The Impact of Loss Estimation on Gibbs Measures

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Abstract

In recent years, the shortcomings of Bayes posteriors as inferential devices has received increased attention. A popular strategy for fixing them has been to instead target a Gibbs measure based on losses that connect a parameter of interest to observed data. While existing theory for such inference procedures relies on these losses to be analytically available, in many situations these losses must be stochastically estimated using pseudo-observations. The current paper fills this research gap, and derives the first asymptotic theory for Gibbs measures based on estimated losses. Our findings reveal that the number of pseudo-observations required to accurately approximate the exact Gibbs measure depends on the rates at which the bias and variance of the estimated loss converge to zero. These results are particularly consequential for the emerging field of generalised Bayesian inference, for estimated intractable likelihoods, and for biased pseudo-marginal approaches. We apply our results to three Gibbs measures that have been proposed to deal with intractable likelihoods and model misspecification by Turner and Sederberg (2014), Ghosh and Basu (2016), and Chérief-Abdellatif and Alquier (2020).

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1 Introduction

Bayesian inference has long been the gold standard for principled statistical methodology reliant on the inclusion of prior knowledge. Beyond that, the Bayes posterior provides a natural way for quantifying uncertainty about the data-generating process under study (e.g. Robert et al., 2007; Bernardo and Smith, 2009). However, orthodox Bayesianism also suffers from various pathologies in modern data-rich environments, complex models, and machine learning tasks (see Berger et al., 1994; Bissiri et al., 2016; Knoblauch et al., 2022).

For example, under model misspecification, the Bayes posterior concentrates towards sub-optimal parameter values and provides miscalibrated parameter inference (Draper, 1995; Bunke and Milhaud, 1998; Walker, 2013). Worse still, it produces brittle inferences in the presence of outliers and heterogeneous observations (Owhadi et al., 2015; Jewson et al., 2018). An entirely different disadvantage emerges in models with intractable likelihood functions (Lyne et al., 2015; Dellaporta et al., 2022). For example, since traditional methods for Bayesian computation rely on analytically available likelihood functions, they cannot be deployed for simulation-based models. To circumnavigate this, methods like approximate Bayesian computation (ABC) obtain posterior distributions by approximating the likelihood function itself (Beaumont, 2010; Fearnhead and Prangle, 2012; Frazier et al., 2018, 2020). A similar problem arises for likelihoods that are only known up to an intractable normalisation constant: once again, standard Markov chain Monte Carlo (MCMC) methods cannot be deployed without first approximating the likelihood function itself (Turner and Sederberg, 2014; Papamakarios et al., 2019; Durkan et al., 2020).

While model misspecification and intractable likelihoods are unrelated problems, various solutions proposed to both problems take the same form: directly changing the inference

target. More specifically, rather than focusing on the Bayes posterior, a host of contemporary proposals instead base parameter inferences on alternative Gibbs measures (Hooker and Vidyashankar, 2014; Knoblauch et al., 2018; Matsubara et al., 2022, 2023; Altamirano et al., 2023a,b). While algorithmically straightforward, this leads to a new and hitherto unaddressed theoretical challenge. Specifically, many of the desired Gibbs measures must be constructed from losses that cannot be computed without stochastic estimation based on simulations from a statistical model for the data. Notable examples include β -divergence losses (see Section 2.3.1) and the maximum mean discrepancy (see Section 2.3.2), both of which are popular in the literature on generalised Bayes (Ghosh and Basu, 2016; Knoblauch et al., 2018; Chérief-Abdellatif and Alquier, 2020; Pacchiardi and Dutta, 2021). Much like in ABC, these losses lead to a discrepancy between the nominal target measure and the de-facto target from which an MCMC algorithm produces samples.

This poses several practical problems: firstly, it is completely unclear what level of computational effort is required to make the de-facto target sufficiently close to the nominally targeted Gibbs measure. In fact, our lack of understanding about the approximation quality raises the question under which conditions this new class of algorithms and Gibbs measures constitutes a preferable alternative to Bayes posteriors. The existing literature provides little beyond heuristic arguments. The current paper addresses this research gap, and provides the first suite of theoretical guarantees for algorithms that sample from Gibbs measures based on estimated losses.

2 Motivation and Setup

Throughout, we consider the finite-dimensional data set $y_{1:n} = (y_1, \ldots, y_n)^{\top}$ with $y_i \in \mathcal{Y}$ for all $i = 1, \ldots, n$, and where we assume $y_{1:n} \sim P_0$. While the true data-generating measure P_0 will generally depend on n, we suppress this for notational convenience. Using our data,

we model P_0 through the class of parametric models

$$\{P_{\theta}: \theta \in \Theta\} \subset \mathcal{P}(\mathcal{Y}^n) \text{ for } \Theta \subseteq \mathbb{R}^d,$$

where we write p_{θ} to denote the density of P_{θ} , and once again suppress dependence of P_{θ} and p_{θ} on n. In a Bayesian approach to statistical modelling, we express our prior beliefs about the true state of the world probabilistically through the prior measure $\Pi \in \mathcal{P}(\Theta)$ with density π . Given p_{θ} and π , the Bayesian approach for quantifying uncertainty about the value of θ that best describes the observed data proceeds via Bayes' Rule, which yields the Bayes posterior measure with density

$$\pi(\theta \mid y_{1:n}) = \frac{p_{\theta}(y_{1:n})\pi(\theta)}{\int_{\Theta} p_{\theta}(y_{1:n})\pi(\theta)d\theta}.$$

Conceptually, the Bayes posterior updates our prior beliefs about the world with data, and represents an internally coherent belief update. Crucially, our beliefs about the world are expressed *exclusively* in terms of θ —a direct consequence of the fact that the Bayesian approach forces us to explicitly posit a model for how the data $y_{1:n}$ was generated.

2.1 New Posterior Belief Distributions

It is easy to generalise the Bayes posterior: letting $\omega > 0$ be a scalar and $\mathsf{L}_n : \Theta \times \mathcal{Y}^n \to \mathbb{R}$ a loss function expressing the information about the parameter θ contained in $y_{1:n}$, then so long as $\int_{\Theta} \exp\{-\omega \cdot \mathsf{L}_n(\theta, y_{1:n})\}\pi(\theta) d\theta < \infty$, we can define the Gibbs measure

$$\pi(\theta \mid \mathsf{L}_n) := \frac{\exp\{-\omega \cdot \mathsf{L}_n(\theta, y_{1:n})\}\pi(\theta)}{\int_{\Theta} \exp\{-\omega \cdot \mathsf{L}_n(\theta, y_{1:n})\}\pi(\theta)d\theta},\tag{1}$$

which recovers the Bayes posterior $\pi(\theta \mid y_{1:n})$ when $\mathsf{L}_n(\theta, y_{1:n}) = -\log p_\theta(y_{1:n})$ and $\omega = 1$. Gibbs measures make appearances in many areas of Bayesian methodology, including ABC (Beaumont, 2010; Frazier et al., 2018; Frazier, 2020), Provably Approximately Correct (PAC) Bayes (Germain et al., 2016; Alquier, 2024), and generalised Bayesian methods (Bissiri et al., 2016; Knoblauch et al., 2022).

2.1.1 Approximate Bayesian Computation (ABC)

The motivation of ABC algorithms is the approximation of Bayes posteriors for simulator models with intractable likelihoods, where p_{θ} is not available analytically, but can be sampled with low computational overhead using simulated data $U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ for each $\theta \in \Theta$. See Sisson et al. (2019) for a handbook of these methods, and Martin et al. (2024) for a recent review. Given a vector of summary statistics $S(y_{1:n})$, a distance d, and a positive tolerance ε , ABC methods can be interpreted as sampling algorithms targeting a Gibbs measure given by (1) with $\mathsf{L}_n^{\varepsilon}(\theta,y_{1:n}) = -\log \mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \left[\mathbb{1}_{\{d\{S(y_{1:n}),S(U_{1:m})\} \leq \varepsilon\}} \cdot p_{\theta}(U_{1:m}) \right]$ and $\omega = 1$. This measure is used as an approximation to the Bayes posterior $\pi(\theta \mid y_{1:n})$ with its accuracy depending on the choice of S, d, and ε . Exemplifying the more general problem under study in the current paper, the expectation in $\mathsf{L}_n^{\varepsilon}$ is unavailable. ABC approximates this expectation by drawing pseudo-observations $u_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ to construct an estimator for $\mathsf{L}_n^{\varepsilon}$.

An alternative to ABC in the setting of simulation-based inference is Bayesian synthetic likelihood (Wood, 2010; Price et al., 2018; Frazier et al., 2022). The corresponding target posterior is a Gibbs measure as in (1) with loss

$$\mathsf{L}_n(\theta, y_{1:n}) = -\frac{1}{2} \ln \det \{ \Sigma(\theta) \} - \frac{1}{2} \{ S_n(y_{1:n}) - b(\theta) \}^{\top} \Sigma(\theta)^{-1} \{ S_n(y_{1:n}) - b(\theta) \}.$$

Here, $\Sigma(\theta) = \operatorname{Var}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \{S(U_{1:m})\}$ and $b(\theta) = \mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \{S(U_{1:m})\}$. As with ABC, the expectations defining L_n are generally intractable, so that Bayesian synthetic likelihood approximates them with m samples from P_{θ} .

2.1.2 Provably Approximately Correct (PAC) Bayes

Provably approximately correct (PAC) Bayesian results have direct links with Bayesian inference, but have distinct origins and motivations (see e.g. Shawe-Taylor and Williamson, 1997; Germain et al., 2016). In particular, despite their name and the re-appearance of

a 'prior' measure Π , they do *not* rely on Bayes' Theorem. Rather, they are justified through generalisation error bounds. To illustrate this, we adapt Theorem 2.1 in Alquier (2024) which itself is a version of the bound in Catoni (2003). Taking $\mathcal{P}(\Theta)$ as the set of probability measures on Θ , assuming that $y_{1:n}$ are sampled independently from P_0 , and supposing $0 \leq \mathsf{L}_n(\theta, y_{1:n}) \leq C$ for all $y_{1:n} \in \mathcal{Y}^n$ and $\theta \in \Theta$, it holds with probability at least $1 - \varepsilon$ that for all $Q \in \mathcal{P}(\Theta)$,

$$\int \mathbb{E}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} \left[\omega \cdot \mathsf{L}_n(\theta, Y_{1:n}) \right] Q(d\theta) \leq \int \omega \cdot \mathsf{L}_n(\theta, Y_{1:n}) Q(d\theta) + \mathrm{KL}(Q \| \Pi) + \frac{\omega^2 C^2}{8n} - \log \varepsilon,$$

where KL denotes the Kullback-Leibler divergence. Minimizing the right hand side of this bound with respect to $Q \in \mathcal{P}(\Theta)$, one recovers (1). A proof can be found in Theorem 1 of Knoblauch et al. (2022), which follows by the arguments of Donsker and Varadhan (1975). For other significant PAC-Bayes bounds that recover Gibbs measures, see also Catoni (2007), Zhang (2006), or Germain et al. (2016).

2.1.3 Generalised Bayes

To provide reliable and well-calibrated uncertainty quantification, the Bayes posterior requires that the model is well-specified. If this condition is violated, $\pi(\theta \mid y_{1:n})$ lacks robustness, and produces miscalibrated representations of parameter uncertainty. A recent line of work has therefore argued for *generalised* Bayesian beliefs that are the outcome of an optimisation problem over $\mathcal{P}(\Theta)$ (see e.g. Knoblauch et al., 2022), recovering Gibbs measures as

$$\pi(\theta \mid \mathsf{L}_n) = \underset{q \in \mathcal{P}(\Theta)}{\operatorname{argmin}} \left\{ \int_{\Theta} \omega \cdot \mathsf{L}_n(\theta, y_{1:n}) \mathrm{d}q(\theta) + \mathrm{KL}(q||\pi) \right\}. \tag{2}$$

When L_n is additive over the data so that $L_n(\theta, y_{1:n}) = \sum_{i=1}^n \ell_i(\theta, y_i)$ for some sequence of losses $\ell_i : \Theta \times \mathcal{Y} \to \mathbb{R}$, such Gibbs measures coincide with general Bayesian updating as advocated for by Bissiri et al. (2016). The implication of (2) is significant: as the Bayes

posterior is the special case associated to $L_n(\theta) = -\log p_{\theta}(y_{1:n})$, some of the drawbacks of relying on Bayes' Theorem can be addressed by choosing different losses L_n .

In particular, noting the brittleness of the standard Bayes posterior $\pi(\theta \mid y_{1:n})$ under model misspecification, various generalised posteriors are based on losses L_n that guarantee robustness. Most of these are built on discrepancy measures between P_{θ} and P_0 such as scoring rules (Jewson et al., 2018; Pacchiardi and Dutta, 2021), α -, β -, and γ -divergences (Hooker and Vidyashankar, 2014; Ghosh and Basu, 2016; Nakagawa and Hashimoto, 2020; Fujisawa et al., 2021), or the maximum mean discrepancy (Chérief-Abdellatif and Alquier, 2020; Dellaporta et al., 2022). More recent work has also combined this desire for robustness with improved computational properties: for example, Stein discrepancies (Matsubara et al., 2022, 2023) and generalisations of score matching (Altamirano et al., 2023a,b) provide conjugate posteriors for exponential family likelihoods with intractable normalisers.

2.2 Computational Challenges

While many methodologically interesting Gibbs measures are straightforward to compute, several expose a computational hurdle whose impact on asymptotic behaviour has consistently been left unaddressed in existing contributions (see e.g. Hooker and Vidyashankar, 2014; Ghosh and Basu, 2016; Chérief-Abdellatif and Alquier, 2020; Miller, 2021; Pacchiardi and Dutta, 2021). In particular, many methods in the literature rely on losses L_n that are intractable, and have to be estimated through a proxy $L_{m,n}$ that relies on m samples $u_{1:m}$ with $u_j \sim P_\theta$ for j = 1, 2, ... m. Yet, prior theory operates directly on the idealised target posterior $\pi(\theta \mid L_n)$ —rather than the approximate version based on the estimated loss $L_{m,n}$ —several examples of which are given in the following section. Not accounting for the randomness in $L_{m,n}$ means that no existing results tell us how to choose m in practice, and that existing asymptotic results often do not apply to the Gibbs measure we compute in practice.

In the current paper, we address this issue head-on, and provide comprehensive results on asymptotic behaviours in both m and n. The results are the first of their kind, and essential in understanding exactly how large m should be to ensure that $L_{m,n}$ leads to reliable approximations of $\pi(\theta \mid L_n)$. While our findings are most immediately relevant for the emerging field of generalised Bayesian posteriors, they also inform a host of other settings. For example, they provide the first set of generic asymptotic results for MCMC-based inference based on biased likelihood estimates, extending existing results on the behavior of pseudo-marginal methods to the case where the likelihood estimator is biased.

2.3 Applicability of Results

To provide a clear exposition to the research gap our results are closing, we provide three examples of posteriors that suffer from computationally intractable losses. The first and second are generalised Bayes posteriors based on the β -divergence and the maximum mean discrepancy (MMD), respectively. These two generalised posteriors are popular in the literature, and have not only been proposed for general methodology (Ghosh and Basu, 2016; Jewson et al., 2018; Chérief-Abdellatif and Alquier, 2020; Jewson et al., 2023), but have also found specific use for filtering problems (Boustati et al., 2020), changepoint detection (Knoblauch et al., 2018), differential privacy (Jewson et al., 2023), Bayesian deep learning (Futami et al., 2018; Knoblauch, 2019), and simulation-based inference (Pacchiardi and Dutta, 2021; Dellaporta et al., 2022). While asymptotic results were derived by Pacchiardi and Dutta (2021) and Ghosh and Basu (2016), they are based on the exact loss L_n , rather than the posterior calculated from the estimate $L_{m,n}$ that must be used in practice. The third is a Bayes posterior whose intractable likelihood model is approximated using kernel density estimation (KDE) introduced by Turner and Sederberg (2014), and which has similarities with more contemporary—albeit much harder to analyse—black box density estimation methods (see Lueckmann et al., 2017; Papamakarios et al., 2019; Greenberg et al., 2019). The use of KDE for intractable models can deliver more informative inferences than classical ABC based on summary statistics (see Drovandi and Frazier, 2022), but the behaviour of the resulting posteriors has not been studied previously.

