A NOTE ON THE MAXIMAL OPERATOR ON BANACH FUNCTION SPACES

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ABSTRACT. In this note we answer positively to two conjectures proposed by Nieraeth [13] about the maximal operator on rescaled Banach function spaces. We also obtain a new criterion saying when the maximal operator bounded on a Banach function space X is also bounded on the associate space X'.

1. INTRODUCTION

Let X be a Banach function space over $\mathbb{R}^n$. Denote by X' its associate space. Next, for $p > 0$ denote by X^p the space with finite semi-norm

$$\|f\|_{X^p} := \||f|^{1/p}\|_X^p.$$

Let $s > r \geq 1$. Assume that X is r-convex and s-concave. Define the (r, s)-rescaled Banach function space of X as

$$X_{r,s} := \Big[\big[(X^r)'\big]^{(\frac{s}{r})'}\Big]'.$$

This space was introduced in a recent work by Nieraeth [13]. The factorization formula (see [13, Cor. 2.12])

$$X = (X_{r,s})^{\frac{1}{r}-\frac{1}{s}} \cdot L^s(\mathbb{R}^n)$$

makes the space $X_{r,s}$ an important tool in the extrapolation theory for general Banach function spaces (see [13] and also [12]).

Let M be the Hardy–Littlewood maximal operator defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point x.

In [13, Conjectures 2.38, 2.39] the following conjectures were stated.

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Conjecture 1.1. *Let $s > r \geq 1$ and let X be an r-convex and s-concave Banach function space over $\mathbb{R}^n$. Suppose that M is bounded on $\left[(X^r)'\right]^{(\frac{s}{r})'}$. Then the following are equivalent:*

(i) M is bounded on $X_{r,s}$;
(ii) M is bounded on X^r.

Conjecture 1.2. *Let $s > r \geq 1$ and let X be an r-convex and s-concave Banach function space over $\mathbb{R}^n$. Then the following are equivalent:*

(i) M is bounded on $X_{r,s}$ and on $(X_{r,s})'$;
(ii) M is bounded on X^r and $(X')^{s'}$.

Observe that actually both above conjectures are formulated in [13] in a more general setting of quasi-Banach function spaces and abstract maximal operators. We restrict ourselves to Banach function spaces and the most standard maximal operator. Note also that the implications (i) $\Rightarrow$ (ii) were shown in [13] for both conjectures and, hence, the question is about the converse implications (ii) $\Rightarrow$ (i).

In this note we show that both Conjectures 1.1 and 1.2 are true. A useful tool in our proofs will be a new criterion about the interplay between the boundedness of M on X and X', which is perhaps of some independent interest. This criterion is formulated in terms of the local maximal operator m_λ acting on measurable functions on $\mathbb{R}^n$ by

$$m_\lambda f(x) := \sup_{Q \ni x} (f\chi_Q)^*(\lambda|Q|), \quad \lambda \in (0,1),$$

where f^* stands for the standard non-increasing rearrangement of f, and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point x.

Given a Banach function space X, we associate with it the function φ_X defined by

$$\varphi_X(\lambda) := \inf_{\|f\|_X=1} \|m_\lambda f\|_X, \quad \lambda \in (0,1).$$

Observe that since $m_\lambda f \geq |f|$ a.e. (see, e.g., [8, Lemma 6]), we have $\varphi_X(\lambda) \geq 1$ for all $\lambda \in (0,1)$. Also, it is immediate that the function $\varphi_X(\lambda)$ is non-increasing.

We have the following result.

Theorem 1.3. *Let X be a Banach function space over $\mathbb{R}^n$, and assume that the maximal operator M is bounded on X. Then*

(i) the function $\varphi_{X'}(\lambda)$ is unbounded;
(ii) if there exists $\lambda_0 \in (0,1)$ such that $\varphi_X(\lambda_0) > 1$, then M is bounded on X'.

An immediate consequence of Theorem 1.3 is the following criterion.

Theorem 1.4. *Let X be a Banach function space over $\mathbb{R}^n$, and assume that the maximal operator M is bounded on X. The following statements are equivalent:*

(i) M is bounded on X';
(ii) the function $\varphi_X(\lambda)$ is unbounded;
(iii) there exists $\lambda_0 \in (0,1)$ such that $\varphi_X(\lambda_0) > 1$.

