4-dimensional Space forms as determined by the volumes of small geodesic balls

JeongHyeong Park

Sungkyunkwan University, Suwon, Korea

Abstract

Gray-Vanhecke conjectured that the volumes of small geodesic balls could determine if the manifold is a space form, and provided a proof for the compact 4-dimensional manifold, and some cases. In this paper, similar results for the 4-dimensional case are obtained, based upon tensor calculus and classical theorems rather than the topological characterizations in [6].

Mathematics Subsect Classification (2020) : 53C21, 53B20
Keywords : geodesic balls, space forms

1 Introduction

Let M be a Riemannian manifold. Let $p \in M$ and $B_r(p) = \{q \in M \mid d(q,p) \leq r\}$ be a geodesic ball centered at p of radius r and $V_M(p,r)$ be the volume of $B_r(p)$. In this study, we investigate how the volume of a small geodesic ball determines the geometry of the manifold.

One interesting avenue of inquiry is to investigate how the volume of a small geodesic ball is related to the curvature of the underlying manifold. There is a vast literature on this subject; we refer to [1, 3, 4, 5, 6, 10] and the references cited therein for further details. In this context, Gray and Vanhecke [6] posed the following conjecture:

Conjecture. Let M be an n-dimensional Riemannian manifold and suppose that $V_M(p,r)$ is the same as that of an n-dimensional manifold of constant sectional curvature c for all p and all sufficiently small $r > 0$, then M is also a space of constant sectional curvature c.

By rescaling the metric, we let $\mathbb{S}^4 := (\mathbb{S}^4, g_{+1})$ be the sphere of constant positive sectional curvature 1, $\mathbb{H}^4 := (\mathbb{H}^4, g_{-1})$ be the hyperbolic space of constant negative sectional curvature -1 and $\mathbb{T}^4$ be the flat torus. Let $\chi(M)$ be a Euler characteristic of M. In this paper, we shall prove the following Theorem.

Theorem 1 *Let $(M,\, g)$ be a 4-dimensional compact Riemannian manifold.*
(1) Let $B_r(p) \subset M$, $B_r(q) \subset \mathbb{T}^4$. If $V_M(p,r) = V_{\mathbb{T}^4}(q,r)$ for all sufficiently small r and for all $p \in M$, $q \in \mathbb{T}^4$, and if $\chi(M) \geq 0$, then M is flat.
(2) Let $B_r(p) \subset M$, $B_r(q) \subset \mathbb{S}^4$. If $V_M(p,r) = V_{\mathbb{S}^4}(q,r)$ for all sufficiently small r and for all $p \in M$, $q \in \mathbb{S}^4$, and if $\chi(M) \geq \frac{3}{4\pi^2} vol(M,g)$, then (M,g) is a space of constant sectional curvature 1. In addition, if $vol(M,g) \geq vol(\mathbb{S}^4)$, then (M,g) is $\mathbb{S}^4$.
(3) Let $B_r(p) \subset M$, $B_r(q) \subset \mathbb{H}^4$. If $V_M(p,r) = V_{\mathbb{H}^4}(q,r)$ for all sufficiently small r and for all $p \in M$, $q \in \mathbb{H}^4$, and if $\chi(M) \geq \frac{3}{4\pi^2} vol_g(M)$, then (M,g) is a space of constant sectional curvature -1, so isometric to $\mathbb{H}^4/\Gamma$, where Γ is a discrete cocompact torsion-free subgroup of isometries on $\mathbb{H}^4$.

In this paper, the author provide a simple proof using tensor calculus and classical theorems rather than using topological characterizations in [6].

2 Preliminaries

In this section, we prepare some notations. Let $M = (M,g)$ be an n-dimensional Riemannian manifold and $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on M. We denote the Levi-Civita connection, the curvature tensor, the Ricci tensor, and the scalar curvature of M by ∇, R, ρ, and τ, respectively. The curvature tensor is defined by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

for X, Y, $Z \in \mathfrak{X}(M)$. The Weyl tensor is defined by

$$\begin{aligned} W_{abcd} = R_{abcd} &- \frac{1}{n-2}(\rho_{ac}g_{bd} + \rho_{bd}g_{ac} - \rho_{ad}g_{bc} - \rho_{bc}g_{ad}) \\ &+ \frac{\tau}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}). \end{aligned}$$