2.3.1 β -divergence

Suggested for parameter estimation by Basu et al. (1998), the β -divergence is also often called the *density power divergence*. As the second name suggests, it is a discrepancy that relies on the existence of densities, and owes its name to the fact that these densities are raised to a power $\beta > 0$. For a given β , this divergence is written as

$$d_{\beta}(p_0, p_{\theta}) = \int \left\{ p_{\theta}^{1+\beta}(y) - \left(1 + \frac{1}{\beta}\right) p_0(y) p_{\theta}(y)^{\beta} + \frac{1}{\beta} p_0^{1+\beta}(y) \right\} dy.$$
 (3)

If we have access to a sample $y_{1:n}$ of n data points for which $y_i \stackrel{\text{iid}}{\sim} p_0$, and if we wish to minimise this expression over θ , we can ignore the last term and obtain the loss function

$$\mathsf{L}_n^{\beta}(\theta, y_{1:n}) = n \int p_{\theta}^{1+\beta}(u) \mathrm{d}u - \left(1 + \frac{1}{\beta}\right) \sum_{i=1}^n p_{\theta}(y_i)^{\beta}.$$

Minimising this loss yields an estimator for θ that converges to the true data-generating parameter as $n \to \infty$ if the model is well-specified, and which exhibits robustness under under model misspecification (Basu et al., 1998). Here, β determines the degree of robustness: the loss becomes more robust as $\beta \to \infty$, but less robust as $\beta \to 0$. As $\beta \to 0$, its limit eventually recovers the non-robust negative log likelihood $-\log p_{\theta}(y_{1:n})$. This makes L_n^{β} ideally suited for the computation of robust generalised Bayes posteriors, as it recovers the standard Bayes posterior for $\beta \to 0$. However, outside a small subset of exponential families, the integral $\int p_{\theta}^{1+\beta}(u) \mathrm{d}u$ needs to be approximated. As a result, one generally has to draw samples $u_j \stackrel{\mathrm{iid}}{\sim} p_{\theta}$ for $j = 1, 2, \ldots m$, and then use the approximation

$$\mathsf{L}_{m,n}^{\beta}(\theta, y_{1:n}) = \frac{n}{m} \sum_{j=1}^{m} p_{\theta}^{\beta}(u_j) - \left(1 + \frac{1}{\beta}\right) \sum_{i=1}^{n} p_{\theta}(y_i)^{\beta}.$$

This means that the actual target posterior depends on $\mathsf{L}_{m,n}^{\beta}$ rather than L_{n}^{β} . This is not taken into account in existing theory, and prior work has opted to study the idealised target $\pi(\theta \mid \mathsf{L}_{n}^{\beta})$ instead (Ghosh and Basu, 2016; Miller, 2021).

2.3.2 Maximum Mean Discrepancy

The maximum mean discrepancy (MMD) is a popular distance for testing and parameter estimation (see e.g. Gretton et al., 2012; Alquier et al., 2023; Alquier and Gerber, 2024). There are generally two reasons for using the MMD as a loss in the context of generalised Bayes posteriors. The first is robustness, which can be formally shown to hold whenever the kernel $k: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$ it is based on is bounded (see e.g. Briol et al., 2019; Chérief-Abdellatif and Alquier, 2020). The second relates to computation: unlike most other losses, the MMD does not require an analytically tractable likelihood function. Instead, all we need is access to a generative model for P_{θ} from which we can draw samples, which makes the MMD particularly useful in simulation-based inference (see Park et al., 2016; Pacchiardi and Dutta, 2021; Dellaporta et al., 2022).

For a positive-definite and symmetric kernel function $k: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$ associated to the reproducing kernel Hilbert space \mathcal{H} with norm $\|\cdot\|_{\mathcal{H}}$, the corresponding MMD between the empirical measure $P_n = \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$ and P_{θ} is given by

$$\mathrm{MMD}_{k}(P_{n}, P_{\theta}) = \sup_{f \in \mathcal{H}: ||f||_{\mathcal{H}} \le 1} \left| \mathbb{E}_{U \sim P_{\theta}} \left[f(U) \right] - \frac{1}{n} \sum_{i=1}^{n} f(y_{i}) \right|.$$

This is an integral probability metric (see Müller, 1997; Sriperumbudur et al., 2012), and appears intractable at first glance. Fortunately, one can solve the underlying optimisation problem analytically. After squaring, this becomes

$$\mathrm{MMD}_{k}(P_{n}, P_{\theta})^{2} = \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{i'=1}^{n} k(y_{i}, y_{i'}) - 2 \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{U \sim P_{\theta}} \left[k(U, y_{i}) \right] + \mathbb{E}_{U \sim P_{\theta}, U' \sim P_{\theta}} \left[k(U, U') \right].$$

As the first term does not depend on θ , a natural loss to construct from the above is

$$\mathsf{L}_{n}^{k}(\theta, y_{1:n}) = -2\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{U \sim P_{\theta}} \left[k(U, y_{i}) \right] + \mathbb{E}_{U \sim P_{\theta}, U' \sim P_{\theta}} \left[k(U, U') \right],$$

which is also often referred to as the *kernel score* (e.g. Székely and Rizzo, 2005; Gneiting and Raftery, 2007), and which relies on a number of intractable expectations. Just as for the expectations featuring in the β -divergence, one can approximate them by taking $u_j \stackrel{\text{iid}}{\sim} P_{\theta}$ for j = 1, 2, ...m, which results in the estimator

$$\mathsf{L}_{m,n}^{k}(\theta, y_{1:n}) = -2\frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} k(u_j, y_i) + \frac{1}{m^2} \sum_{j=1}^{m} \sum_{j'=1}^{m} k(u_j, u_{j'}). \tag{4}$$

Similarly to the case of generalised Bayes posteriors based on β -divergences, MMD-based posteriors are computed using $\mathsf{L}_{m,n}^k$, while existing asymptotic theory is applicable only to the idealised version $\pi(\theta \mid \mathsf{L}_n^k)$ (Chérief-Abdellatif and Alquier, 2020; Pacchiardi and Dutta, 2021; Miller, 2021).

2.3.3 Approximating Intractable Likelihoods via Kernel Density Estimation

Our third and last example relates to a Bayes posterior whose likelihood function p_{θ} is not analytically available, but where for each θ , it is easy to generate samples from the corresponding model P_{θ} . In such situations, working with the Bayes posterior $\pi(\theta \mid y_{1:n})$ demands estimation of the normalised likelihood. One way of doing so is via kernel density estimation (KDE) with a suitable isotropic kernel function $k: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$, and bandwidth $h \geq 0$. Given simulated data $u_j \stackrel{\text{iid}}{\sim} P_{\theta}$ for $j = 1, 2, \dots m$, we can estimate the intractable likelihood using the density estimate:

$$\hat{p}_{\theta}(y) = \frac{1}{mh} \sum_{j=1}^{m} k_h (y - u_j), \quad k_h(x) = k(x/h),$$

where we have followed conventional practice and overloaded notation as k(y, y') = k(y - y'), for $y, y' \in \mathcal{Y}$. From the above, we estimate the negative log likelihood $\mathsf{L}_n^h(\theta, y_{1:n}) = -\sum_{i=1}^n \log p_\theta(y_i)$ as

$$\mathsf{L}_{m,n}^{h}(\theta, y_{1:n}) = -\sum_{i=1}^{n} \log \hat{p}_{\theta}(y_{i}).$$

Turner and Sederberg (2014) and Drovandi and Frazier (2022) use $\mathsf{L}^h_{m,n}(\theta,y_{1:n})$ within MCMC to conduct inference on θ even thought the actual density is intractable.¹

More recently, alternative density estimation methods have been proposed that replace the KDE-based estimator with approaches that instead estimate the likelihood using neural networks (Lueckmann et al., 2017; Papamakarios et al., 2019; Greenberg et al., 2019). While the reliance on black box models makes it impossible to explicitly verify our theory in this setting, the results we derive for the KDE-based estimator should be understood as the best case scenario we could hope for with these sequential neural likelihood approaches.

2.3.4 Wider Applicability

The three estimators presented above are constructed using three distinct principles (averages, U-statistics, and kernel density estimators), and our theory applies to all of them. While we will confine ourselves to these three examples throughout the remainder of the paper, it is important to note our setup applies to a much larger class of posteriors which rely on approximating some idealised loss L_n with an estimator $L_{m,n}$. This includes posteriors based on Hellinger distances (Hooker and Vidyashankar, 2014) and γ -divergences (Nakagawa and Hashimoto, 2020), sliced distances (Kolouri et al., 2019; Gong et al., 2020), as well as the so-called ρ -Bayes approach (Baraud and Birgé, 2020). In the context of ABC, it also applies to the Wasserstein-ABC posterior (Bernton et al., 2019a), approaches based on general integral probability metrics (Legramanti et al., 2022), as well as other distance-based variants of ABC (Park et al., 2016; Fujisawa et al., 2021; Drovandi and Frazier, 2022). For all of these cases, previous work has exclusively studied theoretical properties of the idealised posterior $\pi(\theta \mid L_n)$, not to the computationally feasible version

¹The above estimator differs slightly from the one suggested by Turner and Sederberg (2014), which is based on simulating $m = M \cdot n$ data points for some $M \in \mathbb{N}$. The two approaches are equivalent when the simulated data are iid.

based on $L_{m,n}$. In addition, the setting also includes posteriors computed using MCMC based on biased likelihood estimates (Andrieu and Roberts, 2009).

2.4 Motivating Example

Before providing our theoretical results, we motivate their necessity using the losses described in the previous section and demonstrate the practical ramifications of using $L_{m,n}$ instead of L_n in posterior computations. To this end, we simulate n = 100 observations from a Student's t location model with fixed unit variance and 10 degrees of freedom. Inference is performed only on the location parameter, which is set to $\theta = 1$. We then construct generalised Bayes posteriors based on the losses introduced in Section 2.3 for different m, and compare their accuracy. The results are collected in Figure 1, and demonstrate that even large values of m can lead to poor approximations.

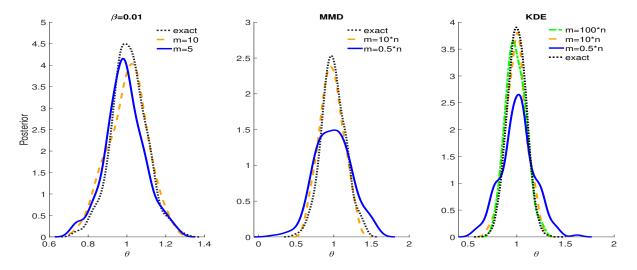


Figure 1: Posteriors for a location model across three different losses covered by our theory. For each, we use 25,000 Metropolis-Hastings MCMC draws with starting value $\theta = 1$. Here, the density labelled as *exact* refers to the exact Bayes posterior in the KDE panel. For the β -divergence and MMD panels, the exact posterior is not available. In these panels, the label refers to a Gibbs measure computed using sufficiently large m for the estimators $\mathsf{L}_{m,n}^{\beta}$ and $\mathsf{L}_{m,n}^{k}$, specifically m=250 for the β -divergence and $m=100 \cdot n$ for the MMD.

Beyond that, the results show that across all three loss functions, the posteriors become more concentrated around the true value $\theta=1$ as m increases. For the MMD and KDE losses, we see a marked increase in estimation accuracy when using more draws, while this effect is less pronounced for the β -divergence loss. Put together, this simple example illustrates that for accurate approximations of the target Gibbs measure, we must carefully choose the number of draws m used in our loss estimates. In particular, in the MMD and KDE cases it is clear that location and variability of the posteriors change as m increases. This suggests that the choice of m will itself also influence concentration of the posterior, an empirical finding that we will verify theoretically in the next section.

3 A New Asymptotic Theory

To set the stage, we first note the clear parallels with pseudo-marginal MCMC: exactly as in that setting, using $L_{m,n}$ instead of L_n to construct our posterior implies that we do not actually target $\pi(\theta \mid L_n)$. Instead, when we construct a sampling algorithm based on $L_{m,n}$, we obtain an unbiased estimator for the *implicit* target posterior

$$\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \propto \pi(\theta) \cdot \mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \left[\exp\{-\omega \cdot \mathsf{L}_{m,n}(\theta, y_{1:n})\} \right].$$

Here, $\mathbb{E}_{U_{1:m} \text{ iid}} P_{\theta}$ [exp $\{-\omega \cdot \mathsf{L}_{m,n}(\theta, y_{1:n})\}$] is the expected exponentiated loss exp $\{-\omega \cdot \mathsf{L}_{m,n}(\theta, y_{1:n})\}$. The expectation is taken over the distribution of the samples $u_{1:m}$ used in constructing $\mathsf{L}_{m,n}(\theta,y_{1:n})$. In the case of the β -divergence and MMD losses of Sections 2.3.1 and 2.3.2 for instance, these samples are used to approximate an intractable integral. The expectation over $U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ is crucial when defining the implicit target: it reflects the fact that using sampling methods like MCMC to approximately sample from Gibbs measures $\pi(\theta \mid \mathsf{L}_n)$ would recompute $\mathsf{L}_{m,n}$ based on a new sample $u_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ at each iteration of the sampling algorithm—rather than re-using a fixed sample at each iteration.

This resulting target posterior $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ is at best a biased estimator for $\pi(\theta \mid \mathsf{L}_n)$, even

if the loss estimator is unbiased so that $\mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} [\mathsf{L}_{m,n}(\theta, y_{1:n})] = \mathsf{L}_{n}(\theta, y_{1:n})$. In particular, whenever $\mathsf{L}_{m,n}(\theta, y_{1:n})$ is non-constant in $u_{1:m}$, Jensen's Inequality implies

$$\mathbb{E}_{U_{1:m} \overset{\text{iid}}{\sim} P_{\theta}} \left[\exp\{-\omega \mathsf{L}_{m,n}(\theta, y_{1:n})\} \right] \neq \exp\left\{-\omega \cdot \mathbb{E}_{U_{1:m} \overset{\text{iid}}{\sim} P_{\theta}} \left[\mathsf{L}_{m,n}(\theta, y_{1:n}) \right] \right\}.$$

This inequality raises two questions:

- 1. For a fixed n, how different are $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ and $\pi(\theta \mid \mathsf{L}_n)$ as a function of m?
- 2. As $n \to \infty$, how fast does m need to increase for $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ to behave like $\pi(\theta \mid \mathsf{L}_n)$? In the remainder, we address this by deriving the rate at which $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ converges to $\pi(\theta \mid \mathsf{L}_n)$ (Theorem 1), deriving the posterior concentration of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ as a function of both n and m (Theorem 2, Corollary 1), and by establishing the size of m required to ensure that a Bernstein-von-Mises type result for $\pi(\theta \mid \mathsf{L}_n)$ transfers to $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ (Corollary 2).

3.1 Preliminaries

To state our assumptions and results, we make use of several common notational short-hands. For two sequences of scalars $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, we take $a_n \lesssim b_n$ to mean that for all n large enough and a constant C > 0, we have $a_n \leq Cb_n$. Moreover, we take $a_n \approx b_n$ to mean that both $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

Similarly, for given sequences of random variables $\{X_n\}_{n\in\mathbb{N}}$ and scalars $\{a_n\}_{n\in\mathbb{N}}$, we take $X_n \lesssim a_n$ to mean that the sequence $\{X_n/a_n\}_{n\in\mathbb{N}}$ is stochastically bounded by a constant C>0 almost surely, so that $\mathbb{P}(\lim_{n\to\infty}X_n/a_n\leq C)=1$. Similarly, we write $X_n\asymp a_n$ if $a_n\lesssim X_n$ and $X_n\lesssim a_n$. It will be clear from context whether we use this notation for asymptotics in m only, or for joint asymptotics in n with m=m(n). In addition, we also use $X_n=O_p(a_n)$ to mean that there is a C>0 so that $\lim_{n\to\infty}\mathbb{P}(X_n/a_n\leq C)=1$, and $X_n=o_p(a_n)$ to mean that for all $\varepsilon>0$, we have $\lim_{n\to\infty}\mathbb{P}(X_n/a_n<\varepsilon)=1$.

Further, for $\Theta \subseteq \mathbb{R}^d$, and a function $\ell : \Theta \to \mathbb{R}$, we denote $\nabla_{\theta} \ell(\theta_0)$ as the vector of first partial derivatives of ℓ evaluated at θ_0 , and $\nabla_{\theta}^2 \ell(\theta_0)$ as the $d \times d$ matrix of its second partial derivatives. We will also use ||x|| to denote the usual Euclidean norm for any vector $x \in \mathbb{R}^d$, and $\min\{z,y\}$ (max $\{z,y\}$) for the minimum (maximum) between two scalars $z,y \in \mathbb{R}$.

3.2 Approximation Quality (fixed n, large m)

To ensure that $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ is a good approximation for $\pi(\theta \mid \mathsf{L}_n)$ as $m \to \infty$, a minimal condition to impose is that the loss estimate $\mathsf{L}_{m,n}$ is pointwise consistent for L_n . While it is easy to demonstrate that $|\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)|$ vanishes for all $\theta \in \Theta$ as $m \to \infty$ under this assumption, such a result would have little practical use for two main reasons: it tells us nothing about the speed of convergence, and it does not necessarily even imply convergence in distribution of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ to $\pi(\theta \mid \mathsf{L}_n)$. We address this by deriving the rate at which $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ converges to $\pi(\theta \mid \mathsf{L}_n)$ in total variation as n is held fixed and $m \to \infty$ (cf. Theorem 1). To achieve this, we require some regularity conditions for the estimation quality of $\mathsf{L}_{m,n}$ stated below.

