flex conjecture, for any infinitesimal flex $\hat{\mathbf{p}}'$ of of $(\bar{G}, \hat{\mathbf{p}})$, we have the last row of $\Omega \hat{\mathbf{p}}'$ is equal to zero. Using Lemma 4.1 we must have $\text{tr}(\mathbf{p}'^t \Omega \mathbf{p}') \geq 0$.

Next we show that if $\text{tr}(\mathbf{p}'^t \Omega \mathbf{p}') = 0$, then $\mathbf{p}'$ must be a trivial flex. From Lemma 4.1, if $\text{tr}(\mathbf{p}'^t \Omega \mathbf{p}') = 0$ then we must have (each of the d spatial coordinates of) $\mathbf{p}'$ in the kernel of Ω. From the structure of the kernel of Ω, this makes $\mathbf{p}'$ an "affine flex" of $\mathbf{p}$, i.e. for all i, $\mathbf{p}'_i = A\mathbf{p}_i + \mathbf{t}$ for some matrix A and vector $\mathbf{t}$.

Suppose this affine flex is not trivial. From [1, Lemmas A.3, A.7], this implies that $(\bar{G}^*, \hat{\mathbf{p}})$ has its "edges on a conic at infinity" (see that paper for definitions). Since this is a coned framework, from [3, Lemma 4.10] the framework must be "ruled" (see that paper for definitions). But the one skeleton of a convex polytope cannot be ruled ([3, Prop 3.4].)

Thus for a non-trivial flex $\hat{\mathbf{p}}$, the inequality of Equation (1) must hold, making $(G^*, \hat{\mathbf{p}})$ prestress stable. □

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