4-connected 1-planar chordal graphs are Hamiltonian-connected

Licheng Zhang,*[1] Yuanqiu Huang,†[2] Shengxiang Lv,‡[3] Fengming Dong§[4]

[1]School of Mathematics, Hunan University, China
[2]College of Mathematics and Statistics, Hunan Normal University, China
[3]School of Mathematics and Statistics,Hunan University of Finance and Economics, China
[4]National Institute of Education, Nanyang Technological University, Singapore

Abstract

Tutte proved that 4-connected planar graphs are Hamiltonian. It is unknown if there is an analogous result on 1-planar graphs. In this paper, we characterize 4-connected 1-planar chordal graphs, and show that all such graphs are Hamiltonian-connected. A crucial tool used in our proof is a characteristic of 1-planar 4-trees.

Keywords: 1-planar graph, chordal graph, Hamiltonian-connected
MSC: 05C10, 05C40, 05C45, 05C62

1 Introduction

Only finite simple connected graphs are considered in this paper. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set. For any subset V_0 of $V(G)$, the subgraph of G induced by V_0, denoted by $G[V_0]$, is the graph with vertex set V_0 and edge set $\{uv \in E(G) : u, v \in V(G)\}$. Let $G - V_0$ denote the subgraph of G induced by $V(G) \setminus V_0$ when $V_0 \neq V(G)$. A subset S of $V(G)$ is called a *separator* if $G - S$ is disconnected, and a separator S is called a *k-separator* if $|S| = k$. For a non-complete graph G, its *connectivity*, denoted by $\kappa(G)$, is defined to be the minimum value of $|S|$ over all separators S of G, and $\kappa(G) = |V(G)| - 1$ if G is connected.

A *Hamiltonian path (resp. cycle)* in G is a path (resp. cycle) in G that visits every vertex of G exactly once. A graph G is called *Hamiltonian* if G contains a Hamiltonian cycle. A graph G is

*Email: lczhangmath@hnu.edu.cn
†Corresponding author. Email: hyqq@hunnu.edu.cn.
‡Email:lvsxx23@126.com
§Email:fengming.dong@nie.edu.sg

Hamiltonian-connected if every two vertices in G are connected by a Hamiltonian path. Clearly, all Hamiltonian-connected graphs of orders at least three are Hamiltonian.

A *drawing* of a graph $G = (V, E)$ is a mapping $D(G)$ that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc connecting $D(u)$ and $D(v)$. A drawing is said to be *good* if no edge crosses itself, no two edges cross more than once, and no two edges incident with a vertex cross each other. All drawings considered in this paper are good ones. A graph is called *planar* if has a drawing so that no edges cross each other. The problem of Hamiltonicity of planar graphs has received considerable attention. Whitney [32] showed that every 4-connected plane triangulation is Hamiltonian, and Tutte [30] extended this conclusion to every 4-connected planar graph. Tutte's result has been strengthened in various ways. For example, Thomassen [29] proved that every 4-connected planar graph is Hamiltonian-connected, and Thomas and Yu [28] proved that 4-connected projective-planar graphs are Hamiltonian. For more references, the reader may refer to [11, 20, 23, 24].

A graph is called 1-*planar* if has a drawing so that any edge is crossed at most once, and such a drawing is also called a 1-*plane graph*. 1-planar graphs are introduced by Ringel (1965) [25] in the connection with the problem of the simultaneous colouring of the vertices and faces of plane graphs. Since then, many properties of 1-planar graphs have been widely investigated (see [4, 8, 12, 15, 21, 27] for example). It is known that 1-planar graphs have minimum degree at most 7, and thus they are at most 7-connected [13]. People try to establish a foundational result on Hamiltonicity of highly connected 1-planar graphs, akin to Tutte's work on 4-connected planar graphs. A 1-planar graph G with $|V(G)| \geq 3$ vertices has at most $4|V(G)|-8$ edges [5]. A 1-planar graph is called *optimal* if $|E(G)| = 4|V(G)|-8$. Fujisawa et al. [14, 27] showed that every optimal 1-planar graphs has connectivity 4 or 6. Hudák et al. [16] proved that every optimal 1-planar is Hamiltonian. Fabrici et al. [12] proved that a 3-connected locally maximal 1-planar graph G is Hamiltonian if it has at most three 3-vertex-cuts, implying that all 4-connected maximal 1-planar graphs are Hamiltonian. However, there are still many non-Hamiltonian 1-planar graphs with connectivity 4 or 5 [2, 12]. Actually, it is even unknown if 1-planar graphs with connectivity 6 or 7 are Hamiltonian.

Problem 1 ([2, 12]). *Is every* 6*-connected (or* 7*-connected)* 1*-planar graph is Hamiltonian?*

A *hole* is an induced cycle of length at least four. A graph is *chordal* if it does not contain holes. In this article, we will establish the following conclusion on the Hamiltonicity of 1-planar graphs.

Theorem 1. *Let G be a* 1*-planar graph. If G is* 4*-connected and choral, then G is Hamiltonian-connected.*

Note that the conclusion in Theorem 1 does not hold if some condition is replaced by a weaker one. Some details are given in following remark.

Remark 1. (i). *There are non-Hamiltonian* 4*-connected* 1*-planar graphs (see [16]);*

(ii). *It is easy to construct* 4*-connected chordal graphs that are not Hamiltonian. For example, the graph obtained from a graph* H *isomorphic to* K_4 *by adding* r *vertices* $v_1, \ldots, v_r$*, where* $r \geq 5$*, and adding edges joining each* v_i *to every vertex in* H *is not Hamiltonian.*

(iii). *There exist* 3*-connected planar chordal graphs that are not Hamiltonian [22]. We can also construct* 3*-connected* 1*-planar chordal graphs that are neither planar nor Hamiltonian. An example is shown in Fig. 1. It is easy to verify that this graph* G_0 *is chordal,* 1*-planar and* 3*-connected, but not* 4*-connected. This graph is not Hamiltonian due to the fact that the subgraph of* G_0 *obtained by removing vertices* $0, 1, 3, 5$ *and* 7 *has six components. Taking any one of such* 1*-planar graphs with a* 3*-region* f *whose boundary edges are not crossed, we can obtained a new one by identifying the three vertices of* f *with vertices* $0, 11$ *and* 12 *in* G_0*. Hence there are infinity many* 3*-connected* 1*-planar chordal graphs that are neither planar nor Hamiltonian.*

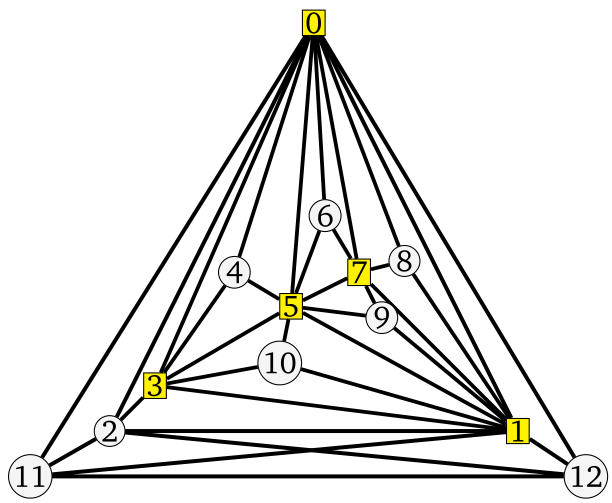


Fig. 1. A chordal and 1-planar graph G_0 that has connectivity 3 and is not Hamiltonian.