Assumption 1. For constants $\kappa_1, \kappa_2 > 0$ and a sequence of functions $\{\sigma_n^2(\theta)\}_{n \in \mathbb{N}}$ with $\sigma_n(\theta) : \Theta \to \mathbb{R}_+$ allowed to depend on $y_{1:n}$, it holds for any fixed $y_{1:n}$ that

(i)
$$\mathbb{E}_{U_{1:m} \stackrel{iid}{\sim} P_{\theta}} \left[\mathsf{L}_{m,n}(\theta, y_{1:n}) \right] - \mathsf{L}_{n}(\theta, y_{1:n}) \lesssim \sigma_{n}^{2}(\theta) m^{-\kappa_{1}} \text{ as } m \to \infty;$$

$$(ii) \ \mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\left\{ \mathsf{L}_{m,n}(\theta,y_{1:n}) - \mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\mathsf{L}_{m,n}(\theta,y_{1:n}) \right] \right\}^2 \right] \lesssim \sigma_n^2(\theta) m^{-\kappa_2}.$$

The interpretation of Assumption 1 is straightforward: (i) allows $L_{m,n}$ to be a biased estimator for L_n , but requires that the pointwise bias goes to zero like $m^{-\kappa_1}$ as $m \to \infty$ for some κ_1 . Similarly, (ii) ensures that the estimation error of $L_{m,n}(\theta, y_{1:n})$ goes to zero as $m^{-\kappa_2}$ for some κ_2 and as $m \to \infty$. On a technical level, these requirements allow the derivation of a pointwise asymptotic bound of the moment-generating function of

the random variable $-\mathsf{L}_{m,n}(\theta,y_{1:n})$ in $U_{1:m}$ with the constant $\exp\left\{-\mathsf{L}_n(\theta,y_{1:n})\right\}$, so that $\mathbb{E}_{U_{1:m}\overset{\text{iid}}{\sim}P_{\theta}}\left[\exp\left\{-\mathsf{L}_{m,n}(\theta,y_{1:n})\right\}\right]\lesssim \exp\left\{-\mathsf{L}_n(\theta,y_{1:n})\right\}+\sigma_{m,n}^2(\theta)m^{-\kappa_2}$. This explains why Assumption 1 looks superficially similar to classic conditions required for concentration inequalities for sub-Gaussian random variables (see e.g. Boucheron et al., 2013). Note however that Assumption 1 is much weaker: classic sub-Gaussianity requires the left-hand side of (i) to be zero for all m, and that (ii) holds for all finite m (rather than asymptotically as $m \to \infty$). In this sense, Assumption 1 is akin to an asymptotic sub-Gaussianity assumption, with $\sigma_n^2(\theta)$ serving as a form of asymptotic variance proxy (see Appendix A.1).

Having upper bounded bias and variance in terms of $\sigma_n^2(\theta)$, it is straightforward to define interpretable conditions that allow us to strenghten pointwise to posterior-averaged convergence. In particular, if certain moments of $\pi(\theta)$ and $\pi(\theta \mid \mathsf{L}_n)$ exist, then the posterior-averaged biases and variances will vanish uniformly over Θ at rates κ_1 and κ_2 , respectively.

Assumption 2. For any
$$n \geq 1$$
 and some $p \geq 1$, $\int_{\Theta} \|\theta\|^p \pi(\theta \mid \mathsf{L}_n) d\theta < \infty$, $\int_{\Theta} \|\theta\|^p \sigma_n^2(\theta) \pi(\theta) d\theta < \infty$, $\int_{\Theta} \sigma_n^2(\theta) \|\theta\|^p \pi(\theta \mid \mathsf{L}_n) d\theta < \infty$, $0 < \int_{\Theta} \exp\{-\omega \cdot \mathsf{L}_n(\theta)\} \pi(\theta) d\theta < \infty$, $\int_{\Theta} \|\theta\|^p \pi(\theta) d\theta < \infty$.

Under Assumption 1, and the moment conditions in Assumption 2, we can control the difference term $|\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)|$ uniformly over Θ , for any $n \geq 1$, and as $m \to \infty$.

Theorem 1. If Assumptions 1 and 2 are satisfied, then for all $\xi \in [0,2]$, as $m \to \infty$

$$\int_{\Theta} \|\theta\|^{\xi} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| d\theta \lesssim m^{-\min\{\kappa_1, \kappa_2\}}$$

which also implies that

$$\int_{\Theta} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| \, \mathrm{d}\theta \lesssim m^{-\min\{\kappa_1, \kappa_2\}},$$

$$\left\| \int_{\Theta} \theta \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \, \mathrm{d}\theta - \int_{\Theta} \theta \pi(\theta \mid \mathsf{L}_n) \, \mathrm{d}\theta \right\| \lesssim m^{-\min\{\kappa_1, \kappa_2\}}.$$

The above result holds conditionally for any sample $y_{1:n}$. While Theorem 1 has three parts, the last two equations are direct consequences of the first, and correspond to the

special cases of convergence in total variation ($\xi = 0$) and convergence in mean total variation ($\xi = 1$). Taken together, Theorem 1 tells us that under mild regularity conditions, inferences based on $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ can be made arbitrarily close to those based on the infeasible target $\pi(\theta \mid \mathsf{L}_n)$ by increasing m. Beyond that, Theorem 1 also tells us that the rate at which m increases the approximation quality of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ for $\pi(\theta \mid \mathsf{L}_n)$ depends on the slower of the two rates κ_1 and κ_2 that govern how quickly bias and variance of $\mathsf{L}_{m,n}(\theta,y_{1:n})$ vanish. In other words, it tells us that a biased estimator $\mathsf{L}_{m,n}(\theta,y_{1:n})$ for $\mathsf{L}_m(\theta,y_{1:n})$ will not adversely affect the approximation quality of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ as long as the order κ_1 of the bias is upper-bounded by that of the estimator's variance.

3.3 Joint Asymptotics and Concentration (large n, large m)

While the previous result demonstrates that the behaviour of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ is desirable for any fixed n as $m \to \infty$, it tells us nothing about how $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ behaves as m and n increase. This is an important question since the convergence results in Theorem 1 only hold conditionally for a fixed $y_{1:n}$, and therefore obscure dependence on a constant factor, say $C(y_{1:n})$, that generally depends on both n and $y_{1:n}$. Depending on the nature of $C(y_{1:n})$, ensuring that $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ remains a good approximation may become computationally infeasible: for instance, if $C(y_{1:n}) \approx \exp(n)$, the required number of samples m would have to increase exponentially in n.

To investigate these interactions between n and m, we take m = m(n) in the remainder, and study $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ as $n \to \infty$. To do this, we have to impose additional regularity conditions to control the behaviour of L_n as $n \to \infty$. To this end, define the limiting object

$$\mathsf{L}(\theta) = \lim_{n \to \infty} \mathbb{E}_{Y_{1:n} \sim P_0} \left[n^{-1} \mathsf{L}_n(\theta, Y_{1:n}) \right].$$

We will assume that $L(\theta)$ exists for all $\theta \in \Theta$, and define its minimiser as

$$\theta_0 = \underset{\theta \in \theta}{\operatorname{argmin}} \ \mathsf{L}(\theta).$$

We impose the below assumptions to guarantee posterior concentration of $\pi(\theta \mid \mathsf{L}_n)$.

Assumption 3. For any $\delta > 0$, $L(\theta_0) < \inf_{\|\theta - \theta_0\| \ge \delta} L(\theta)$. There exists a function $\gamma : \Theta \to \mathbb{R}_+$ such that $\int_{\Theta} \gamma^2(\theta) \pi(\theta) d\theta < \infty$, and for some $N \in \mathbb{N}$ and all n > N:

(i)
$$\mathbb{E}_{Y_{1:n} \sim P_0} \left[\left\{ n^{-1} \mathsf{L}_n(\theta, Y_{1:n}) - \mathsf{L}(\theta) \right\}^2 \right] \le C \gamma^2(\theta) / n;$$

(ii)
$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta < \infty$$
, and $\int_{\Theta} \gamma^2(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta < \infty$.

Assumption 3 part (i) is a standard regularity condition for posterior concentration of $\pi(\theta \mid \mathsf{L}_n)$ (see, e.g., Condition 1 of Syring and Martin, 2020 for a similar requirement), while part (ii) and the prior moment condition on γ^2 requires further discussion. Paralleling how $\sigma_n^2(\theta)$ in Assumption 1 controls the pointwise variance in $\mathsf{L}_{m,n}(\theta,y_{1:n})$ due to $U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ and is used for the asymptotics in m (for fixed n), $\gamma^2(\theta)$ in Assumption 3 controls the pointwise variance in $\mathsf{L}_n(\theta;Y_{1:n})$ due to the variability in the observed data $Y_{1:n} \sim P_0$, and will drive asymptotics in n (with m = m(n)). While part (ii) and the prior moment condition on $\gamma^2(\theta)$ are not standard and may be hard to verify in general, they are relatively mild conditions. Whenever $\gamma^2(\theta)$ can be derived explicitly as for our examples in Section 4, the prior and posterior moment conditions can also be verified explicitly and numerically, respectively. For the case where Θ is compact or $\sup_{\theta \in \Theta} \mathsf{L}(\theta) < \infty$, both conditions hold automatically.

To extend our asymptotics to the case where $n \to \infty$ and m = m(n), we also require minor regularity conditions for the prior, as well as some refined bounds for the moment conditions in Assumption 2 that allow $n \to \infty$ and m = m(n). These requirements are jointly summarised in our last set of suppositions.

Assumption 4. Suppose that the prior π is continuous on Θ , and that $\pi(\theta_0) > 0$. Further, assume that m is chosen as m = m(n), and that for constants $\alpha_1, \alpha_2, \eta_1, \eta_2 > 0$,

(i) for
$$\mathcal{B}_n = \{\theta \in \Theta : |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \le \alpha_1 \sqrt{n}\}, \text{ we have } \int_{\mathcal{B}_n} \pi(\theta) d\theta \ge \exp\{-\alpha_1 \sqrt{n}\};$$

(ii)
$$m^{-\kappa_1} \int_{\Theta} \sigma_n^2(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta \asymp m^{-\eta_1} \text{ and } m^{-\kappa_2} \int_{\Theta} \sigma_n^2(\theta) \pi(\theta) d\theta \asymp m^{-\eta_2} \text{ as } n \to \infty;$$

where $\sigma_n^2(\theta)$ is the same variance proxy term as in Assumption 1.

Almost all conditions in Assumption 4 are standard for posterior concentration. The exception is (ii), which extends some of the moment conditions in Assumption 2 for m = m(n) and $n \to \infty$. On a technical level, (ii) ensures that the bounds obtained by integrating out the bias and variance bounds in Assumption 1 do not diverge as $n \to \infty$ with m = m(n).

3.3.1 Posterior Concentration

Together, our assumptions allow the derivation of novel concentration rates for the posterior $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$. They constitute the first results applicable to the setting we address, and provide direct information about the rate at which m = m(n) should scale with n as $n \to \infty$. Our main result shows that if we want $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ to place most of its posterior mass in regions of Θ that correspond to small $\mathsf{L}(\theta)$, we must choose $m \asymp n^{0.5/\min\{\eta_1,\eta_2\}}$.

Theorem 2. If Assumptions 1-4, are satisfied, then for any $M_n \to \infty$, it holds that

$$P_0\left(\int_{\Theta}|\mathsf{L}(\theta)-\mathsf{L}(\theta_0)|\overline{\pi}(\theta\mid\mathsf{L}_{m,n})\mathrm{d}\theta>M_n/\min\{\sqrt{n},m^{\min\{\eta_1,\eta_2\}}\}\right)\longrightarrow 0.$$

The above shows that posterior concentration occurs at rate $\min\{\sqrt{n}, m^{\min\{\eta_1,\eta_2\}}\}$. In contrast, concentration of $\pi(\theta \mid \mathsf{L}_n)$ occurs at rate \sqrt{n} . The implication is clear: for $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ to concentrate at the same rate as $\pi(\theta \mid \mathsf{L}_n)$, we require $m \approx n^{0.5/\min\{\eta_1,\eta_2\}}$.

Under a local Hölder continuity condition for L in a neighbourhood of θ_0 , concentration onto regions of low loss in Theorem 2 extends to open balls around θ_0 . Such conditions are often encountered in the study of Gibbs posteriors, see for instance Frazier et al. (2018), Bernton et al. (2019a), and Legramanti et al. (2022).

Corollary 1. Under Assumptions 1-4, and if on some open neighborhood of θ_0 , $\|\theta - \theta_0\| \le C|\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|^{\alpha}$ for some C > 0 and $\alpha > 0$, then for any $M_n \to \infty$ and the sequence $S_n = \{\theta \in \Theta : \|\theta - \theta_0\| > C(M_n/\min\{\sqrt{n}, m^{\min\{\eta_1, \eta_2\}}\})^{\alpha}\}$, it holds P_0 -almost surely that

$$\int_{S_n} \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \lesssim M_n^{-1}.$$

While local Hölder continuity condition is a mild requirement, it relates to the limiting object L, and is thus generally hard to verify. In most settings of practical interest however, it is unproblematic to simply assert this condition for some $\alpha > 0$, which implies that $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ concentrates around θ_0 at rate $\min\{n^{\alpha/2}, m^{\alpha \min\{\eta_1, \eta_2\}}\}$.

3.3.2 Asymptotic Normality and Posterior Mean

Corollary 1 implies that the contraction rate of $\pi(\theta \mid \mathsf{L}_n)$ is transferred to $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ as long as $m \asymp n^{0.5 \cdot \min\{\eta_1, \eta_2\}}$. A similar result also holds for the posterior mean and asymptotic posterior shape. To state this, let $\mathsf{H} = [\omega \cdot \nabla_{\theta}^2 \mathsf{L}(\theta_0)]^{-1}$, $\hat{\theta}_n := \operatorname{argmin}_{\theta \in \Theta} \mathsf{L}_n(\theta)$, and $\mathcal{N}(\theta; \mu, \Sigma)$ the density of a Gaussian distribution at θ with mean μ and variance Σ .

Corollary 2. If Assumptions 1-4 are satisfied, and if $\int_{\Theta} \left| \pi(\theta \mid \mathsf{L}_n) - \mathcal{N}(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right| d\theta = o_p(1)$, then for $\bar{\theta}_n = \int_{\Theta} \theta \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta$, we have

(i)
$$\int_{\Theta} \left| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \mathcal{N}(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right| d\theta = o_p(1) \text{ if } m^{-\min\{\eta_1, \eta_2\}} \to 0 \text{ as } n \to \infty;$$

(ii)
$$\|\sqrt{n}(\hat{\theta}_n - \bar{\theta}_n)\| = o_p(1)$$
 if $\sqrt{n}m^{-\min\{\eta_1,\eta_2\}} \to 0$ as $n \to \infty$.

Part (i) of the above result is especially useful for calibration of generalised Bayes: such techniques often rely on asymptotic normality of $\pi(\theta \mid \mathsf{L}_n)$ to scale $\mathsf{L}_{m,n}$ by a value of ω in (2) that achieves approximate frequentist coverage. Part (ii) implies that if we choose m large enough, then the posterior mean of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ asymptotically behaves like the computationally infeasible risk minimizer $\hat{\theta}_n$. In order for this behavior to emerge however, we must account for the simulation-induced bias of estimators constructed from $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$. In particular, ensuring that this bias does not pollute the asymptotic distribution of $\overline{\theta}_n$ requires choosing $m \gg n^{0.5/\min\{\eta_1,\eta_2\}}$.

In the last part of this paper, we show that our theory is sufficiently flexible to cover a wide range of settings. This includes the MMD-based loss $\mathsf{L}^k_{m,n}$ (see Section 2.3.2), for which one can apply our results with $\kappa_1 = \kappa_2 = \eta_1 = \eta_2 = 1$, which implies that

the computationally feasible scaling $m \simeq \sqrt{n}$ is sufficient to achieve the standard \sqrt{n} -concentration rate for $\overline{\pi}(\theta \mid \mathsf{L}^k_{m,n})$. In contrast, a guarantee of \sqrt{n} -concentration with the KDE-based loss $\mathsf{L}^h_{m,n}$ (see Section 2.3.3) makes $\kappa_1, \kappa_2, \eta_1, \eta_2$ depend on the kernel bandwidth, and ultimately requires choosing $m \simeq n^{2.25}$ or larger. Similarly, for the losses $\mathsf{L}^\beta_{m,n}$ induced by β -divergences (see Section 2.3.1) we have $\min\{\eta_1, \eta_2\} = 0.2$, so that $m \simeq n^{2.5}$ achieves \sqrt{n} -concentration.

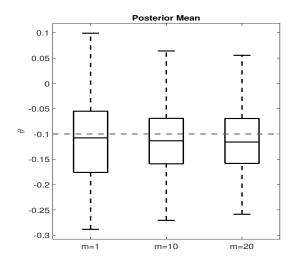
4 Applications

We apply our theory to Gibbs measures based on losses introduced in Section 2.3: the β -divergence, the MMD, and a KDE-based likelihood estimator. We focus on verifying the most crucial conditions, namely Assumptions 1, 3 part (i), and 4 part (ii). We summarise our findings in Lemmas 1 and 2, and conclude that $\mathsf{L}^k_{m,n}$ and $\mathsf{L}^k_{m,n}$ require $m \asymp \sqrt{n}$ and $m \asymp n^{2.25}$ respectively for \sqrt{n} -concentration. We empirically verify this on simulated data using the g-and-k model and a copula model. For the β -divergence, we find min $\{\eta_1, \eta_2\} = 0.2$, so that $m \asymp n^{2.5}$ is required for standard \sqrt{n} -contraction rates (see Appendix D.1).