Indeed, observe that the implication (i) $\Rightarrow$ (ii) follows by taking X' instead of X in Theorem 1.3 and taking into account that, by the Lorentz–Luxembourg theorem, $X'' = X$. Next, (ii) $\Rightarrow$ (iii) is trivial. Finally, (iii) $\Rightarrow$ (i) is contained in item (ii) of Theorem 1.3.

The equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 1.4 can be written in the following form.

Corollary 1.5. *Let X be a Banach function space over $\mathbb{R}^n$, and assume that the maximal operator M is bounded on X. Then either $\varphi_X(\lambda)$ is unbounded or $\varphi_X(\lambda) \equiv 1$ for all $\lambda \in (0,1)$.*

The function φ_X is especially useful when dealing with scaled spaces X^p. Indeed, using that $(|f|^r)^*(t) = f^*(t)^r, r > 0$, we obtain

$$\varphi_{X^p}(\lambda) = \inf_{\||f|^{1/p}\|_X=1} \|m_\lambda(|f|^{1/p})\|_X^p = \varphi_X(\lambda)^p, \quad p > 0.$$

Thus, if $\varphi_{X^{p_0}}$ is unbounded for some $p_0 > 0$, then φ_{X^p} is unbounded for all $p > 0$. This fact combined with the above results will be crucial in proving Conjectures 1.1 and 1.2.

In order to prove Conjecture 1.2, we will also essentially use the notion of A_p-regularity of Banach function spaces. This notion was considered by Rutsky [15, 16].

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we prove Theorem 1.3. Conjectures 1.1 and 1.2 are proved in Section 4. We will also give an alternative proof, based on the function φ_X, of a recent result by Lorist and Nieraeth [12] about the boundedness of M on X and X'.

2. Preliminaries

2.1. **Banach function spaces.** Let $L^0(\mathbb{R}^n)$ denote the space of measurable functions on $\mathbb{R}^n$. A vector space $X \subseteq L^0(\mathbb{R}^n)$ equipped with a norm $\|\cdot\|_X$ is called a Banach function space over $\mathbb{R}^n$ if it satisfies the following properties:

- *Ideal property*: If $f \in X$ and $g \in L^0(\mathbb{R}^n)$ with $|g| \le |f|$, then $g \in X$ and $\|g\|_X \le \|f\|_X$.

- *Fatou property*: If $0 \le f_j \uparrow f$ for $\{f_j\}$ in X and $\sup_j \|f_j\|_X < \infty$, then $f \in X$ and $\|f\|_X = \sup_j \|f_j\|_X$.
- *Saturation property*: For every measurable set $E \subset \mathbb{R}^n$ of positive measure, there exists a measurable subset $F \subseteq E$ of positive measure such that $\chi_F \in X$.

We refer to a recent survey by Lorist and Nieraeth [11] about (quasi)-Banach function spaces, where, in particular, one can find a discussion about the above choice of axioms.

The following statement is an equivalent formulation of the Fatou property (see, e.g., [11, Lemma 3.5]).

Proposition 2.1. *Let X be a Banach function space on $\mathbb{R}^n$. Then for every sequence $f_j \in X$,*

$$\|\liminf_{j\to\infty} f_j\|_X \le \liminf_{j\to\infty} \|f_j\|_X.$$

The next statement is also well known.

Proposition 2.2. *Let X be a Banach function space on $\mathbb{R}^n$, and assume that M is bounded on X. Then $\chi_Q \in X$ for every cube $Q \subset \mathbb{R}^n$.*

Proof. Fix a cube Q. By saturation property, there is a set $F \subseteq Q$ of positive measure such that $\chi_F \in X$. Since M is bounded on X,

$$\frac{|F|}{|Q|}\|\chi_Q\|_X \le \|M\chi_F\|_X \le c\|\chi_F\|_X,$$

which implies $\|\chi_Q\|_X < \infty$. □

Given a Banach function space X, we define the associate space (also called the Köthe dual) X' as the space of all $f \in L^0(\mathbb{R}^n)$ such that

$$\|f\|_{X'} := \sup_{\|g\|_X \le 1} \int_{\mathbb{R}^n} |fg| < \infty.$$

By the Lorentz–Luxembourg theorem (see [17, Th. 71.1]), we have $X'' = X$ with equal norms.