The remaining sections of this paper are organized as follows. In Section 2, we explain some terminology, notations and some preliminary lemmas, and in Section 3, we give some fundamental results on chordal graphs. In Section 4, we introduce a 4-join operation for generating 1-planar 4-trees. In Section 5, we characterize 4-connected and 1-planar chordal graphs. In Section 6, we prove Theorem 1 by the result obtained in Section 5. Finally we leave some interesting open questions.

2 Preliminaries

In this section, we introduce some notations and definitions.

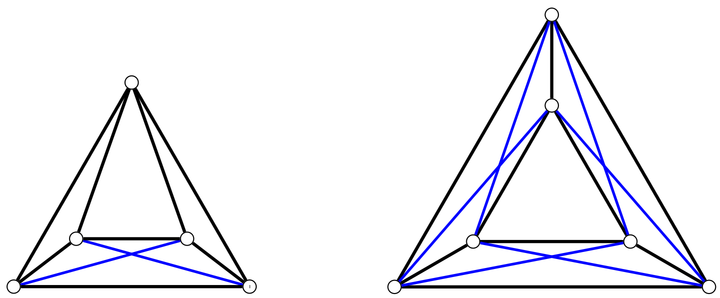
Let G be a graph with a drawing D. For a subgraph H of graph G, the subdrawing of D induced by H is called a *restricted drawing* of D, denoted by $D|H$. Two drawings of a graph are *isomorphic* if there is a homeomorphism of the sphere that maps one drawing to the other. If D is a 1-planar drawing, an edge of D is *crossed* if it crosses another edge, and is *uncrossed* otherwise. A *face* in a 1-planar drawing D is given by a cyclic list of edges and edge segments, where the latter occurs in the case of a crossing. The *planar skeleton* $\mathscr{S}(D)$ of D is the subgraph of G by removing all crossed edges of D. For a face f of D, denote $\partial(f)$ by the boundary of f. We call f is *crossed* if $\partial(f)$ has a crossing, and is *uncrossed* otherwise. All ertices and crossings on $\partial(f)$ are called *corners*. A face f is triangular if $\partial(f)$ has exactly three corners. A *triangulated* 1-plane graph is one in which all faces are triangular.

For other terminology and notations not defined here, we refer to [7], and the following are some auxiliary lemmas in this paper.

Some fundamental results on 1-planar graphs are provided below.

Lemma 1 ([27]). *For any* 1*-planar graph G of order $n \geq 3$, $|E(G)| \leq 4n-8$, and moreover, $|E(G)| \leq 4n-9$ if $n = 7$ or 9.*

Lemma 2 ([27]). *The complete graphs K_5 and K_6 have exactly one (up to isomorphism) 1-planar drawings as shown in Fig. 2, respectively.*



(a) K_5 (b) K_6

Fig. 2. The unique 1-planar drawings of K_5 and K_6.

Lemma 3. *Let D be a 1-planar drawing of G, and let H be a subgraph of G. If some vertex of $G-H$ lies inside a crossed triangular face f of $D|H$, then $\kappa(G) \leq 3$.*

Proof. Let $f = [abca]$. Assume that a is the unique crossing in $\partial(f)$. Let v be a vertex lying inside f. If bc is uncrossed, then $\{b,c\}$ is a 2-separator of G. If bc is crossed by xy, where $x \notin \partial(f)$, then $\{b,c,x\}$ is a 3-separator of G. In either case, $\kappa(G) \leq 3$. □

If e is an uncrossed edge on $\partial(f)$ for some face f in a 1-planar drawing D, let $\widehat{f_e}$ denote the other face in D such that e is also on $\partial(\widehat{f_e})$.

Lemma 4. *Let D be a 1-planar drawing of a 1-planar graph G. Assume that vertex $v \in V(G)$ lies within face f of $D' := D|(G-v)$. Then, the following conclusions hold:*

(i). *the sum of the number of uncrossed corners (i.e., vertices in $G - v$) on $\partial(f)$ and the number of uncrossed edges e on $\partial(f)$ for which $\partial(\widehat{f_e})$ has at least three uncrossed corners is at least $d_G(v)$; and*

(ii). *if D' is triangulated and $d_D(v) = 4$, then f is uncrossed and $\widehat{f_e}$ is also uncrossed for at least one edge e on $\partial(f)$.*

Proof. The result follows directly by the definition of 1-planar drawings. □

A *clique* of a graph G is a subset S of $V(G)$ such that $G[S]$ is complete. A *k-clique* is a clique with exactly k vertices.

Lemma 5. *For a 1-planar graph G of order $n \geq 7$, if G contains a 6-clique, then $\kappa(G) \leq 3$.*

Proof. Let D be a 1-planar drawing of G and let S be a 6-clique of G. By Lemma 2, $D|G[S]$ is unique, which is shown in Fig 2 (b).

Since $n \geq 7$, there is a vertex $u \in V(G) \setminus S$. Then u lies in a triangular face f in $D|G[S]$. If f is crossed, the conclusion follows from Lemma 3. If f is uncrossed, the three vertices on $\partial(f)$ form a 3-separator of G. Hence $\kappa(G) \leq 3$. □

3 Some fundamental results on chordal graphs

In this section, we introduce some basic results on chordal graphs. By the definition of chordal graphs, the following result holds.

Lemma 6 (p51,[31]). *Let G be a chordal graph. Then every induced subgraph of G is chordal.*

A vertex u in G is said to be *simplicial* if either $d(u) = 0$ or $N_G(u)$ is a clique. An ordering $(u_1 u_2 \cdots u_n)$ of the vertices in G is a called *perfect elimination order*, if u_i is a simplicial vertex of $G - \{u_j : 1 \leq j < i\}$ for all $i = 1, 2, \cdots, n-1$. Rose [26] proved that a graph is chordal if and only if it has a perfect elimination ordering. Fix an integer $k \geq 1$, the class of *k-trees* is defined recursively as follows. The complete graph K_k is the smallest k-tree, and a graph G of order n, where $n \geq k+1$, is a *k-tree* if G has a simplicial vertex u of degree k and $G - \{u\}$ is a k-tree. Clearly, from the construction of k-trees, they naturally contain perfect elimination orderings, and so k-trees form a subclass of chordal graphs.