4.1 Maximum Mean Discrepancy (MMD)

If the kernel k used to construct L^k_n and $\mathsf{L}^k_{m,n}$ is bounded and characteristic so that the corresponding MMD satisfies the identity of discernibles, all key assumptions hold. This is unrestrictive, and true for most kernels of interest; including Gaussian and Matérn kernels.

Lemma 1. If the kernel $k: \mathcal{Y}^2 \to \mathbb{R}$ is characteristic, and if $\sup_{y,y'} k(y,y') \leq K$, then for $\mathsf{L}^k_{m,n}$ in (4), Assumption 1 holds for $\sigma^2(\theta) = 16K^4 + 2K$, and $\kappa_1 = \kappa_2 = 1$. If it additionally holds that $y_{1:n}$ is independently and identically distributed, then we also have that Assumption 3 part (i) holds with $\gamma^2(\theta) = 16K^4 + 2K$ and C = 1.



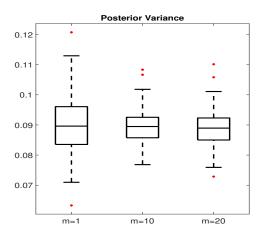


Figure 2: Boxplots of posterior mean and variance for 100 replications of $\overline{\pi}(\theta \mid \mathsf{L}^k_{m,n})$ based on a copula example with n=100 observations. The red dots indicate outliers. The dotted line in the left plot represents the true value used to generate the data. As predicted by theory, $m \times \sqrt{n} = 10$ yields the best trade-off between computational effort and efficiency. In particular, choosing $m > 10 = \sqrt{n}$ does not result in better performance.

Under the conditions of the above result, $\mathsf{L}^k_{m,n}$, L^k_n , and the corresponding limiting loss L^k are all uniformly bounded by K, and both $\sigma^2_n(\theta)$ and $\gamma^2(\theta)$ are constant functions. Thus, Assumption 2 is actually verifiable by elementary arguments. Further, Assumption 4 part (ii) yields $\eta_1 = \kappa_1 = 1$ and $\eta_2 = \kappa_2 = 1$. Hence, Theorem 1 implies that the difference between $\overline{\pi}(\theta \mid \mathsf{L}^k_{m,n})$ and $\pi(\theta \mid \mathsf{L}^k_n)$ vanishes at rate m^{-1} for any fixed n. Lastly, Theorem 2 implies that $m \asymp \sqrt{n}$ is sufficient for achieving the standard \sqrt{n} concentration rate.

4.1.1 MMD-based Copula Model Inference

To verify the practical relevance of our theory, we provide numerical demonstrations using MMD-based inference in a copula model based on the loss $\mathsf{L}^k_{m,n}$ proposed in Alquier et al. (2023), details of which are provided in Appendix A.2. We compute generalised posteriors based on $\mathsf{L}^k_{m,n}$, and compare the accuracy of posterior moments. In particular, we generate n=100 observations from a Student's t copula model with 10 degrees of freedom and

dependence parameter $\theta = -0.1$. Inference is performed on θ only, holding the degrees of freedom fixed at 10. Across 100 replications of this procedure, we compare mean and variance of the distribution $\overline{\pi}(\theta \mid \mathsf{L}^k_{m,n})$ for different choices of m in Figure 2. Our results show that posterior means become more accurate as m increases. Similarly, posterior variances tend to shrink as m increases. As our theory predicts, improvements are substantial for smaller m, but negligible once $m \geq \sqrt{n} = 10$. Indeed, there is little difference in the boxplots for posterior means and variances between m = 10 and m = 20.

4.2 Kernel Density Estimation (KDE) for Intractable Likelihood

When the likelihood is intractable so that $p_{\theta}(y_{1:n})$ cannot be evaluated exactly but we can sample $U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$ for each $\theta \in \Theta$, Turner and Sederberg (2014) propose to estimate the likelihood via KDE. This amounts to computing $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ based on $\mathsf{L}^h_{m,n}$ as introduced in Section 2.3.3 to approximate the exact Bayes posterior $\pi(\theta \mid y_{1:n}) = \pi(\theta \mid \mathsf{L}_n)$ with $\mathsf{L}_n(\theta, y_{1:n}) = -\log p_{\theta}(y_{1:n})$. To show that our theory applies to this setting, we first establish conditions under which Assumption 1 holds.

Lemma 2. Assume that for an isotropic kernel k and a bandwidth h > 0, we have

(i)
$$C_j(\theta) := C_j \sup_{y \in \mathcal{V}} \left| \frac{\partial^j}{\partial y^j} p_{\theta}(y) \right| < \infty \text{ for } C_j < \infty \text{ and } j \in \{0, 1, 2, 3\};$$

(ii) for every $\theta \in \Theta$, there exist $\delta(\theta) > 0$ so that $\inf_{y \in \mathcal{Y}} p_{\theta}(y) \geq \delta(\theta)$;

(iii)
$$\int k(v) dv = 1$$
, $k(v) = k(-v)$, $\int k^2(v) dv < \infty$, and $\int v^2 k(v) dv < \infty$.

Then, as $h \to 0$, and $m \to \infty$ with $mh \to \infty$, it holds for any finite n that

$$\mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\log \hat{p}_{\theta}(y_{1:n}) \right] - \log p_{\theta}(y_{1:n}) \le \frac{\max\{C_{2}(\theta), C_{3}(\theta)\}}{\delta(\theta)} n h^{2};$$

$$\mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\left\{ \log \hat{p}_{\theta}(y_{1:n}) - \log p_{\theta}(y_{1:n}) \right\}^{2} \right] \le \frac{n}{mh} \frac{\{C_{2}(\theta) + C_{0}(\theta)h\}}{\delta(\theta)^{2}} + n h^{4} \frac{\max\{C_{1}(\theta), C_{2}(\theta)\}^{2}}{\delta(\theta)^{2}}.$$

Additional assumptions on p_{θ} and the data-generating process are needed to verify Assumption 3 (i). For instance, if observations are sampled independently and are identically distributed according to P_0 , we can simplify Assumption 3 by noting that

$$\mathbb{E}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} \left[\{ n^{-1} \log p_{\theta}(Y_{1:n}) - \mathsf{L}(\theta) \}^2 \right] = \operatorname{Var}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} \{ n^{-1} \log p_{\theta}(Y_{1:n}) \} = \frac{\operatorname{Var}_{Y_1 \sim P_0} \{ \log p_{\theta}(Y_1) \}}{n}.$$

Hence, we can take $\gamma^2(\theta) = \operatorname{Var}_{Y_1 \sim P_0} \{ \log p_{\theta}(Y_1) \}$, so that Assumption 3 (i) simplifies to requiring that the Bayes posterior-averaged variance and expectation of the log likelihood is finite so that $\int \operatorname{Var}_{Y_1 \sim P_0} \{ \log p_{\theta}(Y_1) \} \pi(\theta \mid y_{1:n}) d\theta < \infty$ and $\int \mathbb{E}_{Y_1 \sim P_0} \{ \log p_{\theta}(Y_1) \} \pi(\theta \mid y_{1:n}) d\theta < \infty$. This generally depends on p_{θ} and P_0 , but will generally hold if we have a second moment for $\log p_{\theta}(Y_1)$ in $Y_1 \sim P_0$, a condition commonly required for proving concentration of generalised posteriors (e.g. Condition 1 in Syring and Martin, 2023).

Given this, Corollary 2(i) implies that $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ behaves as $\pi(\theta \mid y_{1:n})$ so long as $nh^2 \to 0$, and $n/(mh) \to 0$. More specifically, it implies that for $m = m(n) \to \infty$ and as $n \to \infty$,

$$\int_{\Theta} \left| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}^h) - \pi(\theta \mid y_{1:n}) \right| d\theta \lesssim \max\{n(mh)^{-1}, nh^2\}.$$
 (5)

Here, $nh^2 \to 0$ ensures that the bias of $\mathsf{L}^h_{m,n}(\theta,y_{1:n}) = \log \hat{p}_{\theta}(y_{1:n})$ vanishes, and that $h^2 \ll n^{-1/2}$. To control the variance, we also need $n/(mh) \to 0$. For $m=n^a$ and $h=n^{-b}$, a,b>0, the bias condition implies b>1/2, and the variance condition requires a>1+b. As a result, we require $m \asymp n^{2.25}$ for $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ to concentrate at rate \sqrt{n} (see Appendix A for details).

4.2.1 KDE-based Inference for G-and-K-Model

The scaling required for $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ to concentrate at the \sqrt{n} -rate calls the computational feasibility of KDE-based likelihood estimators into question. We demonstrate this by conducting inference on the g-parameter of the intractable g-and-k distribution (see Appendix A.4 for additional details). Throughout our experiments, we compare against

n	m	mean bias	median bias	standard deviation	80%	90%	95%
100	$50 \cdot n$	1.4364	0.9549	1.6560	60	71	87
	$100 \cdot n$	1.0864	0.6821	1.3876	61	80	90
	$1000 \cdot n$	0.6980	0.4086	1.0498	79	90	95
1000	$50 \cdot n$	0.1327	0.1274	0.1271	61	69	78
	$100 \cdot n$	0.1035	0.0988	0.1207	65	75	83
	$\boxed{1000 \cdot n}$	0.0419	0.0392	0.1090	70	86	96

Table 1: Results for 100 repeated simulations for inference on g in the g-and-k model based on $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$. Reported is the average bias of the posterior mean, the average bias of the posterior median, the average posterior's standard deviation, and the coverage associated to the 80%, 90%, and 95% credible set for g. Credible sets for $100 \cdot x\%$ were constructed as $[F_n(0.5x), F_n(1-0.5x)]$ using the empirical CDF F_n of the posterior.

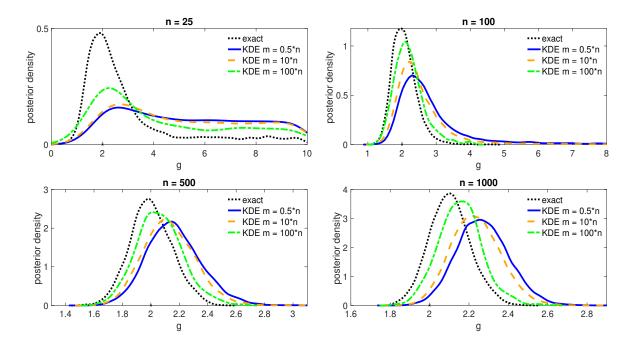


Figure 3: Inference quality for g in the g-and-k model with KDE-based posteriors $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$. Comparisons across dataset sizes n=25,100,500,1000 and $m=L\cdot n$ for L=0.5,10,100.

an almost exact benchmark posterior obtained using the computationally prohibitive likelihood approximation strategy in Rayner and MacGillivray (2002). Figure 3 plots the results for a single run with different choices for m and n. The results reflect our theory: regardless of n, the approximation quality of $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ is poor for smaller choices of m. In particular, $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ is biased and has higher variance than the target posterior. While bias and variance of $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ decrease as m increases, the difference with the target is notable even for large m. Perhaps counter-intuitively, the results also illustrate that for small n, m has to be increased by a larger multiple than for larger n to achieve the same performance. This points to a limitation of our theoretical guarantees: they are asymptotic in nature and thus ignore small-sample effects.²

To supplement the qualitative findings of Figure 3, we replicate these computations 100 times for different choices of n and m, and summarise the results in Table 1 for the model parameter g, and in Appendix A.5 for the remaining parameters. To keep comparisons fair across different n and m, the MCMC algorithm used for posterior computations uses a total of 10 million simulations of the full dataset from the model for each setting.³ The findings validate our theory, and show that m has to be chosen much larger than n for reliable inference.

²To illustrate this, inspect (5). Choosing the MSE-optimal bandwidth rate of $h \approx m^{-2/5}$, we obtain $\kappa_1 = 3/5$, so that the non-asymptotic version of the upper bound in (5) for n = 25 and $m = 100 \cdot n$ becomes $C \max\{0.23, 0.05\}$ for some C. Since the righthand side is also trivially upper bounded by 2, this is vacuous if C > 2/0.23 > 9. While this upper bound will eventually become meaningful as n and m = m(n) diverge, ensuring that this term is small for an arbitrary C may require choosing a very large value of m when n is also small.

³This means that for any m, we use $10 \cdot n$ million simulations from the g-and-k distribution.

5 Discussion

Frequently, a target Gibbs measure $\pi(\theta \mid \mathsf{L}_n)$ is computationally infeasible, and has to be approximated through the measure $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$, which substitutes an intractable loss L_n for the tractable approximation $\mathsf{L}_{m,n}$. While the use of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ is already common in contemporary practice, our results provide the first generic and comprehensive asymptotic theory for such objects. Our key finding is that their approximation quality depends crucially on the quality of the estimator $\mathsf{L}_{m,n}$. In particular, our theory shows that if η_1 and η_2 denote the rates at which posterior-averaged biases and variances of $\mathsf{L}_{m,n}$ vanish, the approximation error of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ vanishes at rate $m^{-\min\{\eta_1,\eta_2\}}$ for any fixed n; see Assumption 4 in Section 3.3 for the definition of η_1, η_2 and further details. To ensure that the approximation quality of $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ stays constant as n increases, our findings further indicate that we need to scale the number of simulations as $m \asymp n^{0.5/\min\{\eta_1,\eta_2\}}$. Note that all results apply regardless of the choice for ω in (2)—a crucial tuning parameter often referred to as the learning rate in the literature on generalised Bayes.

While we focused on working out the technical details for three illustrative examples, the theory derived in the previous section has much wider applicability. For generalised Bayesian methods, it applies immediately for the setting of losses constructed from α - and γ -divergences (Hooker and Vidyashankar, 2014; Nakagawa and Hashimoto, 2020; Fujisawa et al., 2021), total variation (Knoblauch and Vomfell, 2020), Wasserstein distances (Bernton et al., 2019b), and sliced distances (Kolouri et al., 2019; Gong et al., 2020).

Further, the application of our theory to kernel density estimation (KDE) suggests that various newer black-box density estimation methods, like sequential neural likelihood (see e.g. Lueckmann et al., 2017; Papamakarios et al., 2019; Greenberg et al., 2019), for inference with intractable likelihoods may also require a similar level of draws in order to accurately approximate an intractable posterior. To the best of our knowledge, no formal results exist

on the accuracy of such likelihood estimators.

Lastly, our results establish the first general asymptotic theory for posteriors constructed from MCMC methods based on biased likelihood estimators, and thus extend existing results on the large-sample behavior of pseudo-marginal MCMC methods (see e.g. Andrieu and Roberts, 2009; Schmon et al., 2021). Surprisingly, the results show that a biased likelihood estimator may be preferable to an unbiased alternative if computational effort is a major concern. In particular, our theoretical results suggest that instead of using unbiased likelihood estimators with slowly decaying variances, we should consider using biased estimators whose variance decays more quickly. Given that existing methods for pseudo-marginal methods do not allow for bias, our contributions could help to liberate the field from the restrictive requirement of unbiased likelihood estimators.

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Appendix

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A Further Details

A.1 Assumption 1

Importantly, while the moment bounds (i) and (ii) in Assumption 1 are pointwise for each $\theta \in \Theta$, they are tied together via $\sigma_n^2(\theta)$. Hence, $\sigma_n^2(\theta)$ can be thought of as an asymptotic

variance proxy. To illustrate this, define the sequence of functions $\{\sigma_{m,n}^2\}_{n\in\mathbb{N},m\in\mathbb{N}}$ with $\sigma_{m,n}^2:\Theta\to\mathbb{R}_+$ via the identities

$$\mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \left[\left\{ \mathsf{L}_{m,n}(\theta, y_{1:n}) - \mathbb{E}_{U_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}} \left[\mathsf{L}_{m,n}(\theta, y_{1:n}) \right] \right\}^{2} \right] = \sigma_{m,n}^{2}(\theta) m^{-\kappa_{2}}. \tag{6}$$

We can now see that for any fixed n, $\sigma_n^2(\theta)$ is an asymptotic upper bound over $\{\sigma_{m,n}^2(\theta)\}_{m\in\mathbb{N}}$. To make this bound as tight as possible, we could choose $\sigma_n^2(\theta)$ so that $\sigma_n^2(\theta)m^{-\kappa_2} \approx \sigma_{m,n}^2(\theta)m^{-\kappa_2}$ as $m \to \infty$. This also implies an upper bound on the bias term in (i): decomposing the left hand side of (6) into a squared bias and variance term yields

$$\left(\mathbb{E}_{U_{1:m} \overset{\text{iid}}{\sim} P_{\theta}}\left[\mathsf{L}_{m,n}(\theta,y_{1:n})\right] - \mathsf{L}_{n}(\theta,y_{1:n})\right)^{2} + \mathbb{E}_{U_{1:m} \overset{\text{iid}}{\sim} P_{\theta}}\left[\left\{\mathsf{L}_{m,n}(\theta,y_{1:n}) - \mathsf{L}_{n}(\theta,y_{1:n})\right\}^{2}\right],$$

and since the second term in this decomposition is positive, an application of Jensen's inequality reveals that $\sigma_n(\theta)m^{-\kappa_1}$ with $\kappa_1 = 0.5 \cdot \kappa_2$ is a valid asymptotic upper bound on the bias term.⁴ In practice however, $\kappa_1 = 0.5 \cdot \kappa_2$ is typically a much slower rate than what reasonable estimators achieve. As our convergence results depend on the minimum between κ_1 and κ_2 , we thus state our assumptions in terms of both rates.

