Anisotropic conformal change of conic pseudo-Finsler surfaces, I

S. G. Elgendi[1], **Nabil L. Youssef**[2], **A. A. Kotb**[3] **and Ebtsam H. Taha**[2]

[1] Department of Mathematics, Faculty of Science,
Islamic University of Madinah, Madinah, Kingdom of Saudia Arabia

[2]Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt

[3]Department of Mathematics, Faculty of Science,
Damietta University, Damietta , Egypt

salah.ali@fsc.bu.edu.eg, salahelgendi@yahoo.com
nlyoussef@sci.cu.edu.eg, nlyoussef2003@yahoo.fr
alikotb@du.edu.eg, kotbmath2010@yahoo.com
ebtsam.taha@sci.cu.edu.eg, ebtsam.h.taha@hotmail.com

Abstract

The present work is devoted to investigate anisotropic conformal transformation of conic pseudo-Finsler surfaces (M, F), that is, $F(x,y) \longmapsto \overline{F}(x,y) = e^{\phi(x,y)}F(x,y)$, where the function $\phi(x,y)$ depends on both position x and direction y, contrary to the ordinary (isotropic) conformal transformation which depends on position only. If F is a pseudo-Finsler metric, the above transformation does not yield necessarily a pseudo-Finsler metric. Consequently, we find out necessary and sufficient condition for a (conic) pseudo-Finsler surface (M, F) to be transformed to a (conic) pseudo-Finsler surface $(M, \overline{F})$ under the transformation $\overline{F} = e^{\phi(x,y)}F$. In general dimension, it is extremely difficult to find the anisotropic conformal change of the inverse metric tensor in a tensorial form. However, by using the modified Berwald frame on a Finsler surface, we obtain the change of the components of the inverse metric tensor in a tensorial form. This progress enables us to study the transformation of the Finslerian geometric objects and the geometric properties associated with the transformed Finsler function $\overline{F}$. In contrast to isotropic conformal transformation, we have a non-homothetic conformal factor $\phi(x,y)$ that preserves the geodesic spray. Also, we find out some invariant geometric objects under the anisotropic conformal change. Furthermore, we investigate a sufficient condition for $\overline{F}$ to be dually flat or/and projectively flat. Finally, we study some special cases of the conformal factor $\phi(x,y)$. Various examples are provided whenever the situation needs.

Keywords: conic pseudo-Finsler surface; modified Berwald frame; anisotropic conformal change; projectively flat; dually flat

MSC 2020: 53B40, 53C60

Introduction

Conformal transformations have been investigated in different frameworks, since they have some important geometric features [9, 10, 18, 20, 22] and successful applications such as those in physics, biology and ecology [1, 12, 14, 16, 21]. In the conformal transformation theory, it is useful to study the geometric properties and geometric objects that are preserved under conformal transformations. In Riemannian geometry, (isotropic) conformal transformation is an angle-preserving. In pseudo Riemannian metrics with signature (Lorentzian metrics), it preserves causality. In parallelizable manifolds, it preserves both angles and causality as each parallelization structure induces a pseudo-Riemannian (or Lorentzian) metric, cf. [16, 22]. In Finsler geometry, (isotropic) conformal transformation is angle-preserving [9] and leaves the geodesic spray invariant if the conformal factor is a homothety [2, Theorem 3.3]. Further, each Randers space $(M, F = \alpha + \beta)$ has a globally defined nonholonomic frame which is called the Holland frame [11]. This frame is an isotropic conformally invariant in the sense that the transformation $F(x, y) \longmapsto \overline{F}(x, y) = e^{\phi(x)}F(x, y)$ leaves the frame elements fixed [4, Theorem 5.10.1].

Given two pseudo-Finsler metrics $F(x, y)$ and $\overline{F}(x, y)$, the anisotropic conformal transformation $F(x, y) \longmapsto \overline{F}(x, y) = e^{\phi(x,y)}F(x, y)$ preserves the lightcone [14] and lightlike geodesics and their focal points [13, 14]. The anisotropic conformal change in Finsler Geometry is not only valuable in applications, but also it has a great impact in the study of Finsler geometry. For example, in contrast to isotropic conformal change, the anisotropic conformal change of a pseudo-Finsler metric is not necessarily a pseudo-Finsler metric (see Theorem 2.11). In addition, the anisotropic conformal change can send a psudo-Finsler metric to a pseudo-Riemannian one and vice-versa.

Since conic pseudo-Finsler surfaces are used widely in applications, especially in physics, we study here the anisotropic conformal transformation $F(x, y) \longmapsto \overline{F}(x, y) = e^{\phi(x,y)}F(x, y)$ of a conic pseudo-Finsler metric $F(x, y)$. First, under the anisotropic conformal change, we compute the components $\overline{g}_{ij}$ of the metric tensor of $\overline{F}$. Actually, the expression of $\overline{g}_{ij}$ contains the second derivative of the function ϕ with respect to directional arguments. Generally, this term represents a severe obstacle to compute the components $\overline{g}^{ij}$ of the inverse metric tensor. Keeping in mind that the inverse metric tensor is the door to find the geodesic spray, which enables to study the geometry of the transformed space $(M, \overline{F})$, this motivates us to consider the two-dimensional case to make use of the modified Berwald frame. Consequently, we become able to derive some important geometric objects associated with the transformed metric $\overline{F}$, for example, the transformed Cartan tensor, main scalar, geodesic spray and Barthel connection. Further, we find out some invariant objects under anisotropic conformal transformations (see Theorem 3.11).

The present paper is organized in the following manner. In §1, we recall some basic facts on the geometry of sprays and Finsler manifolds. In §2, we first show that the fact that "two Finsler metrics are anisotropically conformally related" is not equivalent to the "proportionality of their associated metric tensors", contrary to what has been mentioned in [20, (11.1.1)]. Then, we investigate the anisotropic conformal transformations of pseudo-Finsler surfaces. In §3, we characterize the anisotropic conformal transformations and study their action on some important Finslerian geometric objects. Some anisotropic conformal invariant geometric properties are found out. In §4, we study the case when the anisotropic conformal transformation is a projective transformation. Moreover, we obtain sufficient conditions for $\overline{F}$ to be projectively flat and dually flat. Finally, in §5, we consider two important special cases: the factor ϕ is a function of position only and the factor ϕ is a function of direction only. Interesting results in both cases are obtained.

It should be noted that, various examples have been provided whenever the situation needs. The Maple's code of these examples is presented in the Appendix at the end of the paper.

1 Notation and preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \longrightarrow M$ the canonical projection of the tangent bundle TM onto M. Let $TM_0 := TM \setminus (0)$ be the slit tangent bundle, (0) being the null section, and (x^i, y^i) the corresponding coordinate system TM. The natural almost-tangent structure J of TM is the vector 1-form defined by $J = \frac{\partial}{\partial y^i} \otimes dx^i$. The vertical vector field $\mathcal{C} = y^i \frac{\partial}{\partial y^i}$ on TM is called the Liouville vector field. The set of smooth functions on TM_0 is denoted by $C^\infty(TM_0)$.

A spray on M is a vector field S on TM such that $JS = \mathcal{C}$ and $[\mathcal{C}, S] = S$. Locally, it is given by [8]

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where the spray coefficients $G^i(x, y)$ are positively 2-homogeneous functions in y (or simply $h(2)$-functions). A nonlinear connection is defined by an n-dimensional distribution $H(TM_0)$ on TM_0 which is supplementary to the vertical distribution $V(TM_0)$. This means that for all $u \in TM_0$, we have

$$T_u(TM_0) = V_u(TM_0) \oplus H_u(TM_0).$$

The local basis of $V_u(TM_0)$ and $H_u(TM_0)$ are given, respectively by, $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$ and $\delta_i := \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - G^j_i(x, y) \dfrac{\partial}{\partial y^j}$. The coefficients of Cartan nonlinear (or Barthel) connection and Berwald connection are defined by $G^j_i(x, y) := \dfrac{\partial G^j}{\partial y^i}$ and $G^h_{ij}(x, y) := \dfrac{\partial G^h_j}{\partial y^i}$, respectively.

In the following we set the definition of a (conic pseudo-) Finsler manifold which will be used throughout the paper.

Definition 1.1. *A conic sub-bundle of TM is a non-empty open subset $\mathcal{A} \subset TM_0$ such that $\pi(\mathcal{A}) = M$ and $\lambda v \in \mathcal{A} \quad \forall v \in \mathcal{A}$ and $\forall \lambda \in \mathbb{R}^+$.*

Definition 1.2. *A* ***conic pseudo-Finsler metric*** *on M is a function $F : \mathcal{A} \longrightarrow \mathbb{R}$ which satisfies the following conditions:*

(i) *F is smooth on $\mathcal{A}$,*

(ii) *$F(x, y)$ is positively homogeneous of degree one in y: $F(x, \lambda y) = \lambda F(x, y)$ for all $(x, y) \in \mathcal{A}$ and $\lambda \in \mathbb{R}^+$,*

(iii) *The Finsler metric tensor $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, y)$ is non-degenerate at each point of $\mathcal{A}$.*

The pair (M, F) is called a conic pseudo-Finsler manifold.

The Finsler metric F is said to be:

- *a* ***conic Finsler metric*** *if the metric tensor g_{ij} in* **(iii)** *is positive definite, or equivalently, F is non-negative, and in this case (M, F) is a conic Finsler manifold.*
- *a* ***pseudo-Finsler metric*** *if $\mathcal{A}$ is replaced by TM, and in this case (M, F) is a pseudo-Finsler manifold.*
- *a* ***Finsler metric*** *if $\mathcal{A}$ is replaced by TM and g_{ij} in* **(iii)** *is positive definite, and in this case (M, F) is a Finsler manifold.*

Each pseudo-Finsler metric F induces a spray S on M, for which the spray coefficients are given by $G^i = \frac{1}{4} g^{ij}(y^k \dot{\partial}_j \partial_k F^2 - \partial_j F^2)$ [8, 20], this spray is called the *geodesic spray* of F.

2 Anisotropic conformal transformation

In this section, we introduce the anisotropic conformal transformation of a conic pseudo-Finsler metric F and investigate its elementary properties.

Conformality is classically defined as follows.

Definition 2.1. [19] *Let F and $\overline{F}$ be two Finsler metric functions defined on a smooth manifold M. The two Finsler metric tensors g_{ij} and $\overline{g}_{ij}$ resulting from F and $\overline{F}$, respectively, are called conformal if there exists a factor of proportionality $\Psi(x,y)$ between the metric tensors:*

$$\overline{g}_{ij}(x,y) = \Psi(x,y)\, g_{ij}(x,y), \tag{2.1}$$

where $\Psi(x,y)$ is a positive smooth function of position x and direction y.