Lemma 7 ([10, 26]). *Let G be a graph. The following statements are equivalent:*

(i). *G is chordal;*

(ii). *every minimal separator of G is a clique; and*

(iii). *G has a perfect elimination ordering.*

Lemma 8. *Let G be a 4-tree with $|V(G)| = 6$. Then $G \cong K_6 \setminus e$. Moreover, G has exactly three non-isomorphic 1-planar drawings, as shown in Fig. 3.*

Proof. By the definition of 4-trees, any 4-tree with five vertices is K_5, and then $G \cong K_6 \setminus e$. Due to Korzhik [19], G has exactly three 1-planar drawings A_1,A_2 and A_3 shown in Fig. 3. □

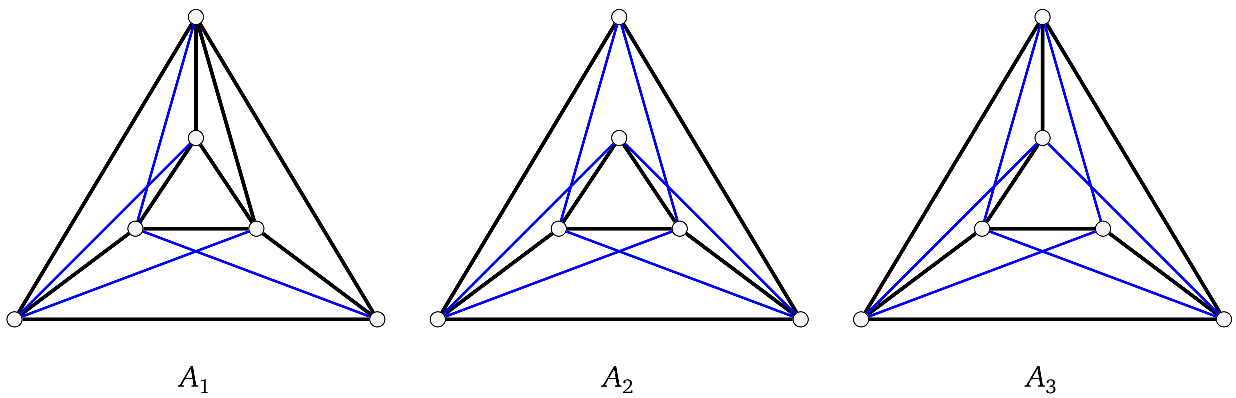


Fig. 3. Three non-isomorphic 1-planar drawings of $K_6 \setminus e$.

Lemma 9 ([17])**.** *Let G be a k-connected chordal graph. If G has no $(k+2)$-clique, then G is a k-tree.*

The following lemma must have appeared somewhere, although we cannot find it. We provide a simple proof.

Lemma 10. *Let G be a chordal graph with n vertices. If G is k-connected, then $|E(G)| \geq kn - \frac{1}{2}k(k+1)$.*

Proof. We can prove this result by induction on n. Since G is k-connected, we have $n \geq k+1$. When $n = k+1$, G is the complete graph K_{k+1}, and the result holds. Now assume that $n \geq k+2$. By Lemma 7, G contains a simplicial vertex u. Since G is k-connected, we have $d(u) \geq k$. Obviously, $G - \{u\}$ is k-connected and chordal. By induction, the result holds for $G - \{u\}$, and thus

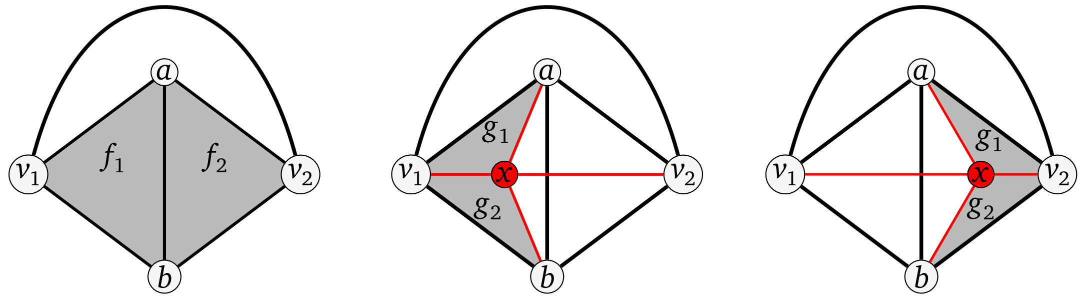
$$|E(G)| = |E(G - \{u\})| + d(u) \geq k(n-1) - \frac{1}{2}k(k+1) + k = kn - \frac{1}{2}k(k+1).$$

Hence the result holds. □

4 4-join operation

In this section, we introduce a graph operation, the 4-join operation, which will be applied to generate 1-planar 4-trees.

Let D be a 1-plane graph. If $f_1 := [abv_1a]$ and $f_2 := [abv_2a]$ are uncrossed triangular faces in D such that $\partial(f_1)$ and $\partial(f_2)$ share exactly one edge (i.e., ab) and $D[\{a, b, v_1, v_2\}] \cong K_4$, then f_1 and f_2 are called **twin faces** in D, as shown in Fig. 4 (a).



(a) (b) (c)

Fig. 4. 4-join operation on twin faces f_1 and f_2

Definition 1 (4-join operation). *Let $f_1 := [abv_1a]$ and $f_2 := [abv_2a]$ be twin faces in a plane graph D, as shown in Fig. 4 (a). A 4-join operation at the order pair (f_1, f_2) is defined as follows:*

(i). *put a new vertex x within face f_1, and*

(ii). *draw edges xa, xb, xv_1 inside f_1, and draw edge xv_2 that crosses ab, as shown in Fig. 4 (b).*

The 1-plane graph drawing obtained by the 4-join operation at (f_1, f_2) is denoted by $D \oplus (f_1, f_2)$, as shown at Fig. 4 (b).[1]

Lemma 11. *Let D be a 1-planar drawing of a 1-planar graph G, and let $v \in V(G)$ is a simplicial vertex in G of degree 4. Assume that $D' := D|(G - v)$ is triangulated and v is within face f of D'. If e is the only edge on $\partial(f)$ such that $f' := \widehat{f_e}$ is uncrossed, then $D = D' \oplus (f, f')$.*

Proof. Let $f = [bcab]$ and $f' = [bcdb]$, where edge bc is on both $\partial(f)$ and $\partial(f')$. Since v is within f of D', by Lemma 4 (ii), $N_D(v) = \{a, b, c, d\}$ is a clique of D. Thus, $D = D' \oplus (f, f')$. □

5 Characterization of 4-connected 1-planar chordal graphs

Clearly, K_5 is the 4-connected chordal graph with the smallest order, and K_6 and $K_6 - e$ are the only 4-connected chordal graph of order 6. Note that $\kappa(K_6) = 5 > 4$. We first prove that K_6 is the only 1-planar chordal graph with connectivity 5, and every 1-planar chordal graph G with $\kappa(G) = 4$ is a 4-tree.

Proposition 1. *Let G be a 1-planar chordal graph that is not K_6. Then $\kappa(G) \leq 4$. Moreover, if $\kappa(G) = 4$, G is a 4-tree.*

[1] Note that $D \oplus (f_2, f_1)$, shown at Fig. 4 (c), is different from $D \oplus (f_1, f_2)$, as $D \oplus (f_2, f_1)$ is the one with the vertex x within f_2.