A.2 Numerical Demonstrations with Copula Model

Motivated by the impact of outliers on likelihood-based copula inference, Alquier et al. (2023) propose a point estimator for the dependence parameter in copula models by minimising the function $\theta \mapsto \mathsf{L}_n^k(\theta,y_{1:n})$. While more robust to misspecification than the negative log likelihood, this function has no closed form, and estimation must instead be based on the approximate version $\mathsf{L}_{m,n}^k$. Following Chérief-Abdellatif and Alquier (2020), we can also use $\mathsf{L}_{m,n}^k$ for a generalised Bayesian version of this estimation problem. Here, our theory predicts that as long as $m \asymp \sqrt{n}$, the resulting posterior $\overline{\pi}(\theta \mid \mathsf{L}_{m,n}^k)$ should be close to $\pi(\theta \mid \mathsf{L}_n^k)$. In practice, this means that we expect $\overline{\pi}(\theta \mid \mathsf{L}_{m,n}^k)$ to provide an excellent approximation to $\pi(\theta \mid \mathsf{L}_n^k)$, even if m is quite small.

⁴This is true since we can always increase $\sigma_n(\theta)$ if necessary to ensure $\sigma_n(\theta) > 1$.

To verify the predictions of our theory, we consider a simple bivariate copula model. Suppose that we observe the bivariate random variables $Y_i = (Y_{i,1}, Y_{i,2})^{\top} \stackrel{\text{iid}}{\sim} F$ from the joint cumulative probability function (CDF) with marginal CDFs F_j . Then, Sklar's Theorem states that F can be modeled by some copula $C: [0,1]^2 \to \mathbb{R}_+$, where for $y = (y_1, y_2)$,

$$F(y_1, y_2) = C(F_1(y_1), F_2(y_2)) = C(u_1, u_2), \text{ where } u_j = F_j(y_j) \text{ for } j = 1, 2.$$
 (7)

Based on this, a common approach for modelling such bivariate observed data $y_{1:n}$ is via some parametric copula function: $C_{\theta}: [0,1]^2 \times \Theta \to \mathbb{R}_+$, where the parameter θ measures the dependence between $y_{i,1}$ and $y_{i,2}$ for each $i=1,2,\ldots n$. In practice, the marginals F_j are of course unknown, so that one must estimate them. A common approach is to do this via the empirical marginal CDFs $F_{n,j}(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{y_{i,j} \leq t\}$. One can then construct a corresponding set of transformed observations $\hat{u}_i = (\hat{u}_{1,i}, \hat{u}_{2,i}) = (F_{n,1}(y_{1,i}), F_{n,2}(y_{1,i}))^{\top}$, where the hat notation signifies that $\hat{u}_{j,i}$ is an estimate of $u_{j,i} = F_j(y_{j,i})$.

Given $\{\hat{u}_i\}_{i=1}^n$, pseudo-likelihood estimation of θ can be carried out via the copula probabilisty density function c_{θ} (see e.g. Genest et al., 1995). While pseudo-likelihood inference is extremely common in copula modelling, the method is known to be non-robust to outliers and contamination bias, due to the behavior of copula densities near the extremes of the support. To make copula-based inference more robust, Alquier et al. (2023) thus propose to conduct inference on θ by minimising the MMD between the empirical distribution of $\{\hat{u}_i\}_{i=1}^n$ (often called the empirical copula process) and the copula model. For some kernel function $k:[0,1]^2\times[0,1]^2\to\mathbb{R}_+$, this gives rise to the loss

$$\mathsf{L}_{n}^{k}(\theta, y_{1:n}) = \mathbb{E}_{U \sim c_{\theta}, U' \sim c_{\theta}} \left[k(U, U') \right] - \frac{2}{n} \sum_{i=1}^{n} \mathbb{E}_{U \sim c_{\theta}} \left[k(U, \hat{u}_{i}) \right].$$

Unfortunately, the integrals featuring in L_n^k are intractable even for the simplest copulas. However, it is straightforward to approximate them via Monte Carlo: defining \mathcal{U}_2 as the distribution of a bivariate uniform distribution on $[0,1]^2$, we can simulate m sets of n independently and identically distributed draws from the copula model via the pushforward $u_j^{\theta} = (u_{1,j}^{\theta}, u_{2,j}^{\theta})^{\top} \stackrel{iid}{\sim} C_{\theta} \circ \mathcal{U}_2$ for $j = 1, \dots m$, and construct

$$\mathsf{L}_{m,n}^k(\theta,y_{1:n}) = \frac{1}{m^2} \sum_{j'=1,j=1}^m k(u_j^{\theta},u_{j'}^{\theta}) - \frac{2}{mn} \sum_{j=1}^m \sum_{i=1}^n k(u_j^{\theta},\hat{u}_i).$$

A.3 Discussion on m and h in the KDE

While Lemma 2 shows that Theorem 1 is satisfied by the KDE likelihood estimator for any fixed n, it is not directly clear what rates on h and m are needed to control the bias and variance of the estimated likelihood as $n \to \infty$. To this end, we parameterise $m \asymp n^a$ with a > 0, and choose the bandwidth $h \asymp n^{-b}$, b > 0. Given these choices, let us explore how we can control the terms nh^2 and $n(mh)^{-1}$ as $n \to \infty$. Clearly, we must choose a and b such that $mh = n^{a-b} \to \infty$, and $nh^2 \asymp n^{1-2b} \to 0$, which requires that a > b > 1/2.

However, if we additionally want to ensure that the desired \sqrt{n} -convergence is satisfied, Theorem 2 implies that we must choose these terms so that

$$nh^2 \ll n^{-1/2}$$
, and $n/(mh) \ll n^{-1/2}$.

Using the parameterisations $m = n^a$ and $h = n^b$, we can ensure this is the case by analysing the values of a and b for which it holds that

$$1 - 2b = -1/2$$
 and $1 + b - a = -1/2$.

The first equation implies b=3/4, and plugging this into the second yields a=2.25. Hence, in order to control the bias of the approximation and ensure that the posterior concentrates at the canonical \sqrt{n} parametric rate, we must choose the bandwidth like $h \ll n^{-3/4}$, and simulate $m \gg n^{2.25}$ data points at each MCMC iteration. Taken together, this showcases that to obtain accurate results using the KDE likelihood estimator, one must simulate a very large number of data points—resulting in a rather large computational cost.

A.4 Numerical Demonstrations with G-and-K Model

To numerically illustrate that choosing m to be sufficiently large is crucial, we use the g-and-k distribution (see e.g. Rayner and MacGillivray, 2002). This is a commonly used example for likelihood-free inference (see e.g. Drovandi and Pettitt, 2011; Dellaporta et al., 2022). It has no closed form density, and is instead defined in terms of its quantiles

$$Q\{z(p);\theta\} = a + b \left[1 + c \frac{1 - \exp\{-gz(p)\}\}}{1 + \exp\{-gz(p)\}} \right] \{1 + z(p)^2\}^k z(p),$$

where p denotes the g-and-k distribution's quantile of interest, z(p) the quantile of a standard normal, and $\theta = (a, b, c, g, k)^{\top}$ the model parameters. Throughout, we follow common practice and fix c = 0.8 (see Rayner and MacGillivray (2002) for a justification). While the likelihood function for the g-and-k distribution is not available in closed form, it can be approximated numerically to arbitrary precision using the strategy outlined in Rayner and MacGillivray (2002). Though far more cumbersome than approximations based on $\mathsf{L}^h_{m,n}$, this provides near-exact computations of the Bayes posterior and can be used as the benchmark against which we compare $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$.

The parameter g is the most challenging parameter for inference in the g-and-k model. Accordingly, in the main text we limit attention to inference on g, and defer results for the remaining parameters to Appendix A.5. Throughout, we generate n data points from the g-and-k model, and then compare $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ against near-exact computations of the Bayes posterior following the likelihood approximation strategy in Rayner and MacGillivray (2002). Figure 3 plots these posteriors for g for different m and n. The results reflect our theory: regardless of n, the approximation quality of $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ is very poor for smaller choices of m. In particular, $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ is biased and has higher variance than the target Bayes posterior. While bias and variance of $\overline{\pi}(\theta \mid \mathsf{L}^h_{m,n})$ decrease as m increases, the difference with the target is notable even for large m.

Though instructive, Figure 3 only shows posterior inferences based on a single data set.

To ascertain that the inflated variance and the bias are a problem across all possible data sets, we compare posteriors computations across 100 datasets and different choices of m and n. MCMC is used for posterior computations in each case so that 10 million model simulations of the full dataset are used in total for each setting. Thus, regardless of m and n the number of MCMC iterations is adjusted to ensure a roughly constant computational burden: for example, for m = 50 (m = 100) we run MCMC for 200,000 (100,000) iterations. Table 1 summarises the results of this analysis for g, and Appendix A.5 contains results on the remaining parameters. These simulations validate the findings of Figure 3 and of Lemma 2 by showing that m has to be chosen much larger than n for reliable posterior inferences.

A.5 Additional Numerical Results: G-and K

This section gives results on the KDE-based posterior for the a, b, k parameters in the gand-k model (see Tables 2, 3, 4, 5). The results show that the other parameters are less
impacted by the choice of m than g. This is due to the fact that their impact on the density
is not as nonlinear as the impact of g, which controls the skewness of the density. That
being said, our theory still applies, and the numerical results show that higher samples
sizes still require larger choices for m to obtain more accurate inferences. However, the
results are not as extreme as those for the g parameter.

n	m	mean bias	median bias	standard deviation	80%	90%	95%
100	$50 \cdot n$	-0.0472	-0.0565	0.1169	76	88	92
	$100 \cdot n$	-0.0313	-0.0398	0.1176	79	86	95
	$\boxed{1000 \cdot n}$	-0.0050	-0.0141	0.1228	83	88	95
1000	$50 \cdot n$	-0.0027	-0.0130	0.0354	85	91	94
	$100 \cdot n$	-0.0091	-0.0099	0.0352	86	92	94
	$1000 \cdot n$	-0.0027	-0.0032	0.0357	86	92	96

Table 2: Same results as in Table 1, but for parameter a.

n	m	mean bias	median bias	standard deviation	80%	90%	95%
100	$50 \cdot n$	0.1048	0.0770	0.2653	83	90	96
	$100 \cdot n$	0.1092	0.0816	0.2643	83	89	97
	$1000 \cdot n$	0.1077	0.0777	0.2628	80	89	97
1000	$50 \cdot n$	0.0171	0.0144	0.0770	86	95	99
	$100 \cdot n$	0.0155	0.0127	0.0758	86	96	99
	$1000 \cdot n$	0.0077	0.0052	0.0740	85	95	99

Table 3: Same results as in Table 1, but for parameter b.

n	m	mean bias	median bias	standard deviation	80%	90%	95%
100	$50 \cdot n$	-0.0409	-0.0516	0.1551	85	89	96
	$100 \cdot n$	-0.0396	-0.0492	0.1513	83	90	97
	$1000 \cdot n$	-0.0344	-0.0433	0.1431	80	90	96
1000	$50 \cdot n$	-0.0129	-0.0138	0.0443	83	92	98
	$100 \cdot n$	-0.0106	-0.0116	0.0435	84	91	98
	$\boxed{1000 \cdot n}$	-0.0043	-0.0052	0.0413	85	92	100

Table 4: Same results as in Table 1, but for parameter k.

n	m	ESS
100	$50 \cdot n$	2884
	$100 \cdot n$	2008
	$1000 \cdot n$	339
1000	$50 \cdot n$	5816
	$100 \cdot n$	4227
	$1000 \cdot n$	738

Table 5: Effective Sample Size (ESS) for the g-and-k example.

B Technical Lemmas

We now state and prove several lemmas used to help prove the main results of the paper. The first result uses Assumptions 2-4 to establish a concentration bound for $\pi(\theta \mid \mathsf{L}_n)$. We note that this result is not for the posterior $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$ but the idealised version $\pi(\theta \mid \mathsf{L}_n)$. To further simplify our notation in the appendix we write $\mathsf{L}_n(\theta)$ for $\mathsf{L}_n(\theta, y_{1:n})$.

B.1 Lemma 3

Lemma 3. Under Assumptions 2-4, for any $M_n \to \infty$ as $n \to \infty$,

$$P_0\left(\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|\pi(\theta \mid \mathsf{L}_n)d\theta > M_n/n^{1/2}\right) \lesssim 1/M_n.$$

Proof. The result follows a similar set of arguments to those in Alquier et al. (2016), Chérief-Abdellatif and Alquier (2020), as well as Theorem 1 in Matsubara et al. (2022). Critically, however, unlike these results we do not assume uniform boundedness of expectations for $L_n(\theta)$. To simplify certain arguments that follow, we assume without loss of generality that $L(\theta_0) = 0$. Note that this is without loss of generality as we can always redefine $L_n(\theta)$ (and $L(\theta)$) as $L_n(\theta) := L_n(\theta) - L(\theta_0)$ (and $L(\theta) := L(\theta) - L(\theta_0)$) without changing the resulting posterior distributions we study.

Firstly, since $0 = L(\theta_0) \le L(\theta)$ for all $\theta \in \Theta$, it follows that

$$\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta = \left| \int_{\Theta} \{\mathsf{L}(\theta) - \mathsf{L}(\theta_0)\} \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \right| = \left| \int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta - \mathsf{L}(\theta_0) \right|.$$

Hence, for any $\delta > 0$,

$$P_0\left(\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta > \delta\right) = P_0\left(\left|\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta - \mathsf{L}(\theta_0)\right| > \delta\right). \tag{8}$$

Now, note that

$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \le \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta + \int_{\Theta} |n^{-1} \mathsf{L}_n(\theta) - \mathsf{L}(\theta)| \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta.$$

From Assumption 3, with probability at least $1 - \delta$, $|n^{-1}\mathsf{L}_n(\theta) - \mathsf{L}(\theta)| \leq C\gamma(\theta)/\sqrt{n}\delta$ by Markov's inequality. By Assumption 3, $\int_{\Theta} \gamma(\theta)\pi(\theta \mid \mathsf{L}_n)\mathrm{d}\theta = C_1 < \infty$, so that we have

$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \le \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta + \frac{C_1}{\sqrt{n}\delta}.$$

Adding $n^{-1}\mathrm{KL}(\rho \| \pi) \geq 0$, and noting that $\pi(\theta \mid \mathsf{L}_n)$ solves (2) and satisfies $\int \gamma(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta < C$ for some constant C > 0 (Assumption 3), we have

$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta \leq \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta + n^{-1} \mathsf{KL} \{ \pi(\cdot \mid \mathsf{L}) \| \pi \} + \frac{C_1}{\sqrt{n} \delta} \\
\leq \inf_{\rho \in \mathcal{P}(\Theta): \int \gamma(\theta) \rho(d\theta) < C} \left\{ \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \rho(d\theta) + n^{-1} \mathsf{KL}(\rho \| \pi) \right\} + \frac{C_1}{\sqrt{n} \delta}. \tag{9}$$

By Assumption 3, it holds with probability at least $1 - \delta$ for the set of all $\rho \in \mathcal{P}(\Theta)$ for which $\int \gamma(\theta) \rho(\mathrm{d}\theta) < C$ simultaneously that

$$\int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \rho(\mathrm{d}\theta) \le \int_{\Theta} \mathsf{L}(\theta) \rho(\mathrm{d}\theta) + \frac{1}{\sqrt{n}\delta} \int_{\Theta} \gamma(\theta) \rho(\mathrm{d}\theta) \le \int_{\Theta} \mathsf{L}(\theta) \rho(\mathrm{d}\theta) + \frac{C_2}{\sqrt{n}\delta}. \tag{10}$$

Using this, we combine (9) and (2) to conclude that

$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \le \inf_{\rho \in \mathcal{P}(\Theta): \int \gamma(\theta) \rho(\mathrm{d}\theta) < C} \left\{ \int_{\Theta} \mathsf{L}(\theta) \rho(\mathrm{d}\theta) + n^{-1} \mathrm{KL}(\rho \| \pi) \right\} + \frac{C_1 + C_2}{\sqrt{n}\delta}.$$

Clearly, $L(\theta) \leq L(\theta_0) + |L(\theta) - L(\theta_0)|$, so that the RHS of the above is bounded above by

$$\int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \leq \inf_{\rho \in \mathcal{P}(\Theta): \int \gamma(\theta) \rho(\mathrm{d}\theta) < C} \left\{ \int_{\Theta} \{ \mathsf{L}(\theta) - \mathsf{L}(\theta_0) \} \rho(\mathrm{d}\theta) + n^{-1} \mathrm{KL}(\rho \parallel \pi) \right\} + \frac{C_1 + C_2}{\sqrt{n}\delta}.$$
(11)