Suppose g_{ij} and $\overline{g}_{ij}$ are conformal, then (2.1) holds. As $g_{ij}y^iy^j = F^2$, we get

$$\overline{F}(x,y) = \sqrt{\Psi(x,y)}\, F(x,y). \tag{2.2}$$

This means that if $g_{ij}(x,y)$ and $\overline{g}_{ij}(x,y)$ are proportional, then $F(x,y)$ and $\overline{F}(x,y)$ are proportional, or, in other words, (2.1) implies (2.2). The converse is not true in general [19]: proportional Finsler metric functions may yield non proportional Finsler metric tensors. Nevertheless, we have

Lemma 2.2. *Let two Finsler metric functions F and $\overline{F}$ on M be proportional, i.e., $\overline{F}(x,y) = \Psi(x,y)\, F(x,y)$. A necessary and sufficient condition for the associated Finsler metric tensors g_{ij} and $\overline{g}_{ij}$ to be proportional is that $\Psi(x,y)$ is a functions of position x only.*

Proof. Let F and $\overline{F}$ be proportional, that is,

$$\overline{F}(x,y) = \Psi(x,y)\, F(x,y).$$

By squaring and differentiating with respect to y^i and y^j, we get

$$\overline{g}_{ij} = \Psi^2 g_{ij} + \{(\dot{\partial}_i\Psi)(\dot{\partial}_j\Psi) + \Psi(\dot{\partial}_i\dot{\partial}_j\Psi)\}F^2 + \Psi\{(\dot{\partial}_j\Psi)\dot{\partial}_iF^2 + (\dot{\partial}_i\Psi)\dot{\partial}_jF^2\}.$$

Consequently, the metric tensors g_{ij} and $\overline{g}_{ij}$ are proportional if and only if

$$\{(\dot{\partial}_i\Psi)(\dot{\partial}_j\Psi) + \Psi(\dot{\partial}_i\dot{\partial}_j\Psi)\}F^2 + \Psi\{(\dot{\partial}_j\Psi)\dot{\partial}_iF^2 + (\dot{\partial}_i\Psi)\dot{\partial}_jF^2\} = 0.$$

Contracting both sides of the above equation by y^j and using the homogeneity properties of F and Ψ, we get

$$\Psi(\dot{\partial}_i\Psi)F^2 = 0,$$

from which $\dot{\partial}_i\Psi(x,y) = 0$ and $\Psi \neq 0$ is a function of x only. □

Remark 2.3. (a) *The above discussion shows that the proportionality of F and $\overline{F}$ **does not imply** automatically that the conformal factor Ψ is independent of direction, contrary to what has been mentioned by Y. Shen and Z. Shen in* [20, (11.1.1)].

(b) *On the other hand, it was proved by Knebelman* [15] *that the proportionality of g_{ij} and $\overline{g}_{ij}$ **does imply** automatically that the conformal factor Ψ is independent of direction, as has been mentioned by H. Rund in* [19, VI,2].

We begin our investigation from the proportionality of F and $\overline{F}$, regardless of the proportionality of g_{ij} and $\overline{g}_{ij}$ and regardless of the independence of the conformal factor of direction. In fact, we are particularly interested in the dependence of the conformal factor on both position and direction, motivated by the current applications in physics and other branches of science, and by the high potentiality of near future applications.

Now, let us set our definition of conformality.

Definition 2.4. *Let (M, F) be a conic pseudo-Finsler manifold. The anisotropic conformal transformation of F is defined by*

$$F \longmapsto \overline{F}(x, y) = e^{\phi(x,y)} F(x, y), \tag{2.3}$$

where $\phi(x, y)$ is an $h(0)$ smooth function on $\mathcal{A}$. In this case, we say that F and $\overline{F}$ are anisotropically conformally related, or $\overline{F}$ is anisotropically conformal to F.
If the conformal factor $\phi(x, y)$ is independent of direction, (2.3) is called isotropic conformal, or, simply, conformal transformation.

One of the benefits of the anisotropic conformal transformation is that, unlike conformal transformation [9], the anisotropic conformal transformation can transform a Riemannian metric to a Finslerian one.

Proposition 2.5. *Let (M, F) be a conic pseudo-Finsler space, then under the anisotropic conforaml transformation (2.3), we have*

$$\overline{g}_{ij} = e^{2\phi} \left[g_{ij} + 2F^2\, \dot{\partial}_i\phi\, \dot{\partial}_j\phi + 2F(\,\ell_j\, \dot{\partial}_i\phi + \ell_i\, \dot{\partial}_j\phi) + F^2\, \dot{\partial}_i\dot{\partial}_j\phi \right]. \tag{2.4}$$

Proof. Since $\overline{F}^2 = e^{2\phi(x,y)} F^2$, we get $\dot{\partial}_i \overline{F}^2 = e^{2\phi}\, (2\, F^2\, \dot{\partial}_i\phi + \dot{\partial}_i F^2)$. Thereby,

$$\begin{aligned} \overline{g}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j\overline{F}^2 &= e^{2\phi}[2F^2(\dot{\partial}_i\phi)(\dot{\partial}_j\phi) + 2F\,(\ell_j\, \dot{\partial}_i\phi + \ell_i\, \dot{\partial}_j\phi) + F^2\, \dot{\partial}_i\dot{\partial}_j\phi + \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2] \\ &= e^{2\phi}[g_{ij} + 2F^2(\dot{\partial}_i\phi)(\dot{\partial}_j\phi) + 2F\,(\ell_j\, \dot{\partial}_i\phi + \ell_i\, \dot{\partial}_j\phi) + F^2\, \dot{\partial}_i\dot{\partial}_j\phi], \end{aligned}$$

where $\ell_i = \dot{\partial}_i F$. □

Remark 2.6. *The existence of the term $F^2\dot{\partial}_i\dot{\partial}_j\phi$ in (2.4) makes calculating the inverse metric $\overline{g}^{ij}$, in a tensorial form, very complicated in general dimension. However, $\overline{g}^{ij}$ can be found out in the two-dimensional case thanks to the existence of Berwald frame in such a case.*

Henceforward, we work in a two-dimensional conic pseudo-Finsler space equipped with a modified Berwald frame (ℓ_i, m_i) [3, 17]. The components g_{ij} of the metric tensor are given by

$$g_{ij} = \ell_i\ell_j + \varepsilon m_i m_j, \tag{2.5}$$

where $\varepsilon = \pm 1$ is called the signature of F. Further, the angular metric coefficients h_{ij} can be expressed as

$$h_{ij} = \varepsilon m_i m_j.$$

Also, we have

$$g^{ij} = \ell^i\ell^j + \varepsilon m^i m^j,$$

where $\ell^i = \frac{y^i}{F}$. The two vector fields $\ell = (\ell^1, \ell^2)$ and $m = (m^1, m^2)$ have been chosen in such a way that they satisfy

$$g(\ell, \ell) = 1,\ g(\ell, m) = 0,\ g(m, m) = \varepsilon.$$

Moreover, the determinant $\mathfrak{g}$ of the matrix (g_{ij}) is given by

$$\mathfrak{g} := \det(g_{ij}) = \varepsilon(\ell_1 m_2 - \ell_2 m_1)^2. \tag{2.6}$$

To calculate m^i and m_i, we give the following lemma.

Lemma 2.7. *The covector m_i is given by*

$$m_1 = -\sqrt{\varepsilon\mathfrak{g}}\ \ell^2, \quad m_2 = \sqrt{\varepsilon\mathfrak{g}}\ \ell^1.$$

Moreover, the vector m^i is given by

$$m^1 = -\frac{1}{\sqrt{\varepsilon\mathfrak{g}}}\,\ell_2, \quad m^2 = \frac{1}{\sqrt{\varepsilon\mathfrak{g}}}\,\ell_1.$$

Proof. By (2.6), we have

$$\ell_1 m_2 - \ell_2 m_1 = \sqrt{\varepsilon\mathfrak{g}}.$$

Also, the property $g(\ell, m) = 0$ gives

$$\ell^1 m_1 + \ell^2 m_2 = 0.$$

Then, the result follows by solving the above two equations for m_1 and m_2. The proof of the formulae of m^i can be calculated in a similar manner. □

Thus, the components C_{ijk} of the Cartan tensor are given by [3]

$$FC_{ijk} = \mathcal{I}\, m_i m_j m_k,$$

where $\mathcal{I}$ is the main scalar of the surface (M, F).

From now on, when we say that (M, F) is a conic pseudo-Finsler surface, this means that (M, F) is a conic pseudo-Finsler surface equipped with the modified Berwald frame.

Lemma 2.8. [17] *Let (M, F) be a conic pseudo-Finsler surface. Then we have the following:*

(i) $\ell^i m_i = \ell_i m^i = 0$,

(ii) $m^i m_i = \varepsilon, \quad \ell^i \ell_i = 1$,

(iii) $\delta_i^j = \ell_i \ell^j + \varepsilon m_i m^j$,

(iv) $F\dot{\partial}_j \ell_i = \varepsilon m_i m_j = h_{ij}, \qquad F\dot{\partial}_j \ell^i = \varepsilon m^i m_j$,

(v) $F\dot{\partial}_j m_i = -(\ell_i - \varepsilon\mathcal{I} m_i) m_j, \qquad F\dot{\partial}_j m^i = -(\ell^i + \varepsilon\mathcal{I} m^i) m_j$,

(vi) *For a smooth function f on $\mathcal{A}$, we have $F\dot{\partial}_i f = f_{;1}\,\ell_i + f_{;2}\,m_i$, where $f_{;1} = F(\dot{\partial}_i f)\,\ell^i$ and $f_{;2} = \varepsilon F\,(\dot{\partial}_i f) m^i$. In particular, if f is $h(r)$, then $f_{;1} = rf$,*

(vii) $F\dot{\partial}_i \partial_k f = (\partial_k f)_{;1}\,\ell_i + (\partial_k f)_{;2}\,m_i$, *where* $\partial_k = \frac{\partial}{\partial x^k}$,

(viii) $\delta_i f = f_{,1}\,\ell_i + f_{,2}\,m_i$, *where* $f_{,1} = (\delta_i f)\ell^i$ *and* $f_{,2} = \varepsilon(\delta_i f)m^i$.

Using the modified Berwald frame, we can write the formulae of the objects $\dot{\partial}_i\phi$ and $\dot{\partial}_i\dot{\partial}_j\phi$ in terms of the frame elements. Consequently, plugging these objects into (2.4), we are able to find the formula of components $\overline{g}^{ij}$ of the inverse metric of $\overline{F}$. More precisely, we have:

Lemma 2.9. *Let (M, F) be a conic pseudo-Finsler surface. The term $\dot{\partial}_i\dot{\partial}_j\phi$ given in (2.4) can be expressed in the form*

$$F^2\dot{\partial}_i\dot{\partial}_j\phi = -\phi_{;2}\,(\ell_i m_j + \ell_j m_i) + (\phi_{;2;2} + \varepsilon\,\mathcal{I}\,\phi_{;2})\,m_j m_i. \tag{2.7}$$

Proof. From Lemma 2.8 **(vi)**, $F\dot{\partial}_i\phi = \phi_{;1}\,\ell_i + \phi_{;2}\,m_i$. Since ϕ is $h(0)$, then $\phi_{;1} = 0$. That is,

$$F\dot{\partial}_i\phi = \phi_{;2}\,m_i. \tag{2.8}$$

Multiplying both sides of (2.8) by F, then differentiating with respect to y^j and use Lemma 2.8 **(v)** and **(vi)**, we get

$$F^2\dot{\partial}_i\dot{\partial}_j\phi = -\phi_{;2}\,(\ell_i m_j + \ell_i m_j) + (\phi_{;2;2} + \varepsilon\,\mathcal{I}\,\phi_{;2})\,m_j m_i. \qquad \square$$

Proposition 2.10. *Let (M, F) be a conic pseudo-Finsler surface, then under the anisotropic conformal transformation (2.3), the metric tensors of $\overline{F}$ and F are related by*

$$\begin{aligned}\overline{g}_{ij} &= e^{2\phi}[g_{ij} + \phi_{;2}(\ell_i m_j + \ell_j m_i) + \sigma m_i m_j] \\ &= e^{2\phi}[\ell_i\ell_j + \phi_{;2}(\ell_i m_j + \ell_j m_i) + (\sigma+\varepsilon)m_i m_j],\end{aligned} \tag{2.9}$$

where $\sigma = \phi_{;2;2} + \varepsilon\mathcal{I}\phi_{;2} + 2\,(\phi_{;2})^2$.

Proof. It follows from (2.4), (2.5), (2.7) and (2.8). $\square$

It should be noted that the anisotropic conformal transformation of a conic pseudo-Finsler surface (M, F) does not yield in general a conic pseudo-Finsler surface, as shown in Example 2.12 below. The following result gives a necessary and sufficient condition for $(M, \overline{F})$ to be a conic pseudo-Finsler surface.