Proof. Suppose $\kappa(G) \geq 5$. Clearly, $|V(G)| \geq 6$. Since $G \ncong K_6$, we have $|V(G)| \geq 7$. By Lemmas 1 and 10, we have

$$5|V(G)| - 15 \leq |E(G)| \leq 4|V(G)| - 8,$$

implying that $|V(G)| \leq 7$. Thus, $|V(G)| = 7$. However, when $|V(G)| = 7$, applying Lemmas 1 and 10 yields that

$$20 = 5 \times 7 - 15 \leq |E(G)| \leq 4 \times 7 - 9 = 19,$$

a contradiction. Hence $\kappa(G) \leq 4$.

Now assume that $\kappa(G) = 4$. Then $|V(G)| \geq 5$. When $|V(G)| = 5$, G is K_5 which is a 4-tree. If $|V(G)| = 6$, by Lemma 10, $|E(G)| \geq 4 \times 6 - \frac{4 \times 5}{2} = 14$. Since $|E(K_6)| = 15$, we have $|E(G)| = 14$. Hence, $G \cong K_6 - e$. By Lemma 8, $K_6 - e$ is a 1-planar 4-tree. For $|V(G)| \geq 7$, since $\kappa(G) = 4$, by Lemma 5, G has no 6-clique. Furthermore, by Lemma 9, G is a 4-tree. □

By Proposition 1, K_6 is the only 5-connected 1-planar chordal graphs, and all 1-planar chordal graphs with connectivity four are 4-trees. However, there exist 4-trees which are not 1-planar. For example, $K_4 + \overline{K_3}$ is a 4-tree, where $G + H$ is the graph obtained from vertex disjoint copies of G and H by adding edges joining every vertex in G to every vertex in H. But $K_4 + \overline{K_3}$ is not 1-planar (see [19]).

An *Apollonian network* is a planar graph obtained by starting with a triangle, and repeatedly picking a triangular face f and subdividing f into three triangles by inserting a new vertex in it. A planar graph is a 3-tree if and only if it is a Apollonian network [3]. In the following, we are going to establish an analogous result for 4-connected and 1-planar chordal graphs by the 4-join operation introduced in the previous section.

By Proposition 1, every 1-planar chordal graph G with $\kappa(G) = 4$ is a 4-tree. Such graphs of orders at most 6 are K_5 and $K_6 - e$. In the following, we first characterize 1-planar 4-trees of order 7, and then all 1-planar 4-trees of order more than 7.

Proposition 2. *The* 1*-planar drawings of all* 1*-planar* 4*-trees of order* 7 *are shown in Fig. 5.*

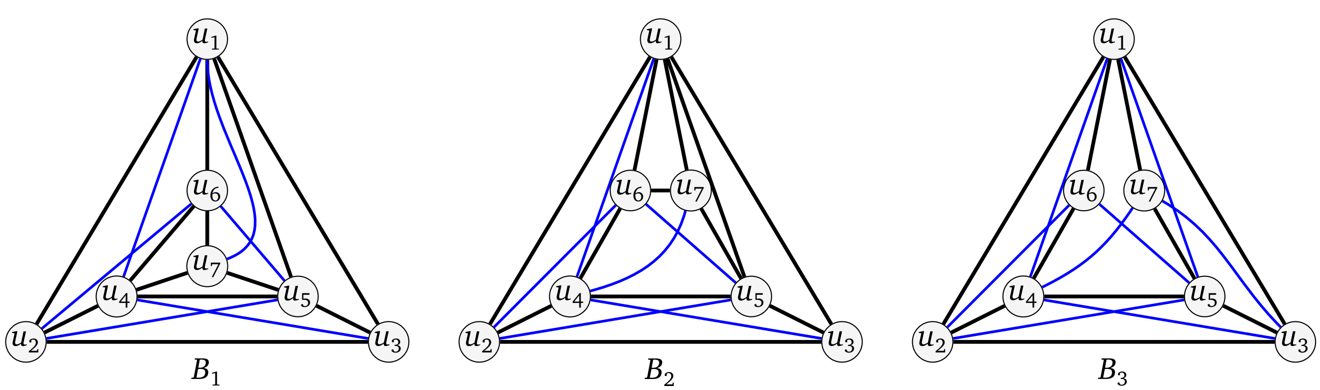


Fig. 5. All 1-planar drawings of 1-planar 4-trees of order 7

Proof. Let D be a 1-planar drawing of a 1-planar 4-tree T of order 7, and let v be a simplicial vertex of T with $d_T(v) = 4$. Then, $N_T(v)$ is a clique in T of size 4 and $D|(T-v)$ must be a 1-planar drawing of $T-v$. Since $\kappa(T) = 4$, by Lemma 3, v cannot be within a crossed triangular face of $D|(T-v)$.

Claim A: $D|(T-v)$ is isomorphic to A_1 shown in Fig. 3 or 6

Note that $T - v \cong K_6 - e$ is a 1-planar 4-tree of order 6. By Lemma 8, $D|(T-v)$ must be one of the three non-isomorphic 1-planar drawings A_1, A_2 or A_3 shown in Fig. 3 or 6.

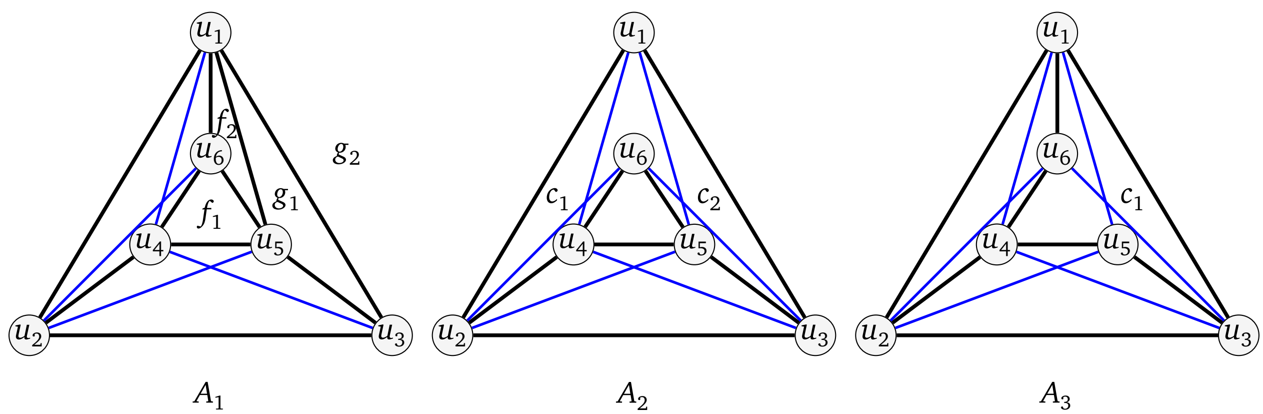


Fig. 6. A_1, A_2 and A_3

We shall show that $D|(T-v)$ must be the 1-planar drawing A_1. If $D|(T-v)$ is A_2, as shown in Fig. 6, then v must be within a uncrossed triangular face of A_2 or a face of A_2 which is not triangular, implying that v must be within the crossed face $[u_1c_1u_6c_2u_1]$, or within one of the uncrossed faces $[u_1u_2u_3u_1], [u_4u_5u_6u_4]$, where c_1 and c_2 are crossings of edges, as shown in A_2 of Fig. 6. However, by Lemma 4 (i), in any one of three situations, connecting v to any one clique in $D|(T-v)$ of size 4 will violate the 1-planarity of T, a contradiction.