Then, by direct computation, we obtain

$$\mid W \mid^2 = \mid R \mid^2 - \frac{4}{n-2} \mid \rho \mid^2 + \frac{2}{(n-1)(n-2)}\tau^2, \tag{2.1}$$

where $|R|^2 = R_{ijkl}R^{ijkl}$, $|W|^2 = W_{ijkl}W^{ijkl}$ and $|\rho|^2 = \rho_{ij}\rho^{kl}$. Let $p \in M$ and let $G_r(p) = \{q \in M \mid d(p,q) = r\}$ and $B_r(p) = \{q \in M \mid d(q,p) \leq r\}$ be a geodesic sphere and a geodesic ball centered at p of radius r, respectively. Volume of $B_r(p)$ is represented by

$$\mathrm{Vol}(\ B_r(p)) = \int_{B_r(p)} dv_g = \int_0^r \int_{G_t(p)} d\theta dt$$

where $dv_g = \sqrt{\det(g_{ij})(p)}dx^1 \cdots dx^n$ is the Riemannian volume element with respect to local coordinates $\{x^1, \cdots, x^n\}$ of M around p and $d\theta$ denotes the volume form on $G_r(p)$ induced from M. Gray proved the the following holds for any Riemannian manifold M and any $p \in M$ [6]:

$$V_M(p,r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!}\{1 - \frac{\tau}{6(n+2)}r^2 + \frac{1}{360(n+2)(n+4)}(-3 \mid R \mid^2 +8 \mid \rho \mid^2 +5\tau^2 - 18\Delta\tau)r^4 + O(r^6)\}_p.$$

Let $\tilde{\rho} = \rho_{ij} - \frac{\tau}{n}g_{ij}$ be the traceless Ricci tensor. Then

$$\mid \tilde{\rho} \mid^2 = \mid \rho \mid^2 - \frac{1}{n}\tau^2. \tag{2.2}$$

Then, by using (2.1) and (2.2), we may express Gray's formula

$$V_M(p,r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!}\{1 - \frac{\tau}{6(n+2)}r^2 + \frac{1}{360(n+2)(n+4)}(-3 \mid W \mid^2 +(8 - \frac{12}{n-2}) \mid \tilde{\rho} \mid^2 + \frac{2}{n(n-1)}\tau^2 - 18\Delta\tau)r^4 + O(r^6)\}_p, \tag{2.3}$$

holds for sufficiently small r.

For the proof of the Theorem 1, we refer the following theorem.

Theorem 2 ([9],[2]) *Let M be a complete Riemannian manifold of even dimension with constant sectional curvature K =1, Then it is isometric to the sphere $\mathbb{S}^n$ or the real projective space $\mathbb{RP}^n$.*

3 Proof of Theorem 1

Now, we let $M = (M, g)$ be a 4-dimensional compact Riemannian manifold. Then, by the Chern-Gauss-Bonnet formula, it is known that Euler characteristic $\chi(M)$ of M is expressed by the following integral formula

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{|R|^2 - 4|\rho|^2 + \tau^2\}dv_g, \tag{3.1}$$

where $|R|^2$ and $|\rho|^2$ are the square norms of the curvature tensor and the Ricci tensor, respectively. By using using (2.1) and (2.2), we get

$$\mid \tilde{\rho} \mid^2 = \mid \rho \mid^2 - \frac{1}{4}\tau^2,$$

$$| W |^2 = | R |^2 - 2 | \tilde{\rho} |^2 - \frac{1}{6}\tau^2. \tag{3.2}$$

We remark that (M, g) has constant curvature if and only if $| W |^2 = 0$ and $| \tilde{\rho} |^2 = 0$. Now, by assumption $V_M(p, r) = V_{\mathbb{T}^4}(q, r)$, by taking account of (2.3), we have

$$\tau_M(p) = \tau_{\mathbb{T}^4} = 0.$$

For $n = 4$, by (2.3), we obtain

$$-3 | W(p) |^2 + 2 | \tilde{\rho}(p) |^2 = 0. \tag{3.3}$$

By using (2.1) and (2.2), the Euler characteristic (3.1) can be expressed

$$\chi(M) = \frac{1}{32\pi^2} \int_M | W |^2 - 2 | \tilde{\rho} |^2 + \frac{1}{6}\tau^2 dvol. \tag{3.4}$$

From (3.4), by taking account of the assumption $\chi(M) \geq 0$, and $\tau_M = \tau_{\mathbb{T}^4} = 0$, then we obtain

$$0 \leq \int_M | W_M |^2 - 2 | \tilde{\rho}_M |^2 \, dvol.$$

By using (3.3), we have

$$0 \leq \int_M (-2 + \frac{2}{3}) | \tilde{\rho}_M |^2 \, dvol.$$

This implies that $|\tilde{\rho}_M|^2 = 0$. Consequently, $|W_M|^2 = 0$ by (3.3). From (3.2), since $\tau_M = 0$, (M, g) is flat.