Theorem 2.11. *Let (M, F) be a conic pseudo-Finsler surface and (2.3) be an anisotropic conformal transformation of F. Then, $(M, \overline{F})$ is a conic pseudo-Finsler surface if and only if*

$$\sigma - (\phi_{;2})^2 + \varepsilon = F^2[\dot{\partial}_i\dot{\partial}_j\phi + (\dot{\partial}_i\phi)(\dot{\partial}_j\phi)]m^i m^j + \varepsilon \neq 0.$$

Proof. Since both $F(x, y)$ and $e^{\phi(x,y)}$ are smooth on $\mathcal{A}$, then $\overline{F}(x, y)$ is smooth on $\mathcal{A}$. Moreover, $\overline{F}(x, y)$ is $h(1)$. Now, from (2.9), we get

$$\det(\overline{g}_{ij}) = e^{4\phi}\begin{vmatrix} \ell_1\ell_1 + 2\phi_{;2}\,\ell_1 m_1 + (\sigma+\varepsilon)\,m_1 m_1 & \ell_1\ell_2 + \phi_{;2}(\ell_1 m_2 + \ell_2 m_1) + (\sigma+\varepsilon)m_1 m_2 \\ \ell_1\ell_2 + \phi_{;2}(\ell_1 m_2 + \ell_2 m_1) + (\sigma+\varepsilon)m_1 m_2 & \ell_2\ell_2 + 2\phi_{;2}\,\ell_2 m_2 + (\sigma+\varepsilon)\,m_2 m_2 \end{vmatrix}.$$

That is, by using (2.6), we have

$$\begin{aligned}\det(\overline{g}_{ij}) &= e^{4\phi}[(\sigma+\varepsilon)\ell_1\ell_1 m_2 m_2 + (\sigma+\varepsilon)\ell_2\ell_2 m_1 m_1 - 2(\sigma+\varepsilon)\ell_1 m_1\ell_2 m_2 \\ &\qquad -(\phi_{;2})^2\ell_1\ell_1 m_2 m_2 - (\phi_{;2})^2\ell_2\ell_2 m_1 m_1 + 2(\phi_{;2})^2\ell_1 m_1\ell_2 m_2] \\ &= e^{4\phi}[\sigma+\varepsilon-(\phi_{;2})^2](\ell_1 m_2 - \ell_2 m_1)^2 \\ &= \varepsilon e^{4\phi}[\sigma - (\phi_{;2})^2 + \varepsilon]\det(g_{ij}).\end{aligned}$$

Therefore, the matrix $(\overline{g}_{ij})$ is non-degenerate if and only if

$$\sigma - (\phi_{;2})^2 + \varepsilon = F^2[\dot{\partial}_i\dot{\partial}_j\phi \; + (\dot{\partial}_i\phi)(\dot{\partial}_j\phi)]m^i m^j + \varepsilon \neq 0. \qquad \square$$

Example 2.12. *This example gives a conic pseudo-Finsler surface F whose anisotropic conformal transformation $\overline{F} = e^{\phi}F$ is not a conic pseudo-Finsler surface for a certain smooth $h(0)$ function ϕ. Let* $F(x,y) = \dfrac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2}$ *be the Klein metric on unit ball $B^n \subset \mathbb{R}^n$. Define*

$$\overline{F}(x,y) = e^{\phi(x,y)}F(x,y) = \frac{\langle x,y\rangle}{1-|x|^2},$$

where $\phi(x,y) = \ln\left(\dfrac{\langle x,y\rangle}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}\right)$.
It is clear that $\overline{F}$ is linear function in y, i.e., $\det(\overline{g}_{ij}) = 0$. That is, $\sigma + \varepsilon - (\phi_{;2})^2 = 0$. Hence, $\overline{F}$ is not a conic pseudo-Finsler metric, by Theorem 2.11.

Proposition 2.13. *Let (M,F) be a conic pseudo-Finsler surface and (2.3) be an anisotropic conformal transformation of F. Then, we have*

$$\begin{aligned}\overline{g}^{ij} &= e^{-2\phi}[\varepsilon\rho g^{ij} + \sigma\rho\ell^i\ell^j - \varepsilon\phi_{;2}\,\rho(\ell^i m^j + \ell^j m^i)] \\ &= e^{-2\phi}[g^{ij} + (\phi_{;2})^2\rho\ell^i\ell^j - \varepsilon\phi_{;2}\,\rho(\ell^i m^j + \ell^j m^i) + (\rho-\varepsilon)m^i m^j],\end{aligned} \tag{2.10}$$

where $\rho := \dfrac{1}{\sigma+\varepsilon-(\phi_{;2})^2}$.

Proof. In a conic pseudo-Finsler surface (M,F), a tensor of type (2,0) can be written as $T^{ij} = A_1\ell^i\ell^j + A_2\ell^i m^j + A_3 m^i\ell^j + A_4 m^i m^j$, where $A_1,\ A_2,\ A_3$ and A_4 are smooth function on $\mathcal{A}$. Consequently, the inverse of the metric tensor $\overline{g}_{ij}$, given by (2.9), can be written as

$$\overline{g}^{ij} = e^{-2\phi}[A_1\ell^i\ell^j + A_2\ell^i m^j + A_3 m^i\ell^j + A_4 m^i m^j]. \tag{2.11}$$

To determine the functions $A_1,\ A_2,\ A_3$ and A_4, we calculate

$$\overline{g}^{ir}\overline{g}_{rj} = \delta^i_j = [A_1\ell^i\ell^r + A_2\ell^i m^r + A_3 m^i\ell^r + A_4 m^i m^r][\ell_r\ell_j + \phi_{;2}(\ell_r m_j + \ell_j m_r) + (\sigma+\varepsilon)m_r m_j].$$

Using Lemma 2.8 (**(i)-(iii)**), we get

$$\begin{aligned}0 =&(A_1 + \varepsilon\phi_{;2}\,A_2 - 1)\ell_j\ell^i + (A_3 + \varepsilon\phi_{;2}\,A_4)\ell_j m^i + (\phi_{;2}\,A_1 + \varepsilon A_2(\sigma+\varepsilon))m_j\ell^i \\ &+ (\phi_{;2}\,A_3 + \varepsilon A_4(\sigma+\varepsilon) - \varepsilon)m_j m^i.\end{aligned} \tag{2.12}$$

Contracting (2.12) by $\ell^i\ell_j, m^i m_j, \ell^i m_j$ and $m^i\ell_j$, respectively, we get the following equations:

$$A_1 + \varepsilon\phi A_2 = 1, \quad A_3 + \varepsilon\phi_{;2}\,A_4 = 0, \quad \phi_{;2}\,A_1 + \varepsilon(\sigma+\varepsilon)A_2 = 0, \quad \phi_{;2}\,A_3 + \varepsilon(\sigma+\varepsilon)A_4 = \varepsilon. \tag{2.13}$$

Solving the system (2.13) yields

$$A_1 = \frac{\sigma+\varepsilon}{\sigma+\varepsilon-(\phi_{;2})^2}, \qquad A_2 = A_3 = \frac{-\varepsilon\phi_{;2}}{\sigma+\varepsilon-(\phi_{;2})^2}, \qquad A_4 = \frac{1}{\sigma+\varepsilon-(\phi_{;2})^2}.$$

Substituting $A_1,\ A_2,\ A_3$ and A_4 into (2.11), noting that $\rho := \frac{1}{\sigma+\varepsilon-(\phi_{;2})^2}$, we obtain

$$\overline{g}^{ij} = e^{-2\phi}[\rho(\sigma+\varepsilon)\ell^i\ell^j - \varepsilon\phi_{;2}\,\rho(\ell^i m^j + \ell^j m^i) + \rho m^i m^j].$$

By (2.5), we get

$$\overline{g}^{ij} = e^{-2\phi}[\varepsilon\rho g^{ij} + \sigma\rho\ell^i\ell^j - \varepsilon\phi_{;2}\,\rho(\ell^i m^j + \ell^j m^i)].$$

∎

Let us end this section by listing some properties of the functions ϕ, ρ, σ and $\mathcal{I}$ for subsequent use.

Remark 2.14. *As* $\phi_{;2}$, σ, ρ *and* $\mathcal{I}$ *are* $h(0)$*-functions, the following relations hold:*

(a) $\sigma = \phi_{;2;2} + \varepsilon\mathcal{I}\phi_{;2} + 2(\phi_{;2})^2, \quad \rho = \dfrac{1}{\sigma + \varepsilon - (\phi_{;2})^2} = \dfrac{1}{\varepsilon + \phi_{;2;2} + \varepsilon\mathcal{I}\phi_{;2} + (\phi_{;2})^2}.$

(b) $F\dot{\partial}_i\phi_{;2} = \phi_{;2;2}\, m_i, \quad F\dot{\partial}_i\sigma = \sigma_{;2}m_i, \quad F\dot{\partial}_i\rho = \rho_{;2}\, m_i, \quad F\dot{\partial}_i\mathcal{I} = \mathcal{I}_{;2}\, m_i.$

3 Transformation of fundamental geometric objects

From now on, we consider the anisotropic conformal transformation of conic pseudo-Finsler metric F given by

$$\overline{F}(x,y) = e^{\phi(x,y)}F(x,y), \quad \text{with } \sigma + \varepsilon - (\phi_{;2})^2 \neq 0. \tag{3.1}$$

We calculate the anisotropic conformal change (3.1) of some important geometric objects. Namely, we find the transformation of modified Berwald frame, angular metric, Cartan tensor, main scalar, geodesic spray, Barthel connection and Berwald connection.

Proposition 3.1. *Under the anisotropic conformal transformation* (3.1)*, we obtain*

(i) $\overline{\ell}_i = e^{\phi}[\ell_i + \phi_{;2}\, m_i], \quad \overline{\ell}^i = e^{-\phi}\ell^i,$

(ii) $\overline{h}_{ij} = e^{2\phi}[h_{ij} + (\sigma - (\phi_{;2})^2)m_im_j],$

(iii) $\overline{m}_i = e^{\phi}\sqrt{\frac{\varepsilon}{\rho}}\, m_i, \quad \overline{m}^j = e^{-\phi}\sqrt{\varepsilon\rho}[m^j - \varepsilon\phi_{;2}\, \ell^j],$

(iv) $\overline{m}_i\overline{m}^i = \varepsilon, \quad \overline{\ell}_i\overline{\ell}^i = 1, \quad \overline{\ell}^i\overline{m}_i = \overline{\ell}_i\overline{m}^i = 0.$

Proof. **(i)** Since $\overline{F} = e^{\phi}F$, by (2.8), we get

$$\dot{\partial}_i\overline{F} = \overline{\ell}_i = e^{\phi}F\dot{\partial}_i\phi + e^{\phi}\,\dot{\partial}_iF = e^{\phi}[\ell_i + F\dot{\partial}_i\phi] = e^{\phi}[\ell_i + \phi_{;2}m_i]. \tag{3.2}$$

Then from (2.10), we obtain

$$\begin{aligned}\overline{\ell}^j := \overline{g}^{ij}\,\overline{\ell}_i &= e^{-\phi}[\rho(\sigma+\varepsilon)\ell^i\ell^j - \varepsilon\phi_{;2}\,\rho(\ell^im^j + \ell^jm^i) + \rho m^im^j](\ell_i + \phi_{;2}\, m_i) \\ &= e^{-\phi}\rho\,\ell^j(\sigma + \varepsilon - (\phi_{;2})^2) = e^{-\phi}\,\ell^j.\end{aligned}$$

(ii) Differentiating (3.2) with respect to y^j, we get

$$\dot{\partial}_i\dot{\partial}_j\overline{F} = e^{\phi}[F(\dot{\partial}_i\phi)(\dot{\partial}_j\phi) + F\dot{\partial}_i\dot{\partial}_j\phi + (\dot{\partial}_iF)\dot{\partial}_j\phi + (\dot{\partial}_jF)\dot{\partial}_i\phi + \dot{\partial}_i\dot{\partial}_jF].$$

From (2.7) and (2.8), we obtain

$$\overline{h}_{ij} = \overline{F}\,\dot{\partial}_i\dot{\partial}_j\overline{F} = e^{2\phi}[h_{ij} + (\sigma - (\phi_{;2})^2)m_im_j].$$

(iii) From **(ii)**, we have $\varepsilon\overline{m}_i\overline{m}_j = e^{2\phi}[\sigma + \varepsilon - (\phi_{;2})^2]m_im_j = \frac{e^{2\phi}}{\rho}\, m_im_j$.
Hence, $\overline{m}_i = \sqrt{\frac{\varepsilon}{\rho}}\, e^{\phi}\, m_i$. But $m^i = g^{ij}m_j$, then from (2.10), we have

$$\overline{m}^j = \sqrt{\varepsilon\rho}\, e^{-\phi}\,[m^j - \varepsilon\phi_{;2}\,\ell^j].$$

(iv) It follows from **(i)** and **(iii)**. □

In view of Proposition 3.1 **(iv)**, we have the following corollary.