Similarly, $D|(T-v)$ cannot be A_3. Hence Claim A holds.

Now we are going to show that D is isomorphic to one of the drawings shown in Fig. 5. By Claim A, D is obtained by inserting v to a face of A_1 and then joining v to all vertices in a 4-clique of A_1.

Note that A_1 is triangular. By Lemma 4 (i), v cannot be within any crossed triangular face of A_1. Thus, v must be in a triangular face of A_1, i.e., f_1, f_2, g_1 or g_2 (see Fig. 6).

Case 1: v is within face $f_1 = [u_4, u_5, u_6, u_4]$.

In this case, by Lemma 4 (ii), $N_T(v)$ must be the set $\{u_4, u_5, u_6, u_1\}$, and D is isomorphic to B_1 shown in Fig. 5.

Case 2: v is within face $f_2 = [u_1, u_5, u_6, u_1]$.

In this case, by Lemma 4 (ii), $N_T(v)$ must be either $\{u_1, u_4, u_5, u_6, u_1\}$ or $\{u_1, u_3, u_4, u_5, u_1\}$, and D is isomorphic to either B_2 or B_3 shown in Fig. 5.

Case 3: v is within face $g_1 = [u_1, u_3, u_5]$.

In this case, by Lemma 4 (ii), $N_T(v)$ must be either $\{u_1, u_3, u_5, u_2\}$ or $\{u_1, u_3, u_5, u_4\}$, and thus D is isomorphic to either B_2 or B_3 shown in Fig. 5.

Case 4: v is within face $g_2 = [u_1, u_2, u_3]$.

In this case, by Lemma 4 (ii), $N_T(v)$ must be the set $\{u_1, u_2, u_3, u_5\}$, and D is thus isomorphic to B_1 shown in Fig. 5.

Hence the result holds. □

Let Φ denote the set of 1-planar drawings of some 1-planar graphs defined below:

(i). the 1-planar drawings B_1 and B_2 in Fig. 5 are the ones in Φ with the smallest order; and

(ii). if $D \in \Phi$, then both $D \oplus (f_1, f_2)$ and $D \oplus (f_2, f_1)$ belong to Φ for twin faces f_1 and f_2 in D.

Lemma 12. *Every* 1*-planar drawing D in Φ represents a* 1*-planar* 4*-tree with exactly two simplicial vertices.*

Proof. We have the following two facts:

(i). Both B_1 and B_2, shown in Fig. 5, represent 1-planar 4-trees with exactly two somplicial vertices and two pairs of twin faces, and each simplicial vertex is located in one pair of twin faces;

(ii). if $D = D' \oplus (f_1, f_2)$ for a pair of twin faces f_1 and f_2 of D' and some simplicial vertex of D is on $\partial(f_1) \cup \partial(f_2)$, then D and D' have the equal number of simplicial vertices.

By the definition of Φ, the result holds. □

In the following, we will show that the 1-planar drawings of all 1-planar 4-trees of order at least 8 belongs to Φ.

For any 1-planar drawing D with at least one uncrossed face, let $E_{uf}(D)$ be the set of edges e in D such that e is on $\partial(f)$ for some uncrossed face f of D, and let V_{uf} be the set of vertices in D which are incident with some edges in $E_{uf}(D)$. Let D_{uf} denote the 1-planar drawing $D|H$, where $H = (V_{uf}, E_{uf}(D))$.

Proposition 3. *Any* 1*-planar drawing of a* 1*-planar* 4*-tree of order at least* 8 *is isomorphic to some* 1*-planar drawing in Φ.*

Proof. Let T be a 1-planar 4-tree of order n (≥ 8), and let v be a simplicial vertex of T. Thus, $N_T(v)$ is a 4-clique in T. Let D be a 1-planar drawing of T. Then, $D|(T - v)$ must be a 1-planar drawing of $T - v$. Since $\kappa(T) = 4$, by Lemma 3, v cannot be within a crossed triangular face of $D|(T - v)$.

In the following, we shall show that D_{uf} is isomorphic to some D_j in the set $\{D_j : 1 \leq j \leq 6\}$, shown in Fig. 7 with the following properties:

P1. in each D_j ($1 \leq j \leq 6$), f_i and f_{i+1} are twin faces[2] of D for each $i \in \{1,3\}$, where f_1, f_2, f_3 and f_4 are faces in D_j, and

P2. v_0v_2, v_1v_2 and v_1v_3 are not edges in D if D_{uf} is D_1 or D_2.[3]

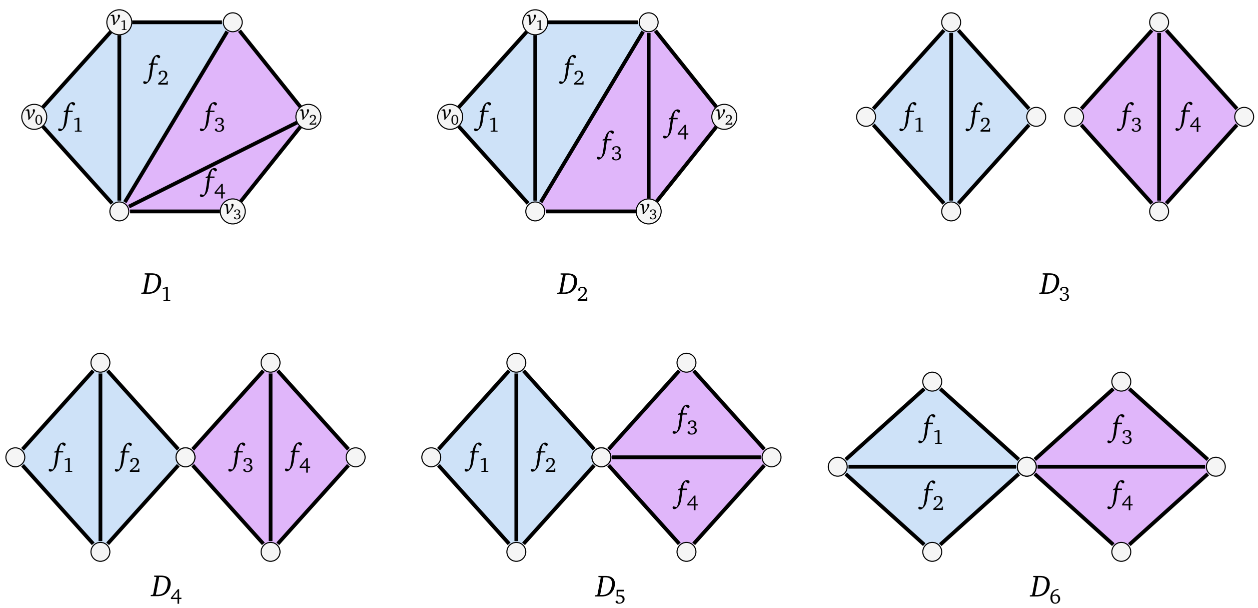


Fig. 7. Possible 1-planar drawings of D'_{uf} with properties P1 and P2

We first prove the following conclusion. Let $D' := D|(T - v)$.