Let X be a Banach function space, and let $1 \le p, q \le \infty$. We say that X is p-convex if

$$\|(|f|^p + |g|^p)^{1/p}\|_X \le (\|f\|_X^p + \|g\|_X^p)^{1/p}, \quad f, g \in X,$$

and we say that X is q-concave if

$$(\|f\|_X^q + \|g\|_X^q)^{1/q} \le \|(|f|^q + |g|^q)^{1/q}\|_X, \quad f, g \in X.$$

2.2. A_p **weights.** By a weight we mean a non-negative locally integrable function on $\mathbb{R}^n$. Given a weight w and a measurable set $E \subset \mathbb{R}^n$, denote $w(E) := \int_E w$ and $\langle w\rangle_E := \frac{1}{|E|}\int_w$.

Recall that a weight w satisfies the A_1 condition if

$$[w]_{A_1} := \Big\|\frac{Mw}{w}\Big\|_{L^\infty} < \infty;$$

a weight w satisfies the $A_p, 1 < p < \infty$, condition if

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{-p'/p}\rangle_Q^{p/p'} < \infty;$$

a weight w satisfies the A_∞ condition if

$$[w]_{A_\infty} := \sup_Q \frac{\int_Q M(w\chi_Q)}{w(Q)} < \infty.$$

It was shown in [6] that if $w \in A_\infty$, then for $r := 1 + \frac{1}{c_n[w]_{A_\infty}}$ and for every cube Q,

$$\Big(\frac{1}{|Q|}\int_Q w^r\Big)^{1/r} \le 2\frac{1}{|Q|}\int_Q w.$$

From this, by Hölder's inequality we obtain that for every cube Q and any measurable subset $E \subset Q$,

$$w(E) \le 2\Big(\frac{|E|}{|Q|}\Big)^\delta w(Q), \tag{2.1}$$

where $\delta := \frac{1}{r'} = \frac{1}{1+c_n[w]_{A_\infty}}$.

2.3. A_p**-regular Banach function spaces.** Let X be a Banach function space, and let $1 \le p \le \infty$. We say that X is A_p-regular if there exist $C_1, C_2 > 0$ such that for every $f \in X$ there is an A_p weight $w \ge |f|$ a.e. with $[w]_{A_p} \le C_1$ and $\|w\|_X \le C_2\|f\|_X$.

Proposition 2.3. *A Banach function space X is A_1-regular if and only if M is bounded on X.*

Proof. Indeed, one direction is trivial, namely, if X is A_1-regular, then

$$\|Mf\|_X \le \|Mw\|_X \le C_1\|w\|_X \le C_1C_2\|f\|_X.$$

Conversely, if M is bounded on X, then there exists $r > 1$ depending only on $\|M\|_{X\to X}$ such that M_r is also bounded on X, where $M_rf := M(|f|^r)^{1/r}$ (see, e.g., [10] for the proof of this result). Combining this with the well known fact that $M_rf \in A_1$ [2] with the A_1-constant depending only on r and n, we obtain that X is A_1-regular. ∎

The following result is an abridged version of the characterization obtained by Rutsky [15, Th. 2].

Theorem 2.4 ([15]). *Let X be a Banach function space, and let $1 < p < \infty$. The following statements are equivalent:*

(i) both $X^{1/p}$ and $(X^{1/p})'$ are A_1-regular;

(ii) X' is A_p-regular.

A difficult part of this result is the implication (i) $\Rightarrow$ (ii). We outline a slightly different proof for the sake of completeness. Essentially this is contained in the following result by Rubio de Francia [14, Section 3].

Theorem 2.5 ([14]). *Let X be a Banach function space, and let $p \in (1, \infty)$. Assume that T is a linear operator bounded on $X^{1/p}(\ell^p)$, namely, there exists $C > 0$ such that for every sequence $\{f_j\}$,*

$$\|(\sum_j |Tf_j|^p)^{1/p}\|_{X^{1/p}} \le C\|(\sum_j |f_j|^p)^{1/p}\|_{X^{1/p}}. \tag{2.2}$$

Then for every $f \in X'$, which is positive almost everywhere, there exists a function $w \ge f$ such that $\|w\|_{X'} \le 2\|f\|_{X'}$ and T is bounded on $L^p(w)$ with the operator norm $\|T\|_{L^p(w)\to L^p(w)}$ depending only on C in (2.2).