Corollary 3.2. *Let (M, F) be a conic pseudo-Finsler surface. The anisotropic conformal transformation of a modified Berwald frame is a modified Berwald frame.*

Now, we find the anisotropic conformal transformation of some non-Riemannian quantities.

Proposition 3.3. *Under the anisotropic conformal transformation (3.1), we get*

(i) $F\,\overline{C}_{ijk} = e^{2\phi} F\,C_{ijk} + e^{2\phi}[\varepsilon\mathcal{I}\sigma + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)]m_i m_j m_k$
$\qquad = e^{2\phi}[\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)]m_i m_j m_k,$

(ii) $\overline{\mathcal{I}} = (\varepsilon\rho)^{\frac{3}{2}}[\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)],$

(iii) $F\overline{C}^i_{jk} = \varepsilon\rho[FC^i_{jk} + \{\varepsilon\mathcal{I}\sigma + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)\}m^i - \varepsilon\phi_{;2}\{\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)\}\ell^i]m_j m_k.$

Proof. **(i)** Differentiating (2.9) with respect to y^k, we have

$$\begin{aligned}2\overline{C}_{ijk} =& \dot{\partial}_k(\overline{g}_{ij})\\ =& 2e^{2\phi}\dot{\partial}_k\phi[\ell_i\ell_j + \phi_{;2}(\ell_i m_j + \ell_j m_i) + (\sigma+\varepsilon)m_i m_j]\\ &+ e^{2\phi}\dot{\partial}_k[g_{ij} + \phi_{;2}(\ell_i m_j + \ell_j m_i) + \sigma m_i m_j].\end{aligned}$$

Multiplying both sides of the above equation by F and using Lemma 2.8 **(iv)** and **(v)**, we have

$$\begin{aligned}2F\overline{C}_{ijk} = & e^{2\phi}[2\phi_{;2}\ell_i\ell_j m_k + 2(\phi_{;2})^2(\ell_i m_j m_k + \ell_j m_i m_k) + 2\phi_{;2}(\sigma+\varepsilon)m_i m_j m_k]\\ &+ 2Fe^{2\phi}C_{ijk} + e^{2\phi}[\phi_{;2;2}(\ell_i m_j m_k + \ell_j m_i m_k) + \phi_{;2}(\varepsilon m_i m_j m_k - \ell_i\ell_j m_k\\ &+ \varepsilon\mathcal{I}\ell_i m_j m_k + \varepsilon m_i m_j m_k - \ell_i\ell_j m_k + \varepsilon\mathcal{I}\ell_j m_i m_k) + \sigma_{;2}m_i m_j m_k\\ &+ \sigma(-\ell_i m_j m_k + \varepsilon\mathcal{I}m_i m_j m_k - \ell_j m_i m_k + \varepsilon\mathcal{I}m_i m_j m_k)].\end{aligned}$$

By Remark 2.14, we get

$$F\overline{C}_{ijk} = Fe^{2\phi}C_{ijk} + e^{2\phi}(\varepsilon\mathcal{I}\sigma + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon))m_i m_j m_k.$$

(ii) Since we have $\overline{F}\,\overline{C}_{ijk} = \overline{\mathcal{I}}\,\overline{m}_i\,\overline{m}_j\,\overline{m}_k$, then by **(i)** and Proposition 3.1 **(iii)**, we obtain

$$\overline{\mathcal{I}}e^{3\phi}\left(\frac{\varepsilon}{\rho}\right)^{\frac{3}{2}} m_i m_j m_k = e^{3\phi}[\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)]m_i m_j m_k.$$

That is,

$$\overline{\mathcal{I}} = (\varepsilon\rho)^{\frac{3}{2}}[\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon)].$$

(iii) By using (2.10) and **(i)**, we can write

$$\begin{aligned}F\overline{C}^i_{jk} =& \overline{g}^{ih}\overline{C}_{hjk} = [\varepsilon\rho g^{ih} + \sigma\rho\ell^i\ell^h - \varepsilon\phi_{;2}\rho(\ell^i m^h + \ell^h m^i)]\\ &\qquad [FC_{ijk} + (\varepsilon\mathcal{I}\sigma + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon))m_i m_j m_k]\\ =& \varepsilon\rho[FC^i_{jk} + (\varepsilon\mathcal{I}\sigma + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon))m^i m_j m_k\\ &- \varepsilon\phi_{;2}(\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon))\ell^i m_j m_k].\end{aligned}$$

□

Remark 3.4. **(a)** *Substituting* $C^i_{jk} = g^{ih}C_{hjk} = \frac{\mathcal{I}}{F}m^i m_j m_k$ *in Proposition* 3.3 **(iii)**, *we get*

$$F\overline{C}^i_{jk} = \rho(\mathcal{I}(1+\varepsilon\sigma) + \frac{1}{2}\sigma_{;2} + \phi_{;2}(\sigma+2\varepsilon))(\varepsilon m^i m_j m_k - \phi_{;2}\ell^i m_j m_k).$$

(b) *From Lemma* 2.8 **(v)** *and Proposition* 3.1, *we can get another equivalent formula of the main scalar* $\overline{\mathcal{I}}$ *of* $\overline{F}$ *as follows*

$$\overline{\mathcal{I}} = \sqrt{\varepsilon\rho}\,[\mathcal{I} + 2\varepsilon\phi_{;2} - \frac{\varepsilon\rho_{;2}}{2\rho}]. \tag{3.3}$$

The equivalence of Proposition 3.3 **(ii)** *and* (3.3) *can be proven easily.*

(c) *From* (3.3), $\overline{\mathcal{I}} = \sqrt{\varepsilon\rho}\mathcal{I}$ *if and only if* $4\rho\phi_{;2} = \rho_{;2}$. *Hence, a necessary and sufficient condition for the property of being Riemannian space to be preserved under anisotropic conformal change is that*

$$4\rho\phi_{;2} = \rho_{;2}.$$

In the following we find the change of the geodesic spray under the anisotropic conformal transformation (3.1) and consequently we find the transformation of the nonlinear connection and Berwald connection.

Proposition 3.5. *Under the anisotropic conformal transformation* (3.1), *the geodesic spray coefficients* $\overline{G}^i$ *and* G^i *of* $\overline{F}$ *and* F, *respectively, are related by*

$$\overline{G}^i = G^i + Q\, m^i + P\, \ell^i, \tag{3.4}$$

where

$$\begin{aligned} P =& \frac{1}{4}[-\phi_{;2}\, A + 2F^2(\partial_k\phi)\ell^k]), \qquad Q = \frac{1}{4}[\varepsilon A + 2\varepsilon F(\partial_k F)m^k - 2F(\partial_k F)_{;2}\ell^k], \\ A =& \{2F\phi_{;2}\,\rho\,(F\partial_k\phi + \partial_k F) + 2F\rho(F\partial_k\phi + \partial_k F)_{;2}\}\ell^k - 2\varepsilon F\rho(F\partial_k\phi + \partial_k F)m^k. \end{aligned} \tag{3.5}$$

Proof. Since $\overline{F}^2 = e^{2\phi}F^2$, we get $\partial_k\overline{F}^2 = 2Fe^{2\phi}[F\,\partial_k\phi + \partial_k F]$. Thus,

$$\dot{\partial}_j\partial_k\overline{F}^2 = e^{2\phi}[4\phi_{;2}(F\partial_k\phi + \partial_k F)m_j + 4F(\partial_k\phi)\ell_j + 2F(\partial_k\phi)_{;2}\, m_j + \dot{\partial}_j\partial_k F^2].$$

Consequently,

$$\begin{aligned} y^k\dot{\partial}_j\partial_k\overline{F}^2 - \partial_j\overline{F}^2 = e^{2\phi}[&y^k\dot{\partial}_j\partial_k F^2 - \partial_j F^2 + (4\phi_{;2}(\,F^2\partial_k\phi + F\partial_k F) + 2F^2(\partial_k\phi)_{;2})m_j\ell^k \\ &+ 4F^2(\partial_k\phi)\ell_j\ell^k - 2F^2\partial_j\phi]. \end{aligned} \tag{3.6}$$

Since $\partial_k F^2$ is $h(2)$, Eqn. (3.6) can be written as

$$\begin{aligned} y^k\dot{\partial}_j\partial_k\overline{F}^2 - \partial_j\overline{F}^2 = e^{2\phi}[&\{4\phi_{;2}(F^2\partial_k\phi + F\partial_k F) + 2F(F\partial_k\phi + \partial_k F)_{;2}\}m_j\ell^k \\ &+ 4F(F\partial_k\phi + \partial_k F)\ell_j\ell^k - 2F(F\partial_j\phi + \partial_j F)]. \end{aligned} \tag{3.7}$$

From (2.10), (3.6) and (3.7), we get

$$\begin{aligned} \overline{G^i} =& \frac{1}{4}\overline{g}^{ij}(y^k\,\dot{\partial}_j\partial_k\overline{F}^2 - \partial_j\overline{F}^2) \\ =& G^i + \frac{1}{4}[\{-2F(\phi_{;2})^2\rho(F\partial_k\phi + \partial_k F) - 2F\phi_{;2}\rho(F\partial_k\phi + \partial_k F)_{;2} + 2F^2\partial_k\phi\}\ell^k \\ &+ 2\varepsilon F\rho\phi_{;2}(F\partial_k\phi + \partial_k F)m^k]\ell^i + \frac{1}{4}[\{2\varepsilon F\phi_{;2}\rho(F\partial_k\phi + \partial_k F) + 2\varepsilon F\rho(F\partial_k\phi + \partial_k F)_{;2} \\ &- 2F(\partial_k F)_{;2}\}\ell^k + \{-2F\rho(F\partial_k\phi + \partial_k F) + 2\varepsilon F\partial_k F\}m^k]m^i. \end{aligned}$$

Hence, the result follows. □

Remark 3.6. *Using $\delta_k\phi = \partial_k\phi - G^i_k\dot{\partial}_i\phi$ and $\delta_k F^2 = 0$, we obtain the following technical equalities:*

$$\left.\begin{aligned}
&\textbf{(a)}\ F^2\ell^k\,\partial_k\phi = F^2\phi_{,1} + 2G^k\phi_{;2}\,m_k,\\
&\textbf{(b)}\ Fm^k\,\partial_k\phi = \varepsilon F\,\phi_{,2} + G^i_k\,\phi_{;2}\,m^k m_i,\\
&\textbf{(c)}\ \ell^k\,\partial_k F^2 = 4\,G^k\,\ell_k,\\
&\textbf{(d)}\ m^k\partial_k F^2 = 2FG^i_k\,\ell_i\,m^k,\\
&\textbf{(e)}\ F^2\ell^k(\partial_k\phi)_{;2} = F^2\,\phi_{,1;2} + \varepsilon F G^k_i\,\phi_{;2}\,m_k m^i + 2G^k\,\phi_{;2;2}\,m_k + 2\varepsilon\phi_{;2}\,\mathcal{I}\,G^k m_k\\
&\qquad\qquad -2G^k\,\phi_{;2}\,\ell_k - F^2\,\phi_{,2}\\
&\textbf{(f)}\ \ell^k(\partial_k F^2)_{;2} = 2\varepsilon F\,G^k_i\,\ell_k\,m^i + 4\varepsilon\,G^k\,m_k.
\end{aligned}\right\}\tag{3.8}$$

Subsituting (3.8) into the expressions of P and Q given by (3.5), we obtain the following short and compact expressions:

$$2Q = \varepsilon\rho F^2(\phi_{;2}\,\phi_{,1} + \phi_{,1;2} - 2\phi_{,2}),\tag{3.9}$$

$$2P = -\rho F^2\phi_{;2}(\phi_{;2}\,\phi_{,1} + \phi_{,1;2} - 2\phi_{,2}) + F^2\phi_{,1}.\tag{3.10}$$

Moreover, from (3.9) and (3.10), we get

$$2\varepsilon\,\phi_{;2}\,Q + 2P = F^2\phi_{,1}.\tag{3.11}$$

Hence, we get the following equivalent form of Proposition 3.5.