Let

$$\widetilde{\rho}_n(\mathrm{d}\theta) := \begin{cases} \pi(\theta)/\Pi(\mathcal{B}_n) & \text{if } \theta \in \mathcal{B}_n, \\ 0 & \text{else,} \end{cases}$$

denote the prior measure normalized by the prior magnitude of the set $\mathcal{B}_n := \{\theta \in \Theta : \{\mathsf{L}(\theta) - \mathsf{L}(\theta_0)\} \le \alpha_1/\sqrt{n}\}$, and note that, by Assumption 4, $\Pi(\mathcal{B}_n) \ge \exp\{-\alpha_2\sqrt{n}\}$. Further,

we note tha $\widetilde{\rho}_n(\theta)$ is in the stated set, since $\int_{\Theta} \gamma(\theta)^2 \pi(\theta) d\theta < C$ by Assumption 3. It then follows that

$$\int_{\Theta} \{ \mathsf{L}(\theta) - \mathsf{L}(\theta_0) \} \widetilde{\rho}_n(\mathrm{d}\theta) \le \alpha_1 / \sqrt{n}, \quad n^{-1} \mathrm{KL} \{ \widetilde{\rho}_n || \pi \} = -\log \Pi(\mathcal{B}_n) \le \alpha_2 / \sqrt{n}.$$

The above terms deliver an upper bound for the first term in equation (11), so that for $0 < \delta < 1$,

$$\int_{\Theta} \{ \mathsf{L}(\theta) - \mathsf{L}(\theta_0) \} \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \le \frac{\alpha_1 + \alpha_2 + C_1 + C_2}{\sqrt{n}\delta},$$

with probability at least $1 - \delta$. From the above, and (8), we have that

$$P_0\left(\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \pi(\theta \mid \mathsf{L}_n) d\theta > \frac{\alpha_1 + \alpha_2 + C_1 + C_2}{\sqrt{n}\delta}\right) \le \delta. \tag{12}$$

Taking $\delta = \{(2C + \alpha_1 + \alpha_2)\}/M_n$ then yields

$$P_0\left(\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta > \frac{M_n}{\sqrt{n}},\right) \leq \frac{(2C + \alpha_1 + \alpha_2)}{M_n} \lesssim M_n^{-1},$$

as desired. \Box

B.2 Lemma 4

We use the following general result to bound the difference between the infeasible posterior $\pi(\theta \mid \mathsf{L}_n)$, in equation (13), and $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$. As far as we are aware, the following result is similar to, but distinct from, arguments in the literature that characterize sub-Gaussian behavior of random variables; see, e.g., Theorem 2.1 in Boucheron et al. (2013).

Lemma 4. Let Z be a positive, scalar-valued random variable with mean μ that satisfies $\mathbb{E}\left[(Z-\mu)^2\right] \leq Bb^2$ for some B,b>0. Then, for any $0\leq \lambda \leq 1/b$,

$$\mathbb{E}\left[\exp(-\lambda Z)\right] \le \exp(-\lambda \mu) + B(\lambda b)^2.$$

Proof of Lemma 4. For any $Z \geq 0$, a Taylor expansion of $\exp(-\lambda Z)$ around $Z = \mu$ yields

$$\exp(-\lambda Z) = \exp(-\lambda \mu) \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} (Z - \mu)^n = \tau(Z) + \mathcal{E}(Z),$$

where $\tau(Z)$ is the truncation of the series

$$\tau(Z) := \exp(-\lambda \mu) \left\{ 1 - \lambda (Z - \mu) \right\}$$

and

$$\mathcal{E}(Z) := \exp(-\lambda \zeta) \frac{\lambda^2}{2!} (Z - \mu)^2$$

is the Lagrange remainder term with ζ a constant between Z and μ .

Boundedness of $\exp(-x)$ over $x \ge 0$, implies that for any $\lambda \ge 0$ and $Z \ge 0$

$$\mathcal{E}(Z) \le \lambda^2 (Z - \mu)^2$$
.

Consequently,

$$\exp(-\lambda Z) = \tau(Z) + \mathcal{E}(Z) \le \exp(-\lambda \mu) \left\{ 1 - \lambda(Z - \mu) \right\} + \lambda^2 (Z - \mu)^2.$$

By hypothesis, $\mathbb{E}\left[(Z-\mu)^2\right] \leq Bb^2 < \infty$ for some B,b>0. Take expectations of both sides to obtain

$$\mathbb{E}\left[\exp(-\lambda Z)\right] \le \exp(-\lambda \mu) \left\{1 - \lambda \mathbb{E}(Z - \mu)\right\} + \lambda^2 \mathbb{E}(Z - \mu)^2$$
$$\le \exp(-\lambda \mu) + B(\lambda b)^2.$$

C Proofs of Main Results

Recall that $\mathsf{L}_{m,n}(\theta,y_{1:n})$ depends on a sequence of random variables $U_{1:m} \stackrel{iid}{\sim} P_{\theta}$, where the dependence on $U_{1:m}$ was previously subsumed via the dependence on the index m. Define $g_n^{\omega}(\theta) := \exp\{-\omega \cdot \mathsf{L}_n(\theta)\}$ and recall the generalized posterior:

$$\pi(\theta \mid \mathsf{L}_n) = \frac{\exp\{-\omega \cdot \mathsf{L}_n(\theta)\}\pi(\theta)}{\int_{\Theta} \exp\{-\omega \cdot \mathsf{L}_n(\theta)\}\pi(\theta)\mathrm{d}\theta} \equiv \frac{g_n^{\omega}(\theta)\pi(\theta)}{\int_{\Theta} g_n^{\omega}(\theta)\pi(\theta)\mathrm{d}\theta},\tag{13}$$

which is the (generalized) posterior that would result if we were able to analytically calculate $L_n(\theta)$. In addition, let us simplify our notation and subsume the dependence of $L_{m,n}(\theta, y_{1:n})$ on $y_{1:n}$ via the index n and write $L_{m,n}(\theta)$ for $L_{m,n}(\theta, y_{1:n})$.

C.1 Theorem 1

Theorem 1. If Assumptions 1 and 2 are satisfied, then for all $\xi \in [0,2]$, as $m \to \infty$

$$\int_{\Theta} \|\theta\|^{\xi} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| d\theta \lesssim m^{-\min\{\kappa_1, \kappa_2\}}$$

which also implies that

$$\int_{\Theta} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| \, \mathrm{d}\theta \lesssim m^{-\min\{\kappa_1, \kappa_2\}},$$

$$\left\| \int_{\Theta} \theta \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \, \mathrm{d}\theta - \int_{\Theta} \theta \pi(\theta \mid \mathsf{L}_n) \, \mathrm{d}\theta \right\| \lesssim m^{-\min\{\kappa_1, \kappa_2\}}.$$

Proof. We first demonstrate that, uniformly over Θ ,

$$\mathbb{E}_{U_{1:m}}[\exp\{-\omega \mathsf{L}_{m,n}(\theta)\}] = \exp\{-\omega \mathsf{L}_{n}(\theta)\} \left[1 + C\left\{\frac{\sigma_{n}^{2}(\theta)}{m^{\kappa}}\right\}\right] + C\left\{\sigma_{n}^{2}(\theta)/m^{\kappa_{2}}\right\}, \quad (14)$$

where the dependence of $\mathbb{E}_{U_{1:m}}$ on θ and $y_{1:n}$ is suppressed for notational simplicity, and where we remind the reader that $U_{1:m} \stackrel{iid}{\sim} P_{\theta}$.

From Assumption 1,

$$\mathbb{E}_{U_{1:m}}\{\mathsf{L}_{m,n}(\theta)-\mathsf{L}_n(\theta)\}\lesssim \frac{\sigma_n^2(\theta)}{m^{\kappa_1}}, \text{ and } \mathbb{E}_{U_{1:m}}[\{\mathsf{L}_{m,n}(\theta)-\mathbb{E}_{U_{1:m}}[\mathsf{L}_{m,n}(\theta)]\}^2]\lesssim \frac{\sigma_n^2(\theta)}{m^{\kappa_2}},$$

where, throughout the remainder of this proof, the notation \lesssim is in relation to sequences in m for fixed n (as the result is true for fixed n). Take $Z = \mathsf{L}_{m,n}(\theta)$, $\mu = \mathbb{E}_{U_{1:m}} \mathsf{L}_{m,n}(\theta)$, $\lambda = \omega$, and apply Lemma 4 to obtain the following upper bound for $\mathbb{E}_{U_{1:m}}[\exp\{-\omega \mathsf{L}_{m,n}(\theta)\}]$: with $b = m^{-0.5 \cdot \kappa_2}$, $m \ge 1$, and $B = C\sigma_n^2(\theta)$, for some C > 0 such that $\mathbb{E}_{U_{1:m}}[\{\mathsf{L}_{m,n}(\theta) - \mathsf{L}_n(\theta)\}^2] \le Bb^2$,

$$\begin{split} \mathbb{E}_{U_{1:m}}[\exp\{-\omega\mathsf{L}_{m,n}(\theta)\}] &\leq \exp\{-\omega\mathbb{E}_{U_{1:m}}[\mathsf{L}_{m,n}(\theta)]\} + B(\lambda b)^2 \\ &\leq \exp[-\omega\mathsf{L}_n(\theta) + C\{\omega\sigma_n^2(\theta)/m^{\kappa_1}\}] + C\{\sigma_n^2(\theta)/m^{\kappa_2}\} \\ &\lesssim \exp\{-\omega\mathsf{L}_n(\theta)\}[1 + \{\omega\sigma_n^2(\theta)/m^{\kappa_1}\}] + \sigma_n^2(\theta)/m^{\kappa_2}. \end{split}$$

Where the first inequality uses Assumption 1 part (ii), and the second holds due to the fact that $\exp\{\omega\sigma_n^2(\theta)/m^{\kappa_1}\}=1+\exp\{\omega\sigma_n^2(\theta)/m^{\kappa_1}\}-1\lesssim 1+\{\omega\sigma_n^2(\theta)/m^{\kappa_1}\}.$

Define $\widehat{g}_n^{\omega}(\theta) = \mathbb{E}_{U_{1:m}}[\exp\{-\omega \mathsf{L}_{m,n}(\theta)\}]$. From equation (14) and the definitions of $\widehat{g}_n^{\omega}(\theta)$ and $g_n^{\omega}(\theta)$,

$$|\widehat{g}_n^{\omega}(\theta) - g_n^{\omega}(\theta)| \le g_n^{\omega}(\theta)C\left\{\sigma_n^2(\theta)/m^{\kappa_1}\right\} + C\left\{\sigma_n^2(\theta)/m^{\kappa_2}\right\}. \tag{15}$$

By Assumption 2, for any $n \ge 1$, $\int_{\Theta} g_n^{\omega}(\theta) \pi(\theta) d\theta < \infty$, so that, applying (15),

$$\left| \int_{\Theta} \widehat{g}_{n}^{\omega}(\theta) \pi(\theta) d\theta - \int_{\Theta} g_{n}^{\omega}(\theta) \pi(\theta) d\theta \right| \leq \int_{\Theta} \left| \widehat{g}_{n}^{\omega}(\theta) - g_{n}^{\omega}(\theta) | \pi(\theta) d\theta \right|$$

$$\lesssim \frac{1}{m^{\kappa_{1}}} \int_{\Theta} \sigma_{n}^{2}(\theta) g_{n}^{\omega}(\theta) \pi(\theta) d\theta + \frac{1}{m^{\kappa_{2}}} \int_{\Theta} \sigma_{n}^{2}(\theta) \pi(\theta) d\theta$$

$$= \frac{1}{m^{\kappa_{1}}} \int_{\Theta} g_{n}^{\omega}(\theta) \pi(\theta) d\theta \int_{\Theta} \sigma_{n}^{2}(\theta) \pi(\theta) | L_{n} d\theta$$

$$+ \frac{1}{m^{\kappa_{2}}} \int_{\Theta} \sigma_{n}^{2}(\theta) \pi(\theta) d\theta$$

where the second line follows from equation (15), and the equality in the third line from

$$\int \sigma_n^2(\theta) g_n^{\omega}(\theta) \pi(\theta) d\theta = \int g_n^{\omega}(\theta) \pi(\theta) d\theta \cdot \left(\frac{\int \sigma_n^2(\theta) g_n^{\omega}(\theta) \pi(\theta) d\theta}{\int g_n^{\omega}(\theta) \pi(\theta) d\theta} \right)
= \int g_n^{\omega}(\theta) \pi(\theta) d\theta \cdot \int \sigma_n^2(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta.$$
(16)

Notice that Assumption 2 ensures that $\int_{\Theta} \sigma_n^2(\theta) \|\theta\|^2 \pi(\theta) d\theta$ and $\int_{\Theta} \sigma_n^2(\theta) \|\theta\|^2 \pi(\theta \mid \mathsf{L}_n) d\theta$ are finite for any n. Since it also holds for any $\xi \in [0,2]$ that $\int_{\Theta} \sigma_n^2(\theta) \|\theta\|^{\xi} \pi(\theta) d\theta \leq \int_{\Theta} \sigma_n^2(\theta) \|\theta\|^2 \pi(\theta) d\theta$ and $\int_{\Theta} \sigma_n^2(\theta) \|\theta\|^{\xi} \pi(\theta \mid \mathsf{L}_n) d\theta \leq \int_{\Theta} \sigma_n^2(\theta) \|\theta\|^2 \pi(\theta \mid \mathsf{L}_n) d\theta$, we may apply the exact same arguments as before to conclude that

$$\left| \int_{\Theta} \|\theta\|^{\xi} \widehat{g}_{n}^{\omega}(\theta) \pi(\theta) d\theta - \int_{\Theta} \|\theta\|^{\xi} g_{n}^{\omega}(\theta) \pi(\theta) d\theta \right| \lesssim \frac{1}{m^{\kappa_{1}}} \int_{\Theta} \|\theta\|^{\xi} \sigma_{n}^{2}(\theta) \pi(\theta \mid \mathsf{L}_{n}) d\theta + \frac{1}{m^{\kappa_{2}}} \int_{\Theta} \|\theta\| \sigma_{n}^{2}(\theta) \pi(\theta) d\theta.$$

$$(17)$$

The above display can be re-expressed as

$$\left| \int_{\Theta} \|\theta\|^{\xi} \widehat{g}_{n}^{\omega}(\theta) \pi(\theta) d\theta - \int_{\Theta} \|\theta\|^{\xi} g_{n}^{\omega}(\theta) \pi(\theta) d\theta \right| \lesssim m^{-\min\{\kappa_{1}, \kappa_{2}\}}. \tag{18}$$

Aplying equation (18) twice with $\xi = 0$, once for the numerator and once for the denominator, yields

$$\frac{\left| \int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta - \int_{\Theta} g_n^{\omega}(\theta) \pi(\theta) d\theta \right|}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta} \lesssim m^{-\min\{\kappa_1, \kappa_2\}}.$$
 (19)

Write $\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L})$ as

$$\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n) = \frac{\widehat{g}_n^{\omega}(\theta)\pi(\theta)}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta)\pi(\theta)d\theta} - \frac{g_n^{\omega}(\theta)\pi(\theta)}{\int_{\Theta} g_n^{\omega}(\theta)\pi(\theta)d\theta} \\
= \left\{ \widehat{g}_n^{\omega}(\theta) - g_n^{\omega}(\theta) \right\} \frac{\pi(\theta)}{\int_{\Theta} g_n^{\omega}(\theta)\pi(\theta)d\theta} \frac{\int_{\Theta} g_n^{\omega}(\theta)\pi(\theta)d\theta}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta)\pi(\theta)d\theta} \\
- g_n^{\omega}(\theta)\pi(\theta) \left(\frac{1}{\int_{\Theta} g_n^{\omega}(\theta)\pi(\theta)d\theta} - \frac{1}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta)\pi(\theta)d\theta} \right),$$

and apply the triangle inequality to obtain

$$|\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| \leq |\widehat{g}_n^{\omega}(\theta) - g_n^{\omega}(\theta)| \frac{\pi(\theta)}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta} + \frac{\left| \int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta - \int_{\Theta} g_n^{\omega}(\theta) \pi(\theta) d\theta \right|}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta} \pi(\theta \mid \mathsf{L}_n).$$

Multiplying by $\|\theta\|^{\xi}$, integrating both sides, and using the exact same line of arguments as above, leads to

$$\int_{\Theta} \|\theta\|^{\xi} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_{n})| \, \mathrm{d}\theta \leq \frac{1}{\int_{\Theta} \widehat{g}_{n}^{\omega}(\theta)\pi(\theta) \, \mathrm{d}\theta} \int_{\Theta} \|\theta\|^{\xi} |\widehat{g}_{n}^{\omega}(\theta) - g_{n}^{\omega}(\theta)| \, \pi(\theta) \, \mathrm{d}\theta
+ \frac{\left|\int_{\Theta} \widehat{g}_{n}^{\omega}(\theta)\pi(\theta) \, \mathrm{d}\theta - \int_{\Theta} g_{n}^{\omega}(\theta)\pi(\theta) \, \mathrm{d}\theta\right|}{\int_{\Theta} \widehat{g}_{n}^{\omega}(\theta)\pi(\theta) \, \mathrm{d}\theta} \int_{\Theta} \|\theta\|^{\xi} \pi(\theta \mid \mathsf{L}_{n}) \, \mathrm{d}\theta.$$
(20)

By (18)-(19), we have that

$$\frac{1}{\int_{\Theta} \widehat{g}_n^{\omega}(\theta) \pi(\theta) d\theta} \int_{\Theta} \|\theta\|^{\xi} |\widehat{g}_n^{\omega}(\theta) - g_n^{\omega}(\theta)| \pi(\theta) \lesssim \frac{m^{-\min\{\kappa_1, \kappa_2\}}}{\int_{\Theta} g_n^{\omega}(\theta) \pi(\theta) d\theta} \lesssim m^{-\min\{\kappa_1, \kappa_2\}}$$

where the second inequality follows since $0 < \int_{\Theta} g_n^{\omega}(\theta) \pi(\theta) d\theta < \infty$ by Assumption 2. For the second term in (20), first note that by Assumption 2, $\int_{\Theta} \|\theta\|^{\xi} \pi(\theta \mid \mathsf{L}_n) d\theta < \infty$, hence, applying again (19) shows that this term is also bounded by a multiple of $m^{-\min\{\kappa_1,\kappa_2\}}$. This completes the proof.