Now observe that by Proposition 2.3, condition (i) of Theorem 2.4 is equivalent to M being bounded on $X^{1/p}$ and on $(X^{1/p})'$. In turn, by [13, Th. 4.9], this implies that (2.2) holds for every standard Calderón–Zygmund operator T. It remains to choose any nondegenerate Calderón–Zygmund operator T, namely, an operator T for which $\|T\|_{L^p(w)\to L^p(w)} < \infty$ implies $w \in A_p$, and an application of Theorem 2.5 completes the proof.

The following result was also obtained by Rutsky [16, Prop. 7].

Proposition 2.6 ([16]). *Let X be a Banach function space over $\mathbb{R}^n$ such that X is A_p-regular for some $1 \le p < \infty$. Suppose that there exists $\delta > 0$ such that M is bounded on X^δ. Then M is bounded on X.*

We sketch the proof of this statement. By the well known property of the A_p weights [1], if $w \in A_p$, then there exist $\alpha, \beta > 0$ depending only on $[w]_{A_p}$ and p such that for every cube Q,

$$\alpha|Q| < |\{x \in Q : w(x) > \beta\langle w\rangle_Q\}|.$$

From this, by Chebyshev's inequality, for every $\delta > 0$,

$$Mw(x) \le C_{\delta,\alpha,\beta} M_\delta w(x).$$

Hence, taking $f \in X$ and the corresponding $w \in A_p$ from the definition of the A_p-regularity, we obtain

$$\begin{aligned} \|Mf\|_X &\le \|Mw\|_X \le C\|M_\delta w\|_X \\ &= C\|M(w^\delta)\|_{X^\delta}^{1/\delta} \le C\|w\|_X \le C\|f\|_X, \end{aligned}$$

which proves the result.

2.4. **Rearrangements and the maximal operator m_λ.** Denote by $S_0(\mathbb{R}^n)$ the space of measurable functions f on $\mathbb{R}^n$ such that

$$\mu_f(\alpha) := |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < \infty$$

for any $\alpha > 0$. Recall that for $f \in S_0(\mathbb{R}^n)$ the non-increasing rearrangement f^* is defined by

$$f^*(t) := \inf\{\alpha > 0 : \mu_f(\alpha) \le t\}, \quad t > 0.$$

It can be easily seen from the definition of the rearrangement that for every measurable function f and any cube Q,

$$(f\chi_Q)^*(\lambda|Q|) > \alpha \Leftrightarrow |Q \cap \{|f| > \alpha\}| > \lambda|Q|, \quad \alpha > 0, \lambda \in (0,1).$$

From this we have that

$$\{x \in \mathbb{R}^n : m_\lambda f(x) > \alpha\} = \{x \in \mathbb{R}^n : M\chi_{\{|f|>\alpha\}}(x) > \lambda\}, \tag{2.3}$$

and, for every measurable set $E \subset \mathbb{R}^n$,

$$m_\lambda(\chi_E)(x) = \chi_{\{M\chi_E > \lambda\}}(x). \tag{2.4}$$

3. Proof of Theorem 1.3

We start with part (i) of Theorem 1.3, and we will show that it even holds under a weaker assumption, namely, instead of assuming that M is bounded on X (which, by Proposition 2.3, says that X is A_1-regular) it suffices to assume the A_∞-regularity of X. This is a simple corollary of the following lemma.

Lemma 3.1. *Let $w \in A_\infty$. Then for all $f \in L^1(w)$ and $\lambda \in (0,1)$,*

$$\int_{\mathbb{R}^n} |f|w \le 2^{n+1}\lambda^\delta \int_{\mathbb{R}^n} (m_\lambda f)w, \tag{3.1}$$

where $\delta := \frac{1}{1+c_n[w]_{A_\infty}}$.

Proof. Observe that (3.1) is equivalent to the same inequality but with $f = \chi_E$, where E is an arbitrary measurable set of finite measure. Indeed, assume that (3.1) is true for $f = \chi_E$. By (2.4), this means that

$$w(E) \le 2^{n+1}\lambda^\delta w\{x : M\chi_E > \lambda\}. \tag{3.2}$$

Taking here $E := \{x : |f| > \alpha\}$ and applying (2.3), we obtain

$$w(\{x : |f| > \alpha\}) \le 2^{n+1}\lambda^\delta w(\{x : m_\lambda f > \alpha\}),$$

which, in turn, implies (3.1) by integrating over $\alpha \in (0,\infty)$ (in order to ensure that the set $\{x : |f| > \alpha\}$ is of finite measure, one can assume first that f is compactly supported and then use the monotone convergence theorem).