Theorem 3.7. *Under the anisotropic conformal transformation (3.1), the geodesic spray coefficients $\overline{G}^i$ and G^i of $\overline{F}$ and F, respectively, are related by*

$$\overline{G}^i = G^i + Qm^i + (\frac{1}{2}F^2\phi_{,1} - \varepsilon\phi_{;2}\,Q)\ell^i,\tag{3.12}$$

where Q is given by (3.9).

Remark 3.8. (a) *As a direct consequence of (3.4), the geodesic spray $\overline{S}$ associated with $\overline{F}$ has the form*

$$\overline{S} = S - 2(P\,\ell^i + Q\,m^i)\,\dot{\partial}_i.$$

(b) *From (3.12), we get $G^i = \overline{G}^i$ if and only if $Q = 0$ and $\phi_{,1} = 0$.*

Proposition 3.9. *Under the anisotropic conformal transformation (3.1), the coefficients of the Barthel connections $\overline{G}^i_j$ and G^i_j associated with $\overline{F}$ and F, respectively, are related by*

$$F\overline{G}^i_j = FG^i_j + \left\{2P\ell^i\ell_j + (P_{;2} - Q)\ell^i m_j + 2Q\ell_j m^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i m_j\right\}.\tag{3.13}$$

Proof. The proof is obtained directly by differentiating (3.4) with respect to y^j and using Lemma 2.8. □

Proposition 3.10. *Under the anisotropic conformal transformation (3.1), the coefficients of the Berwald connection $\overline{G}^i_{jk}$ and G^i_{jk} associated with $\overline{F}$ and F, respectively, are related by*

$$\begin{aligned}F^2\,\overline{G}^i_{jk} =& F^2\,G^i_{jk} + (2P\ell^i + 2Qm^i)\ell_j\ell_k + \{(P_{;2} - Q)\ell^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i\}(\ell_j m_k + \ell_k m_j)\\ &+ \{(\varepsilon P + P_{;2;2} - 2Q_{;2} + \varepsilon\mathcal{I}P_{;2})\ell^i + (2\varepsilon P_{;2} + \varepsilon Q + Q_{;2;2} - \varepsilon\mathcal{I}_{;2}Q - \varepsilon\mathcal{I}Q_{;2})m^i\}m_j m_k.\end{aligned}\tag{3.14}$$

Proof. It follows from (3.13) by differentiating $\overline{G}^i_j$ with respect to y^k and using Lemma 2.8. $\square$

Geometric properties or geometric objects which are preserved under the anisotropic conformal transformation are said to be anisotropic conformal invariants. In general, the geodesic spray is not invariant under the anisotropic conformal change. In the (isotropic) conformal transformation $\overline{F}(x, y) = e^{\phi(x)}F(x, y)$, the homothetic transformation is the only case that leads to unchanged geodesic spray [20, Proposition 11.1]. For the anisotropic conformal transformation (3.1), the geodesic spray is not invariant in general. Nevertheless, we have

Theorem 3.11. *Let (M, F) be a conic pseudo-Finsler surface. Under the anisotropic conformal transformation* (3.1)*, the following assertions are equivalent:*

(i) $\overline{S} = S$*, that is,* $P = Q = 0$*, where P and Q are given by* (3.9) *and* (3.10)*, respectively.*

(ii) $\delta_i\phi = 0$*, or equivalently,* $d_h\phi = 0$*.*

Proof. **(i) $\Longrightarrow$ (ii)**: $\overline{F} = e^{\phi}F$ implies

$$\delta_i\overline{F} = \delta_i(e^{\phi}F) = e^{\phi}F\,\delta_i\phi + e^{\phi}\,\delta_iF = \overline{F}\delta_i\phi + e^{\phi}\,\delta_iF. \tag{3.15}$$

It is known that F is horizontally constant, that is, $\delta_iF = 0$. By assumption $\overline{S} = S$ and so $\overline{\delta}_i = \delta_i$, then

$$\delta_i\overline{F} = \overline{\delta}_i\overline{F} = 0.$$

Thereby, (3.15) gives $\overline{F}\delta_i\phi = 0$. Hence, $\delta_i\phi = 0$.

(ii) $\Longrightarrow$ (i): Suppose $\delta_i\phi = 0$. Thus, Lemma 2.8 **(vii)** gives

$$\phi_{,1}\ell_i + \phi_{,2}m_i = 0. \tag{3.16}$$

Contracting both sides of (3.16) by ℓ^i and m^i, respectively, we obtain $\phi_{,1} = \phi_{,2} = 0$. Then, $P = 0$ and $Q = 0$ by (3.9) and (3.10). From (3.4), $\overline{G}^i = G^i$ and so $\overline{S} = S$. $\square$

Corollary 3.12. *Let (M, F) be a conic pseudo-Finsler surface. Under the anisotropic conformal transformation* (3.1)*, if $\delta_i\phi = 0$, then the Barthel connection G^i_j and Berwald connection G^i_{jk} are invariant.*

Proposition 3.13. *Let (M, F) be a conic pseudo-Finsler surface. Under the anisotropic conformal transformation* (3.1)*, the following assertions are equivalent:*

(i) $\overline{G}^i = G^i$. **(ii)** $\overline{G}^i_j = G^i_j$. **(iii)** $\overline{G}^i_{jk} = G^i_{jk}$.

Proof. **(i)$\Longrightarrow$(ii)** and **(ii)$\Longrightarrow$(iii)** follow directly by differentiating G^i with respect to y^j and G^i_j with respect to y^k, respectively.
(iii)$\Longrightarrow$(i): From (3.14), $\overline{G}^i_{jk} = G^i_{jk}$ implies that

$$\begin{aligned} 0 &= (2P\ell^i + 2Qm^i)\ell_j\ell_k + \{(P_{;2} - Q)\ell^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i\}(\ell_jm_k + \ell_km_j) + \{(\varepsilon P \\ &\quad + P_{;2;2} - 2Q_{;2} + \varepsilon\mathcal{I}P_{;2})\ell^i + (2\varepsilon P_{;2} + \varepsilon Q + Q_{;2;2} - \varepsilon\mathcal{I}_{;2}Q - \varepsilon\mathcal{I}Q_{;2})m^i\}m_jm_k. \end{aligned} \tag{3.17}$$

Contracting (3.17) by $\ell_i\ell^j\ell^k$ and $m_i\ell^j\ell^k$, respectively, we get $Q = P = 0$. Hence, by Theorem 3.11, the coefficients of the geodesic spray are invariant. $\square$

The following example provides a non-trivial anisotropic conformal transformation which leaves the geodesic spray invariant. A Maple's code of the detailed calculations is found in the Appendix at the end of the paper.

Example 3.14. *Let* $M = \mathbb{B}^2 \subset \mathbb{R}^2$, $x \in M$, $y \in T_x\mathbb{B}^2 \cong \mathbb{R}^2$, $a = (a_1, a_2) \in \mathbb{R}^2$ *and* a *is constant vector with* $|a| < 1$. *Let*

$$z^i = \frac{(1 + \langle a, x\rangle)y^i - \langle a, y\rangle x^i}{\langle a, y\rangle}.$$

Define the Finsler metric F *by*

$$F = \frac{\langle a, y\rangle \sqrt{(z^1)^2 + (z^2)^2}}{(1 + \langle a, x\rangle)^2}.$$

The geodesic spray coefficients are given by

$$G^i = -\frac{\langle a, y\rangle}{1 + \langle a, x\rangle} y^i.$$

Now, let $\overline{F} = e^\phi F = \dfrac{\langle a, y\rangle \sqrt{(z^1)^2 + (z^2)^2}}{(1 + \langle a, x\rangle)^2} \ \exp\left(\sqrt{(z^1)^2 + (z^2)^2}\right)$, *where* $\phi = \sqrt{(z^1)^2 + (z^2)^2}$. *One can easily check that* $\overline{F}$ *satisfies Theorem 2.11 (which means that* $\overline{F}$ *is a pseudo-Finsler metric) and* $\delta_i\phi = 0$. *Furthermore,*

$$\overline{G}^i = G^i = -\frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\, y^i.$$

4 Anisotropic conformal and Projective changes

Definition 4.1. [7] *Two sprays* S *and* $\hat{S}$*on* M *are projectively equivalent (or related) if there exists an* $h(1)$*-function* $\mathcal{P} : TM \longrightarrow \mathbb{R}$ *such that* $\hat{S} = S - 2\mathcal{P}\,\mathcal{C}$. *The function* $\mathcal{P}$ *is called the projective factor. Locally,* $\hat{G}^i = G^i + \mathcal{P}y^i$.

The association $S \longmapsto \hat{S} = S - 2\mathcal{P}\,\mathcal{C}$ or $G \longmapsto \hat{G}^i = G^i + \mathcal{P}y^i$ is said to be projective transformation (or projective change).

Theorem 4.2. *Under the anisotropic conformal transformation (3.1), the following assertions are equivalent:*

(i) *the geodesic sprays* S *and* $\overline{S}$ *are projectively equivalent,*

(ii) $\phi_{;2}\phi_{,1} + \phi_{,1;2} = 2\phi_{,2}$,

(iii) $\overline{G}^i = G^i + \frac{1}{2}F\phi_{,1}y^i$.

Proof. From (3.1) and (3.4), the two geodesic sprays S and $\bar{S}$ are projectively equivalent if and only if $\overline{G}^i = G^i + \mathcal{P}y^i$ where $\mathcal{P} = \dfrac{P}{F}$ and $Q = 0$. This is equivalent to $\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2} = 0$, by (3.9). In this case (3.10) takes the form $2P = F^2\phi_{,1}$ and hence the geodesic spray of $\overline{F}$ is given by

$$\overline{G}^i = G^i + \frac{1}{2}F\phi_{,1}\, y^i.$$

□

In view of Theorem 4.2, we have

Corollary 4.3. *Under the anisotropic conformal transformation* (3.1) *with* $\phi_{,1} = 0$, *the coefficients of the geodesic spray are invariant if and only if the two geodesic sprays* S *and* $\bar{S}$ *are projectively equivalent.*

Definition 4.4. [6] *A conic pseudo-Finsler metric* $F = F(x, y)$ *on an open subset* $U \subset \mathbb{R}^n$ *is said to be projectively flat if any one of the following equivalent conditions is satisfied:*

(i) *the geodesics of* F *are straight line segments in* U,

(ii) $y^j \partial_j \dot{\partial}_i F = \partial_i F$ *or equivalently* $\dot{\partial}_i(y^j \partial_j F) = 2\partial_i F$,

(iii) *the geodesic spray coefficients* G^i *are of the form* $G^i = \mathcal{P} y^i$, *with the projective factor* $\mathcal{P}$ *is given by* $\mathcal{P} = \dfrac{y^k \partial_k F}{2F}$.