Claim 1. *If $n = 8$, then D' is triangulated and D'_{uf} is isomorphic to D_1 or D_4 in Fig. 7 with properties P1 and P2.*

Note that D' is a 1-planar drawing of $T - v$. By Proposition 2, D' is isomorphic to B_1, B_2 or B_3 in Fig. 5. By Lemma 4 (i), v cannot be within any face of B_3. Thus, D' is isomorphic to B_1 or B_2. Clearly, D' is triangulated.

If D' is isomorphic to B_1, then D'_{uf} is isomorphic to D_4 in Fig. 7. If D' is isomorphic to B_2, then D'_{uf} is isomorphic to D_1 in Fig. 7 with the property that there is no edge in D' joining v_2 to v_0 or v_1, or v_3 to v_1. In both case, properties P1 and P2 hold.

Claim 1 holds.

Claim 2. *If D' is triangulated and D'_{uf} is isomorphic to D_i in Fig. 7 for some $i \in \{1, 2, \cdots, 6\}$ with properties P1 and P2, then D is obtained from D' by a 4-join operation and D_{uf} is also isomorphic to isomorphic to some D_i in Fig. 7, where $i \in \{1, 2, \ldots, 6\}$, with properties P1 and P2.*

[2]The set of vertices in $\partial(f_i) \cup \partial(f_{i+1})$, where $i \in \{1,3\}$, forms a 4-clique of D, although it does not in D_{uf}.

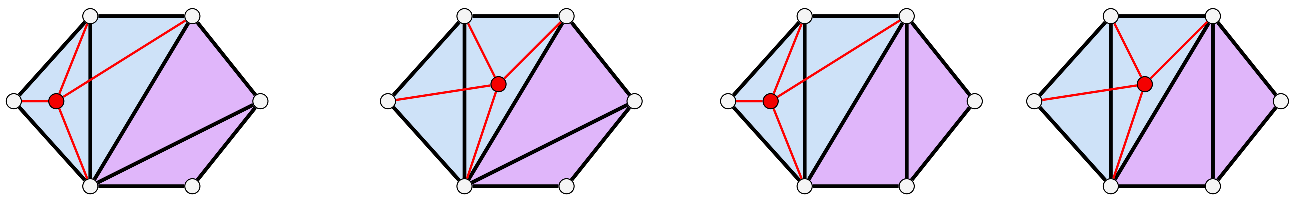
[3]In each D_i, $1 \leq i \leq 2$, Property 2 implies that there are two nonadjacent vertices in D_i of degree 3 each of which is not adjacent to some vertex in D_i of degree 2 in D. P2 also implies that if v is within face f_2, then $v_2 \notin N_D(v)$.

By Lemma 4 (ii), v is within a uncrossed face of D'.

Case 1: D'_{uf} is isomorphic to D_1.

If v is within f_1 or f_4, then, by Lemma 11, $D = D' \oplus (f_1, f_2)$ or $D = D' \oplus (f_4, f_3)$ respectively. Clearly, D_{uf} is isomorphic to D_5 and properties P1 and P2 hold (see Fig. 8 (a)).

If v is within f_2 or f_3, then, by Lemma 11, $D = D' \oplus (f_2, f_1)$ or $D = D' \oplus (f_3, f_4)$ respectively. Clearly, D_{uf} is isomorphic to D_2 and properties P1 and P2 hold (see Fig. 8 (b)).



(a) (b) (c) (d)

Fig. 8. D is obtained from D' by a 4-join operation

Case 2: D'_{uf} is isomorphic to D_2.

If v is within f_1 or f_4, then, by Lemma 11, $D = D' \oplus (f_1, f_2)$ or $D = D' \oplus (f_4, f_3)$ respectively. Clearly, D_{uf} is isomorphic to D_4 and properties P1 and P2 hold (see Fig. 8 (c)).

If v is within f_2 or f_3, then, by Lemma 11, $D = D' \oplus (f_2, f_1)$ or $D = D' \oplus (f_3, f_4)$ respectively. Clearly, D_{uf} is isomorphic to D_1 and properties P1 and P2 hold (see Fig. 8 (d)).

Case 3: D'_{uf} is isomorphic to D_i in Fig. 7 for some $i \in \{3, 4, 5, 6\}$.

By Lemma 11, $D = D' \oplus (f_i, f_{i+1})$ or $D = D' \oplus (f_{i+1}, f_i)$ for some $i \in \{1, 3\}$. Observe that

- if D'_{uf} is isomorphic to D_3, then D_{uf} is also isomorphic to D_3 (see Fig. 9 (a));
- if D'_{uf} is isomorphic to D_4, then D_{uf} is isomorphic to D_3 or D_5 (see Fig. 9 (b) and (c));
- if D'_{uf} is isomorphic to D_5, then D_{uf} is isomorphic to D_3, D_4 or D_6 (see Fig. 9 (d), (e) and (f)), and
- if D'_{uf} is isomorphic to D_6, then D_{uf} is isomorphic to D_5 (see Fig. 9 (g)).

In all cases above, properties P1 and P2 hold.

Thus, Claim 2 holds. By Claims 1 and 2, every 1-planar drawing of a 1-planar 4-tree of order at least 8 belongs to Φ. The result holds. □

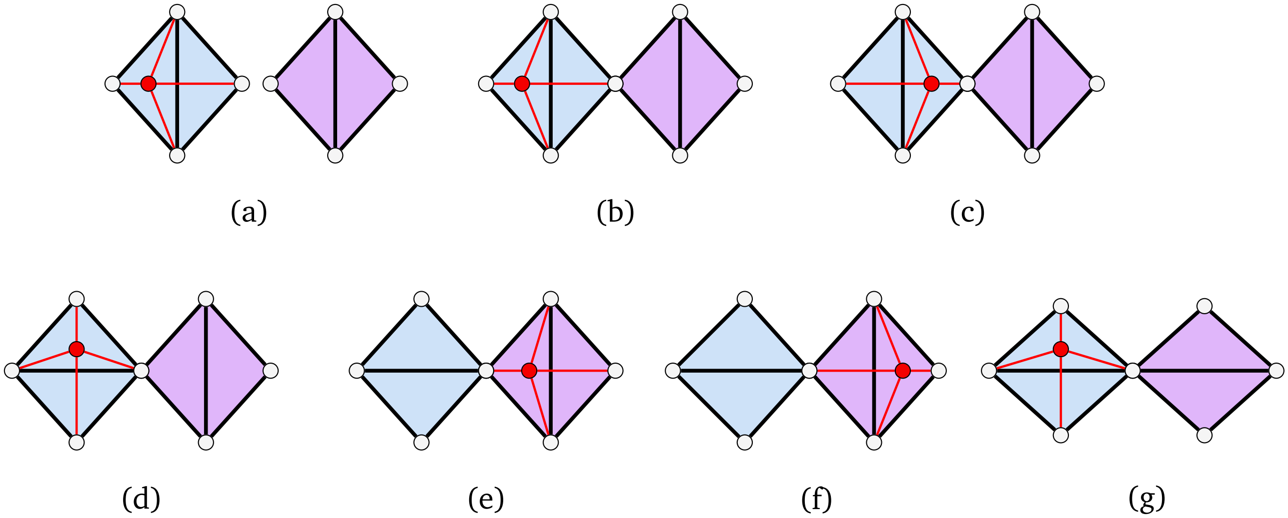


Fig. 9. D is obtained from D' by a 4-join operation

6 Proof of Theorem 1

We first provides a sufficient condition for a graph to be Hamiltonian-connected.