C.2 Theorem 2

Theorem 2. If Assumptions 1-4, are satisfied, then for any $M_n \to \infty$, it holds that

$$P_0\left(\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta > M_n / \min\{\sqrt{n}, m^{\min\{\eta_1, \eta_2\}}\}\right) \longrightarrow 0.$$

Proof. We apply Theorem 1 and Lemma 3 to derive a concentration bound for $\overline{\pi}(\theta \mid \mathsf{L}_{m,n})$. This result relies on asymptotics in (m,n) and so we express m=m(n), but suppress the dependence on n for notational simplicity. As in the proof of Lemma 3, we assume without loss of generality that $\mathsf{L}(\theta_0)=0$, since we can always redefine $\mathsf{L}_n(\theta)$ (and $\mathsf{L}(\theta)$) as $\mathsf{L}_n(\theta):=\mathsf{L}_n(\theta)-\mathsf{L}(\theta_0)$ (and $\mathsf{L}(\theta)=\mathsf{L}(\theta)-\mathsf{L}(\theta_0)$) without altering the posterior distribution we study.

Define $\kappa = \min{\{\kappa_1, \kappa_2\}}$. Using similar arguments to those of Lemma 3,

$$\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta = \left| \int_{\Theta} \mathsf{L}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta - \mathsf{L}(\theta_0) \right|.$$

Similarly, with probability at least $1 - \delta$, we have

$$\int_{\Theta} \mathsf{L}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta \leq \int_{\Theta} \mathsf{L}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta + \int_{\Theta} |\mathsf{L}(\theta) - n^{-1} \mathsf{L}_{n}(\theta)| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta \\
\leq \int_{\Theta} n^{-1} \mathsf{L}_{n}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta + \frac{1}{\sqrt{n}\delta} \int_{\Theta} \gamma(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta.$$

We can decompose the second term into two components that both converge to zero. The first of these vanishes by Theorem 1, and the second by Assumption 2 via

$$\frac{1}{\sqrt{n}} \int_{\Theta} \gamma(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta \leq \frac{1}{\sqrt{n}} \int_{\Theta} \gamma(\theta) |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| d\theta + \frac{1}{\sqrt{n}} \int_{\Theta} \gamma(\theta) \pi(\theta \mid \mathsf{L}_n) d\theta
\leq \frac{C}{\sqrt{n} m^{\kappa} \delta} + \frac{C}{\sqrt{n} \delta},$$
(21)

where the second line utilizes a moment bound for $\gamma(\theta)$ in Assumption 3(i). Focusing on $\int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta$, we obtain the upper bound

$$\int_{\Theta} n^{-1} \mathsf{L}_{n}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta = \int_{\Theta} \mathsf{L}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta + \int_{\Theta} \{n^{-1} \mathsf{L}_{n}(\theta) - \mathsf{L}(\theta)\} \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta
\leq \int_{\Theta} \mathsf{L}(\theta) \{ \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_{n}) \} d\theta + \int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_{n}) d\theta
+ \int_{\Theta} \{n^{-1} \mathsf{L}_{n}(\theta) - \mathsf{L}(\theta)\} \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta. \tag{22}$$

Consider the last term in (22). Similarly as in (21), by Theorem 1, Assumptions 2 and 3,

and with probability at least $1 - \delta$, we have the bound

$$\int_{\Theta} \{n^{-1}\mathsf{L}_{n}(\theta) - \mathsf{L}(\theta)\}\overline{\pi}(\theta \mid \mathsf{L}_{m,n})d\theta \leq \frac{1}{\sqrt{n}\delta} \int_{\Theta} \gamma(\theta)\overline{\pi}(\theta \mid \mathsf{L}_{m,n})d\theta
\leq \frac{1}{\sqrt{n}\delta} \int_{\Theta} \gamma(\theta) \{\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_{n})\} d\theta
+ \frac{1}{\sqrt{n}} \int_{\Theta} \gamma(\theta)\pi(\theta \mid \mathsf{L}_{n})d\theta
\leq \frac{C}{\sqrt{\delta^{2}m^{2\kappa}n}} + \frac{C}{\delta\sqrt{n}}.$$
(23)

Consider the second term in (22), and note that, again with probability at least $1 - \delta$,

$$\begin{split} \int_{\Theta} \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta &= \int_{\Theta} \{ \mathsf{L}(\theta) - n^{-1} \mathsf{L}_n(\theta) \} \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta + \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \\ &\leq \frac{1}{\sqrt{n}\delta} \int_{\Theta} \gamma(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta + \int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta. \end{split}$$

Using similar arguments as those in (10) and Lemma 3, for some C > 0, with probability at least $1 - \delta$

$$\int_{\Theta} n^{-1} \mathsf{L}_n(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \le \frac{C}{\sqrt{n}\delta}.$$
 (24)

Recall that, from Assumption 4, for m = m(n) and $n \to \infty$, $m^{-\kappa_2} \int_{\Theta} \sigma_n^2(\theta) \mathsf{L}(\theta) \pi(\theta) \mathrm{d}\theta \approx m^{-\eta_1}$ and $m^{-\kappa_1} \int_{\Theta} \sigma_n^2(\theta) \mathsf{L}(\theta) \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \approx m^{-\eta_2}$ for some $\eta_1, \eta_2 > 0$. Under this assumption, using similar arguments as those that led to (20) in the second half of the proof of Theorem 1 we obtain the bound

$$\int_{\Theta} \mathsf{L}(\theta) \left| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_{n}) \right| d\theta$$

$$\lesssim m^{-\min\{\eta_{1},\eta_{2}\}} + m^{-\min\{\eta_{1},\eta_{2}\}} \cdot \frac{\int_{\Theta} g_{n}^{\omega} (y_{1:n} \mid \theta) \pi(\theta) d\theta}{\int_{\Theta} \widehat{g}_{n}^{\omega} (y_{1:n} \mid \theta) \pi(\theta) d\theta} \left\{ \int_{\Theta} \mathsf{L}(\theta) \pi (\theta \mid \mathsf{L}_{n}) d\theta \right\}$$

$$\approx m^{-\min\{\eta_{1},\eta_{2}\}}.$$
(25)

Collecting the terms in (21)-(25), we have that, with probability at least $1 - \delta$, for some C > 0,

$$\int_{\Theta} \mathsf{L}(\theta) \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta \lesssim \mathsf{L}(\theta_0) + \frac{1}{\delta} \left(\frac{C}{\sqrt{nm^{2\kappa}}} + \frac{3C}{\sqrt{n}} + \frac{C}{m^{\min\{\eta_1,\eta_2\}}} \right).$$

For $0 < \delta < 1$, we then have that

$$P_0\left\{\int_{\Theta}|\mathsf{L}(\theta)-\mathsf{L}(\theta_0)|\overline{\pi}(\theta\mid\mathsf{L}_{m,n})\mathrm{d}\theta>\frac{1}{\delta}\left(\frac{C}{\sqrt{nm^{2\kappa}}}+\frac{C}{\sqrt{n}}+\frac{C}{m^{\min\{\eta_1,\eta_2\}}}\right)\right\}\leq\delta.$$

Noting that $\sqrt{n} \geq \sqrt{nm^{2\kappa}}$, and taking $\delta = C/M_n$,

$$P_0\left\{\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta > \frac{M_n}{\min\{\sqrt{n}, m^{\min\{\eta_1, \eta_2\}}\}}\right\} \leq C/M_n \lesssim 1/M_n.$$

C.3 Corollary 1

Corollary 1. Under Assumptions 1-4, and if on some open neighborhood of θ_0 , $\|\theta - \theta_0\| \le C|\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|^{\alpha}$ for some C > 0 and $\alpha > 0$, then for any $M_n \to \infty$ and the sequence $S_n = \{\theta \in \Theta : \|\theta - \theta_0\| > C(M_n/\min\{\sqrt{n}, m^{\min\{\eta_1, \eta_2\}}\})^{\alpha}\}$, it holds P_0 -almost surely that

$$\int_{S_n} \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \lesssim M_n^{-1}.$$

Proof. The result follows by appropriately modifying the argument used in, e.g., Corollary 2 of Bernton et al. (2019a). Define $m_{\eta,n} := \min\{m^{\min\{\eta_1,\eta_2\}}, \sqrt{n}\}$. Firstly, recall that, for $A \subset \Theta$,

$$\overline{\Pi}(\theta \in A \mid \mathsf{L}_{m,n}) = \int_{A} \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) \mathrm{d}\theta.$$

By Markov's inequality, and Theorem 2, we have that

$$\overline{\Pi}\left\{|\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \ge M_n/m_{\eta,n} \mid \mathsf{L}_{m,n}\right\} \le \frac{\mathbb{E}\left[\int_{\Theta} |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta\right]}{M_n/m_{\eta,n}} \lesssim M_n^{-1}. \quad (26)$$

Let $\delta > 0$ be such that $\{\theta \in \Theta : \|\theta - \theta_0\| \le \delta\} \subset V := \{\theta \in \Theta : \|\theta - \theta_0\| \le C |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|^{\alpha}\}$. By Assumption 3, for any $\delta > 0$ such that $\|\theta - \theta_0\| > \delta$, there exists ϵ such that $0 < \epsilon \le |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|$. Let $\epsilon_n = \{M_n/m_{\eta,n}\} \to 0$ as $n \to \infty$, and let n be large enough such that $\epsilon_n := \{M_n/m_{\eta,n}\} < \epsilon$, which implies that

$$\{\theta \in \Theta : |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \le \epsilon_n\} \subseteq \{\theta \in \Theta : \|\theta - \theta_0\| \le \delta\} \subset V.$$

Hence, from equation (26) in the proof of Theorem 2, for any $M_n \to \infty$,

$$1 - M_n^{-1} \gtrsim \overline{\Pi} \left\{ |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)| \le M_n / m_{\eta,n} \mid \mathsf{L}_{m,n} \right\} = \overline{\Pi} \left\{ C |\mathsf{L}(\theta) - \mathsf{L}(\theta_0)|^{\alpha} \le C \left\{ M_n / m_{\eta,n} \right\}^{\alpha} \mid \mathsf{L}_{m,n} \right\}$$
$$\ge \overline{\Pi} \left\{ \|\theta - \theta_0\| \le C \left\{ M_n / m_{\eta,n} \right\}^{\alpha} \mid \mathsf{L} \right\}.$$

C.4 Corollary 2

Corollary 2. If Assumptions 1-4 are satisfied, and if $\int_{\Theta} \left| \pi(\theta \mid \mathsf{L}_n) - \mathcal{N}(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right| d\theta = o_p(1)$, then for $\bar{\theta}_n = \int_{\Theta} \theta \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta$, we have

(i)
$$\int_{\Theta} \left| \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \mathcal{N}(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right| d\theta = o_p(1) \text{ if } m^{-\min\{\eta_1, \eta_2\}} \to 0 \text{ as } n \to \infty;$$

(ii)
$$\|\sqrt{n}(\hat{\theta}_n - \bar{\theta}_n)\| = o_p(1)$$
 if $\sqrt{n}m^{-\min\{\eta_1,\eta_2\}} \to 0$ as $n \to \infty$.

Proof. By the triangle inequality

$$\int_{\Theta} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \mathcal{N}(\theta; \hat{\theta}_n, \mathsf{H}^{-1}/n)| d\theta \leq \int_{\Theta} |\overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_n)| d\theta + \int_{\Theta} |\pi(\theta \mid \mathsf{L}_n) - \mathcal{N}(\theta; \hat{\theta}_n, \mathsf{H}^{-1}/n)| d\theta.$$

Taking $L(\theta) = 1$ in the proof of Theorem 2, see equation (25), we can show that $\int_{\Theta} |\overline{\pi}(\theta)| = L_{m,n} - \pi(\theta \mid L_n) |d\theta = O_p(m^{-\min\{\eta_1,\eta_2\}})$, which implies that $\int_{\Theta} |\overline{\pi}(\theta \mid L_{m,n}) - \mathcal{N}(\theta; \hat{\theta}_n, \mathsf{H}^{-1}/n)|d\theta \le O_p(m^{-\min\{\eta_1,\eta_2\}}) + o_p(1)$. Hence, for $m^{-\min\{\eta_1,\eta_2\}} \to 0$, the first stated result follows.

To obtain the second result, decompose $\bar{\theta}_n$ as

$$\bar{\theta}_{n} = \int \theta \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) d\theta = \int \theta \left\{ \overline{\pi}(\theta \mid \mathsf{L}_{m,n}) - \pi(\theta \mid \mathsf{L}_{n}) \right\} d\theta + \int \theta \pi(\theta \mid \mathsf{L}_{n}) d\theta.$$

Applying Theorem 1, we have

$$\begin{split} \bar{\theta}_n = & O_p(m^{-\min\{\eta_1,\eta_2\}}) + \int \theta \pi(\theta \mid \mathsf{L}_n) \mathrm{d}\theta \\ = & O_p(m^{-\min\{\eta_1,\eta_2\}}) + \int \theta \left\{ \pi(\theta \mid \mathsf{L}_n) - N(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right\} \mathrm{d}\theta + \int \theta N(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \mathrm{d}\theta \\ = & \hat{\theta}_n + O_p(m^{-\min\{\eta_1,\eta_2\}}) + \int \theta \left\{ \pi(\theta \mid \mathsf{L}_n) - N(\theta; \hat{\theta}_n, n^{-1}\mathsf{H}^{-1}) \right\} \mathrm{d}\theta. \end{split}$$

Applying the change of variable $t = \sqrt{n}(\theta - \hat{\theta}_n)$,

$$\bar{\theta}_n - \hat{\theta}_n = O_p(m^{-\min\{\eta_1,\eta_2\}}) + \frac{1}{\sqrt{n}} \int t \left\{ \pi(t \mid \mathsf{L}_n) - N(t;0,\mathsf{H}^{-1}) \right\} dt$$

Since the total variation distance is invariant to changes of location and scale, from the hypothesis of Corollary 2, we have that $\int t \{\pi(t \mid \mathsf{L}_n) - N(t; 0, \mathsf{H}^{-1})\} dt = o_p(1)$, and we can re-arrange the above equation as

$$\sqrt{n}(\bar{\theta}_n - \hat{\theta}_n) = O_p(\sqrt{n}m^{-\min\{\eta_1,\eta_2\}}) + o_p(1).$$

So long as $\sqrt{n}m^{-\min\{\eta_1,\eta_2\}} \to 0$ as $n \to \infty$, the stated result follows.

D Applications

D.1 β -divergence

Recalling the definitions of L_n^β and $\mathsf{L}_{m,n}^\beta$ in Section 2.3.1, it is straightforward to verify Assumption 1. In particular, it is immediately clear that $\mathbb{E}_{U_{1:m}\overset{\text{iid}}{\sim}P_{\theta}}\left[\mathsf{L}_{m,n}^\beta(\theta,y_{1:n})\right] = \mathsf{L}_n^\beta(\theta,y_{1:n})$, so that part (i) holds automatically for any $\kappa_1 > 0$ and $\sigma_n^2(\theta) = 0$. For part (ii), note that

$$\mathbb{E}_{U_{1:m} \overset{\text{iid}}{\sim} P_{\theta}} \left[\left\{ \mathsf{L}_{m,n}^{\beta}(\theta, y_{1:n}) - \mathsf{L}_{n}^{\beta}(\theta, y_{1:n}) \right\}^{2} \right] = \frac{n^{2}}{m} \mathbb{E}_{U_{j} \sim P_{\theta}} \left[p_{\theta}^{2\beta}(U_{j}) \right] = \frac{n^{2}}{m} \int p_{\theta}^{1+2\beta}(u) du,$$

so that Assumption 1 part (ii) holds for $\sigma_n^2(\theta) = n^2 \int p_{\theta}^{1+2\beta}(u) du$ and $\kappa_2 = 1$.