Hence, it suffices to prove (3.2). Let M^d denote the dyadic maximal operator. By the Calderón–Zygmund decomposition, the set $\{x : M^d\chi_E > \lambda\}$ can be written as the union of pairwise disjoint cubes Q_j satisfying

$$\lambda < \frac{|Q_j \cap E|}{|Q_j|} \le 2^n\lambda.$$

From this, by (2.1),

$$w(Q_j \cap E) \le 2^{1+n\delta}\lambda^\delta w(Q_j) \le 2^{n+1}\lambda^\delta w(Q_j).$$

Summing up this inequality yields (3.2), and therefore the proof is complete. □

Proof of Theorem 1.3, part (i). We will show that if X is A_∞-regular, then there exist $C, \delta > 0$ such that $\varphi_{X'}(\lambda) \ge C\lambda^{-\delta}$ for all $\lambda \in (0, 1)$.

Given $g \in X$, take an A_∞ weight w such that $|g| \le w$ a.e., where $[w]_{A_\infty} \le C_1$ and $\|w\|_X \le C_2\|g\|_X$. By Lemma 3.1, there exists $\delta > 0$ (one can take $\delta := \frac{1}{1+c_nC_1}$) such that

$$\begin{aligned}\int_{\mathbb{R}^n} |fg| &\le \int_{\mathbb{R}^n} |f|w \le 2^{n+1}\lambda^\delta \int_{\mathbb{R}^n} (m_\lambda f)w \\ &\le 2^{n+1}\lambda^\delta \|m_\lambda f\|_{X'}\|w\|_X \le 2^{n+1}C_2\lambda^\delta \|m_\lambda f\|_{X'}\|g\|_X.\end{aligned}$$

From this, taking the supremum over all $g \in X$ with $\|g\|_X = 1$ yields

$$\|f\|_{X'} \le 2^{n+1}C_2\lambda^\delta \|m_\lambda f\|_{X'}.$$

Hence, $\varphi_{X'}(\lambda) \ge \frac{1}{2^{n+1}C_2}\lambda^{-\delta}$, and the proof is complete. □

Turn to part (ii) of Theorem 1.3. This part is a simple combination of several known results, which we will describe below.

Define the Fefferman–Stein sharp function $f^\#$ by

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f - \langle f\rangle_Q|,$$

where the supremum is taken over all cubes Q containing the point x. It was proved by Fefferman and Stein [4] that for every $p > 1$ and for all $f \in S_0(\mathbb{R}^n)$,

$$\|f\|_{L^p} \le C_{n,p}\|f^\#\|_{L^p}.$$

Having this result in mind, we say that a Banach function space X has the Fefferman–Stein property if there exists $C > 0$ such that

$$\|f\|_X \le C\|f^\#\|_X$$

for all $f \in S_0(\mathbb{R}^n)$.

The following characterization was obtained in [9, Cor. 4.3].

Theorem 3.2 ([9]). *Let X be a Banach function space over $\mathbb{R}^n$, and assume that M is bounded on X. Then M is bounded on X' if and only if X has the Fefferman–Stein property.*

Further, we will use the following pointwise estimate obtained in [7, Th. 2].

Theorem 3.3 ([7]). *For any locally integrable function f and for all $x \in \mathbb{R}^n$,*

$$m_\lambda(Mf)(x) \le C_{n,\lambda} f^\#(x) + Mf(x).$$

Now, the second part of Theorem 1.3, modulo some technicalities, is just a combination of two above results.

Proof of Theorem 1.3, part (ii). The proof is almost identical to the proof of a similar result proved in [9, Th. 4.1].

Our goal is to show that if $\varphi_X(\lambda_0) > 1$ for some $\lambda_0 \in (0,1)$, then X has the Fefferman–Stein property. Thus, by Theorem 3.2, we would obtain that M is bounded on X'.