Proposition 4. *Let G be a graph with two vertices a and u such that $d_G(u) \geq 3$, $N_G[u] \subseteq N_G[a]$ and $|N_G(a) \setminus N_G[u]| \leq 1$. If both $G - u$ and $G - \{u, a\}$ are Hamiltonian-connected, then G is also Hamiltonian-connected.*

Proof. By the given condition, we may assume that $\{a, b, c\} \subseteq N_G(u)$. Since $N_G[u] \subseteq N_G[a]$, $\{u, b, c\} \subseteq N_G(a)$, as shown in Fig. 10. Let $G' = G - u$ and $G'' = G - \{u, a\}$. We only need to prove that for any two vertices x and y in G, there is a Hamiltonian path in G connecting x and y. By the given conditions, G' has a Hamiltonian path P connecting b and y.

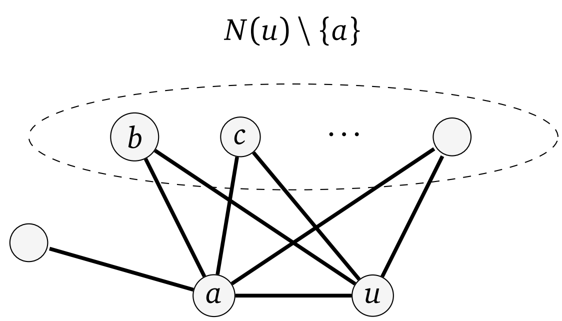


Fig. 10. $\{a, b, c\} \subseteq N_G(u) \subseteq N_G[a]$ and $|N_G(a) \setminus N_G[u]| \leq 1$

We divide the proof into three cases.

Case 1. $u \in \{x, y\}$.

By symmetry, assume that $x = u$. Then $y \in V(G')$. Obviously, $y \neq a$ or $y \neq b$. Assume that $y \neq b$. Combining P and edge ub produces a Hamiltonian path in G between u (i.e., x) and y.

Case 2. *$u \notin \{x,y\}$ and $a \notin \{x,y\}$.*

Since $|N_G(a) \setminus N_G[u]| \le 1$, at least one edge e^* in the set $\{av : v \in N_G(u)\}$ is on P. Without loss of generality, assume that $e^* = ab$. Let P' be the path in G obtained from P by replacing edge ab by the path aub. Clearly, P' is a Hamiltonian path in G connecting x and y.

Case 3. *$u \notin \{x,y\}$ and $a \in \{x,y\}$.*

By symmetry, assume that $a = x$. By the given condition, G'' is Hamiltonian-connected. Then, G'' has Hamiltonian paths P_1 connecting b and y when $b \ne y$, and P_2 connecting c and y when $c \ne y$. If $b \ne y$, then combining path P_1 and path aub yields a Hamiltonian path in G connecting a (i.e., x) and y. If $b = y$, combining path P_2 and path auc yields a Hamiltonian path in G connecting a (i.e., x) and y.

Hence the result holds. □

By the definition of k-trees, some conclusions on simplicial vertices of k-trees can be obtained.

Lemma 13. *Let G be a k-tree of order n. If $n \ge k+2$, then G has the following properties:*

(i). *G has at least two simplicial vertices; and*

(ii). *any two simplicial vertices in G are not adjacent, and*

(iii). *for any simplicial vertex v of G, if $n \ge k+3$, then $G-v$ does not have more simplcial vertices than G has.*

Proof. The result is obvious when $n = k+2$. For $n > k+2$, let v be a simplicial vertex. Then $G-v$ is a k-tree of order $n-1 \ge k+2$. By induction, $G-v$ has all these three properties.

Let $S(G)$ denote the set of simplicial vertices in G. If no vertex in $N_G(v)$ belongs to $S(G-v)$, then $S(G) = S(G-v) \cup \{v\}$. If some vertex $u \in N_G(v)$ belongs to $S(G-v)$, then $S(G) = (S(G-v) \setminus \{u\}) \cup \{v\}$.

Therefore G also has these three properties. □

Applying Proposition 4, we can obtain the following conclusion on k-trees.

Proposition 5. *For any k-tree G, where $k \ge 3$, if $|V(G)| \ge k+2$ and G has only two simplicial vertices, then G is Hamiltonian-connected.*

Proof. Let G be a k-trees. Then $|V(G)| \ge k+1$. If $k \le k+1$, then G is a complete graph and thus it is Hamiltonian-connected. If $|V(G)| = k+2$, then $G = K_{k+2} - e$, which is also Hamiltonian-connected.

Now assume the result holds for all k-trees of order less than n, where $n \ge k+3$. Let G be any k-tree of order n. By Lemma 13 (i), G has at least two non-adjacent simplicial vertices. By the given conditions, G has exactly two simplicial vertices, say u and v.

By the definition of k-tree, $G' := G - u$ is a k-tree of order $n-1$ $(\ge k+2)$. By Lemma 13 (iii), G' has exactly two simplcial vertices and one of them must be v. Let a be another simplicial vertex of G'. Clearly, $a \in N_G(u)$. By the inductive assumption, both $G-u$ and $G'-a$ $(= G-\{u,a\})$ are Hamiltonian connected.

Since $ua \in E(G)$, u is a simplicial vertex of G with $d_G(u) = k \geq 3$ and a is a simplicial vertex of G' with $d_{G'}(a) = k$, we have $N_G[u] \subseteq N_G[a]$ and $|N_G(a) \setminus N_G[u]| = 1$. By Proposition 4, G is Hamiltonian-connected.

Hence the result holds. $\square$

Now we are going to prove Theorem 1.

Proof of Theorem 1: Let G be a 1-planar graph which is 4-connected and chordal.

Let n be the order of G. Then $n \geq 5$. If $n = 5$, then $G \cong K_5$. If $n = 6$, then $G \cong K_6 - e$ or K_6. Clearly, G is Hamiltonian-connected when $n \in \{5, 6\}$. Now assume that $n \geq 7$. By Proposition 1, G is a 4-tree. In the following, we first show that G has exactly two simplicial vertices.

If $n = 7$, by Proposition 2, G has a 1-planar drawing isomorphic to B_1, B_2 or B_3 shown in Fig. 5, implying that G is a 4-tree with exactly two simplicail vertices.

When $n \geq 8$, by Lemma 12 and Proposition 3, G has exactly two two simplicail vertices.