To prove the assertion 2, If volumes of small balls $V_M(p, r)$ equal spherical balls $V_{\mathbb{S}^4}(q, r)$, by (2.3), we obtain $\tau_M(p) = \tau_{\mathbb{S}^4} = 12$. Consequently τ_M is constant and the terms $\Delta\tau$ and τ^2 play no role. Since $W = \tilde{\rho} = 0$ on $\mathbb{S}^4$, for $n = 4$, by (2.3), we again obtain

$$-3 | W_M(p) |^2 + 2 | \tilde{\rho_M}(p) |^2 = 0. \tag{3.5}$$

Now, from (3.4),

$$32\pi^2\chi(M) = \int_M (-2 + \frac{2}{3})|\tilde{\rho_M}|^2 dvol + 24 \; vol(M, g). \tag{3.6}$$

So, in (3.6), by taking account of the assumption $32\pi^2\chi(M) \geq 24vol(M, g)$, then we get $|\tilde{\rho}|^2 = 0$. So, $|W_M|^2 = 0$ by (3.5). Thus, (M, g) has constant sectional curvature 1. By Theorem 2, then (M, g) is a round sphere $\mathbb{S}^4$.

To prove assertion 3, we note that if (M, g_{-1}) is compact Riemannian manifold of negative constant sectional curvature -1, then (M, g_{-1}) is isometric to $\mathbb{H}^4/\Gamma$, where Γ is a discrete cocompact torsion-free subgroup of isometries on $\mathbb{H}^4$ [7, 8]. From (3.1), the Euler characteristic $(\mathbb{H}^4/\Gamma, g_{-1})$ is given by

$$\chi(\mathbb{H}^4/\Gamma) = \frac{3}{4\pi^2} vol(H^4, g_{-1}). \tag{3.7}$$

Now, if volumes of small balls equal volumes of hyperbolic balls, then $\tau = \tau_{-1} = -12$, and we obtain again (3.5) and (3.6). Similarly as in the proof of assertion (2), we can prove the assertion (3). Considering assumption $32\pi^2\chi(M) \geq 24vol(M, g)$,

$$32\pi^2\chi(M) = \int_M (-2 + \frac{2}{3})|\tilde{\rho_M}|^2 dvol + 24vol(M, g) \geq 24vol(M, g).$$

then we have that equality hold and (M, g) is a compact hyperbolic manifold of sectional curvature -1. This completes the proof of Theorem 1.

4 Acknowledgements

The author would like to thank Professors Alice Chang and Michael Anderson for helpful comments and discussions. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2019R1A2C1083957).

References

[1] A.L. Besse, Manifolds all of whose geodesics are closed, Ergebnisse der Mathematik und ihrer Grenzgebiete 93, Springer-Verlag, Berlin 1978.

[2] do Carmo, M. Riemannian geometry, Basel, Birkhäuser, 1992.

[3] P. Gilkey and J. H. Park, Harmonic spaces and density functions, Results Math. 75: 121 (2020) no. 3, 121.

[4] P. Gilkey, J. H. Park, R. Vázquez-Lorenzo, Aspects of differential geometry II Synth. Lect. Math. Stat., 16 Morgan & Claypool Publishers, Williston, VT, 2015, xiv+143 pp.

[5] A. Gray, The volume of a small geodesic ball of a Riemannian manifold, *Michigan Math. J.* **20** (1973), 329–344.

[6] A. Gray and L. Vanhecke, Riemannian geometry as determined by the volumes of small geodesic balls, Acta Math. 142 (1979) 157-198.

[7] M. Kapovich, Hyperbolic manifolds and discrete groups, Modern Birkhäuser Classics, Boston, MA: Birkhäuser Boston, 2009.

[8] J. G. Ratcliffe, Foundations of hyperbolic manifolds, Graduate Texts in Mathematics, vol. 149 (2nd ed.), Berlin, New York: Springer-Verlag 2006.

[9] J. A. Wolf, Spaces of constant curvature, McGraw-Hill Hill Book Company, 1967.

[10] T. J. Willmore, Riemanniangeometry. Oxford Science Publications, The Clarendon Press, Oxford University Press,New York,1993.

JeongHyeong Park
Department of Mathematics, Sungkyunkwan University, Suwon 16419, Korea.
E-mail: parkj@skku.edu