Since $(|f|)^\#(x) \le 2f^\#(x)$ (see, e.g., [5, p. 155]), we will assume that $f \ge 0$. By Theorem 3.3,

$$\varphi_X(\lambda_0)\|Mf\|_X \le \|m_{\lambda_0}(Mf)\|_X \le C_{n,\lambda_0}\|f^\#\|_X + \|Mf\|_X.$$

Assuming that $f \in X$, and using that M is bounded on X, we obtain that $\|Mf\|_X < \infty$. Therefore, by the above estimate,

$$\|f\|_X \le \|Mf\|_X \le \frac{C_{n,\lambda_0}}{\varphi_X(\lambda_0) - 1}\|f^\#\|_X. \tag{3.3}$$

Now, in order to show that X has the Fefferman–Stein property, it remains to extend (3.3) from $f \in X$ to $f \in S_0(\mathbb{R}^n)$. Assume first that $f \in S_0(\mathbb{R}^n) \cap L^\infty$. We will use the fact proved in [9, Lemma 4.5] and saying that there is a sequence $\{f_j\}$ of bounded and compactly supported functions such that $f_j \to f$ a.e. and $(f_j)^\#(x) \le c_n f^\#(x)$.

Observe that, by Proposition 2.2, each f_j belongs to X. Therefore, by (3.3),

$$\|f_j\|_X \le C\|(f_j)^\#\|_X \le C'\|f^\#\|_X.$$

From this, by Proposition 2.1,

$$\|f\|_X \le C'\|f^\#\|_X.$$

It remains to extend this estimate from $f \in S_0(\mathbb{R}^n) \cap L^\infty$ to $f \in S_0(\mathbb{R}^n)$. For $f \in S_0(\mathbb{R}^n)$ and $N > 0$ define $f_N := \min(f, N)$. Then

$f_N \in S_0(\mathbb{R}^n) \cap L^\infty$. Using that $(f_N)^\#(x) \le \frac{3}{2} f^\#(x)$ (see [5, p. 155]) and applying the previous estimate, we obtain

$$\|f_N\|_X \le C\|f^\#\|_X.$$

Applying again Proposition 2.1 proves the Fefferman–Stein property of X, and therefore the proof is complete. $\square$

4. Proof of Conjectures 1.1 and 1.2

We start with the following statement which collects some standard properties related to the boundedness of M on a Banach function space X.

Proposition 4.1. *Let X be a Banach function space, and assume that M is bounded on X. Then*

(i) M is bounded on X^r for all $r \in (0,1)$;
(ii) there exists $r > 1$ such that M is bounded on X^r;
(iii) M is bounded on $(X')^r$ for all $r \in (0,1)$.

Proof. Observe that M is bounded on X^r if and only if M_r is bounded on X. Therefore, part (i) follows trivially by Hölder's inequality. In turn, part (ii) is just a reformulation of the result [10] saying that if M is bounded on X, then there exists $r > 1$ depending only on $\|M\|_{X\to X}$ such that M_r is also bounded on X.

Part (iii) is an immediate consequence of the Fefferman–Stein inequality [3] saying that for all locally integrable f, g and for all $p > 1$,

$$\int_{\mathbb{R}^n} (Mf)^p |g| dx \le C_{n,p} \int_{\mathbb{R}^n} |f|^p (Mg) dx.$$

From this

$$\int_{\mathbb{R}^n} (Mf)^p |g| dx \le C_{n,p} \||f|^p\|_{X'} \|Mg\|_X \le C\||f|^p\|_{X'} \|g\|_X.$$

Hence, by duality, for all $p > 1$,

$$\|(Mf)^p\|_{X'} \le C\||f|^p\|_{X'},$$

which finishes the proof. $\square$

As we mentioned in the Introduction, the implications (i) $\Rightarrow$ (ii) are known for both Conjectures 1.1 and 1.2. For the sake of the completeness we give different proofs based on Theorem 1.3.

Proof of Conjecture 1.1. Observe that since X is r-convex and s-concave, the space $X_{r,s}$ is a Banach function space (see [13]). Therefore, by the Lorentz–Luxembourg theorem, $\big[(X^r)'\big]^{(\frac{s}{r})'} = X'_{r,s}$.