To verify Assumption 3, notice that if the data is independently and identically distributed, part (i) is easy to establish: since $\mathbb{E}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} \left[\mathsf{L}_n^{\beta}(\theta; Y_{1:n}) \right] = \mathsf{L}(\theta) = n \cdot d_{\beta}(p_0, p_{\theta}),$ the expression on the left of (i) is equivalent to $\operatorname{Var}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} (n^{-1} \mathsf{L}_n(\theta, Y_{1:n}))$. Further,

$$\operatorname{Var}_{Y_{1:n} \stackrel{\text{iid}}{\sim} P_0} \left(n^{-1} \mathsf{L}_n(\theta, Y_{1:n}) \right) = \left(1 + \frac{1}{\beta} \right)^2 n^{-1} \operatorname{Var}_{Y_i \sim P_0} \left(p_{\theta}(Y_i) \right),$$

so that Assumption 3 part (i) holds for $C = \left(1 + \frac{1}{\beta}\right)^2$ and $\gamma(\theta) = \sqrt{\operatorname{Var}_{Y_i \sim P_0}(p_{\theta}(Y_i))}$.

Notice that the moment conditions on γ^2 in Assumption 3 now become equivalent to the mild assumption that $\operatorname{Var}_{Y_i \sim P_0}(p_{\theta}(Y_i))$ has finite prior and posterior expectation.

For Assumption 4 part (ii), recall that $\sigma_n^2(\theta) = n^2 \int p_{\theta}^{1+2\beta}(u) du$ and $\kappa_2 = 1$. Therefore, provided that $\int (\int p_{\theta}^{1+2\beta}(u) du) \pi(\theta) d\theta < \infty$, we require $m^{-1} \cdot n^2 \approx m^{-\eta_2}$. Taking $m \approx n^{\xi}$, this leads to the requirement $2 - \xi = -\xi \eta_2$. To make posterior computation simultaneously as accurate and inexpensive as possible, we wish to pick the smallest ξ that still achieves the fastest possible posterior concentration rate of $\overline{\pi}(\theta \mid \mathsf{L}_{n,m}^{\beta})$. Since $m \approx n^{\xi}$, Theorem 2 tells us that such a ξ must satisfy $n^{\xi\eta_2} \approx n^{0.5}$, which leads to the relation $\xi\eta_2 = 0.5$. Plugging this into the first relation, we thus find that $\xi = 2.5$, which yields $\eta_2 = 0.2$.

D.2 Lemma 1

Lemma 1. If the kernel $k: \mathcal{Y}^2 \to \mathbb{R}$ is characteristic, and if $\sup_{y,y'} k(y,y') \leq K$, then for $\mathsf{L}^k_{m,n}$ in (4), Assumption 1 holds for $\sigma^2(\theta) = 16K^4 + 2K$, and $\kappa_1 = \kappa_2 = 1$. If it additionally holds that $y_{1:n}$ is independently and identically distributed, then we also have that Assumption 3 part (i) holds with $\gamma^2(\theta) = 16K^4 + 2K$ and C = 1.

Proof. We provide the argument for Assumption 1 only, as the arguments for Assumption 3 part (i) follow along the exact same lines. First, decomposing the mean square error into squared bias and variance, we find

$$\begin{split} & \mathbb{E}_{U}\left[\left(\mathsf{L}_{m,n}(\theta,y_{1:n}) - \mathsf{L}_{n}(\theta,y_{1:n})\right)^{2}\right] \\ & = \left\{\mathbb{E}_{U}\left[\mathsf{L}_{m,n}(\theta,y_{1:n})\right] - \mathsf{L}_{n}(\theta,y_{1:n})\right\}^{2} + \mathbb{E}_{U}\left[\left\{\mathsf{L}_{m,n}(\theta,y_{1:n}) - \mathbb{E}_{U}\left[\mathsf{L}_{m,n}(\theta,y_{1:n})\right]\right\}^{2}\right]. \end{split}$$

Since both terms in the last line are non-negative, this means that to complete our proof, it suffices to show that

$$\mathbb{E}_{U}\left[\left(\mathsf{L}_{m,n}(\theta,y_{1:n})-\mathsf{L}_{n}(\theta,y_{1:n})\right)^{2}\right] \leq \frac{16K^{4}}{m}.$$

To show that this bound indeed holds, note that by definition, $\mathsf{L}^k_{m,n}(\theta,y_{1:n}) = \mathrm{MMD}^2(P_{m,\theta},P_n)$ and $\mathsf{L}^k_n(\theta,y_{1:n}) = \mathrm{MMD}^2(P_{\theta},P_n)$, where $P_n = \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$ and $P_{m,\theta} = \frac{1}{m}\sum_{j=1}^m \delta_{u_j}$ for $u_{1:m} \stackrel{\text{iid}}{\sim} P_{\theta}$. Using this and $|a^2 - b^2| = (a+b)|a-b|$, it is straightforward to show that

$$\begin{aligned} & \left| \mathsf{L}_{m,n}^{k}(\theta, y_{1:n}) - \mathsf{L}_{m}^{k}(\theta, y_{1:n}) \right| \\ &= \left(\mathsf{MMD}(P_{\theta}, \mathbb{P}_{n}) + \mathsf{MMD}(P_{m,\theta}, \mathbb{P}_{n}) \right) \left| \mathsf{MMD}(P_{\theta}, P_{n}) - \mathsf{MMD}(P_{m,\theta}, P_{n}) \right| \\ &\leq 4K \cdot \mathsf{MMD}(P_{\theta}, P_{m,\theta}), \end{aligned}$$

where we have used $\sup_{y,y'} k(y,y') \leq K$ for the first term, and the triangle inequality for the second term in the last line. Notice that the triangle inequality is applicable because k was assumed to be characteristic, so that the MMD is a proper metric on $\mathcal{P}(\mathcal{Y})$. We now complete the proof by squaring the above on both sides, taking the expectation, and then using that $\mathbb{E}_{U}\left[\mathrm{MMD}^{2}(P_{\theta}, P_{m,\theta})\right] \leq \frac{K^{2}}{m}$ (cf. Lemma 5 of Chérief-Abdellatif and Alquier (2020)).

D.3 Lemma 2

Lemma 2. Assume that for an isotropic kernel k and a bandwidth h > 0, we have

(i)
$$C_j(\theta) := C_j \sup_{y \in \mathcal{Y}} \left| \frac{\partial^j}{\partial u^j} p_{\theta}(y) \right| < \infty \text{ for } C_j < \infty \text{ and } j \in \{0, 1, 2, 3\};$$

(ii) for every $\theta \in \Theta$, there exist $\delta(\theta) > 0$ so that $\inf_{y \in \mathcal{Y}} p_{\theta}(y) \geq \delta(\theta)$;

(iii)
$$\int k(v) dv = 1$$
, $k(v) = k(-v)$, $\int k^2(v) dv < \infty$, and $\int v^2 k(v) dv < \infty$.

Then, as $h \to 0$, and $m \to \infty$ with $mh \to \infty$, it holds for any finite n that

$$\mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\log \hat{p}_{\theta}(y_{1:n}) \right] - \log p_{\theta}(y_{1:n}) \le \frac{\max\{C_{2}(\theta), C_{3}(\theta)\}}{\delta(\theta)} n h^{2};$$

$$\mathbb{E}_{U_{1:m} \overset{iid}{\sim} P_{\theta}} \left[\left\{ \log \hat{p}_{\theta}(y_{1:n}) - \log p_{\theta}(y_{1:n}) \right\}^{2} \right] \le \frac{n}{mh} \frac{\left\{ C_{2}(\theta) + C_{0}(\theta) h \right\}}{\delta(\theta)^{2}} + n h^{4} \frac{\max\{C_{1}(\theta), C_{2}(\theta)\}^{2}}{\delta(\theta)^{2}}.$$

Proof. First, for each $y \in \mathcal{Y}$, we have

$$\log \widehat{p}_{\theta}(y) - \log p_{\theta}(y) = \frac{1}{p_{\theta}(y)} \{ \widehat{p}_{\theta}(y) - p_{\theta}(y) \} - \frac{1}{2} \frac{1}{\zeta^{2}} \{ \widehat{p}_{\theta}(y) - p_{\theta}(y) \}^{2},$$

for some real number ζ between $p_{\theta}(y)$ and $\widehat{p}_{\theta}(y)$. Note that, since from the definition of ζ , we have that $\zeta > 0$, and

$$\log \widehat{p}_{\theta}(y) - \log p_{\theta}(y) \leq \frac{1}{p_{\theta}(y)} \{ \widehat{p}_{\theta}(y_i) - p_{\theta}(y) \}$$
$$\leq \delta(\theta)^{-1} \{ \widehat{p}_{\theta}(y_i) - p_{\theta}(y) \}, \tag{27}$$

where we recall that $\delta(\theta) < \inf_{y \in \mathcal{Y}} p_{\theta}(y)$. Since (27) holds for all $y \in \mathcal{Y}$, we can use this, and the fact that

$$\hat{p}_{\theta}(y_{1:n}) = \prod_{i=1}^{n} \left\{ \frac{1}{mh} \sum_{l=1}^{m} k_h(y_i - u_l) \right\}.$$

to obtain the stated results. To this end, we first derive the mean and variance of $\widehat{p}_{\theta}(y_i)$ wrt $u_i = (u_1, \dots, u_m)$, where we recall that $u_i \perp u_j$ for all $j \neq i$.

Term $\mathbb{E}_{U_i}\{\widehat{p}_{\theta}(y_i)\}$

From the definition of $\widehat{p}_{\theta}(y_i)$, and since $u_i \stackrel{iid}{\sim} P_{\theta}$,

$$\mathbb{E}_{U}\left\{\frac{1}{mh}\sum_{i=1}^{m}k\left(\frac{u_{i}-y}{h}\right)\right\} - p_{\theta}(y) = \frac{1}{h}\int p(z\mid\theta)k\left(\frac{u-y}{h}\right)du - p_{\theta}(y) \tag{28}$$

Use the change of variables u - y = hv, and a second-order Taylor series expansion to obtain

$$\frac{1}{h} \int p(u \mid \theta) k \left(\frac{u - y}{h}\right) du$$

$$= \int p(y + hv \mid \theta) k(v) dv$$

$$= \int \left\{ p_{\theta}(y) + \frac{\partial p_{\theta}(y)}{\partial y} hv + 2^{-1} \frac{\partial^{2} p_{\theta}(y)}{\partial y^{2}} (hv)^{2} + 6^{-1} \frac{\partial^{3} p(\overline{y} \mid \theta)}{\partial y^{3}} (hv)^{3} \right\} k(v) dv,$$

for some intermediate value \overline{y} . Considering the last term, we have that, from our assumption on $\partial^3 p_{\theta}(y)/\partial y^3$,

$$6^{-1} \int \frac{\partial^2 p(\overline{y} \mid \theta)}{\partial y^2} (hv)^3 k(v) dv \le \frac{h^3}{6} \sup_{v \in \mathcal{V}} \left| \frac{\partial^2 p(\overline{y} \mid \theta)}{\partial y^2} \right| \int v^3 k(v) dv = C_3(\theta).$$

Noting that $\int vk(v)dv = 0$ by assumption, and that $\int v^2k(v) < \infty$, we can apply the above into equation (28) to obtain

$$\mathbb{E}_{U}\left\{\frac{1}{mh}\sum_{i=1}^{m}k\left(\frac{u_{i}-y}{h}\right)\right\}-p_{\theta}(y) \leq h^{3}C_{3}(\theta)+\frac{h^{2}\int v^{2}k(v)dv}{4}\left|\frac{\partial^{2}p_{\theta}(y)}{\partial y^{2}}\right|.$$

Writing $C_2(\theta) = \frac{\int v^2 k(v) dv}{6} \sup_{y \in \mathcal{Y}} |\partial^2 p_{\theta}(y)/\partial y^2|$, we have the individual upper bound,

$$\mathbb{E}_{U} \left\{ \frac{1}{Lh} \sum_{j=1}^{L} k \left(\frac{z_{j} - y}{h} \right) \right\} - p_{\theta}(y) \le \{ C_{3}(\theta) \lor C_{2}(\theta) \} h^{2}, \tag{29}$$

since $h^2 \gg h^3$ for all m large enough, and which is valid for any $y \in \mathcal{Y}$.

Term $\operatorname{var}_{U_i} \{ \widehat{p}_{\theta}(y) \}$

A similar change of variable and a first-order Taylor expansion yields

$$\operatorname{var}_{U_{i}}\left\{\widehat{p}_{\theta}(y)\right\} = \frac{1}{mh^{2}} \left[\int p(y+hv \mid \theta)k(v)^{2}hdv - \left\{ h \int p(y+hv \mid \theta)k(v)dv \right\}^{2} \right]$$

$$= \frac{1}{mh^{2}} \left[\int \left\{ p_{\theta}(y) + \frac{\partial p(\overline{y} \mid \theta)}{\partial y}(hv) \right\} k(v)^{2}hdv - \left\{ h \int p(y+hv \mid \theta)k(v)dv \right\}^{2} \right]$$

$$= \frac{1}{mh} p_{\theta}(y) \int k^{2}(v)dv + C_{1}(\theta) \frac{1}{m} - \frac{1}{m} \left\{ \int p(y+hv \mid \theta)k(v)dv \right\}^{2}$$

$$\leq \frac{p_{\theta}(y)\kappa_{2}}{mh} + \frac{C_{1}(\theta)}{m} + m^{-1}p_{\theta}(y)$$

$$\leq \frac{p_{\theta}(y)\left\{1 + \kappa_{2}\right\}}{mh} + m^{-1}C_{0}(\theta)$$

$$\leq \left\{ C_{0}(\theta) + C_{1}(\theta)h \right\} / (mh), \tag{30}$$

where $C_1(\theta) = \sup_{y \in \mathcal{Y}} |\partial p_{\theta}(y)/\partial y| \kappa_2$, $\kappa_2 = \int k^2(v) dv$, $C_0 = \{1 + \kappa_2\} \sup_{y \in \mathcal{Y}} p_{\theta}(y)$ and since $\{\int p(y + hv \mid \theta)k(v) dv\}^2 \le p_{\theta}(y) + hC_1(\theta) \int vk(v) dv = p_{\theta}(y)$.

Now, taking the expectation on both sides of (27), and applying (28), we have, uniformly for $y \in \mathcal{Y}$,

$$\mathbb{E}_{U_i} \left\{ \log \widehat{p}_{\theta}(y) - \log p_{\theta}(y) \right\} \leq \delta(\theta)^{-1} \mathbb{E}_{U_i} \left\{ \widehat{p}_{\theta}(y) - p_{\theta}(y) \right\}$$
$$\leq \delta(\theta)^{-1} \left\{ C_2(\theta) \vee C_3(\theta) \right\} h^2 \tag{31}$$

In addition,

$$\mathbb{E}_{U_i} \left\{ \log \widehat{p}_{\theta}(y) - \log p_{\theta}(y) \right\}^2 \le \delta(\theta)^{-2} \mathbb{E}_{U_i} \left\{ \widehat{p}_{\theta}(y) - \mathbb{E}_{U_i} \widehat{p}_{\theta}(y) \right\}^2$$
$$+ \delta(\theta)^{-2} \left[\left\{ \mathbb{E}_{U_i} \widehat{p}_{\theta}(y) - p_{\theta}(y) \right\} \right]^2$$

Applying equations (28) and (29) to the above then yields

$$\mathbb{E}_{U_i} \left\{ \log \widehat{p}_{\theta}(y) - \log p_{\theta}(y) \right\}^2 \le \frac{1}{mh} \frac{\left\{ C_0(\theta) + C_1(\theta)h \right\}}{\delta(\theta)^2} + \frac{\left\{ C_2(\theta) \vee C_3(\theta) \right\}^2 h^4}{\delta(\theta)^2}. \tag{32}$$

Overall Mean: Since each U_i is independent, and since (31) is independent of y, the first result follows.

Overall Variance: Since each U_i is independent,

$$\mathbb{E}_{U} \left\{ \log \widehat{p}_{h}(y_{1:n} \mid \theta, U) - \log p(y_{1:n} \mid \theta) \right\}^{2} = \sum_{i=1}^{n} \mathbb{E}_{U_{i}} \left\{ \log \widehat{p}_{\theta}(y_{i}) - \mathbb{E}_{U_{i}} \log \widehat{p}_{\theta}(y_{i}) \right\}^{2} + \sum_{i=1}^{n} \left[\mathbb{E}_{U_{i}} \left\{ \log \widehat{p}_{\theta}(y_{i}) - \log p(y_{i} \mid \theta) \right\} \right]^{2}$$

For each i, we can now apply the bounds in (31) and (32) to obtain the upper bound

$$\mathbb{E}_{U} \left\{ \log \widehat{p}_{h}(y_{1:n} \mid \theta, U) - \log p(y_{1:n} \mid \theta) \right\}^{2} \leq \frac{n \left\{ C_{0}(\theta) + C_{1}(\theta)h \right\}}{\delta(\theta)^{2} m h} + \delta(\theta)^{-2} \left\{ C_{2}(\theta) \vee C_{3}(\theta) \right\}^{2} h^{4} n.$$