Theorem 4.5. *Under the anisotropic conformal transformation* (3.1), *we have the following:*

(i) *a necessary condition for* $\overline{F}$ *to be projectively flat is that* $Q + \varepsilon G^k m_k = 0$.

(ii) *a sufficient condition for* $\overline{F}$ *to be projectively flat is that* $F\partial_j \phi + \partial_j F = 0$.

Proof. Since $\overline{F} = e^\phi F$, we get

$$\partial_j \overline{F} = e^\phi [F\partial_j \phi + \partial_j F] \tag{4.1}$$

and $y^j \partial_j \overline{F} = e^\phi [F^2(\partial_j \phi)\ell^j + F(\partial_j F)\ell^j]$. Thereby,

$$\begin{aligned}\dot{\partial}_i(y^j \partial_j \overline{F}) = e^\phi [&\{\phi_{;2} F\partial_j \phi + \phi_{;2}\partial_j F + F(\partial_j \phi)_{;2} + (\partial_j F)_{;2}\}\ell^j m_i + \{2F\partial_j \phi + 2\partial_j F\}\ell_i \ell^j \\ &+ \{\varepsilon F\partial_j \phi + \varepsilon \partial_j F\} m^j m_i].\end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we get

$$\begin{aligned}\dot{\partial}_i(y^j \partial_j \overline{F}) - 2\partial_i \overline{F} =& e^\phi [\{\phi_{;2}(F\partial_j \phi + \partial_j F) + (F\partial_j \phi + \partial_j F)_{;2}\}\ell^j m_i + 2(F\partial_j \phi + \partial_j F)\ell_i \ell^j \\ &+ \varepsilon(F\partial_j \phi + \partial_j F) m^j m_i - 2(F\partial_i \phi + \partial_i F)].\end{aligned} \tag{4.3}$$

(i) Let $\overline{F}$ be projectively flat, i.e., $\dot{\partial}_i(y^j \partial_j \overline{F}) - 2\partial_i \overline{F} = 0$. Then, the RHS of (4.3) vanishes. Contracing the later by m^i and using (3.8), we obtain

$$\frac{2}{F\rho}(Q + \varepsilon G^k m_k) = 0,$$

which implies that $Q + \varepsilon G^k m_k = 0$.

(ii) The proof is obtained directly from (4.3). □

Remark 4.6. **(a)** [17, §3.1] *Since* $0 = \delta_i F = \partial_i F - G^r_i \ell_r$, *we get*

$$2G^i \ell_i = y^i \partial_i F \;\; and \;\; 2G^r m_r = \frac{\varepsilon F^2 \mathfrak{M}}{\mathrm{h}}, \tag{4.4}$$

where $\mathrm{h} := \sqrt{\varepsilon \mathfrak{g}}$ *and* $\mathfrak{M} := \dot{\partial}_2 \partial_1 F - \dot{\partial}_1 \partial_2 F$. *From* (4.4), *we get*

$$2G^i = y^r(\partial_r F)\ell^i + \frac{F^2 \mathfrak{M}}{\mathrm{h}} m^i. \tag{4.5}$$

(b) *From* **(a)**, *the conic pseudo-Finsler surface is projectively flat if and only if* $G^i m_i = 0$, *which is equivalent to* $\mathfrak{M} = \dot{\partial}_2 \partial_1 F - \dot{\partial}_1 \partial_2 F = 0$ *(Hammel equation in two-dimensional spaces) and the geodesic spray coefficients have the form* $G^i = \frac{y^r(\partial_r F)}{2F} y^i$.

Theorem 4.7. *Let F be a projectively flat conic pseudo-Finsler metric and let $\overline{F} = e^{\phi}F$ be the anisotropic conformal transformation (3.1). Then, the Finsler metric $\overline{F}$ is projectively flat if and only if $\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2} = 0$.*

Proof. By Theorem 4.2, we have $\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2} = 0$ is equivalent to $\overline{G}^i = G^i + \frac{1}{2}F\phi_{,1}y^i$. Since F is projectively flat, then we get

$$\overline{G}^i = \frac{y^k\partial_k F + F^2\phi_{,1}}{2F}\, y^i. \tag{4.6}$$

By using (a) of (3.8), we obtain

$$\frac{y^k\partial_k\overline{F}}{2\overline{F}} = \frac{y^k\partial_k F + y^k F\partial_k\phi}{2F} = \frac{y^k\partial_k F + F^2\phi_{,1} + 2G^k\phi_{;2}\, m_k}{2F} = \frac{y^k\partial_k F + F^2\,\phi_{,1}}{2F}, \tag{4.7}$$

since $G^k\, m_k = 0$, by Remark 4.6. From (4.6) and (4.7), we get $\overline{G}^i = \overline{\mathcal{P}}\, y^i$, with $\overline{\mathcal{P}} = \dfrac{y^k\partial_k\overline{F}}{2\overline{F}}$. This is equivalent to the projective flatness of the metric $\overline{F}$. □

Proposition 4.8. *Let (M, F) be a conic pseudo-Finsler surface and let $\overline{F} = e^{\phi}F$ be the anisotropic conformal transformation (3.1) with $\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2} = 0$. Assume that $\overline{F}$ is projectively flat. Then, the Finsler metric F is projectively flat if and only if either $G^k m_k = 0$ or the conformal factor ϕ is a function of x only.*

Proof. From Theorem 4.2, we have $\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2} = 0$ is equivalent to $\overline{G}^i = G^i + \frac{1}{2}F\phi_{,1}y^i$. Since $\overline{F}$ is projectively flat, we obtain

$$G^i = \overline{G}^i - \frac{1}{2}F\phi_{,1}\, y^i = \frac{y^k\partial_k F + y^k F\partial_k\phi - F^2\,\phi_{,1}}{2F}\, y^i.$$

By using (a) of (3.8), we get

$$G^i = \frac{y^k\partial_k F + 2G^k\,\phi_{;2}\, m_k}{2F}\, y^i. \tag{4.8}$$

From (4.8), F is projectively flat if and only if $G^k\phi_{;2}m_k = 0$, which is equivalent to either $G^k m_k = 0$ or the conformal factor ϕ is a function of x only. □

Definition 4.9. [5] *A conic pseudo-Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is said to be dually flat if it satisfies the following equations:*

$$y^j\dot{\partial}_i\partial_j F^2 = 2\partial_i F^2.$$

Theorem 4.10. *Under the anisotropic conformal transformation (3.1), the following hold:*

(i) *a necessary condition for $\overline{F}$ to be dually flat is that*

$$\frac{2}{F\rho}(Q + \varepsilon G^k m_k) + \frac{2\varepsilon\phi_{;2}}{F}G^k m_k - G^i_k\ell_i m^k - \phi_{;2}G^i_k m_i m^k + \varepsilon F(\phi_{;2}\phi_{,1} - \phi_{,2}) = 0.$$

(ii) *a sufficient condition for $\overline{F}$ to be dually flat is that $F\partial_j\phi + \partial_j F = 0$.*

Proof. As $\overline{F}^2 = e^{2\phi}F^2$, then $\partial_j\overline{F}^2 = 2e^{2\phi}[F^2\partial_j\phi + F\partial_j F]$. Consequently, one can show that

$$y^j\dot{\partial}_i(\partial_j\overline{F}^2) - 2\partial_i\overline{F}^2 = 2Fe^{2\phi}[\{2\phi_{;2}(F\partial_j\phi + \partial_j F) + (F\partial_j\phi + \partial_j F)_{;2}\}\ell^j m_i + 2(F\partial_j\phi + \partial_j F)\ell_i\ell^j - 2(F\partial_i\phi + \partial_i F)]. \tag{4.9}$$

(i) Let $\overline{F}$ be dually flat, that is, $y^j\dot{\partial}_i(\partial_j\overline{F}^2) - 2\partial_i\overline{F}^2 = 0$. Then, the RHS of (4.9) vanishes. Contracting the later by m^i and using (3.8) and (3.9), we get

$$\frac{2}{F\rho}(Q + \varepsilon G^k m_k) + \frac{2\varepsilon\phi_{;2}}{F}G^k m_k - G^i_k\ell_i m^k - \phi_{;2}G^i_k m_i m^k + \varepsilon F(\phi_{;2}\phi_{,1} - \phi_{,2}) = 0.$$

(ii) The proof is obtained directly From (4.9). □

The following example shows that the condition $F\partial_j\phi + \partial_j F = 0$ is not a necessary condition for neither projective flatness nor dual flatness of $\overline{F}$ (A Maple's code of the detailed calculations is found in the Appendix). On the other hand, the same condition $F\partial_j\phi + \partial_j F = 0$ is a sufficient condition for both projective flatness and dual flatness of $\overline{F}$

Example 4.11. *Let F be the Klein metric on the unit ball $B^n \subset \mathbb{R}^n$, $F = \dfrac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}$. We get*

$$\overline{F} = e^{\phi}F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2},$$

where

$$\phi = \ln\left(1 + \frac{\langle x, y\rangle}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}\right).$$

$\overline{F}$ is both projectively flat and dually flat [5], but $F\partial_j\phi + \partial_j F \neq 0$.

5 Special cases for the anisotropic conformal factor

Based on the importance of the conformal factor $\phi(x, y)$ in studying the geometric objects associated with $\overline{F}$, we focus our attention on studying some special cases of ϕ. We consider two cases: the function ϕ is a function of y only and the function ϕ is a function of x only. In the later case the anisotropic conformal transformation reduces to the (isotropic) conformal transformation [9].

Proposition 5.1. *Let (M, F) be a conic pseudo-finsler surface and consider the anisotropic conformal transformation (3.1) with ϕ a function of y only. Assume that F is projectively flat. Then, $\overline{F}$ is projectively flat if and only if $\phi_{,2} = 0$.*

Proof. Since ϕ is a function of y only, we get

$$\delta_i\phi = \phi_{,1}\ell_i + \phi_{,2}m_i = -\frac{\phi_{;2}}{F}G^r_i m_r. \tag{5.1}$$

Multiple both sides of (5.1) by ℓ^i, we get

$$F^2\phi_{,1} = -2\phi_{;2}G^r m_r. \tag{5.2}$$

As F is projectively flat, from Remark 4.6 and (5.2), we have

$$\phi_{,1} = 0. \tag{5.3}$$

By Theorem 4.7 and (5.3), $\overline{F}$ is projectively flat if and only if $\phi_{,2} = 0$. □

Proposition 5.2. *Let (M, F) be a conic pseudo-finsler surface and consider the anisotropic conformal transformation (3.1) with ϕ a function of y only. Assume that $\overline{F}$ is projectively flat. Then, a necessary condition for $\overline{F}$ to be dually flat is that*

$$\varepsilon F(\phi_{;2}\phi_{,1} - \phi_{,1}) - G^i_k \ell_i m^k = 0.$$

Proof. Since $\overline{F}$ is projectively flat, then by Theorem 4.5, we have

$$Q + \varepsilon G^k m_k = 0. \tag{5.4}$$

Also, ϕ is function of y only, from (a) and (b) of (3.8), we get

$$\frac{2\varepsilon\phi_{;2}}{F} G^k m_k - \phi_{;2} G^i_k m_i m^k + \varepsilon F(\phi_{,1} - \phi_{,2}) = 0. \tag{5.5}$$

From (5.4), (5.5) and Theorem 4.10 if $\overline{F}$ is dually flat, then $\varepsilon F(\phi_{;2}\phi_{,1} - \phi_{,1}) - G^i_k \ell_i m^k = 0$. □