Let us start with the implication (i) $\Rightarrow$ (ii). Since M is bounded on $X_{r,s}$, by Theorem 1.3, the function $\varphi_{\left[(X^r)'\right]^{(\frac{s}{r})'}}$ is unbounded. Hence, $\varphi_{(X^r)'}$ is unbounded as well. Next, since M is bounded on $\left[(X^r)'\right]^{(\frac{s}{r})'}$, by the first part of Proposition 4.1, M is bounded on $(X^r)'$. This along with unboundedness of $\varphi_{(X^r)'}$ implies, by Theorem 1.4, that M is bounded on $(X^r)'' = X^r$.

Turn to (ii) $\Rightarrow$ (i). Since M is bounded on X^r, by Theorem 1.3, $\varphi_{(X^r)'}$ is unbounded. Hence, $\varphi_{\left[(X^r)'\right]^{(\frac{s}{r})'}}$ is unbounded as well. This coupled with the boundedness of M on $\left[(X^r)'\right]^{(\frac{s}{r})'}$ implies, by Theorem 1.4, that M is bounded on $X_{r,s}$. □

Proof of Conjecture 1.2. Let us start with the implication (i) $\Rightarrow$ (ii). In the previous proof we showed that if M is bounded on $X_{r,s}$ and $X'_{r,s}$, then M is bounded on X^r. Using the fact that $X'_{r,s} = (X')_{s',r'}$ [13, Pr. 2.14], in a similar way we obtain that M is bounded on $(X')^{s'}$.

Turn to (ii) $\Rightarrow$ (i). Since M is bounded on $(X')^{s'}$, by the first part of Proposition 4.1, M is bounded on X'. Hence, by Theorem 1.3, φ_X is unbounded, and so φ_{X^r} is unbounded as well. This coupled with the boundedness of M on X^r implies, by Theorem 1.4, that M is bounded on $(X^r)'$. In a similar way we obtain that M is bounded on $[(X')^{s'}]'$.

Setting $Y := \left[(X^r)'\right]^{(\frac{s}{r})'}$ and $q := (\frac{s}{r})'$, we obtain that $Y^{1/q}$ and $(Y^{1/q})'$ are A_1-regular. Therefore, by Theorem 2.4, $Y' = X_{r,s}$ is A_q-regular. Further, it was shown in the proof of [13, Pr. 2.14] that $X^{\theta}_{r,s} = [(X')^{s'}]'$ for $\theta := \frac{1}{(r'/s')'}$. Hence, M is bounded on $X^{\theta}_{r,s}$, which, along with the A_q-regularity of $X_{r,s}$, proves, by Proposition 2.6, that M is bounded on $X_{r,s}$. By symmetry, using that $X'_{r,s} = (X')_{s',r'}$ and applying the same argument, we obtain that M is bounded on $X'_{r,s}$, and, therefore, the proof is complete. □

In order to further illustrate the method based on Theorem 1.3, we give an alternative proof of the following recent result of Lorist and Nieraeth [12].

Theorem 4.2 ([12])**.** *Let $r^* \in (1, \infty)$ and let X be an r^*-convex Banach function space over $\mathbb{R}^n$. Then the following are equivalent:*

(i) We have $M : X \to X$ and $M : X' \to X'$;
(ii) There is an $r_0 \in (1, r^*]$ so that for all $r \in (1, r_0)$ we have

$$M : X^r \to X^r, \quad M : (X^r)' \to (X^r)';$$

(iii) There is an $r \in (1, r^*]$ so that $M : (X^r)' \to (X^r)'$.

Proof. We start with (i) $\Rightarrow$ (ii). By a combination of the first two parts of Proposition 4.1, there is an $r_0 \in (1, r^*]$ so that for all $r \in (1, r_0)$ we have $M : X^r \to X^r$. Next, by Theorem 1.3, the function φ_X is unbounded. Hence, φ_{X^r} is unbounded as well. Therefore, applying Theorem 1.4, we have $M : (X^r)' \to (X^r)'$. Next, the implication (ii) $\Rightarrow$ (iii) is trivial.

Turn to (iii) $\Rightarrow$ (i). By Theorem 1.3, φ_{X^r} is unbounded. Hence, φ_X is unbounded as well. Further, by the third part of Proposition 4.1, $M : (X^r)' \to (X^r)'$ implies $M : X \to X$. It remains to apply Theorem 1.4 in order to conclude that $M : X' \to X'$. $\square$

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