The result then follows from Proposition 5. $\square$

7 Unsolved problems

The toughness of a graph is closely associated with Hamiltonicity. The *toughness* of a graph G, denoted by $\tau(G)$, is the minimum value of $\frac{|X|}{c(G-X)}$ over all non-empty subsets X of $V(G)$ with $c(G-X) > 1$, where $c(H)$ is the number of components of a graph H. The toughness of a complete graph is defined as being infinite. We say that a graph is *t-tough* if its toughness is at least t. There are some known results on the Hamiltonicity of chordal graphs in terms of their toughness. For example, every 10-tough chordal graph is Hamiltonian [18], and every chordal planar graph of order at least three and toughness greater than one is Hamiltonian [6].

Note that a t-tough graph is always $\lceil 2t \rceil$-vertex-connected [9]. So the corollary below follows directly from Theorem 1.

Corollary 1. *Every 1-planar chordal graph G with $\tau(G) > \frac{3}{2}$ is Hamiltonian-connected.*

Chvátal (1973) [9] conjectured that all graphs which are more than $\frac{3}{2}$-tough are Hamiltonian, but this was disproved by Bauer et al. (2000), who showed that not every 2-tough graph is Hamiltonian [1]. Chvátal's toughness conjecture posits that there exists a toughness threshold t_0 above which t_0-tough graphs are always Hamiltonian; its truth remains unresolved.

Corollary 1 states that every 1-planar chordal graph with toughness greater than $\frac{3}{2}$ is Hamiltonian-connected. Naturally, we pose the following problem:

Problem 2. *Is there a 1-planar chordal graph with toughness $\frac{3}{2}$ non-Hamiltonian?*

Notice that neither the examples in [22] nor our examples in Remark 1 can be used for Problem 2. This is because the toughness of every chordal planar non-Hamiltonian graph in [22] is always 1, while our examples can be shown to be at most $\frac{5}{6}$.

8 Acknowledgment

The work is supported by the National Natural Science Foundation of China (Grant No. 12271157, 12371340, 12371346), the Natural Science Foundation of Hunan Province, China (Grant No. 2022JJ30028), Hunan Provincial Department of Education(No. 21A0590) and Natural Science Foundation of Changsha (No. kq2208001).

We claim that there is no conflict of interest in our paper. No data was used for the research described in the article.

References

[1] D. Bauer, H.J. Broersma, H.J. Veldman, Not every 2-tough graph is Hamiltonian, *Discrete Appl. Math.*, **99** (2000), 317-321.

[2] T. Biedl, Are Highly Connected 1-planar Graphs Hamiltonian, preprint, https://arxiv.org/abs/1911.02153v1, 2019.

[3] T. Biedl, L.E. Ruiz Velázquez, Drawing planar 3-trees with given face areas, *Comput. Geom.* **46** (2013), 276-285.

[4] T. Biedl, J. Wittnebel, Matchings in 1-planar graphs with large minimum degree, *J. Graph Theory*, **99** (2022), 217–230.

[5] R. Bodendiek, H. Schumacher, K. Wagner, Bemerkungen zu einem Sechsfarbenproblem von G. Ringel. Abh. Math. Semin. Univ. Hamb., **53** (1983), 41–52.

[6] T. Böhme, J. Harant, M. Tkáč, More than one tough chordal planar graphs are hamiltonian, *J. Graph Theory*, **32** (1999), 405–410.

[7] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.

[8] F.J. Brandenburg, Straight-line drawings of 1-planar graphs. *Comput. Geom.*, **116** (2024), 102036.

[9] V. Chvátal, Tough Graphs and Hamiltonian Circuits, *Discrete Math.*, **5** (1973), 215-228.

[10] G.A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg*, **25** (1961), 71-76.

[11] M.N. Ellingham, E.A. Marshall, K. Ozeki, S. Tsuchiya, Hamiltonicity of planar graphs with a forbidden minor, *J. Graph Theory*, **90** (2019), 459-483.

[12] I. Fabrici, J. Harant, T. Madaras, S. Mohr, R. Soták, and C. T. Zamfirescu, Long cycles and spanning subgraphs of locally maximal 1-planar graphs, *J. Graph Theory*, **95** (2020), 125-137.

[13] I. Fabrici, T. Madaras, The structure of 1-planar graphs, *Discrete Math.*, **307** (2007), 854–865

[14] J. Fujisawa, K. Segawa, Y. Suzuki, The matching extendability of optimal 1-planar graphs, *Graphs Combin.*, **34** (2018), 1089-1099.

[15] J.P. Gollin, K. Hendrey, A. Methuku, C. Tompkins, X. Zhang, Counting cliques in 1-planar graphs, *Eur. J. Comb.*, **109** (2023), 103654.

[16] D. Hudák, T. Madaras, Y. Suzuki, On properties of maximal 1-planar graphs, *Discuss. Math. Graph Theory*, **32** (2012), 737–747.

[17] A. Kabela, Long paths and toughness of k-trees and chordal planar graphs, *Discrete Mathematics*, **342** (2019), 55-63.

[18] A. Kabela, T. Kaiser, 10-tough chordal graphs are Hamiltonian, *J. Comb. Theory, Ser. B*, **122** (2017), 417–427.

[19] V.P. Korzhik, Minimal non-1-planar graphs, *Discrete Math.*, **308** (2008), 1319-1327.

[20] H.J. Lai, Every 4-connected line graph of a planar graph is hamiltonian, *Graphs Combin.*, **10** (1994), 249-253.

[21] J. Liu, Y.Q. Wang, W.F. Wang, Light edges in 1-planar graphs, *J. Graph Theory*, **101** (2022), 746-768

[22] T. Nishizeki, A 1-tough nonhamiltonian maximal planar graph, *Discrete Math.*, **30** (1980), 305-307.

[23] K. Ozeki, Hamiltonicity of graphs on surfaces in terms of toughness and scattering number - a survey (2021), 74–95, in: Lect. Notes Comput. Sci, *Springer*.

[24] K. Ozeki, P. Vrána, 2-edge-Hamiltonian-connectedness of 4-connected plane graphs, *Eur. J. Comb.*, **35** (2014), 432-448.

[25] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, *Abh. Math. Semin. Univ. Hambg.*, **29** (1965), 107–117.

[26] D.J. Rose, Triangulated graphs and the elimination process, *J. Math. Anal. Appl.*, **32** (1970), 597-609.

[27] Y. Suzuki, Re-embeddings of maximum 1-planar graphs, *SIAM J. Discrete Math.*, **24** (2010), 1527–1540.

[28] R. Thomas, X.X. Yu, 4-connected projective-planar graphs are Hamiltonian, *J. Comb. Theory, Ser. B*, **62** (1994), 114-132.

[29] C. Thomassen, A theorem on paths in planar graphs, *J. Graph Theory*, **7** (1983), 169–176.

[30] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.*, **82** (1956), 99–116.

[31] V.I. Voloshin, Introduction to Graph and Hypergraph theory, *Nova Kroshka Books*, 2013.

[32] H. Whitney, A theorem on graphs, *Ann. Math.*, **32** (1931), 378–390.