Lemma 5.3. *Let (M, F) be a conic pseudo-Finsler surface and $\overline{F} = e^{\phi} F$ be the anisotropic conformal transformation (3.1). If ϕ is function of x only, then*

$$\sigma = 0, \quad \rho = \varepsilon. \tag{5.6}$$

Proof. Since ϕ is independent of y, then $\phi_{;2} = 0$ and $\sigma = \phi_{;2;2} + \varepsilon\mathcal{I}\phi_{;2} + 2(\phi_{;2})^2 = 0$. Also we have $\rho = \dfrac{1}{\sigma + \varepsilon - (\phi_{;2})^2} = \dfrac{1}{\varepsilon} = \varepsilon$. □

Lemma 5.4. *Let (M, F) be a conic pseudo-Finsler surface and $\overline{F}(x, y) = e^{\phi(x,y)} F(x, y)$. Then, $\overline{g}_{ij} = e^{2\phi} g_{ij}$ if and only if ϕ is either a function of x only or is a constant (homothetic).*

Proof. Let ϕ be a function of x only and $\overline{F} = e^{\phi} F$. It is obvious that $\overline{g}_{ij} = e^{2\phi} g_{ij}$ (with no further condition). Conversely, let $\overline{g}_{ij} = e^{2\phi} g_{ij}$. Then, by (2.9), we get

$$e^{2\phi}[\phi_{;2}(\ell_i m_j + \ell_j m_i) + \sigma m_i m_j] = 0.$$

Contracting both sides of the previous equation by $\ell^i m^j$, we obtain $\phi_{;2} = 0$. Hence, by (2.8), ϕ is either a function of x only or is a constant. □

Proposition 5.5. *Let $\overline{F} = e^{\phi} F$ be an anisotropic conformal transformation. If ϕ is a function of x only, then*

(i) $\overline{g}_{ij} = e^{2\phi} g_{ij}, \qquad \overline{g}^{ij} = e^{-2\phi} g^{ij},$

(ii) $\overline{\ell}^i = e^{-\phi} \ell^i, \qquad \overline{\ell}_i = e^{\phi} \ell_i,$

(iii) $\overline{m}_i = e^{\phi} m_i, \qquad \overline{m}^i = e^{-\phi} m^i, \qquad \overline{h}_{ij} = e^{2\phi} h_{ij},$

(iv) $\overline{C}_{ijk} = e^{2\phi} C_{ijk}, \qquad \overline{C}^i_{jk} = C^i_{jk}, \qquad \overline{\mathcal{I}} = \mathcal{I},$

(v) $\overline{G}^i = G^i + \frac{1}{2} F^2(\phi_{,1}\ell^i - \phi_{,2} m^i).$

Proof. The proof of **(i)-(iv)** follow by substituting (5.6) into Propositions 3.1 and 3.3.
(v) If ϕ is a function of x only, then by (3.5), we have

$$P = \frac{1}{2}F^2\ell^i\partial_i\phi, \qquad Q = -\frac{1}{2}\varepsilon F^2 m^i\partial_i\phi.$$

By (1) and (2) of (3.8) and (3.4), where $\phi_{;2} = 0$, we get

$$\overline{G}^i = G^i + \frac{1}{2}F^2(\phi_{,1}\ell^i - \phi_{,2}m^i).$$

□

Remark 5.6. *In view of Proposition 5.5, under the anisotropic conformal transformation with ϕ is a function of x only, we conclude the following*

(a) *$\overline{S}$ and S coincide if and only if ϕ is homothetic.*

(b) *From* (v) *and* (4.5)*, we get*

$$\overline{G}^i = \frac{1}{2}F[(\ell^r\partial_r F + F\phi_{,1})\ell^i + F(\frac{\mathfrak{M}}{h} - \phi_{,2})m^i].$$

(c) *From* **(b)** *the geodesic spray of $\overline{F}$ is flat if and only if $\ell^r\partial_r F + F\phi_{,1} = 0$ and $\frac{\mathfrak{M}}{h} = \phi_{,2}$.*

6 Concluding remarks

We end the present paper with the following comments and remarks. In this paper, we have investigated the notion of anisotropic conformal transformation of conic pseudo-Finsler metrics in two-dimensional manifolds using the associated modified Berwald frame. The following points are to be singled out:

- The anisotropic conformal change of a pseudo-Finsler metric $F(x, y)$ does not necessarily yield a pseudo-Finsler metric $\overline{F}(x, y)$. Consequently, we find out the necessary and sufficient condition for $\overline{F}(x, y) = e^{\phi(x,y)}F(x, y)$ to be a pseudo-Finsler metric. This is a crucial result which ensures that two (conic) pseudo-Finsler surfaces may or may not be anisotropically conformally related, contrary to the "pure" Finslerian case where any two Finsler metrics are anisotropically conformally related ($\overline{F} = (\frac{\overline{F}}{F})F$). This justifies our choice of the object of study and certifies that our study is nontrivial.

- We study some geometric objects associated with the transformed metric such as, the Berwald frame, metric tensor, Cartan tensor, inverse metric tensor, main scalar, geodesic spray, Barthel connection and Berwald connection.

- It has been proved that the geodesic spray is an anisotropic conformal invariant if and only if the conformal factor $\phi(x, y)$ has the property that $\delta_i\phi = 0$ or equivalently is d_h-closed. On the other hand, it is well-known that in the (isotropic) conformal case, the geodesic spray is an (isotropic) conformal invariant if and only if the conformal factor $\phi(x)$ is homothetic [2, 20]. As the condition of being d_h-closed is more general than the condition of being homothetic, our result includes as a special case the (isotropic) conformal result. On the other hand, as a consequence of our result, the anisotropic conformal invariance of any one of the geodesic spray, Cartan nonlinear connection or Berwald connection implies the aniotropic conformal invariance of the other ones.

- Under the anisotropic conformal transformation $\overline{F} = e^{\phi} F$, for two projectively related conic pseudo-Finsler surfaces, the pojective flatness property is preserved. Moreover, the condition $F\partial_j\phi + \partial_j F = 0$ is a sufficient condition for both projective flatness and dual flatness of $\overline{F}$, but is neither a necessary condition for projective flatness nor dual flatness of $\overline{F}$.

- Two interesting special cases are considered: (i) when the anisotropic conformal factor depends on position only, (ii) when the anisotropic conformal factor depends on direction only. In the first case our anisotropic conformal transformation reduces to the well-known (isotropic) conformal transformation which was initiated in [9] and in the second case some interesting results are obtained.

This work will be continued in a forthcoming paper: "Anisotropic conformal transformation of conic pseudo-Finsler surface, II", where further geometric properties along with special Finsler spaces will be investigated.

References

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The theory of sprays and pseudo-Finsler spaces with applications in physics and biology*, Kluwer Acad. Publ., 1993.

[2] S. Bacso and C. Xinyue, *Finsler conformal transformations and the curvature invariances*, Publ. Math. Debrecen, **70** (2007), 221-231.

[3] L. Berwald, *On Finsler and Cartan geometries. III Two-dimensional Finsler spaces with rectilinear extremals*, Ann. Math., **42** (1941), 84-112.

[4] I. Bucataru and R. Miron, *Finsler-Lagrange geometry: Applications to dynamical systems*, Editura Academiei Romane, 2007.

[5] X. Cheng, Z. Shen and Y. Zhou, *On locally dually flat Finsler metrics*, Int. J. Math., **21** (2010), 1531–1543.

[6] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientific publ., Singapore, 2005.

[7] S. Elgendi and Z. Muzsnay, *Metrizability of holonomy invariant projective deformation of sprays*, Can. Math. Bull., **66** (2020), 1–14. arXiv: 1912.02257v1 [math.DG].

[8] J. Grifone, *Structure présque-tangente et connexions*, Ann. Inst. Fourier, Grenoble, **22** (1) (1972), 287-334.

[9] M. Hashiguchi, *On conformal transformations of Finsler metrics,* J. Math. Kyoto Univ., **16** (1976), 25–50.

[10] B. T. Hassan and F. Mesabah, *Conformal transformations in Finsler Geometry,* In: M. Anastasiei and P. L. Antonelli (Eds), Finsler and Lagrange Geometry, Kluwer Acad. Publ., 2003.

[11] P. R. Holland, *Electromagnetism, Particles and Anholonomy*, Phys. Lett., **91** (6) (1982), 275–278.

[12] M. A. Javaloyes and M. Sánchez, *On the definition and examples of cones and Finsler spacetimes*, Rev. R. Acad. Cienc., **114** (2020), 30- 46. arXiv: 1805.06978v4 [math.DG].

[13] M. A. Javaloyes and L. B. Soares, *Geodesics and Jacobi fields of pseudo-Finsler manifolds*, Publ. Math. Debrecen, **87** (2014), 57-78. arXiv: 1401.8149v3 [math.DG].

[14] M. A. Javaloyes and L. B. Soares, *Anisotropic conformal invariance of lightlike geodesics in pseudo-Finsler manifolds*, Class. Quantum Grav., **38** (2021), 025002-025017. arXiv: 2006.14959v1 [math.DG].

[15] M. S. Knebelman, *Conformal geometry of generalized metric spaces*, Proc. Nat, Acad. Sci. USA, **15** (1929), 376-379.

[16] J. W. Maluf and F. Faria, *Conformally invariant teleparallel theories of gravity*, Phys. Rev. D., **85** (2012), 027502-027506. arXiv: 1110.3095v3 [gr-qc].

[17] M. Matsumoto, *Finsler geometry in the 20th-century*, In: Handbook of Finsler geometry, Kluwer. Acad. Publ., 2003.

[18] C. Pfeifer, S. Heefer and A. Fuster, *Identifying Berwald Finsler geometries*, Differ. Geom. Appl., **79** (2021), 101817-101828. arXiv: 1909.05284v3 [math.DG].

[19] H. Rund, *Differential geometry of Finsler spaces,* Springer, Berlin, 1959.

[20] Y. Shen and Z. Shen, *Introduction to modern Finsler geometry*, World Scientific Publ., Singapore, 2016.

[21] N. Voicu, *Conformal maps between pseudo-Finsler spaces,* Int. J. Geom. Methods Mod. Phys., **15** (2018), 1850003. arXiv: 1706.01569v2 [math.DG].

[22] Nabil L. Youssef, A. Soleiman and Ebtsam H. Taha, *New conformal invariants in absolute parallelism geometry*, Int. J. Geom. Methods Mod. Phys., **15** (1) (2018), 1850012-1850021. arXiv: 1604.00474v2 [math.DG].

Appendix : *Maple's code with an example* (*Example* 3.14)

```
> restart
> with(LinearAlgebra) :
```

Inserting The Finsler function

$$> z1 := \frac{(a1\,x1 + a2\,x2 + 1)\,y1 - (a1\,y1 + a2\,y2)\,x1}{a1\,y1 + a2\,y2} :$$

$$> z2 := \frac{(a1\,x1 + a2\,x2 + 1)\,y2 - (a1\,y1 + a2\,y2)\,x2}{a1\,y1 + a2\,y2} :$$

$$> F := \frac{(a1\,y1 + a2\,y2)}{(1 + a1\,x1 + a2\,x2)^2}\sqrt{z1^2 + z2^2} :$$

Inserting the factor φ

$$> \varphi := \sqrt{z1^2 + z2^2} :$$

The components of the metric tensor

$$> g11 := \frac{1}{2}\,diff(F^2, y1, y1) : g12 := \frac{1}{2}\,diff(F^2, y1, y2) : g22 := \frac{1}{2}\,diff(F^2, y2, y2) :$$

$$> MetricTensor := simplify(Matrix([[g11, g12], [g12, g22]]));$$

$$MetricTensor := \begin{bmatrix} \frac{a1^2\,x2^2 + a2^2\,x2^2 + 2\,a2\,x2 + 1}{(a1\,x1 + a2\,x2 + 1)^4} & -\frac{a1^2\,x1\,x2 + a2^2\,x1\,x2 + a1\,x2 + a2\,x1}{(a1\,x1 + a2\,x2 + 1)^4} \\ -\frac{a1^2\,x1\,x2 + a2^2\,x1\,x2 + a1\,x2 + a2\,x1}{(a1\,x1 + a2\,x2 + 1)^4} & \frac{a1^2\,x1^2 + a2^2\,x1^2 + 2\,a1\,x1 + 1}{(a1\,x1 + a2\,x2 + 1)^4} \end{bmatrix} \quad \textbf{(1)}$$

The determanint of the metric tensor

$$> g := simplify(Determinant(MetricTensor));$$

$$g := \frac{1}{(a1\,x1 + a2\,x2 + 1)^6} \quad \textbf{(2)}$$

Since the metric g_ij is positive definite, then we set ε = 1

$$> \varepsilon := 1;$$

$$\varepsilon := 1 \quad \textbf{(3)}$$

Berwald frame & coframe ($\ell u1$and mu1 means ℓ^1 and m^1 resp.)

$$> \ell 1 := diff(F, y1) : \ell 2 := diff(F, y2) :$$

$$> \ell u1 := \frac{1}{g}\,g22\;\ell 1 - \frac{1}{g}\,g12\;\ell 2 : \ell u2 := -\frac{1}{g}\,g12\;\ell 1 + \frac{1}{g}\,g11\;\ell 2 :$$

$$> mu1 := -\frac{1}{\sqrt{\varepsilon\,g}}\;\ell 2 : mu2 := \frac{1}{\sqrt{\varepsilon\,g}}\;\ell 1 :$$

$$> m1 := -\sqrt{\varepsilon\,g}\;\ell u2 : m2 := \sqrt{\varepsilon\,g}\;\ell u1 :$$

Main scalar

$$> \mathrm{I} := simplify\Big(\frac{1}{2}\;F\,\big(diff(g11, y1)\;\mu 1^3 + 3\,diff(g11, y2)\;\mu 1^2\,mu2 + 3\,diff(g12, y2)\;\mu 1\;\mu 2^2 + diff(g22, y2)\;\mu 2^3\big)\Big)$$

$$I := 0 \tag{4}$$

Condition* on phi so that *Fbar* is a *Finsler surface

> $\varphi 2 := \varepsilon\, F\, mu1\, diff(\varphi, y1) + \varepsilon\, F\, mu2\, diff(\varphi, y2)$:

> $\varphi 22 := \varepsilon\, F\, mu1\, diff(\varphi 2, y1) + \varepsilon\, F\, mu2\, diff(\varphi 2, y2)$:

> $Cond := \varphi 22 + \varepsilon\, I\, \varphi 2 + \varphi 2^2 + \varepsilon$:

To avoid complications of formulae, we find the "cond" at a point:

> $simplify(subs([x1 = 1, x2 = 2, y1 = 1, y2 = 2, a1 = 1, a2 = 1, \varepsilon = 1], Cond))$

$$\frac{86}{81} + \frac{11}{27}\sqrt{5} \tag{5}$$

Change of the main scalar

> $\sigma := \varphi 22 + 2\, \varphi 2^2 + \varepsilon\, I\, \varphi 2$:

> $\rho := \dfrac{1}{\varepsilon + \sigma - \varphi 2^2}$:

> $\sigma 2 := \varepsilon\, F\, mu1\, diff(\sigma, y1) + \varepsilon\, F\, mu2\, diff(\sigma, y2)$:

> $Ibar := (\varepsilon\, \rho)^{\frac{3}{2}} \left(I\,(1 + \varepsilon\, \sigma) + \frac{1}{2}\, \sigma 2 + \varphi 2\, (\sigma + 2\, \varepsilon) \right)$:

> $simplify(subs([x1 = 1, x2 = 1, y1 = 1, y2 = 1, a1 = 1, a2 = 0, \varepsilon = 1], Ibar))$

$$\frac{1}{18}\, \frac{\sqrt{3}\, (36 + 23\sqrt{2})}{(1 + \sqrt{2})^{3/2}} \tag{6}$$

Change of the spray coefficients

> $A := (2\, \varphi 2\, \rho\, F^2\, diff(\varphi, x1) + \varphi 2\, \rho\, diff(F^2, x1) + 2\, \varepsilon\, \rho\, F^2\, (F\, mu1\, diff(\varphi, y1, x1) + F\, mu2\, diff(\varphi, y2, x1)) + \varepsilon\, \rho\, (F\, mu1\, diff(F^2, y1, x1) + F\, mu2\, diff(F^2, y2, x1)))\, \ell u1 + (2\, \varphi 2\, \rho\, F^2\, diff(\varphi, x2) + \varphi 2\, \rho\, diff(F^2, x2) + 2\, \varepsilon\, \rho\, F^2\, (F\, mu1\, diff(\varphi, y1, x2) + F\, mu2\, diff(\varphi, y2, x2)) + \varepsilon\, \rho\, (F\, mu1\, diff(F^2, y1, x2) + F\, mu2\, diff(F^2, y2, x2)))\, \ell u2 - \varepsilon\, (2\, \rho\, F^2\, diff(\varphi, x1) + \rho\, diff(F^2, x1))\, mu1 - \varepsilon\, (2\, \rho\, F^2\, diff(\varphi, x2) + \rho\, diff(F^2, x2))\, mu2$:

> $Q := simplify\left(\frac{1}{4}\, (\varepsilon\, A + \varepsilon\, diff(F^2, x1)\, mu1 + \varepsilon\, diff(F^2, x2)\, mu2 - \varepsilon\, (F\, mu1\, diff(F^2, y1, x1) + F\, mu2\, diff(F^2, y2, x1))\, \ell u1 - \varepsilon\, (F\, mu1\, diff(F^2, y1, x2) + F\, mu2\, diff(F^2, y2, x2))\, \ell u2)\right)$

$$Q := 0 \tag{7}$$

> $P := simplify\left(\frac{1}{4}\, (-\varphi 2\, A + 2\, F^2\, diff(\varphi, x1)\, \ell u1 + 2\, F^2\, diff(\varphi, x2)\, \ell u2)\right)$

$$P := 0 \tag{8}$$

> $G1 := simplify\left(\frac{1}{4\, g}\, g22\, (y1\, diff(F^2, y1, x1) + y2\, diff(F^2, y1, x2) - diff(F^2, x1)) - \frac{1}{4\, g}\, g12\, (y1\, diff(F^2, y2, x1) + y2\, diff(F^2, y2, x2) - diff(F^2, x2))\right)$

$$G1 := -\frac{(a1\, y1 + a2\, y2)\, y1}{a1\, x1 + a2\, x2 + 1} \tag{9}$$

> $G2 := simplify\left(-\frac{1}{4\ g}\ g12\ \left(y1\ diff\left(F^2, y1, x1\right) + y2\ diff\left(F^2, y1, x2\right) - diff\left(F^2, x1\right)\right) + \frac{1}{4\ g}\ g11\ \left(y1\ diff\left(F^2, y2, x1\right) + y2\ diff\left(F^2, y2, x2\right) - diff\left(F^2, x2\right)\right)\right)$

$$G2 := -\frac{(a1\ y1 + a2\ y2)\ y2}{a1\ x1 + a2\ x2 + 1} \quad \textbf{(10)}$$

> $G1bar := simplify(G1 + P\ \ell u1 + Q\ mu1)$

$$G1bar := -\frac{(a1\ y1 + a2\ y2)\ y1}{a1\ x1 + a2\ x2 + 1} \quad \textbf{(11)}$$

> $G2bar := simplify(G2 + P\ \ell u2 + Q\ mu2)$

$$G2bar := -\frac{(a1\ y1 + a2\ y2)\ y2}{a1\ x1 + a2\ x2 + 1} \quad \textbf{(12)}$$

Dually flat conditions (Example 4.11) :

> $restart$

> $F := \frac{\sqrt{y1^2 + y2^2 - \left(\left(x1^2 + x2^2\right)\cdot\left(y1^2 + y2^2\right) - (x1\ y1 + x2\ y2)^2\right)}}{1 - x1^2 - x2^2}$;

$$F := \frac{\sqrt{y1^2 + y2^2 - \left(x1^2 + x2^2\right)\ \left(y1^2 + y2^2\right) + (x1\ y1 + x2\ y2)^2}}{-x1^2 - x2^2 + 1} \quad \textbf{(13)}$$

> $\varphi := \ln\left(1 + \frac{x1\ y1 + x2\ y2}{\sqrt{y1^2 + y2^2 - \left(\left(x1^2 + x2^2\right)\cdot\left(y1^2 + y2^2\right) - (x1\ y1 + x2\ y2)^2\right)}}\right)$;

$$\varphi := \ln\left(1 + \frac{x1\ y1 + x2\ y2}{\sqrt{y1^2 + y2^2 - \left(x1^2 + x2^2\right)\ \left(y1^2 + y2^2\right) + (x1\ y1 + x2\ y2)^2}}\right) \quad \textbf{(14)}$$

> $Fbar := simplify(\exp(\varphi)\cdot F)$;

$$Fbar := -\frac{x1\ y1 + x2\ y2 + \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}}{x1^2 + x2^2 - 1} \quad \textbf{(15)}$$

> $simplify\left(y1\ diff\left(Fbar^2, x1, y1\right) + y2\ diff\left(Fbar^2, x2, y1\right) - 2\ diff\left(Fbar^2, x1\right)\right)$

$$0 \quad \textbf{(16)}$$

> $simplify\left(y1\ diff\left(Fbar^2, x1, y2\right) + y2\ diff\left(Fbar^2, x2, y2\right) - 2\ diff\left(Fbar^2, x2\right)\right)$

$$0 \quad \textbf{(17)}$$

> $simplify(F\ \ diff(\varphi, x1) + diff(F, x1))$

$$\Big(-x1^3\ y2^2 + 3\ x1^2\ x2\ y1\ y2 - 2\ x1\ x2^2\ y1^2 + x1\ x2^2\ y2^2 - x2^3\ y1\ y2 + \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}\ x1^2\ y1 + 2\ \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}\ x1\ x2\ y2 - \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}\ x2^2\ y1 + 2\ x1\ y1^2 + x1\ y2^2 + x2\ y1\ y2 + y1\ \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}\Big) \Big/ \Big(\left(x1^2 + x2^2 - 1\right)^2\ \Big(x1\ y1 + x2\ y2 + \sqrt{-x1^2\ y2^2 + 2\ x1\ x2\ y1\ y2 - x2^2\ y1^2 + y1^2 + y2^2}\Big)\Big) \quad \textbf{(18)}$$

> $simplify(F\ \ diff(\varphi, x2) + diff(F, x2))$

$$
\begin{aligned}
&- \Big(x1^3\, y1\, y2 - x1^2\, x2\, y1^2 + 2\, x1^2\, x2\, y2^2 - 3\, x1\, x2^2\, y1\, y2 + x2^3\, y1^2 \\
&\quad + \sqrt{-x1^2\, y2^2 + 2\, x1\, x2\, y1\, y2 - x2^2\, y1^2 + y1^2 + y2^2}\; x1^2\, y2 \\
&\quad - 2\sqrt{-x1^2\, y2^2 + 2\, x1\, x2\, y1\, y2 - x2^2\, y1^2 + y1^2 + y2^2}\; x1\, x2\, y1 \\
&\quad - \sqrt{-x1^2\, y2^2 + 2\, x1\, x2\, y1\, y2 - x2^2\, y1^2 + y1^2 + y2^2}\; x2^2\, y2 - x1\, y1\, y2 - x2\, y1^2 - 2\, x2\, y2^2 \\
&\quad - y2\sqrt{-x1^2\, y2^2 + 2\, x1\, x2\, y1\, y2 - x2^2\, y1^2 + y1^2 + y2^2} \Big) \Big/ \Big(\left(x1^2 + x2^2 - 1\right)^2 \Big(x1\, y1 \\
&\quad + x2\, y2 + \sqrt{-x1^2\, y2^2 + 2\, x1\, x2\, y1\, y2 - x2^2\, y1^2 + y1^2 + y2^2} \Big) \Big)
\end{aligned} \tag{19}
$$